LOWER BOUNDS FOR POWER MOMENTS OF L-FUNCTIONS

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ABSTRACT. Let π be an irreducible unitary cuspidal representation of $\mathrm{GL}_d(\mathbb{Q}_{\mathbb{A}})$. Let $L(\pi, s)$ be the L-function attached to π. For real $k \geq 0$ let

$$I_k(\pi, T) = \int_1^T |L(\pi, 1/2 + it)|^{2k} \, dt$$

be the k-th (power) moment of $L(\pi, s)$. Let $\alpha_\pi(p, j) \in \mathbb{C}$ ($1 \leq j \leq d$) be the local parameters at prime p. We prove that if

$$|\alpha_\pi(p, j)| \leq p^\gamma$$

for unramified primes and for a fixed $0 \leq \gamma < 1/4$, then

$$I_k(\pi, T) \gg T (\log T)^{k^2}$$

for any rational $k \geq 0$. As a corollary of this result we establish unconditional lower bounds of the conjectured order of magnitude for the fractional k-th moments of Dirichlet L-functions, modular L-functions, twisted modular L-functions, and Maass L-functions. We derive our results as corollaries of more general theorems related to the lower bounds for fractional moments of analytic functions which have Dirichlet series representations on a complex half-plane. We also establish lower bounds for the k-th moments of Artin L-functions and Dedekind zeta functions of number fields.

1. INTRODUCTION

Let $s = \sigma + it$ denote a point in the complex plane. Let $g(s) = \sum_{n=1}^\infty \frac{a_g(n)}{n^s}$ be a Dirichlet series absolutely convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$. For fixed $\sigma \geq 1/2$ and non-negative real k, we define the k-th (power) moment of g at σ as

$$I_k(g, \sigma, T) = \int_1^T |g(\sigma + it)|^{2k} \, dt.$$

For $\sigma = 1/2$, we denote $I_k(g, 1/2, T)$ by $I_k(g, T)$ and we call it the k-th (power) moment of g. We are interested in studying the behaviour of $I_k(g, \sigma, T)$ as $T \to \infty$.

If $(g(s))^k$ has a representation on $\sigma > 1$ as an absolutely convergent Dirichlet series $\sum_{n=1}^\infty \frac{g_k(n)}{n^s}$, then from the mean value theorem for Dirichlet series [19, Theorem 7.1] we know that

$$(1.1) \qquad I_k(g, \sigma, T) \sim T \sum_{n=1}^\infty \frac{|g_k(n)|^2}{n^{2\sigma}},$$

for $\sigma > 1$, as $T \to \infty$. Very little is known about the behaviour of $I_k(g, \sigma, T)$ for $1/2 \leq \sigma \leq 1$. However for several classes of Dirichlet series arising in number

theory it is conjectured that (1.1) holds for $1/2 < \sigma \leq 1$. Another interesting feature is the close connection of the behaviour of $I_k(g, \sigma, T)$ with the size of $I_k(g, T)$. In fact for certain g it is known that if $I_k(g, T) \ll T^{1+\epsilon}$ for any $\epsilon > 0$, then (1.1) holds for $\sigma > 1/2$ (see [18, Section 9.51]).

In this paper, we are interested in finding lower bounds of $I_k(g, T)$ in terms of T for certain number theoretical L-functions g which have Dirichlet series representations on the half plane $\sigma > 1$. The special case when g is the Riemann zeta function $\zeta(s)$ is classical and has been under investigation for a long time. For $k = 1$ and 2 the asymptotic formulas are known. One of the first results on this subject which treats all integral values of k is due to Titchmarsh [19, Theorem 7.19], who showed that

$$\int_o^\infty \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} e^{-t/T} dt \gg T(\log T)^{k^2}.$$

The first general lower bound for the k-th moment of $\zeta(s)$ was given by Ramachandra. In [14, Theorem 1] he proved that

$$I_k(\zeta, T) \gg T(\log T)^{k^2} (\log \log T)^{-c_k}$$

for real $k \geq 1/2$, where c_k is a constant depending possibly on k. Moreover he proved that under the assumption of the Riemann Hypothesis this lower bound holds for any real $k \geq 0$. Later in [15] Ramachandra studied the general problem of finding lower bounds for the moments of Dirichlet series and in this direction obtained several important results. One of his results is the following.

Theorem 1.1 (Ramachandra). *Let $F(s)$ be a Dirichlet series convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$, and in this region $|F(s)| \leq \exp((\log t)^{c_0})$ where c_0 is a fixed positive constant. Moreover assume that $F(s)$ has a representation in the form*

$$F(s) = \left(\sum_{n=1}^\infty \frac{a(n)}{n^s} \right)^2$$

for $\sigma > 1$, where for $1/2 < \sigma < 1$ the series $\sum_{n=1}^\infty \frac{|a(n)|^2}{n^{2\sigma}}$ converges and

$$\frac{1}{(\sigma - 1/2)^\alpha} \ll \sum_{n=1}^\infty \frac{|a(n)|^2}{n^{2\sigma}} \ll \frac{1}{(\sigma - 1/2)^\alpha}$$

for some positive constant α. Then

$$I_{1/2}(F, T) \gg T(\log T)^\alpha.$$

Proof. This is a special case of [15, Theorem 2]. $\qquad\square$

Letting $F(s) = ((g(s))^{k/2})^2 = (g(s))^k$ in Ramachandra's theorem we get the following.

Corollary 1.2. *Let $g(s)$ be a Dirichlet series convergent for $\sigma > 1$ that can be continued analytically to the region $\sigma \geq 1/2$, $t \geq 1$, and in this region $|g(s)| \leq \exp((\log t)^{c_0})$ where c_0 is a fixed positive constant. Let $k \geq 0$ be an integer. Suppose that $(g(s))^{k/2}$ in $\sigma > 1$ can be represented by a Dirichlet series*

$$(g(s))^{k/2} = \sum_{n=1}^\infty \frac{g_{k/2}(n)}{n^s}.$$

Moreover suppose that for $1/2 < \sigma < 1$ the series $\sum_{n=1}^{\infty} \frac{|g_{k/2}(n)|^2}{n^{2\sigma}}$ converges and

$$\frac{1}{(\sigma - 1/2)^\alpha} \ll \sum_{n=1}^{\infty} \frac{|g_{k/2}(n)|^2}{n^{2\sigma}} \ll \frac{1}{(\sigma - 1/2)^\alpha}$$

for some positive constant α. Then

$$I_{k/2}(g, T) \gg T(\log T)^\alpha.$$

Also, if the above conditions hold for real $k \geq 0$ and in addition $g(s)$ does not have any zero in the region $\sigma > 1/2$, $t \geq 1$, then

$$I_k(g, T) \gg T(\log T)^\alpha,$$

for any non-negative real number k.

A special case of the above corollary is the following.

Corollary 1.3 (Ramachandra). *For the Riemann zeta function $\zeta(s)$, and for a non-negative integer k we have*

$$I_{k/2}(\zeta, T) \gg T(\log T)^{k^2/4}.$$

Moreover, under the assumption of the Riemann hypothesis, for non-negative real k we have

$$I_k(\zeta, T) \gg T(\log T)^{k^2}.$$

In [4], Heath-Brown extended the above result to the fractional moments of the Riemann zeta function.

Theorem 1.4 (Heath-Brown). *For the Riemann zeta function $\zeta(s)$, and for a non-negative rational number k we have*

$$I_k(\zeta, T) \gg T(\log T)^{k^2}.$$

Under the assumption of the Riemann hypothesis, Heath-Brown's method also establishes the same lower bound for any non-negative real k.

In recent years several authors have employed the Heath-Brown method to study the fractional moments of other L-functions. Let χ be a primitive Dirichlet character mod q and let $I_k(\chi, T)$ denote the k-th moment of the Dirichlet L-function $L(\chi, s)$. Then Kacenas, Laurincikas, and Zamarys [6] proved that for $k = 1/n$, $n \geq 2$, we have

$$I_k(\chi, T) \gg T(\log T)^{k^2}.$$

In [10] Laurincikas and Steuding proved that if f is a holomorphic cusp form of weight ℓ and level 1, and $I_k(f, T)$ is the k-th moment of the modular L-function attached to f, then for $k = 1/n$, $n \geq 2$ one has

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

Zamarys [20] has generalized this result to the case of the k-th moment of twisted modular L-functions for positive rational $k \leq 1/2$.

The proofs of the above results closely follow [4]. Our first goal in this paper is to follow the method of Heath-Brown to derive a general theorem in the spirit of Corollary 1.2. More specifically we prove the following:

Theorem 1.5. *Let* $g(s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s}$ *be a Dirichlet series absolutely convergent for* $\sigma > 1$ *that has an analytic continuation to the half plane* $\sigma \geq 1/2$ *with a pole of degree* m *at* $s = 1$ *(with* m *possibly equal to* 0*) and suppose that*

$$(1.2) \qquad\qquad g(\sigma + it) \ll |t|^A$$

for a fixed $A > 0$, $\sigma \geq 1/2$ *and* $|t| \geq 1$ *(the implied constant is independent of* σ *and* t*). For rational* $k \geq 0$ *and* $\sigma > 1$, *we suppose that* $(g(s))^k$ *may be represented by an absolutely convergent Dirichlet series*

$$(g(s))^k = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^s},$$

where

$$(1.3) \qquad\qquad g_k(n) \ll n^\eta$$

for some fixed η *with* $0 \leq \eta < 1/2$. *In addition, assume that there exist fixed constants* $c > 0$ *and* $\alpha \geq 0$ *such that*

$$(1.4) \qquad\qquad \frac{1}{(\sigma - 1/2)^\alpha} \ll \sum_{n=1}^{T} \frac{|g_k(n)|^2}{n^{2\sigma}} \ll (\log T)^\alpha$$

uniformly for $\frac{1}{2} + \frac{c}{\log T} \leq \sigma \leq 1$ *and* $T \geq 2$. *Then for any rational* $k \geq 0$ *we have*

$$(1.5) \qquad\qquad I_k(g, T) \gg T(\log T)^\alpha.$$

Moreover if the above conditions (1.3) *and* (1.4) *hold for real* $k \geq 0$ *and in addition* $g(s)$ *does not have any zeros in the half plane* $\sigma > 1/2$ *then* (1.5) *holds for any non-negative real* k.

We next employ the above theorem to establish lower bounds for moments of certain number theoretical L-functions. The L-functions we consider include the following:

(a) *Principal automorphic L-functions*

Let π be a principal automorphic L-functions whose local parameters $\alpha_\pi(p, j)$ at unramified primes satisfy

$$|\alpha_\pi(p, j)| \leq p^\gamma,$$

for a fixed $0 \leq \gamma < 1/4$ (see Section 2 for terminology). Then, for rational k, we prove that

$$I_k(\pi, T) \gg T(\log T)^{k^2}.$$

As a corollary of this general result we deduce unconditional lower bounds of the conjectured order of magnitude for the fractional moments of classical Dirichlet L-functions, modular L-functions, and the twisted modular L-functions. Our results generalize the lower bound results of Kacenas-Laurincikas-Zamarys [6] and Laurincikas-Steuding [10] to rational values of $k \geq 0$, and the lower bound result of Zamarys [20] to rational values of $k \geq 1/2$.

Moreover, we derive an unconditional lower bound for the fractional k-th moment of Maass L-functions. Let $L(f, s)$ be a Maass cusp newform of weight zero and level N with nebentypus ψ. Then for any rational $k \geq 0$ we prove that

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

To our knowledge this result is the first established unconditional lower bound for the k-th moment of an L-function for which the truth of the Generalized Ramanujan Conjecture (GRC) (see Section 2 for terminology) is not known.

Remark 1.6. Ji [5] has established lower bounds for k-th moments of principal automorphic L-functions over short intervals. As a consequence of [5, Theorem 1.1], for real non-negative k, one has

$$I_k(\pi, \sigma, T) \gg T$$

uniformly for $\sigma \geq 1/2$. Our results give conditional improvements of this lower bound for $\sigma = 1/2$.

(b) *Artin L-functions*

Under the assumption of the Artin holomorphy conjecture we deduce a lower bound for the fractional k-th moment $I_k(K/\mathbb{Q}, \rho, T)$ of the Artin L-function $L(K/\mathbb{Q}, \rho, s)$ associated to a representation ρ of the Galois group of a number field $K/\mathbb{Q}$. We prove that for any non-negative rational k,

$$I_k(K/\mathbb{Q}, \rho, T) \gg T(\log T)^{\langle \varphi, \varphi \rangle k^2},$$

where φ is the character associated to ρ and $\langle \varphi, \varphi \rangle$ is the inner product of φ and φ as defined in Proposition 2.11.

As a direct corollary of this result we derive an unconditional lower bound for the fractional k-th moment $I_k(\zeta_K, T)$ of the Dedekind zeta function $\zeta_K(s)$ of a number field K. More precisely, for non-negative rational k we prove that

$$I_k(\zeta_K, T) \gg T(\log T)^{nk^2},$$

where n is the degree of the Galois extension $K/\mathbb{Q}$.

Our result improves the following theorem of Ramachandra [14, Theorem 2] for rational values of k.

Theorem 1.7 (Ramachandra). *Let K be a degree n Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the Dedekind zeta function of K. Then for real $k \geq 1/2$ we have*

$$I_k(\zeta_K, T) \gg T\frac{(\log T)^{nk^2}}{(\log \log T)^c}$$

where c is a positive constant.

Remarks 1.8. (a) The Generalized Riemann Hypothesis (GRH) for an L-function implies that the L-function does not have any zeros in the half plane $\sigma > 1/2$. Thus by Theorem 1.5, upon the assumption of GRH for an L-function g, the result for the fractional k-th moments of g can be extended to non-negative real values of k as long as (1.3) and (1.4) hold for $(g(s))^k$.

(b) In 2005, Rudnick and Soundararajan [17] devised a method that establishes lower bounds of the conjectured order of magnitude for integral k-th moments of several families of L-functions. They also commented that their method generalizes to the fractional k-th moments for $k \geq 1$. This procedure is also applicable to the integral k-th (power) moment of the Riemann zeta function. Using this method, Chandee [1, Theorem 1.2] has established a lower bound of the conjectured order of magnitude for the integral shifted (power) moment of the Riemann zeta function.

(c) In [4], Heath-Brown also considered the upper bound of the k-th (power) moment of the Riemann zeta function, and by the method similar to the one used in establishing lower bound, he proved that

$$I_k(\zeta, T) \ll T(\log T)^{k^2},$$

unconditionally for $k = 1/n > 0$, where n is an integer, and under the assumption of the Riemann Hypothesis for all real values of $0 \leq k \leq 2$. It would be possible

to generalize this method to obtain upper bounds for the k-th (power) moments of general L-functions. However, other than the case of the Dirichlet L-functions, the results would be conditional upon GRH and only true for some specific values of $k < 1$. See [10, Theorem 1] for an example of such a result.

The structure of the paper is as follows. In Section 2 we describe some applications of Theorem 1.5. To do this we first observe that if the sequence $\{|a_g(p)|^2\}$ has regular distribution then the technical condition (1.4) holds. We prove this result in Section 3. Using this observation in Section 2, we establish a new version (Theorem 2.3) of our main Theorem. Our results on lower bounds for moments of principal automorphic L-functions, Artin L-functions and Dedekind zeta functions are simple consequences of Theorem 2.3. Finally in Sections 4 and 5 we prove Theorem 1.5.

Notation 1.9. $s = \sigma + it$ denotes a point in the complex plane and p denotes a prime number. We use the notation $\ll$, O(.), and $\sim$ with their usual meaning in analytic number theory. All constants implied by the $\ll$ notation may depend on k and the function g. However, the constants are independent of the variables σ and T. By abuse of notation and for simplicity we denote the k-th moment of an L-function $L(f, s)$ by $I_k(f, T)$ instead of $I_k(L(f, .), T)$.

2. Applications of Theorem 1.5

We first identify some classes of Dirichlet series for which the technical condition (1.4) holds. To do this we employ a classical theorem on the average values of multiplicative functions and derive the following proposition.

Proposition 2.1. *Let $h(n)$ be a non-negative multiplicative function. Suppose that for some fixed real $\alpha > 0$ we have*

$$\sum_{p \leq x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$, and

$$\sum_{p \leq x} \frac{h(p)}{p} = \alpha \log \log x + \mathrm{O}(1).$$

Suppose also that

$$h(p^r) \ll (2p)^{r\theta} \text{ for some } 0 \leq \theta < 1/2 \text{ and for all } r \geq 2.$$

Let $T \geq 2$ and let $0 < c \leq \frac{1}{2}\log 2$ be fixed. Then we have

$$\frac{1}{(\sigma - 1/2)^\alpha} \ll \sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} \ll (\log T)^\alpha$$

uniformly in the range $\frac{1}{2} + \frac{c}{\log T} \leq \sigma \leq 1$.

The preceding proposition is proved in Section 3.

As a consequence of this proposition we prove a version of Theorem 1.5 for Dirichlet series with certain Euler products.

To simplify our exposition, from now on we consider the class $\mathcal{C}$ of Dirichlet series $g(s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s}$ that are absolutely convergent for $\sigma > 1$ and satisfy the following:

Except for a pole of degree m at $s = 1$ (with m possibly equal to 0), $g(s)$ extends to an entire function for $\sigma \geq 1/2$ and $|g(\sigma + it)| \ll |t|^A$ for a fixed $A > 0$, $\sigma \geq 1/2$ and $|t| \geq 1$.

For any real k we define the multiplicative function $\tau_k(.)$ by

$$\tau_k(p^r) = \frac{\Gamma(k+r)}{r!\Gamma(k)} = \frac{k(k+1)(k+2)\cdots(k+r-1)}{r!}.$$

Note that $\tau_k(n) \geq 0$ when $k \geq 0$. The following lemma summarizes some properties of $\tau_k(n)$.

Lemma 2.2. (i) *For fixed $k \geq 0$ and $\epsilon > 0$ we have $\tau_k(n) \ll n^\epsilon$.*
(ii) *If j is a positive integer then*

$$\tau_{kj}(n) = \sum_{n=n_1 n_2 \cdots n_j} \tau_k(n_1)\tau_k(n_2)\cdots\tau_k(n_j).$$

Proof. See [4, Lemma 1] $\qquad\square$

Theorem 2.3. *Let $g(s) \in \mathcal{C}$. Suppose $g(s)$ has an Euler product of the form*

$$(2.1) \qquad g(s) = \sum_{n=1}^{\infty} \frac{a_g(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_g(p,j)}{p^s}\right)^{-1}$$

for $\sigma > 1$, where $\alpha_g(p,j) \in \mathbb{C}$ and

$$|\alpha_g(p,j)| \leq (2p)^\gamma, \text{ for a fixed } 0 \leq \gamma < 1/4.$$

Suppose we have

$$(2.2) \qquad \sum_{p \leq x} |a_g(p)|^2 \sim \beta \frac{x}{\log x}$$

as $x \to \infty$, and

$$(2.3) \qquad \sum_{p \leq x} \frac{|a_g(p)|^2}{p} = \beta \log\log x + O(1)$$

for some fixed constant β, where $a_g(p) = \sum_{j=1}^{d} \alpha_g(p,j)$. Then for any rational $k \geq 0$ we have

$$I_k(g,T) \gg T \left(\log T\right)^{\beta k^2}.$$

Proof. Since $g(s)$ is represented by an absolutely convergent Euler product and the Euler factors are nonvanishing on $\sigma > 1$, we have $g(s) \neq 0$ for $\sigma > 1$. Letting

$$b_g(p^r) = \sum_{j=1}^{d} \alpha_g(p,j)^r,$$

for $\sigma > 3/2$ we define a branch of $\log g(s)$ by

$$\log g(s) = \sum_p \sum_{r=1}^{\infty} \frac{b_g(p^r)}{rp^{rs}},$$

and a branch of $g^k(s)$ by

$$g^k(s) = \exp(k \log g(s))$$

for any real k. For $\sigma > 3/2$ we have

$$g^k(s) = \exp\left(k \sum_p \sum_{r=1}^{\infty} \frac{b_g(p^r)}{rp^{rs}}\right)$$

$$= \prod_p \exp\left(-k \sum_{j=1}^{d} \log\left(1 - \frac{\alpha_g(p,j)}{p^s}\right)\right)$$

$$= \prod_p \left(1 - \frac{\alpha_g(p,1)}{p^s}\right)^{-k} \cdots \left(1 - \frac{\alpha_g(p,d)}{p^s}\right)^{-k}.$$

Using the Taylor expansion

$$(1-z)^{-k} = \sum_{r=0}^{\infty} \frac{\Gamma(k+r)}{r!\Gamma(k)} z^r$$

on the region $|z| < 1$ we have

$$g^k(s) = \prod_p \prod_{j=1}^{d} \left(\sum_{r=0}^{\infty} \tau_k(p^r) \frac{\alpha_g(p,j)^r}{p^{rs}}\right)$$

$$= \prod_p \sum_{r=0}^{\infty} \frac{g_k(p^r)}{p^{rs}} = \sum_{n=1}^{\infty} \frac{g_k(n)}{n^s}.$$

Here $g_k(n)$ is the multiplicative function given by

$$g_k(p^r) = \sum_{p^r = p^{j_1} \cdots p^{j_d}} \tau_k(p^{j_1}) \alpha_g(p,1)^{j_1} \tau_k(p^{j_2}) \alpha_g(p,2)^{j_2} \cdots \tau_k(p^{j_d}) \alpha_g(p,d)^{j_d}.$$

By employing $|\alpha_g(p,j)| \leq (2p)^{\gamma}$ and Lemma 2.2 we have

$$(2.4) \quad |g_k(p^r)| \leq (2p)^{r\gamma} \sum_{p^r = p^{j_1} \cdots p^{j_d}} \tau_k(p^{j_1}) \cdots \tau_k(p^{j_d}) = (2p)^{r\gamma} \tau_{kd}(p^r) \ll (2p)^{r(\gamma+\epsilon)},$$

for any $\epsilon > 0$. Applying this bound we have

$$\sum_p \sum_{r=2}^{\infty} \frac{|g_k(p^r)|}{p^{r\sigma}} < \infty,$$

for $\sigma > 1$. On the other hand

$$\sum_p \frac{|g_k(p)|}{p^{\sigma}} = k \sum_p \frac{|a_g(p)|}{p^{\sigma}} \leq kg(\sigma) < \infty,$$

for $\sigma > 1$. Thus $\sum_{n=1}^{\infty} \frac{g_k(n)}{n^s}$ is absolutely convergent for $\sigma > 1$, which together with (2.4) implies that (1.3) holds.

It remains to show that (1.4) holds with $\alpha = \beta k^2$. Let $h(n) = |g_k(n)|^2$, so that $h(p) = k^2 |a_g(p)|^2$. We note that (2.2), (2.3), and (2.4) show that the conditions of Proposition 2.1 hold for $\alpha = \beta k^2$, and thus we may use Proposition 2.1 to see that (1.4) holds.

Therefore for rational k Theorem 1.5 holds with $\alpha = \beta k^2$, which completes the proof of the theorem. □

Next we describe several applications of Theorem 2.3 in the cases of automorphic L-functions, Artin L-functions, and Dedekind zeta functions.

Principal Automorphic L-functions

Let π be an irreducible unitary cuspidal representation of $\mathrm{GL}_d(\mathbb{Q}_{\mathbb{A}})$. The global L-function at finite places attached to π is given by the Euler product of local factors for $\sigma > 1$:

$$(2.5) \qquad L(\pi, s) = \sum_{n=1}^{\infty} \frac{a_{\pi}(n)}{n^s} = \prod_p L_p(\pi_p, s),$$

where

$$L_p(\pi_p, s) = \prod_{j=1}^{d} \left(1 - \frac{\alpha_{\pi}(p, j)}{p^s} \right)^{-1},$$

and $\alpha_{\pi}(p, j) \in \mathbb{C}$ ($1 \leq j \leq d$) are the local parameters at prime p. It is known that ([16, Formula (2.3)])

$$(2.6) \qquad |\alpha_p(p, j)| \leq p^{\frac{1}{2} - \frac{1}{d^2+1}}.$$

It is conjectured that for all primes except finitely many $|\alpha_p(p, j)| = 1$.

To each π there is a finite set of primes associated that are called the ramified primes. The local parameters for primes outside the set of ramified primes (i.e. unramified primes) can be give as eigenvalues of a matrix in $\mathrm{GL}_d(\mathbb{C})$.

We say that an irreducible unitary cuspidal representation π of $\mathrm{GL}_d(\mathbb{Q}_{\mathbb{A}})$ satisfies the *Generalized Ramanujan Conjecture* (GRC) if for local parameters $\alpha_{\pi}(p, j)$ ($1 \leq j \leq d$) at unramified primes p we have $|\alpha_{\pi}(p, j)| = 1$.

Our aim is to apply Theorem 2.3 to $L(\pi, s)$. We call such an L-function a principal automorphic L-function. It is known that $L(\pi, s) \in \mathcal{C}$, and we clearly have an Euler product of the form (2.1). We next investigate sufficient conditions under which formulas analogous to (2.2) and (2.3) hold. To do this, for primes p and integers $r \geq 1$, we define

$$b_{\pi}(p^r) = \begin{cases} \sum_{j=1}^{d} \alpha_{\pi}(p, j)^r & p \text{ unramified}, \\ 0 & p \text{ ramified}. \end{cases}$$

Note that for an unramified prime p,

$$b_{\pi}(p) = a_{\pi}(p).$$

We define $b_{\pi}(n) = 0$ if n is not a prime power. We consider the following hypothesis on the size of the coefficients $b_{\pi}(p^r)$ for $r \geq 2$.

Hypothesis H_0 There is $\epsilon > 0$ such that for any fixed $r \geq 2$,

$$\sum_{p^r \leq x} (\log p) |b_{\pi}(p^r)|^2 \ll x^{1-\epsilon}.$$

H_0 is a mild hypothesis and expected to be true for all π in general. It is true under the assumption of GRC. More generally if the local parameters satisfy

$$|\alpha_{\pi}(p, j)| \leq p^{\gamma}, \text{ for a fixed } 0 \leq \gamma < 1/4$$

then H_0 holds. Employing (2.6) we have

$$\sum_{r \geq \alpha} \sum_{p^r \leq x} (\log p) |b_{\pi}(p^r)|^2 \ll x^{1-\epsilon},$$

where $0 < \epsilon < 2/(d^2 + 1)$ and $\alpha(2/(d^2 + 1) - \epsilon) > 1$. Hence, assuming H_0, there is $\epsilon > 0$ such that

$$(2.7) \qquad \sum_{r \geq 2} \sum_{p^r \leq x} (\log p) |b_{\pi}(p^r)|^2 \ll x^{1-\epsilon}.$$

Finally observe that $\mathbf{H_0}$ implies the hypothesis $\mathbf{H}$ of Rudnick and Sarnak [16, Page 281]. That is, under the assumption of $\mathbf{H_0}$ we have

$$(2.8) \qquad \sum_{r \geq 2} \sum_{p^r \leq x} \frac{(\log p)^2 |b_\pi(p^r)|^2}{p^r} < \infty.$$

In the proof of the following lemma we use some properties of the Rankin-Selberg L-function $L(\pi \times \tilde{\pi}, s)$. See [16, Section 2.4] for a review of these properties.

Lemma 2.4. *Under the assumption of* $\mathbf{H_0}$ *we have*
(i)

$$\sum_{p \leq x} |b_\pi(p)|^2 \sim \frac{x}{\log x}$$

as $x \to \infty$, *and*
(ii)

$$\sum_{p \leq x} \frac{|b_\pi(p)|^2}{p} = \log \log x + \mathrm{O}(1).$$

Proof. (i) Let S be the product of all ramified primes for π and $L_S(\pi \times \tilde{\pi}, s) = \prod_{p \nmid S} L_p(\pi_p \times \tilde{\pi}_p, s)$ be the partial Rankin-Selberg L-function. Then it is known that for $\sigma > 1$,

$$f(s) = -\frac{L_S'}{L_S}(\pi \times \tilde{\pi}, s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)|b_\pi(n)|^2}{n^s}$$

where $\Lambda(n) = \log p$ if $n = p^r$ and zero otherwise (see [16, Page 281, formula (2.22)]). From the properties of the Rankin-Selberg L-functions we deduce that $f(s)$ has a meromorphic continuation to $\sigma \geq 1$ with a simple pole of residue 1 at $s = 1$. By the Wiener-Ikehara Tauberian theorem we have

$$(2.9) \qquad \sum_{n \leq x} \Lambda(n)|b_\pi(n)|^2 \sim x.$$

For simplicity in our exposition and in a similar fashion to the classical number theoretical functions we define

$$\Psi(x) = \sum_{n \leq x} \Lambda(n)|b_\pi(n)|^2, \quad \Theta(x) = \sum_{p \leq x} (\log p)|b_\pi(p)|^2, \text{ and } \Pi(x) = \sum_{p \leq x} |b_\pi(p)|^2.$$

We have

$$(2.10) \qquad \Theta(x) \leq \Psi(x) \leq (\log x)\Pi(x) + C_\epsilon x^{1-\epsilon},$$

where C_ϵ is a fixed constant (here we have used (2.7)). On the other hand, from (2.9), for large x we have

$$\Pi(x) \leq \Psi(x) \leq (1 + \epsilon)x.$$

Employing this upper bound for $\Pi(x)$, for $0 < \alpha < 1$ we have

$$(2.11) \qquad \Theta(x) \geq \sum_{x^\alpha \leq p \leq x} (\log p)|b_\pi(p)|^2 \geq (\Pi(x) - \Pi(x^\alpha)) \log x^\alpha$$

$$\geq \alpha \left(\Pi(x) - (1 + \epsilon)x^\alpha \right) \log x.$$

Since we can choose α arbitrary close to 1, from (2.10), (2.11), and (2.9) we have

$$\lim_{x \to \infty} \frac{\Pi(x)}{x/\log x} = \lim_{x \to \infty} \frac{\Theta(x)}{x} = \lim_{x \to \infty} \frac{\Psi(x)}{x} = 1.$$

(ii) Under the assumption of (2.8) Rudnick and Sarnak [16, Proposition 2.3] have shown that

$$\sum_{n \leq x} \frac{|\Lambda(n) b_\pi(n)|^2}{n} = \frac{(\log x)^2}{2} + \mathrm{O}(\log x).$$

Employing (2.8) and partial summation in the above formula yields the desired result. □

Proposition 2.5. *Let $L(\pi, s)$ be a principal automorphic L-function. Suppose that for the local parameters at unramified primes we have*

(2.12) $$|\alpha_\pi(p, j)| \leq p^\gamma, \text{ for a fixed } 0 \leq \gamma < 1/4.$$

Then

$$I_k(\pi, T) \gg T(\log T)^{k^2}$$

for rational $k \geq 0$.

Proof. Let S be the product of all ramified primes and $g(s) = L_S(\pi, s) = \prod_{p \nmid S} L_p(s, \pi_p)$ be the partial *L*-function. It is clear that if $\gcd(n, S) \neq 1$ then the n-th coefficient of $L_S(\pi, s)$ is zero; otherwise $a_\pi(n)$ is the n-th Dirichlet coefficient of $L_S(\pi, s)$. We have $L_S(\pi, s) \in \mathcal{C}$ (see [2, Section 5.12] for details). Moreover (2.12) implies that $\mathbf{H_0}$ in Lemma 2.4 is satisfied, and so (2.2) and (2.3) hold for $\beta = 1$. Thus the conditions of Theorem 2.3 hold and therefore we deduce the desired lower bound for the fractional k-th moment of $L_S(\pi, s)$. Since $L_S(\pi, s)$ and $L(\pi, s)$ differ only by finitely many factors (bounded from below on $s = 1/2$), the same lower bound holds for $I_k(\pi, T)$. □

Corollary 2.6. *The conclusion of Proposition* 2.5 *remains true if GRC holds for π.*

Since Dirichlet *L*-functions associated to primitive characters, modular *L*-functions associated to newforms, and twisted modular *L*-functions associated to newforms and primitive Dirichlet characters are all examples of principal automorphic *L*-functions that satisfy GRC (see [2, Sections 5.11 and 5.12] for details), as an immediate corollary of the above proposition we have the following extensions of the lower bound results of [6], [10], and [20].

Corollary 2.7. *For the Dirichlet L-function $L(\chi, s)$ attached to a primitive Dirichlet character and for any non-negative rational k we have*

$$I_k(\chi, T) \gg T(\log T)^{k^2}.$$

Corollary 2.8. *Let $L(f, s)$ be the L-function associated to a newform of weight ℓ, level N, and nebentypus ψ. Then for any non-negative rational k we have*

$$I_k(f, T) \gg T(\log T)^{k^2}.$$

Corollary 2.9. *Let $L(f, \chi, s)$ be the L-function associated to a newform of weight ℓ, level N, nebentypus ψ, and a primitive Dirichlet character χ mod q where $(q, N) = 1$. Then for any non-negative rational k we have*

$$I_k(f, \chi, T) \gg T(\log T)^{k^2},$$

where $I_k(f, \chi, T)$ denotes the k-th (power) moment of $L(f, \chi, s)$.

The following is also a direct corollary of Proposition 2.5.

Corollary 2.10. *Let $L(f, s)$ be a Maass cusp newform of weight zero and level N with nebentypus ψ. Then for any rational $k \geq 0$ we have*

$$I_k(f, T) \gg T \left(\log T\right)^{k^2}.$$

Proof. Let $\alpha_f(p, 1)$ and $\alpha_f(p, 2)$ be the local parameters of f at an unramified prime p. From a result of Kim and Sarnak [7, Proposition 2] we know that

$$|\alpha_f(p, 1)| \leq p^{7/64} \text{ and } |\alpha_f(p, 2)| \leq p^{7/64}.$$

The result follows from Proposition 2.5 as $7/64 < 1/4$. $\square$

Artin L-functions

Let $K/\mathbb{Q}$ be a Galois extension with the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_d(\mathbb{C})$$

be a representation of $\mathrm{Gal}(K/\mathbb{Q})$. Let $L(K/\mathbb{Q}, \rho, s)$ be the Artin L-function associated to ρ, which is defined as an Euler product $\prod_p L_p(K/\mathbb{Q}, \rho, s)$ of the local factors $L_p(K/\mathbb{Q}, \rho, s)$ for $\sigma > 1$.

For unramified primes p the local factor of the Artin L-function is defind by

$$L_p(K/\mathbb{Q}, \rho, s) = (\det(I_d - p^{-s}\rho(\sigma_p)))^{-1},$$

where I_d denotes the identity matrix, σ_p is the Frobenius conjugacy class, and $\rho(\sigma_p)$ is the value of ρ at any element belonging to σ_p. Letting $\alpha_\rho(p, j)$ $(1 \leq j \leq d)$ be the eigenvalues of the unitary matrix $\rho(\sigma_p)$, we have

$$L_p(K/\mathbb{Q}, \rho, s) = \prod_{j=1}^{d} \left(1 - \frac{\alpha_\rho(p, j)}{p^s}\right)^{-1},$$

where $|\alpha_\rho(p, j)| = 1$.

As a consequence of the Brauer induction theorem it is known that any Artin L-function has a meromorphic continuation to the whole complex plane. Moreover *Artin's conjecture* asserts that $L(K/\mathbb{Q}, \rho, s)$ has an analytic continuation for all s except possibly for a pole at $s = 1$ of order equal to the multiplicity of the trivial representation in ρ.

Proposition 2.11. *Assume Artin's conjecture for the Artin L-function $L(K/\mathbb{Q}, \rho, s)$. Then for any non-negative rational k, we have*

$$I_k(K/\mathbb{Q}, \rho, T) \gg T(\log T)^{\langle \varphi, \varphi \rangle k^2},$$

where $I_k(K/\mathbb{Q}, \rho, T)$ denotes the k-th moment of $L(K/\mathbb{Q}, \rho, s)$. Here φ is the character associated to ρ and

$$\langle \varphi, \varphi \rangle = \frac{1}{|G|} \sum_{g \in G} |\varphi(g)|^2$$

where $G = \mathrm{Gal}(K/\mathbb{Q})$.

Proof. Set

$$L_{\mathrm{ur}}(K/\mathbb{Q}, \rho, s) = \prod_{\text{unramified } p} L_p(K/\mathbb{Q}, \rho, s) = \sum_{n=1}^{\infty} \frac{a_\rho(n)}{n^s}.$$

Under the assumption of the Artin conjecture we know that $L_{\mathrm{ur}}(K/\mathbb{Q}, \rho, s) \in \mathcal{C}$ (see [2, Section 5.13] for details), and for $\sigma > 1$, $L_{\mathrm{ur}}(K/\mathbb{Q}, \rho, s)$ has an Euler product satisfying the conditions given in Theorem 2.3. Note that for unramified p we have $a_\rho(p) = \varphi(p)$. To apply Theorem 2.3 we need to study the asymptotic

behavior of $\sum_{p \leq x} |a_\rho(p)|^2$. By decomposing this sum according to the conjugacy class C of G that contains σ_p, we have

$$(2.13) \qquad \sum_{p \leq x} |a_\rho(p)|^2 = \sum_{\text{unramified } p \leq x} |\varphi(p)|^2$$

$$= \sum_C |\varphi(g_C)|^2 \left(\#\{\text{unramified } p \leq x;\ \sigma_p \subseteq C\}\right),$$

where g_C is any element of C. From the version of the Chebotarev density theorem with the remainder [8, Theorem 1.3] we get

$$\#\{\text{unramified } p \leq x;\ \sigma_p \subseteq C\} = \frac{|C|}{|G|}\text{Li}(x) + O\left(x \exp\left(-c_0\sqrt{\log x/n}\right)\right),$$

where $\text{Li}(x) = \int_2^\infty dt/\log t \sim x/\log x$, c_0 is an absolute constant and n is the degree of K over $\mathbb{Q}$. Applying this in (2.13), we get

$$\sum_{p \leq x} |a_\rho(p)|^2 = \langle \varphi, \varphi \rangle \text{Li}(x) + O\left(\left(\sum_C |\varphi(g_C)|^2\right) x \exp\left(-c_0\sqrt{\log x/n}\right)\right).$$

It is clear that this formula implies (2.2) for $\beta = \langle \varphi, \varphi \rangle$. Moreover we can derive (2.3) from this formula by partial summation. Thus the conditions of Theorem 2.3 are satisfied and we get the desired lower bound for the k-th moment of $L_{\text{ur}}(K/\mathbb{Q}, \rho, s)$. Since $L_{\text{ur}}(K/\mathbb{Q}, \rho, s)$ and $L(K/\mathbb{Q}, \rho, s)$ differ only by finitely many factors (bounded from below on $s = 1/2$), the same lower bound holds for $I_k(K/\mathbb{Q}, \rho, T)$. □

As a direct corollary of Proposition 2.11 we have the following unconditional improvement of Ramachandra's Theorem 1.7 for non-negative rational values of k.

Corollary 2.12. *Let K be a degree n Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the Dedekind zeta function of K. Then for non-negative rational k we have*

$$I_k(\zeta_K, T) \gg T(\log T)^{nk^2}.$$

Proof. Let ρ be the regular representation of $\text{Gal}(K/\mathbb{Q})$. In this case we have $L(K/\mathbb{Q}, \rho, s) = \zeta_K(s)$ and Artin's conjecture holds. Thus the result follows since $\langle \varphi, \varphi \rangle = n$, where φ is the character of the regular representation of $\text{Gal}(K/\mathbb{Q})$. □

Remark 2.13. R. Murty [13, Theorem 3.1] has shown that the Selberg orthogonality conjecture implies the Artin conjecture. Thus in Proposition 2.11 we can replace the Artin conjecture with the Selberg orthogonality conjecture.

3. Proof of Proposition 2.1

We start by recalling a classical theorem on the average values of multiplicative functions.

Lemma 3.1 (Levin and Fainleib). *Let $h(n)$ be a complex multiplicative function for which*

$$\sum_{p \leq x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$,

$$\sum_{p \leq x} |h(p)| = O\left(\frac{x}{\log x}\right),$$

and

$$h(p^r) \ll (2p)^{r\theta}, \text{ for some } 0 \le \theta < 1/2, \text{ and for all } r \ge 2.$$

Then, as $x \to \infty$,

$$\sum_{n \le x} h(n) \sim \frac{e^{-\gamma_0 \alpha}}{\Gamma(\alpha)} \frac{x}{\log x} \Pi_h(x),$$

where α is a fixed number, $\gamma_0 = 0.5772\ldots$ is Euler's constant and

$$\Pi_h(x) = \prod_{p \le x} \left(1 + \sum_{r=1}^{\infty} \frac{h(p^r)}{p^r}\right).$$

Proof. See [9, Theorem 3]. □

We are ready to prove Proposition 2.1.

Proof. **(Proof of Proposition 2.1)** With the notation of Lemma 3.1 we have

$$\Pi_h(x) = \prod_{p \le x} \left(1 + \frac{h(p)}{p}\right) \prod_{p \le x} \left(1 + \frac{1}{1 + \frac{h(p)}{p}} \sum_{r=2}^{\infty} \frac{h(p^r)}{p^r}\right).$$

Since $h(n)$ is non-negative and

$$\sum_{p \le x} h(p) \sim \alpha \frac{x}{\log x}$$

as $x \to \infty$, we have

$$\frac{h(p)}{p} \le \frac{1}{2}$$

for large enough p. This together with the bound given on $h(p^r)$ for $r \ge 2$ imply that

$$\prod_{p \le x} \left(1 + \frac{h(p)}{p}\right) \ll \Pi_h(x) \ll \prod_{p \le x} \left(1 + \frac{h(p)}{p}\right).$$

Moreover, by employing the bound $\frac{h(p)}{p} \le \frac{1}{2}$ (for large p) we can deduce that

$$\exp\left(\sum_{p \le x} \frac{h(p)}{p} + O(1)\right) \ll \Pi_h(x) \ll \exp\left(\sum_{p \le x} \frac{h(p)}{p} + O(1)\right).$$

Combining

$$\sum_{p \le x} \frac{h(p)}{p} = \alpha \log \log x + O(1)$$

with the previous inequality results in

$$(\log x)^{\alpha} \ll \Pi_h(x) \ll (\log x)^{\alpha}.$$

Thus by applying these bounds in Lemma 3.1, we have

$$(3.1) \qquad x(\log x)^{\alpha-1} \ll \sum_{n \le x} h(n) \ll x(\log x)^{\alpha-1}.$$

Next note that by partial summation we have

$$(3.2) \qquad \sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} = \frac{1}{T^{2\sigma}} \sum_{n \le T} h(n) + 2\sigma \int_1^T \left(\sum_{n \le u} h(n)\right) \frac{du}{u^{2\sigma+1}}.$$

By application of the upper bound of (3.1) in (3.2) and considering that $\frac{1}{2} + \frac{c}{\log T} \leq \sigma \leq 1$ we deduce that

$$\sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} \ll e^{-2c} (\log T)^{\alpha-1} + \int_{1}^{T} (\log u)^{\alpha-1} \frac{du}{u^{2\sigma}}$$

$$\ll (\log T)^{\alpha-1} + (2\sigma - 1)^{-\alpha} \int_{0}^{(2\sigma-1)\log T} e^{-t} t^{\alpha-1} dt$$

$$\ll (\log T)^{\alpha-1} + (\sigma - 1/2)^{-\alpha}$$

$$\ll (\log T)^{\alpha}.$$

Similarly by employing the lower bound of (3.1) in (3.2), and considering that $\frac{1}{2} + \frac{c}{\log T} \leq \sigma \leq 1$ we have

$$\sum_{n=1}^{T} \frac{h(n)}{n^{2\sigma}} \gg (2\sigma - 1)^{-\alpha} \int_{0}^{(2\sigma-1)\log T} e^{-t} t^{\alpha-1} dt$$

$$\gg (2\sigma - 1)^{-\alpha} \int_{0}^{2c} e^{-t} t^{\alpha-1} dt$$

$$\gg (\sigma - 1/2)^{-\alpha}.$$

The proof of the proposition is now completed. $\qquad\square$

4. Proof of Theorem 1.5

In this section, we aim to prove Theorem 1.5. The proof closely follows the method devised by Heath-Brown in [4]. We assume that (1.2) holds for g, and moreover (1.3) and (1.4) hold for rational $k \geq 0$. Note that from (1.4) follows that

$$(4.1) \qquad (\log T)^{\alpha} \ll \sum_{n=1}^{T} \frac{|g_k(n)|^2}{n} \ll (\log T)^{\alpha}.$$

For $T \geq 2$ and $-\infty < t < \infty$, let

$$w(t, T) = \int_{T}^{2T} \exp\left(-(t - \tau)^2\right) d\tau.$$

The next lemma summarizes some properties of $w(t, T)$ that will be used later.

Lemma 4.1. *Let $T \geq 2$ and $-\infty < t < \infty$. We have*

$$w(t, T) \ll 1$$

for all t and T,

$$w(t, T) \ll \exp\left(-\frac{1}{36}\left(t^2 + T^2\right)\right)$$

for $t \leq 0$ and $t \geq 3T$, and

$$w(t, T) \gg 1$$

for all $\frac{4T}{3} \leq t \leq \frac{5T}{3}$.

Proof. See [4]. $\qquad\square$

Define

$$J_k(g, \sigma, T) = \int_{-\infty}^{\infty} |g(\sigma + it)|^{2k} w(t, T) dt.$$

The next lemma shows that when finding a lower bound for $I_k(g, T)$ it is enough to find a lower bound for $J_k(g, 1/2, T)$.

Lemma 4.2. *We have*

$$I_k(g, 3T) + e^{-T^2/37} \gg J_k(g, 1/2, T).$$

Proof. By employing (1.2) and Lemma 4.1 we have

$$\begin{aligned}
J_k(g, 1/2, T) &= \int_{-\infty}^{\infty} |g(1/2 + it)|^{2k} w(t, T) dt \\
&\ll \int_0^{3T} |g(1/2 + it)|^{2k} dt \\
&\quad + \left(\int_{-\infty}^{0} + \int_{3T}^{\infty} \right) \left((1 + |t|)^{2kA} \exp\left(-\frac{1}{36}(t^2 + T^2) \right) \right) dt \\
&\ll \int_1^{3T} |g(1/2 + it)|^{2k} dt + e^{-T^2/37}.
\end{aligned}$$

$\square$

Let $0 < \lambda < \sqrt{3}/24\pi$ be fixed. We now define

$$(4.2) \qquad\qquad s_k(g, s, T) = \sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^s},$$

and let

$$S_k(g, \sigma, T) = \int_{-\infty}^{\infty} |s_k(g, \sigma + it, T)|^2 w(t, T) dt.$$

Observe that, by (1.3), for $\sigma \geq 1/2$ we have

$$(4.3) \qquad |s_k(g, s, T)| \leq \sum_{n=1}^{\lambda T} \frac{|g_k(n)|}{n^{1/2}} \ll \sum_{n=1}^{\lambda T} n^{\eta - 1/2} \ll T^{\eta + 1/2} \ll T.$$

We find upper and lower bounds for $S_k(g, \sigma, T)$.

Lemma 4.3. *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $0 < \lambda < \sqrt{3}/24\pi$. Then*

$$T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}} \ll S_k(g, \sigma, T) \ll T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}.$$

Proof. Recall the following mean-value theorem for Dirichlet polynomials. We have

$$\int_0^T \left| \sum_{n=1}^{N} a_n n^{-it} \right|^2 dt = \left(T + \xi \frac{4\pi}{\sqrt{3}} N \right) \sum_{n=1}^{N} |a_n|^2,$$

where $-1 \leq \xi \leq 1$ (see [11, Page 50]).

We use Lemma 4.1, (4.3), and the above mean value theorem for $N = \lambda T$ to deduce that

$$S_k(g, \sigma, T) = \int_0^{3T} |s_k(g, \sigma + it, T)|^2 \, w(t, T) dt + \mathrm{O}(1)$$

$$\ll \int_0^{3T} \left| \sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^{\sigma+it}} \right|^2 dt + 1$$

$$= (3T + \mathrm{O}(T)) \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}$$

$$\ll T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}.$$

We again use Lemma 4.1 and the above mean value theorem for $N = \lambda T$ to deduce that

$$S_k(g, \sigma, T) \gg \int_{\frac{4T}{3}}^{\frac{5T}{3}} \left| \sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^{\sigma+it}} \right|^2 dt$$

$$= \left(\frac{T}{3} + \lambda \xi' \frac{4\pi}{\sqrt{3}} T \right) \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}$$

$$\gg T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}.$$

Note that in the above formula $-2 \leq \xi' \leq 2$. This completes the proof of the lemma. $\qquad\square$

Next we obtain an upper bound on $J_k(g, \sigma, T)$.

Lemma 4.4. *Let* $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ *and* $T \geq 2$. *Then*

$$J_k(g, \sigma, T) \ll T^{\sigma - \frac{1}{2}} J_k\left(g, 1/2, T\right)^{\frac{3}{2} - \sigma} + e^{-T^2/7}.$$

The proof of this lemma is given in Section 5.

Write $k = \frac{u}{v}$ where u and v are positive coprime integers. We define

$$d_k(g, s, T) = g^u(s) - s_k^v(g, s, T)$$

and

$$D_k(g, \sigma, T) = \int_{-\infty}^{\infty} |d_k(g, \sigma + it, T)|^{\frac{2}{v}} \, w(t, T) dt.$$

We find an upper bound for $D_k(g, \sigma, T)$.

Lemma 4.5. *Let* $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k = \frac{u}{v}$, *and* $T \geq 2$. *Then*

$$D_k(g, \sigma, T) \ll D_k\left(g, 1/2, T\right)^{\frac{5+2\eta-4\sigma}{3+2\eta}} \left(T^{1-(1+\epsilon)/v} \right)^{\frac{4\sigma-2}{3+2\eta}}$$

$$+ D_k\left(g, 1/2, T\right)^{\frac{7+2\eta-8\sigma}{3+2\eta}} e^{-\frac{2\sigma-1}{9+6\eta}T^2},$$

where $0 \leq \eta < 1/2$ *is given in (1.3) and* $0 < \epsilon < 1/2 - \eta$.

The proof of this lemma is given in Section 5.

We now use the bounds found in Lemmas 4.4, 4.5 and 4.3 to find a lower bound for $J_k\left(g, 1/2, T\right)$.

Proposition 4.6. *We have*

$$J_k\left(g, 1/2, T\right) \gg T(\log T)^\alpha.$$

Proof. Notice that by the definition of $d_k(g, s, T)$ we have

$$|s_k(g, s, T)|^2 = |s_k^v(g, s, T)|^{\frac{2}{v}} = |g^u(s) - d_k(g, s, T)|^{\frac{2}{v}}$$
$$\ll |g(s)|^{2k} + |d_k(g, s, T)|^{\frac{2}{v}}.$$

Thus

$$(4.4) \qquad\qquad S_k(g, \sigma, T) \ll J_k(g, \sigma, T) + D_k(g, \sigma, T).$$

We also have

$$|d_k(g, s, T)|^{\frac{2}{v}} = |g(s)^u - s_k^v(g, s, T)|^{\frac{2}{v}}$$
$$\ll |g(s)|^{2k} + |s_k(g, s, T)|^2.$$

Thus

$$(4.5) \qquad\qquad D_k\left(g, 1/2, T\right) \ll J_k\left(g, 1/2, T\right) + S_k\left(g, 1/2, T\right).$$

We now consider two cases, and show that the lemma follows from either case.

We first consider the case $D_k\left(g, 1/2, T\right) \leq T$. In this case, Lemma 4.3, (4.4) with $\sigma = 1/2$, and (4.1) imply

$$T(\log T)^\alpha \ll T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n} \ll J_k\left(g, 1/2, T\right) + T,$$

from which the lemma follows.

We next consider the case $D_k\left(g, 1/2, T\right) > T$. Then Lemma 4.5 yields

$$(4.6) \qquad D_k(g, \sigma, T) \ll D_k\left(g, 1/2, T\right) \left(T^{\frac{(1+\epsilon)(2-4\sigma)}{v(3+2\eta)}} + e^{-\frac{2\sigma-1}{9+6\eta}T^2}\right)$$
$$\ll D_k\left(g, 1/2, T\right) T^{\frac{1-2\sigma}{2v}},$$

the last line following because $\eta < 1/2$ by (1.3) and $1 - 2\sigma \leq 0$. We see that (4.4) followed by (4.6) and (4.5) implies

$$S_k(g, \sigma, T) \ll J_k(g, \sigma, T) + \left(J_k\left(g, 1/2, T\right) + S_k\left(g, 1/2, T\right)\right) T^{\frac{1-2\sigma}{2v}}.$$

So there exists a constant $c(k) > 0$ such that for any given σ and T we have either

$$(4.7) \qquad\qquad S_k(g, \sigma, T) \leq c(k) \left(S_k\left(g, 1/2, T\right) T^{\frac{1-2\sigma}{2v}}\right)$$

or

$$(4.8) \qquad S_k(g, \sigma, T) \leq c(k) \left(J_k(g, \sigma, T) + J_k\left(g, 1/2, T\right) T^{\frac{1-2\sigma}{2v}}\right).$$

We claim that (4.7) cannot hold for all σ and T. The reason is that (4.7) together with Lemma 4.3 and (1.4) yield

$$\frac{(\sigma - 1/2)^{-\alpha}}{(\log T)^\alpha} \ll \frac{\displaystyle\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n^{2\sigma}}}{\displaystyle\sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n}} \ll \frac{S_k(g, \sigma, T)}{S_k(g, 1/2, T)} \ll T^{\frac{1-2\sigma}{2v}}$$

uniformly for $\frac{1}{2} + \frac{c}{\log \lambda T} \leq \sigma \leq \frac{3}{4}$, $\lambda T \geq 2$. Letting $\sigma = \frac{1}{2} + \frac{\delta}{\log T}$ in the above inequality yields

$$\delta^{-\alpha} \ll e^{-\frac{\delta}{v}}$$

which cannot be true for large values of δ. Thus (4.7) is false for a value δ_0, and so (4.8) must hold for $\sigma_0 = \frac{1}{2} + \frac{\delta_0}{\log T}$ and T large enough. Using these values of σ and T in (4.8) and applying Lemma 4.4 yields

$$(4.9) \quad S_k(g, \sigma_0, T) \ll T^{\sigma_0 - 1/2} J_k(g, 1/2, T)^{\frac{3}{2} - \sigma_0} + T^{\frac{1 - 2\sigma_0}{2v}} J_k(g, 1/2, T) + e^{-T^2/7}.$$

Then by the lower bound for $S_k(g, \sigma_0, T)$ given in Lemma 4.3, (4.9), and (4.1), we deduce that

$$T(\log T)^\alpha \ll T \sum_{n=1}^{\lambda T} \frac{|g_k(n)|^2}{n} \ll S_k(g, \sigma_0, T) \ll J_k(g, 1/2, T).$$

This is the desired result. $\qquad\square$

We are ready to prove our main result.

Proof. **(Proof of Theorem 1.5)** We assume that (1.3) and (1.4) hold for rational $k \geq 0$. Lemma 4.2 and Proposition 4.6 yield

$$I_k(g, T) + e^{-(T/3)^2/37} \gg J_k(g, 1/2, T/3) \gg T \sum_{n=1}^{\lambda T/3} \frac{|g_k(n)|^2}{n} \gg T(\log T)^\alpha.$$

Next assume that (1.3) and (1.4) hold for real $k \geq 0$ and $g(s)$ does not have any zero in the half plane $\sigma > 1/2$. Then it is clear that Lemmas 4.2, 4.3, and 4.4 extend to real $k \geq 0$. Moreover since $g(s)$ has no zero in the half plane $\sigma > 1/2$, $(g(s))^k$ is analytic on the half plane $\sigma > 1/2$ (except possibly at $s = 1$) and so Lemma 4.5 holds for $u = k$ and $v = 1$. Thus the theorem follows similar to the case for rational k. $\qquad\square$

5. Proofs of Lemmas 4.4 and 4.5

We state a result which we will utilize in the proofs of this section.

Lemma 5.1 (Gabriel). *Let $f(s)$ be analytic in the infinite strip $\alpha < \sigma < \beta$, and continuous for $\alpha \leq \sigma \leq \beta$. Suppose $f(s) \to 0$ as $|\text{Im}(s)| \to \infty$ uniformly for $\alpha \leq \sigma \leq \beta$. Then for $\alpha \leq \gamma \leq \beta$ and any $q > 0$ we have*

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^q dt \leq \left(\int_{-\infty}^{\infty} |f(\alpha + it)|^q dt \right)^{\frac{\beta - \gamma}{\beta - \alpha}} \left(\int_{-\infty}^{\infty} |f(\beta + it)|^q dt \right)^{\frac{\gamma - \alpha}{\beta - \alpha}}.$$

Proof. See [3, Theorem 2]. $\qquad\square$

Lemma 4.4 *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $T \geq 2$. Then*

$$J_k(g, \sigma, T) \ll T^{\sigma - \frac{1}{2}} J_k\left(g, \frac{1}{2}, T\right)^{\frac{3}{2} - \sigma} + e^{-T^2/7}.$$

Proof. Let $f(s) = (s-1)^m g(s) \exp\left(\frac{1}{2k}(s-i\tau)^2\right)$, $\gamma = \sigma$, $\alpha = 1/2$, $\beta = 3/2$, $q = 2k$ and $1/2 \leq \sigma \leq 3/4$.

We use Lemma 5.1 to see that

$$(5.1) \qquad \int_{-\infty}^{\infty} |f(\sigma+it)|^{2k}\, dt \leq \left(\int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\, dt\right)^{\frac{3}{2}-\sigma}$$

$$\cdot \left(\int_{-\infty}^{\infty} |f(3/2+it)|^{2k}\, dt\right)^{\sigma-\frac{1}{2}}.$$

Next we consider

$$(5.2) \quad \int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\, dt$$

$$= \int_{-\infty}^{\infty} |-1/2+it|^{2mk} |g(1/2+it)|^{2k} \left|e^{(1/2+it-i\tau)^2}\right|\, dt.$$

Since

$$\left|e^{(1/2+it-i\tau)^2}\right| \leq e^{\frac{1}{4}} e^{-(t-\tau)^2},$$

by employing (1.2) we have

$$\left(\int_{-\infty}^{\frac{\tau}{2}} + \int_{\frac{3\tau}{2}}^{\infty}\right) \left(|f(1/2+it)|^{2k}\right)\, dt \ll e^{-\tau^2/5}.$$

Using this with (5.2) yields

$$(5.3) \quad \int_{-\infty}^{\infty} |f(1/2+it)|^{2k}\, dt \ll \tau^{2mk} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\, dt + e^{-\tau^2/5}.$$

In a similar fashion as above, we may show that

$$(5.4) \qquad \int_{-\infty}^{\infty} |f(3/2+it)|^{2k}\, dt \ll \tau^{2mk}.$$

Using (5.3) and (5.4) with (5.1) yields

$$(5.5) \quad \int_{-\infty}^{\infty} |f(\sigma+it)|^{2k}\, dt$$

$$\ll \tau^{2mk} \left(\int_{-\infty}^{\infty} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\, dt\right)^{\frac{3}{2}-\sigma} + e^{-\tau^2/6}.$$

Moreover, since $e^{-(t-\tau)^2} \leq \left|e^{(\sigma+it-i\tau)^2}\right|$, we have

$$(5.6) \quad \int_{-\infty}^{\infty} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\, dt \ll \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\, dt + e^{-\tau^2/5}$$

$$\ll \tau^{-2mk} \int_{-\infty}^{\infty} |f(\sigma+it)|^{2k}\, dt + e^{-\tau^2/5}.$$

(5.5) and (5.6) together imply

$$\int_{-\infty}^{\infty} |g(\sigma+it)|^{2k} e^{-(t-\tau)^2}\, dt \ll \left(\int_{-\infty}^{\infty} |g(1/2+it)|^{2k} e^{-(t-\tau)^2}\, dt\right)^{\frac{3}{2}-\sigma} + e^{-\tau^2/6}.$$

Integrating over $T \leq \tau \leq 2T$ and using Holder's inequality completes the proof of the lemma. $\square$

In the subsequent proof, we will require the use of another theorem due to Gabriel.

Lemma 5.2 (Gabriel). *Let R be a rectangle with vertices s_0, $\bar{s}_0$, $-s_0$ and $-\bar{s}_0$. Let $F(s)$ be continuous on R and analytic on the interior of R. Then*

$$\int_L |F(s)|^q\,|ds| \leq \left(\int_{P_1} |F(s)|^q\,|ds|\right)^{1/2} \left(\int_{P_2} |F(s)|^q\,|ds|\right)^{1/2}$$

for any $q \geq 0$, where L is the line segment from $\frac{1}{2}\left(\bar{s}_0 - s_0\right)$ to $\frac{1}{2}\left(s_0 - \bar{s}_0\right)$, P_1 consists of the three line segments connecting $\frac{1}{2}\left(\bar{s}_0 - s_0\right)$, $\bar{s}_0$, s_0 and $\frac{1}{2}\left(s_0 - \bar{s}_0\right)$, and P_2 is the mirror image of P_1 in L.

Proof. See [3, Theorem 1]. $\qquad\square$

We also need the following mean-value theorem for Dirichlet series.

Lemma 5.3 (Montgomery-Vaughn). *If $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$, then*

$$\int_0^T \left|\sum_{n=1}^{\infty} a_n n^{-it}\right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2(T + O(n)),$$

the implied constant being absolute.

Proof. See [12, Corollary 3]. $\qquad\square$

We are ready to prove Lemma 4.5.

Lemma 4.5 *Let $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, $k = \frac{u}{v}$, and $T \geq 2$. Then*

$$D_k(g,\sigma,T) \ll D_k\left(g,1/2,T\right)^{\frac{5+2\eta-4\sigma}{3+2\eta}} \left(T^{1-(1+\epsilon)/v}\right)^{\frac{4\sigma-2}{3+2\eta}}$$

$$+ D_k\left(g,1/2,T\right)^{\frac{7+2\eta-8\sigma}{3+2\eta}} e^{-\frac{2\sigma-1}{9+6\eta}T^2},$$

where $0 \leq \eta < 1/2$ is given in (1.3) and $0 < \epsilon < 1/2 - \eta$.

Proof. Using Lemma 5.1 with

$$f(s) = d_k(g,s,T) \exp\left(\frac{v}{2}\left(s - i\tau\right)^2\right),$$

$\gamma = \sigma$, $\alpha = 1/2$, $3/4 < \beta < 1$, and $q = 2/v$, we get

$$(5.7) \qquad \int_{-\infty}^{\infty} |f\left(\sigma + it\right)|^{\frac{2}{v}}\,dt \leq \left(\int_{-\infty}^{\infty} |f\left(1/2 + it\right)|^{\frac{2}{v}}\,dt\right)^{\frac{\beta-\sigma}{\beta-1/2}}$$

$$\cdot \left(\int_{-\infty}^{\infty} |f\left(\beta + it\right)|^{\frac{2}{v}}\,dt\right)^{\frac{\sigma-1/2}{\beta-1/2}}.$$

By (1.2) and (4.3), we have

$$(5.8) \qquad d_k(g,s,T) \ll T^v + |t|^{Au} \ll \left(T + |t|^A\right)^{u+v}.$$

Thus (5.8) yields

$$(5.9) \quad \int_{-\infty}^{\infty} |f\left(\beta + it\right)|^{\frac{2}{v}}\,dt = \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f\left(\beta + it\right)|^{\frac{2}{v}}\,dt$$

$$+ O\left(T^{(2+2k)}\left(\int_{-\infty}^{\frac{\tau}{2}} + \int_{\frac{3\tau}{2}}^{\infty}\right)\left(\left(1 + \frac{|t|^A}{T}\right)^{2+2k} e^{-(t-\tau)^2}\right)dt\right).$$

We observe that

$$(5.10) \qquad \left(\int_{-\infty}^{\frac{\tau}{2}} + \int_{\frac{3\tau}{2}}^{\infty} \right) \left(\left(1 + \frac{|t|^A}{T} \right)^{2+2k} e^{-(t-\tau)^2} \right) dt \ll e^{-\frac{\tau^2}{5}}.$$

Putting (5.9) and (5.10) together, we have

$$(5.11) \qquad \int_{-\infty}^{\infty} |f(\beta + it)|^{\frac{2}{v}} dt = \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(\beta + it)|^{\frac{2}{v}} dt + O\left(T^{(2+2k)} e^{-\frac{\tau^2}{5}} \right).$$

We will now use Lemma 5.2 with $F(s) = f(s + \beta + i\tau)$, $s_0 = \beta - 1/2 + (1/2)i\tau$ and $q = \frac{2}{v}$. This allows us to avoid the possible pole of $g(s)$. On one hand, we have

$$(5.12) \qquad \int_L |F(s)|^q \, |ds| = \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(\beta + it)|^{\frac{2}{v}} dt.$$

We also have

$$(5.13) \quad \int_{P_1} |F(s)|^q \, |ds| = \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(2\beta - 1/2 + it)|^{\frac{2}{v}} dt$$

$$+ \int_{\beta}^{2\beta - 1/2} \left(\left| f\left(\mu + \frac{1}{2}i\tau \right) \right|^{\frac{2}{v}} + \left| f\left(\mu + \frac{3}{2}i\tau \right) \right|^{\frac{2}{v}} \right) d\mu$$

and

$$(5.14) \qquad \int_{P_2} |F(s)|^q \, |ds| = \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(1/2 + it)|^{\frac{2}{v}} dt$$

$$+ \int_{\frac{1}{2}}^{\beta} \left(\left| f\left(\mu + \frac{1}{2}i\tau \right) \right|^{\frac{2}{v}} + \left| f\left(\mu + \frac{3}{2}i\tau \right) \right|^{\frac{2}{v}} \right) d\mu.$$

However, (5.8) yields

$$(5.15) \qquad f\left(\mu + \frac{1}{2}i\tau \right) = d_k\left(g, \mu + \frac{1}{2}i\tau, T \right) \exp\left(\frac{v}{2} \left(\mu - \frac{1}{2}i\tau \right)^2 \right)$$

$$\ll (T + \tau^A)^{u+v} e^{-v\tau^2/8}$$

and similarly

$$(5.16) \qquad f\left(\mu + \frac{3}{2}i\tau \right) \ll (T + \tau^A)^{u+v} e^{-v\tau^2/8}.$$

Thus (5.15) and (5.16) show that the second integrals in (5.13) and (5.14) are $\ll T^{(2+2k)} e^{-\tau^2/5}$. Using this, along with Lemma 5.2, (5.12), (5.13) and (5.14), we have

$$(5.17)$$

$$\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(\beta + it)|^{\frac{2}{v}} dt \ll \left(\int_{-\infty}^{\infty} |f(1/2 + it)|^{\frac{2}{v}} dt \right)^{1/2}$$

$$\cdot \left(\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |f(2\beta - 1/2 + it)|^{\frac{2}{v}} dt \right)^{1/2} + T^{(2+2k)} e^{-\tau^2/11}.$$

From (5.11) and (5.17) together with (5.7) and the definition of $f(s)$ we have

(5.18) $\displaystyle\int_{-\infty}^{\infty} |d_k(g, \sigma + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt$

$$\ll \left(\int_{-\infty}^{\infty} |d_k(g, 1/2 + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{\beta - \sigma/2 - 1/4}{\beta - 1/2}}$$

$$\cdot \left(\int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |d_k(g, 2\beta - 1/2 + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{\sigma/2 - 1/4}{\beta - 1/2}}$$

$$+ \left(\int_{-\infty}^{\infty} |d_k(g, 1/2 + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt \right)^{\frac{\beta - \sigma}{\beta - 1/2}}$$

$$\cdot \left(T^{(2+2k)} e^{-\tau^2/11} \right)^{\frac{\sigma - 1/2}{\beta - 1/2}}.$$

Integrating (5.18) for $T \leq \tau \leq 2T$ and applying Hölder's inequality yields

(5.19) $\displaystyle D_k(g, \sigma, T) \ll D_k(g, 1/2, T)^{\frac{\beta - \sigma/2 - 1/4}{\beta - 1/2}}$

$$\cdot \left(\int_{T}^{2T} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |d_k(g, 2\beta - 1/2 + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt d\tau \right)^{\frac{\sigma/2 - 1/4}{\beta - 1/2}}$$

$$+ D_k(g, 1/2, T)^{\frac{\beta - \sigma}{\beta - 1/2}} e^{-\frac{2\sigma - 1}{12(2\beta - 1)} T^2}.$$

Since $w(t, T) \ll 1$, we have

$$\int_{T}^{2T} \int_{\frac{\tau}{2}}^{\frac{3\tau}{2}} |d_k(g, 2\beta - 1/2 + it, T)|^{\frac{2}{v}} e^{-(t-\tau)^2} dt d\tau$$

$$\ll \int_{\frac{T}{2}}^{3T} |d_k(g, 2\beta - 1/2 + it, T)|^{\frac{2}{v}} dt.$$

Now choose $\beta > 3/4$ so that $2\beta - 1/2 > 1$. By this choice of β, $d_k(g, 2\beta - 1/2 + it, T)$ is represented by an absolutely convergent Dirichlet series. Also note that

$$g_u(n) = g_{kv}(n) = \sum_{n = n_1 \cdots n_v} g_k(n_1) \cdots g_k(n_v).$$

Thus, for $\sigma > 1$, we have

(5.20) $\displaystyle d_k(g, s, T) = g^u(s) - s_k^v(g, s, T)$

$$= \sum_{n=1}^{\infty} \frac{g_u(n)}{n^s} - \left(\sum_{n=1}^{\lambda T} \frac{g_k(n)}{n^s} \right)^v$$

$$= \sum_{\lambda T}^{\infty} \frac{a(n)}{n^s}$$

where

$$(5.21) \qquad |a(n)| = \left| g_u(n) - \sum_{\substack{n = n_1 \cdots n_v \\ n_j \le \lambda T \text{ for } j = 1, \ldots, v}} g_k(n_1) \cdots g_k(n_v) \right|$$

$$= \left| \sum_{\substack{n = n_1 \cdots n_v \\ \exists j \text{ such that } n_j > \lambda T}} g_k(n_1) \cdots g_k(n_v) \right|$$

$$\le \sum_{n = n_1 \cdots n_v} |g_k(n_1)| \cdots |g_k(n_v)|$$

$$\le n^\eta \tau_v(n)$$

$$\ll n^{\eta + \epsilon_0}.$$

In the last inequality ϵ_0 is a positive constant that will be chosen appropriately later. Thus, (5.20) and (5.21) allow us to use Lemma 5.3 to get

$$(5.22) \qquad \int_{\frac{T}{2}}^{3T} |d_k(g, 2\beta - 1/2 + it, T)|^2 \, dt = \sum_{\lambda T}^{\infty} \frac{|a(n)|^2}{n^{4\beta - 1}} \left(\frac{5}{2} T + O(n) \right)$$

$$\ll T \int_{\lambda T}^{\infty} t^{2\eta + 2\epsilon_0 - 4\beta + 1} dt + \int_{\lambda T}^{\infty} t^{2\eta + 2\epsilon_0 - 4\beta + 2} dt$$

$$\ll T^{2(\eta + \epsilon_0) - 4\beta + 3}.$$

The last inequality holds if $\beta > (\eta + \epsilon_0)/2 + 3/4$. We now use Holder's inequality and (5.22) to see that

$$(5.23) \qquad \int_{\frac{T}{2}}^{3T} |d_k(g, 2\beta - 1/2 + it, T)|^{\frac{2}{v}} \, dt \le \left(\frac{5}{2} T \right)^{1 - \frac{1}{v}} \left(T^{2(\eta + \epsilon_0) - 4\beta + 3} \right)^{\frac{1}{v}}$$

$$\ll T^{1 + (2(\eta + \epsilon_0) - 4\beta + 2)/v}.$$

Applying the bound (5.23) in (5.19) and choosing $\epsilon_0 = (1/2)(1/2 - \eta) - \epsilon/2$ and $\beta = (7/8) + (1/4)\eta$ completes the proof of the lemma. $\qquad \square$

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