

FOURIER COEFFICIENTS OF AUTOMORPHIC L -FUNCTIONS OVER PRIMES IN RAY CLASSES

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ABSTRACT. We prove Siegel-Walfisz type theorems (over long and short intervals) for the Fourier coefficients of certain automorphic L -functions and Rankin-Selberg L -functions over number fields.

1. INTRODUCTION

Let p and γ denote primes and positive real numbers, respectively, and let $q > 0$ and a be coprime integers. The uniform version of the prime number theorem in arithmetic progressions, known as the Siegel-Walfisz theorem, provides the existence of a constant $c := c(\gamma) > 0$, depending only on γ , such that if $q \leq (\log x)^\gamma$, then

$$(1.1) \quad \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \log p = \frac{x}{\phi(q)} + O\left(x \exp\left(-c(\log x)^{\frac{1}{2}}\right)\right),$$

where $\phi(\cdot)$ is Euler's totient function (see [5, p. 133]). The short interval version of this theorem states that for $y = x^\theta$, with $\theta > 7/12$, one has

$$(1.2) \quad \sum_{\substack{x-y < p \leq x \\ p \equiv a \pmod{q}}} \log p \sim \frac{y}{\phi(q)},$$

as $x \rightarrow \infty$, uniformly for $q \leq (\log x)^\gamma$ (see [26, p. 316]). Our goal in this paper is to prove theorems analogous to the above results for the Fourier coefficients of automorphic L -functions and Rankin-Selberg L -functions over number fields. Previous work on automorphic extensions of (1.1) (for example [22] and [11]) treats only the classical modular forms or as [27] is under the assumption of the Generalized Ramanujan Conjecture. Here, we prove unconditionally extensions of (1.1) and bounds of correct order of magnitude for the sum in (1.2), for certain L -functions of degree less than or equal to four. Our focus here is on non-abelian L -functions. For results related to degree one L -functions, see [18] and [7]. In order to state our results, we start with introducing some terminology and notation.

Let F be a number field, and set $n_F = [F : \mathbb{Q}]$. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ with unitary central character. We shall call such representations, for short, *cuspidal representations* of $\mathrm{GL}_m(\mathbb{A}_F)$. Associated to π , there is an integer $A_\pi \geq 1$, called the *conductor* of π , and a collection of complex numbers $\alpha_\pi(j, \mathfrak{p})$, for $1 \leq j \leq m$, called the *local parameters*, such that for any $\mathfrak{p} \nmid A_\pi$, we have $\alpha_\pi(j, \mathfrak{p}) \neq 0$. We call a prime $\mathfrak{p} \nmid A_\pi$ an *unramified prime*. The *Generalized Ramanujan Conjecture* (GRC) states that, for $1 \leq j \leq m$, $|\alpha_\pi(j, \mathfrak{p})| = 1$ for unramified primes \mathfrak{p} , and $|\alpha_\pi(j, \mathfrak{p})| \leq 1$ for ramified primes \mathfrak{p} . The truth of GRC is known for $m = 1$ and for cuspidal representations that can be associated to Galois representations (for instance, the cuspidal representations arising from modular forms). Corresponding to each cuspidal representation π , there is a cuspidal representation $\tilde{\pi}$, the *contragredient* representation.

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The collection of the local parameters for $\tilde{\pi}$ coincides with the collection of the complex conjugates of the local parameters for π (i.e., $\{\alpha_{\tilde{\pi}}(j, \mathfrak{p})\} = \{\overline{\alpha_{\pi}(j, \mathfrak{p})}\}$). For integer $k \geq 1$, we set

$$a_{\pi}(\mathfrak{p}^k) = \sum_{j=1}^m \alpha_{\pi}(j, \mathfrak{p})^k.$$

For a cuspidal representation π and an idèle class character ψ , the *twist of π by ψ* , denoted $\pi \otimes \psi$, is the representation of $\mathrm{GL}_n(\mathbb{A}_F)$ defined by $(\pi \otimes \psi)(g) = \psi(\det(g))\pi(g)$ for $g \in \mathrm{GL}_n(\mathbb{A}_F)$.

Let π and π' , respectively, be cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_{m'}(\mathbb{A}_F)$ of conductor A_{π} and $A_{\pi'}$. Let $L(s, \pi \times \pi')$ denote the Rankin-Selberg L -function associated with π and π' , where $s = \sigma + it$ is a complex variable. For $\Re(s) > 1$, we have

$$-\frac{L'}{L}(s, \pi \times \pi') = \sum_{\mathfrak{n} \neq \mathfrak{o}} \frac{\Lambda(\mathfrak{n})a_{\pi \times \pi'}(\mathfrak{n})}{N\mathfrak{n}^s},$$

where $N\mathfrak{n}$ is the norm of the ideal \mathfrak{n} , $\Lambda(\mathfrak{n})$ is the number field analogue of the von Mangoldt function (i.e., $\Lambda(\mathfrak{p}^k) = \log N\mathfrak{p}$ and $\Lambda(\mathfrak{n}) = 0$ if \mathfrak{n} is not a power of a prime ideal), and

$$a_{\pi \times \pi'}(\mathfrak{p}^k) = a_{\pi}(\mathfrak{p}^k)a_{\pi'}(\mathfrak{p}^k)$$

for $\mathfrak{p} \nmid (A_{\pi}, A_{\pi'})$. It is known that $L(s, \pi \times \pi')$ has an analytic continuation to the whole complex plane with possible simple poles at $s = i\tau$ or $s = 1 + i\tau$ for some $\tau \in \mathbb{R}$, where poles exist if and only if $\pi' \simeq \tilde{\pi} \otimes |\cdot|^{-i\tau}$. In particular, $L(s, \pi \times \tilde{\pi})$ has only a simple pole at $s = 1$. Throughout this paper we assume, unless otherwise stated, that π and π' are normalized such that their central characters are trivial on the diagonally embedded copy of the positive reals. This normalization will ensure that the possible simple pole of $L(s, \pi \times \pi')$ at $s = 1 + i\tau$ can only occur at $s = 1$. (For a review of the basic properties of the Rankin-Selberg L -functions, see [12, Chapter 5].) We set $L(s, \pi) = L(s, \pi \times 1)$, where 1 is the trivial representation.

If a representation π is isomorphic to $\bigoplus_{i=1}^k \pi_i$, for cuspidal representations π_i of $\mathrm{GL}_{m_i}(\mathbb{A}_F)$, then π is called an (*isobaric*) *automorphic representation* of $\mathrm{GL}_{m_1 + \dots + m_k}(\mathbb{A}_F)$. For π and π' as above with $1 \leq m \leq 2$ and $1 \leq m' \leq 3$, it is known that there exist an automorphic representation of $\mathrm{GL}_{mm'}(\mathbb{A}_F)$, denoted by $\pi \boxtimes \pi'$, such that $L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi')$ and $a_{\pi \boxtimes \pi'} = a_{\pi \times \pi'}$ (see [23] and [13]).

We call a cuspidal representation π *self-dual* if $\pi \simeq \tilde{\pi}$. Also, a cuspidal representation π is called *essentially self-dual* if $\pi \simeq \tilde{\pi} \otimes \psi$ for some idèle class character ψ of F . A cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ is called *dihedral* if it admits a non-trivial self-twist (i.e., $\pi \simeq \pi \otimes \psi$ for some non-trivial idèle class character ψ of F). We say that two cuspidal representations π and π' are *twist-equivalent* if $\pi' \simeq \pi \otimes \psi$ for some idèle class character ψ of F .

We are ready to state our first result.

Theorem 1.1. *Assume that for an automorphic representation Π , one of the following hold:*

- (a) $\Pi \simeq \pi$, where π is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$ for $2 \leq m \leq 3$, or π is a cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$ that is not essentially self-dual.
- (b) $\Pi \simeq \pi \boxtimes \tilde{\pi}$, where π is a non-dihedral cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$.
- (c) $\Pi \simeq \pi \boxtimes \pi'$, where π and π' are non-dihedral cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ that are not twist-equivalent.

Then, for any $\gamma > 0$, there exists $c := c(\Pi) > 0$ such that for any ideal \mathfrak{q} , with $N\mathfrak{q} \leq (\log x)^\gamma$, and any ideal \mathfrak{a} relatively prime to \mathfrak{q} , we have

$$(1.3) \quad \sum_{\substack{N\mathfrak{p} \leq x \\ \mathfrak{p} \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(\mathfrak{p})a_{\Pi}(\mathfrak{p}) = \frac{\delta(\Pi)}{h(\mathfrak{q})}x + O_{\Pi, \gamma} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right),$$

where, as later, the superscript “*” in the above sum means that the sum is taken over primes \mathfrak{p} that $\mathfrak{p} \nmid (A_\pi, A_{\pi'})$, $\mathfrak{p} \sim \mathfrak{a} \pmod{\mathfrak{q}}$ means that \mathfrak{p} and \mathfrak{a} belong to the same ray class of the ray class group modulo \mathfrak{q} , $h(\mathfrak{q})$ is the number of ray classes modulo \mathfrak{q} , $\delta(\Pi) = 0$ in (a) and (c), and $\delta(\Pi) = 1$ in (b).

Remarks 1.2. (i) In the above theorem, $\delta(\Pi)$ is the order of $-\frac{L'}{L}(s, \Pi)$ at $s = 1$.

(ii) In the case $F = \mathbb{Q}$, the asymptotic formula (1.3) becomes

$$\sum_{\substack{p \leq x \\ p \equiv \mathfrak{a} \pmod{q}}}^* (\log p) a_\Pi(p) = \frac{\delta(\Pi)}{\phi(q)} x + O_{\Pi, \gamma} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right)$$

for $q \leq (\log x)^\gamma$. In particular, if π is a non-dihedral cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, we have, unconditionally,

$$\sum_{\substack{p \leq x \\ p \equiv \mathfrak{a} \pmod{q}}}^* (\log p) |a_\pi(p)|^2 = \frac{1}{\phi(q)} x + O_{\pi, \gamma} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right),$$

for $q \leq (\log x)^\gamma$. This removes the assumption of the GRC in [27, Theorem 1].

(iii) An examination of the proof of Theorem 1.1 shows that, in accordance with (1.1), it would be possible to remove the dependence in γ of the implied constant in the error term of (1.3), by making the constant c in the error term to be dependent in γ . We prefer (1.3) as it provides an error formula independent of γ .

(iv) The main obstacles for proving a Siegel-Walfisz type result for general automorphic L -functions are the lack of information on the size of their coefficients and the absence of the Siegel-type bounds for their possible exceptional zeros. The current known bounds towards the GRC together with the non-existence of exceptional zeros for degrees two and three L -functions, Siegel-type bounds for degree one L -functions, and the theory of Rankin-Selberg L -functions provide us with the needed tools in proving such a theorem for certain automorphic L -functions.

The proof of Theorem 1.1 is done along the classical lines by studying the analytic properties of the Rankin-Selberg L -functions twisted by the characters of the ray class groups. It is known that if π is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$, then $\pi \otimes \psi$ is also a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$ for any idèle class character ψ of F . Thus, we can define

$$(1.4) \quad L(s, \pi \times \pi' \times \psi) := L(s, (\pi \otimes \psi) \times \pi').$$

Throughout the paper, we let $L(s, \pi \times \pi' \times \chi)$ be the L -function (1.4) attached to π , π' , and the idèle class character associated with the ray class character χ . Theorem 1.1 is, in fact, a consequence of a more general theorem which we state now.

Theorem 1.3. *Let π and π' be cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_{m'}(\mathbb{A}_F)$, respectively. Let χ denote a ray class character modulo \mathfrak{q} . Assume the following hold:*

(i) *There is $\epsilon_0 := \epsilon_0(\pi, \pi') > 0$ such that*

$$\sum_{x < \mathfrak{N}\mathfrak{n} \leq x+u}^* \Lambda(\mathfrak{n}) |a_{\pi \times \pi'}(\mathfrak{n})| \ll_{\pi, \pi'} u \log x$$

for $x^{1-\epsilon_0} \leq u \leq x$.

(ii) *The L -functions $L(s, \pi \times \pi' \times \chi)$ are holomorphic everywhere except possibly having a simple pole at $s = 1$ for exactly one ray class character $\chi = \eta$ modulo \mathfrak{q} .*

(iii) *There is a positive constant $c_{\pi, \pi'}$, depending only on π and π' , such that for any ray class character χ modulo \mathfrak{q} , the L -function $L(s, \pi \times \pi' \times \chi)$ has either no zeros or possibly only*

one simple real zero $\beta_\chi := \beta(\pi, \pi', \chi)$ in the region

$$(1.5) \quad \sigma \geq 1 - \frac{c_{\pi, \pi'}}{\log((N\mathfrak{q})(|t| + 3))}.$$

Then there exists $c := c(\pi, \pi') > 0$ such that for any ideal \mathfrak{q} , with $N\mathfrak{q} \leq \exp((\log x)^{1/2})$, and any ideal \mathfrak{a} relatively prime to \mathfrak{q} , we have

$$(1.6) \quad \sum_{\substack{Nn \leq x \\ n \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(n) a_{\pi \times \pi'}(n) = \frac{\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})}{h(\mathfrak{q})} x - \frac{1}{h(\mathfrak{q})} \sum_{\chi \pmod{\mathfrak{q}}} \bar{\chi}(\mathfrak{a}) \frac{x^{\beta_\chi}}{\beta_\chi} \\ + O_{\pi, \pi'} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right),$$

where the term $\frac{x^{\beta_\chi}}{\beta_\chi}$ should be omitted if β_χ does not exist. Here $\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a}) = \bar{\eta}(\mathfrak{a})\delta(\pi \times \pi' \times \eta)$, where $\delta(\pi \times \pi' \times \eta)$ is the order of $-\frac{L'}{L}(s, \pi \times \pi' \times \eta)$ at the pole $s = 1$ for the possible unique ray class character η modulo \mathfrak{q} described in (ii), and $\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a}) = 0$ otherwise.

Moreover, under the additional condition to (iii):

(iv) If such β_χ exists, then for any $\epsilon > 0$, there is a constant $\kappa(\epsilon, \pi, \pi')$, depending on ϵ, π , and π' , such that

$$(1.7) \quad \beta_\chi \leq 1 - \frac{\kappa(\epsilon, \pi, \pi')}{N\mathfrak{q}^\epsilon}.$$

Then, for any $\gamma > 0$, there exists $c := c(\pi, \pi') > 0$ such that for any ideal \mathfrak{q} , with $N\mathfrak{q} \leq (\log x)^\gamma$, and any ideal \mathfrak{a} relatively prime to \mathfrak{q} , we have

$$(1.8) \quad \sum_{\substack{Nn \leq x \\ n \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(n) a_{\pi \times \pi'}(n) = \frac{\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})}{h(\mathfrak{q})} x + O_{\pi, \pi', \gamma} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right),$$

where $\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})$ is as defined above.

For simplicity of referring to (1.5), throughout the paper, the region given by (1.5) is called the *classical zero-free region* of $L(s, \pi \times \pi' \times \chi)$. Also the bound for β_χ given in (1.7) is called a *Siegel-type bound* for the exceptional zero β_χ .

In this paper, we shall also prove an estimate of correct order of magnitude towards (1.2) for automorphic L -functions. In [21], Motohashi proved such an estimate for the sum of Hecke-Maass eigenvalues $\tau_V(p)$, associated with an irreducible representation V of $\mathrm{PSL}_2(\mathbb{R})$, with spectral data ν_V , squared over primes in short intervals. The following is [21, Theorem 1].

Theorem 1.4 (Motohashi). *There exist constants $c_0, \theta_0 > 0$ such that uniformly for $(\log x)^{-1/2} \leq \theta \leq \theta_0$, $|\nu_V|^{1/\theta} \leq x$, one has*

$$\sum_{x-y \leq p \leq x} \tau_V^2(p) = \frac{y}{\log x} (1 + O(e^{-c_0/\theta})), \quad y = x^{1-\theta}.$$

For results similar to the above in the context of automorphic L -functions, see [2] and [14]. The main ingredients of the proof of such results are an explicit formula similar to the classical explicit formula for the prime counting function $\psi(x)$, a classical zero-free region, and a *log-free zero-density estimate*. The possibility of obtaining such estimates using a log-free zero-density estimate was first noted by Moreno [19].

Our next result is inspired by Theorem 1.4.

Theorem 1.5. *Let Π be as described in Theorem 1.1 and let $\gamma, \nu > 0$. Then there are positive constants $c_0 := c_0(\Pi)$, $c := c(\Pi)$, and $\theta_0 := \theta_0(\Pi)$ such that, for*

$$(\log x)^{-1/2} \leq \theta \leq \theta_0,$$

$y = x^{1-\theta}$, any ideal \mathfrak{q} with $N\mathfrak{q} \leq (\log x)^\gamma$, and any ideal \mathfrak{a} relatively prime to \mathfrak{q} , we have

$$(1.9) \quad \sum_{\substack{x-y < N\mathfrak{p} \leq x \\ \mathfrak{p} \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(\mathfrak{p}) a_\Pi(\mathfrak{p}) = y \left(\frac{\delta(\Pi)}{h(\mathfrak{q})} + O_{\Pi, \gamma, \nu}(\exp(-c(\log x)^{1-\nu})) + O_\Pi(e^{-c_0/\theta}) \right),$$

where $\delta(\Pi)$ is as defined in Theorem 1.1.

The above theorem is also a consequence of the following more general assertion.

Theorem 1.6. *Let π and π' be cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_{m'}(\mathbb{A}_F)$, respectively. Assume the following hold:*

(i) *There is $\epsilon_0 := \epsilon_0(\pi, \pi') > 0$ such that*

$$\sum_{x < N\mathfrak{n} \leq x+u}^* \Lambda(\mathfrak{n}) |a_{\pi \times \pi'}(\mathfrak{n})| \ll_{\pi, \pi'} u \log x$$

for $x^{1-\epsilon_0} \leq u \leq x$.

(ii) *The L -functions $L(s, \pi \times \pi' \times \chi)$ are holomorphic everywhere except possibly having a simple pole at $s = 1$ for exactly one ray class character $\chi = \eta$ modulo \mathfrak{q} .*

(iii) *There is a positive constant $c_{\pi, \pi'}$, depending only on π and π' , such that for any ray class character χ modulo \mathfrak{q} , the L -function $L(s, \pi \times \pi' \times \chi)$ has either no zeros or possibly only one simple real zero $\beta_\chi := \beta(\pi, \pi', \chi)$ in the region*

$$\sigma \geq 1 - \frac{c_{\pi, \pi'}}{\log((N\mathfrak{q})(|t| + 3))}.$$

(iv) *There is a positive constant $d_{\pi, \pi'}$ such that for $T \geq 1$ and $0 \leq \sigma \leq 1$, we have*

$$N(\sigma, T, \pi \times \pi' \times \chi) \ll_{\pi, \pi'} ((N\mathfrak{q})T)^{d_{\pi, \pi'}(1-\sigma)},$$

where

$$N(\sigma, T, \pi \times \pi' \times \chi) = \#\{\rho = \Re(\rho) + i\Im(\rho) \mid L(\rho, \pi \times \pi' \times \chi) = 0, \Re(\rho) \geq \sigma, |\Im(\rho)| \leq T\}.$$

Then there exists a positive constant $c_0 := c_0(\pi, \pi')$ such that, for

$$(\log x)^{-1/2} \leq \theta \leq \min \left\{ \frac{1}{10d_{\pi, \pi'}}, \frac{\epsilon_0}{4} \right\}$$

and $y = x^{1-\theta}$, we have

$$(1.10) \quad \sum_{\substack{x-y < N\mathfrak{n} \leq x \\ \mathfrak{n} \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) = y \left(\frac{\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})}{h(\mathfrak{q})} + O \left(\frac{1}{h(\mathfrak{q})} \sum_{\chi \pmod{\mathfrak{q}}} x^{\beta_\chi - 1} \right) + O_{\pi, \pi'}(e^{-c_0/\theta}) \right),$$

uniformly for all \mathfrak{q} , with $N\mathfrak{q} \leq x^\theta$, where $\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})$ is as defined in Theorem 1.3.

Moreover, under the additional condition to (iii):

(v) *If the possible exceptional zero β_χ of $L(s, \pi \times \pi' \times \chi)$ exists, then for any $\epsilon > 0$, there is a constant $\kappa(\epsilon, \pi, \pi')$, depending on ϵ, π , and π' , such that*

$$\beta_\chi \leq 1 - \frac{\kappa(\epsilon, \pi, \pi')}{N\mathfrak{q}^\epsilon}.$$

Then, for any $\gamma, \nu > 0$, there exists $c := c(\pi, \pi') > 0$ such that

(1.11)

$$\sum_{\substack{x-y < N\mathfrak{n} \leq x \\ \mathfrak{n} \sim \mathfrak{a} \pmod{\mathfrak{q}}}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) = y \left(\frac{\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})}{h(\mathfrak{q})} + O_{\pi, \pi', \gamma, \nu} \left(\exp(-c(\log x)^{1-\nu}) \right) + O_{\pi, \pi'} \left(e^{-c_0/\theta} \right) \right),$$

uniformly for all \mathfrak{q} , with $N\mathfrak{q} \leq (\log x)^\gamma$, where $\delta(\pi, \pi', \mathfrak{q}, \mathfrak{a})$ is as defined in Theorem 1.3.

Remark 1.7. We note that by the work of Soundararajan and Thorner [25, Theorem 2.4 and Corollary 2.6], the condition (i) of Theorem 1.3 and conditions (i) and (iv) of Theorem 1.6 hold, unconditionally, whenever $F = \mathbb{Q}$. Also, recently, Humphries and Thorner [10, Theorem 2.4] showed that the classical zero-free region (the condition (iii) in Theorems 1.3 and 1.6) and the Siegel-type bound (the condition (iv) of Theorem 1.3 and the condition (v) of Theorem 1.6) are valid if $\pi' = \tilde{\pi}$, $(q, A_\pi) = 1$, and $F = \mathbb{Q}$. Thus, by Theorem 1.3, for any cuspidal representation π of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ and $\gamma > 0$, there exists $c := c(\pi) > 0$ such that

$$\sum_{\substack{p^k \leq x \\ p^k \equiv \mathfrak{a} \pmod{q}}}^* (\log p) |a_\pi(p^k)|^2 = \frac{1}{\phi(q)} x + O_{\pi, \gamma} \left(x \exp \left(-c(\log x)^{\frac{1}{2}} \right) \right)$$

for $q \leq (\log x)^\gamma$ with $(q, A_\pi) = 1$. Moreover, it follows from Theorem 1.6 that there are positive constants $c_0 := c_0(\pi)$, $c := c(\pi)$, and $\theta_0 := \theta_0(\pi)$ such that, for $(\log x)^{-1/2} \leq \theta \leq \theta_0$, $y = x^{1-\theta}$, $\gamma, \nu > 0$, and any $q \leq (\log x)^\gamma$ with $(q, A_\pi) = 1$, we have

$$\sum_{\substack{x-y < p^k \leq x \\ p^k \equiv \mathfrak{a} \pmod{q}}}^* (\log p) |a_\pi(p^k)|^2 = y \left(\frac{1}{\phi(q)} + O_{\pi, \gamma, \nu} \left(\exp(-c(\log x)^{1-\nu}) \right) + O_\pi \left(e^{-c_0/\theta} \right) \right).$$

In addition, by Proposition 2.2, for $1 \leq m \leq 4$, the same estimates hold for the corresponding sums *only* supported over primes.

In the rest of the paper, we prove Theorems 1.1 and 1.5. The structure of the paper is as follows. In Section 2, we start by reviewing some facts and results from the theory of automorphic forms that will be used in the proofs of our main assertions. In Section 3 we show that Theorem 1.1 is a consequence of Theorem 1.3 and then in Section 4 we give a proof of Theorem 1.3. Similarly we describe in Section 5 that Theorem 1.6 implies Theorem 1.5 and then in Section 6 we prove Theorem 1.6.

Notation 1.8. Throughout the paper F is a number field of degree n_F , \mathfrak{p} is a prime ideal of F , $N\mathfrak{q}$ is the norm of an ideal \mathfrak{q} of F , π, π' are cuspidal representations, A_π and $q(\pi)$ are respectively the conductor and the analytic conductor of π , $L(s, \pi)$ is the automorphic L -function associated with π , $\pi \otimes \psi$ is the twist of π by an idèle class character ψ , $\pi \boxplus \pi'$ is the isobaric sum of two cuspidal representations, $\pi \boxtimes \pi'$ is the automorphic representation associated to two cuspidal representations if it exists, $L(s, \pi \times \pi')$ is the Rankin-Selberg L -function associated with π and π' , $A_{\pi \times \pi'}$ and $q(\pi \times \pi')$ are respectively the conductor and the analytic conductor of $\pi \times \pi'$, $\Lambda(\cdot)$ is the number field von Mangoldt function, $h(\mathfrak{q})$ is the number of ray classes modulo \mathfrak{q} , χ and η are ray class characters modulo \mathfrak{q} , and ψ is an idèle class character of F . We use the Landau big- O and Vinogradov \ll notations with their usual meanings. The dependence of the implied constants on the parameter t is denoted by $O_t(\cdot)$ or \ll_t . Throughout the paper we have suppressed the dependence of the constants to the base field F .

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2. PRELIMINARIES

In this section, we review some results from the theory of automorphic representations which we will need in the proof of our theorems. Let π be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$ with local parameters $\alpha_\pi(j, \mathfrak{p})$. For cuspidal representations π and π' and integer $k \geq 1$, we set

$$a_{\pi \times \pi'}(\mathfrak{p}^k) = \sum_{i=1}^m \sum_{j=1}^{m'} \alpha_{\pi \times \pi'}(i, j, \mathfrak{p})^k.$$

Here, $\{\alpha_{\pi \times \pi'}(i, j, \mathfrak{p}) \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq m'\}$ is the collection of local parameters of $L(s, \pi \times \pi')$ at the prime \mathfrak{p} . For $\mathfrak{p} \nmid (A_\pi, A_{\pi'})$, $1 \leq i \leq m$ and $1 \leq j \leq m'$, we have

$$\alpha_{\pi \times \pi'}(i, j, \mathfrak{p}) = \alpha_\pi(i, \mathfrak{p})\alpha_{\pi'}(j, \mathfrak{p}).$$

The best general known bound for $a_{\pi \times \pi'}(\mathfrak{p}^k)$ is

$$(2.1) \quad |a_{\pi \times \pi'}(\mathfrak{p}^k)| \leq mm'(\mathbb{N}\mathfrak{p}^k)^{1 - \frac{1}{m^2+1} - \frac{1}{(m')^2+1}},$$

see [4, Formula (4)]. We denote by $A_{\pi \times \pi'}$ the conductor of $\pi \times \pi'$. The analytic conductor of $\pi \times \pi'$ is

$$\mathfrak{q}(\pi \times \pi') = A_{\pi \times \pi'} \prod_{i=1}^m \prod_{j=1}^{m'} \prod_{v \in S_\infty} (|\kappa_{\pi \times \pi'}(i, j, v)| + 3),$$

where S_∞ is the collection of infinite places of F and the $\kappa_{\pi \times \pi'}(i, j, v)$'s are the parameters of $L(s, \pi \times \pi')$ at infinity. We set $\mathfrak{q}(\pi) := \mathfrak{q}(\pi \times 1)$. It is shown in [9, Lemma A.2] that

$$(2.2) \quad \mathfrak{q}(\pi \times \pi') \leq C_0^{m+m'} \mathfrak{q}(\pi)^{m'} \mathfrak{q}(\pi')^m,$$

for an absolute constant $C_0 > 0$. The parameters at infinity satisfy the bound

$$(2.3) \quad |\Re(\kappa_{\pi \times \pi'}(i, j, v))| \leq 1 - \frac{1}{m^2 + 1} - \frac{1}{(m')^2 + 1}$$

analogous to (2.1) (see [4, Formula (6)]). Also, from the functional equation of the ray class L -function $L(s, \chi)$, we know that the parameters at infinity of χ are either zero or one (see [12, p. 129]). Thus, following an analysis of the parameters at infinity of Rankin-Selberg L -functions, as done in [9, pp. 1119-1121], we have that

$$(2.4) \quad \kappa_{\pi \times \pi' \times \chi}(i, j, v) = \kappa_{\pi \times \pi'}(i, j, v) + c,$$

where c belongs to a finite set depending only on π and π' .

The following estimate on the average size of $a_{\pi \times \pi'}(\mathfrak{n})$ over a short interval will play an important role in the proofs of Theorems 1.1 and 1.5.

Proposition 2.1. *Let π and π' be cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_{m'}(\mathbb{A}_F)$, respectively. Assume that there is $\epsilon_{\pi, \pi'} > 0$ such that*

$$(2.5) \quad \sum_{r \geq 2} \sum_{\mathbb{N}\mathfrak{p}^r \leq x}^* \Lambda(\mathfrak{p}^r) |a_{\pi \times \pi'}(\mathfrak{p}^r)| \ll_{\pi, \pi'} x^{1 - \epsilon_{\pi, \pi'}}.$$

Suppose that $x^{1-\epsilon} \leq u \leq x$, where

$$0 < \epsilon < \min \left\{ \frac{1}{n_F \max\{m, m'\}^2 + 1}, \epsilon_{\pi, \pi'} \right\}.$$

Then

$$\sum_{x < \mathbb{N}\mathfrak{n} \leq x+u}^* \Lambda(\mathfrak{n}) |a_{\pi \times \pi'}(\mathfrak{n})| \ll_{\pi, \pi', \epsilon} u \log x.$$

Proof. Let $b_{\pi \times \pi'}(\mathbf{n})$ be the coefficients coming from the formal identity

$$\sum_{\mathbf{n} \neq \mathbf{0}} \frac{b_{\pi \times \pi'}(\mathbf{n})}{N\mathbf{n}^s} = \prod_{\mathfrak{p}} \prod_{i=1}^m \prod_{j=1}^{m'} \left(1 - \frac{\alpha_{\pi \times \pi'}(i, j, \mathfrak{p})}{N\mathfrak{p}^s} \right)^{-1}.$$

From the number field analogue of [17, Eq. (1.10)], we have

$$(2.6) \quad \sum_{N\mathbf{n} \leq x} b_{\pi \times \pi'}(\mathbf{n}) = c_{\pi} x + O_{\pi, \lambda} \left(x^{\frac{n_F m^2 - 1}{n_F m^2 + 1} + \lambda} \right),$$

for some $c_{\pi} > 0$ and any $\lambda > 0$. We note that by [8, Lemma a], each $b_{\pi \times \pi'}(\mathbf{n})$ is non-negative, also $b_{\pi \times \pi'}(\mathfrak{p}) = a_{\pi}(\mathfrak{p})a_{\pi'}(\mathfrak{p})$ for $\mathfrak{p} \nmid (A_{\pi}, A'_{\pi})$. Thus, by employing (2.5), the Cauchy-Schwarz inequality, and (2.6), we have

$$\begin{aligned} \sum_{x < N\mathfrak{p}^m \leq x+u} (\log N\mathfrak{p}) |a_{\pi \times \pi'}(\mathfrak{p}^m)| &\ll (\log x) \sum_{x < N\mathfrak{p} \leq x+u}^* |a_{\pi}(\mathfrak{p})a_{\pi'}(\mathfrak{p})| + x^{1-\epsilon_{\pi, \pi'}} \\ &\ll (\log x) \left(\sum_{x < N\mathfrak{n} \leq x+u} b_{\pi \times \pi'}(\mathbf{n}) \right)^{\frac{1}{2}} \left(\sum_{x < N\mathfrak{n} \leq x+u} b_{\pi' \times \pi'}(\mathbf{n}) \right)^{\frac{1}{2}} + x^{1-\epsilon_{\pi, \pi'}} \\ &\ll (\log x) u^{\frac{1}{2}} u^{\frac{1}{2}} + x^{1-\epsilon_{\pi, \pi'}} \\ &\ll_{\pi, \pi', \epsilon} u \log x, \end{aligned}$$

as long as $x^{1-\epsilon} \leq u \leq x$. \square

We next note that the truth of (2.5) is known for some small m and m' . The following is a number field adaptation of [28, Lemma 3.1] (combined with the Cauchy-Schwarz inequality).

Proposition 2.2. *The assertion (2.5) holds for $m, m' \in \{1, 2, 3, 4\}$.*

We continue with an important theorem on the automorphy of $L(s, \pi \times \pi')$ for GL_2 representations.

Theorem 2.3. (i) *Let π and π' be cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Then there is an automorphic representation $\pi \boxtimes \pi'$ of $\mathrm{GL}_4(\mathbb{A}_F)$ for which $L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi')$. In addition, if π and π' are non-dihedral, then $\pi \boxtimes \pi'$ is cuspidal whenever π and π' are not twist-equivalent.*

(ii) *Let π be a non-dihedral cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then there is a cuspidal representation $\mathrm{Ad}(\pi)$ of $\mathrm{GL}_3(\mathbb{A}_F)$ such that*

$$L(s, \pi \times \pi' \times \psi) = L(s, \mathrm{Ad}(\pi) \otimes \psi) L(s, \psi)$$

for any idèle class character ψ of F .

Proof. Part (i) is contained in Theorem M of [23]. Part (ii) is a consequence of [6, Theorem 9.3]. \square

We next state several results on the existence of the classical zero-free region for $L(s, \pi \times \pi')$. The following theorem is Theorem A.1 of [9].

Theorem 2.4. *Let π and π' be cuspidal representations of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_{m'}(\mathbb{A}_F)$, respectively. Assume as usual that both π and π' are normalized such that their central characters are trivial on the diagonally embedded copy of the positive reals. Assume that π' is self-dual. Then there is an effective absolute constant $c > 0$ such that $L(s, \pi \times \pi')$ is not vanishing for all $s = \sigma + it \in \mathbb{C}$ satisfying*

$$\sigma \geq 1 - \frac{c}{(m + m')^3 \log(\mathfrak{q}(\pi)\mathfrak{q}(\pi')(|t| + 3)^{mn_F})},$$

with the possible exception of one real zero whenever π is also self-dual.

Proof. The proof is given in [9, Appendix A]. \square

Corollary 2.5. *If π is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$, then $L(s, \pi \times \chi)$ satisfies a classical zero-free region for any given ray class character χ modulo \mathfrak{q} . Moreover, if $\pi \otimes \chi$ is not self-dual, then $L(s, \pi \times \chi)$ admits no exceptional zero.*

Proof. Recall that π is normalized such that its central character ω_π is trivial on the diagonally embedded copy of positive reals. Note that $\pi \otimes \chi$ is a cuspidal representation with the central character $\omega_{\pi \otimes \chi}$, where

$$(2.7) \quad \omega_{\pi \otimes \chi} = \omega_\pi \chi^m.$$

Now, since χ is trivial on the diagonally embedded copy of positive reals, then (2.7) shows that $\pi \otimes \chi$ is normalized. Thus, by replacing π in Theorem 2.4 with $\pi \otimes \chi$ and π' with 1 and noting that, by (2.2),

$$(2.8) \quad \mathfrak{q}(\pi \times \chi) \ll_\pi N(\mathfrak{q})^m,$$

we will have the desired result. \square

Remark 2.6. We note that the L -functions $L(s, \pi \otimes \chi)$, $L(s, \pi \times \chi)$, and $L(s, \pi \times 1 \times \chi)$ are the same. We shall use this fact throughout our discussion.

We also need a result on the classical zero-free region for $L(s, \pi)$ when π is not necessarily normalized.

Proposition 2.7. *Let π be a cuspidal representation (not necessarily normalized) of $\mathrm{GL}_m(\mathbb{A}_F)$. Assume, further, that $L(s, \pi \times \pi)$ is entire if $\pi \not\cong \tilde{\pi}$. Then there is an effective absolute constant $c > 0$ such that $L(s, \pi)$ is non-vanishing for all $s = \sigma + it \in \mathbb{C}$ satisfying*

$$\sigma \geq 1 - \frac{c}{(m+1)^3 \log(\mathfrak{q}(\pi)(|t|+3)^{mn_F})},$$

with the possible exception of one real zero whenever π is self-dual.

Proof. The proof is the same as the proof of Theorem A.1 in [9], when $\pi' = 1$. The main facts used in the proof are that $L(s, \pi \times \pi)$ is entire if $\pi \not\cong \tilde{\pi}$ and $L(s, \pi \times \tilde{\pi})$ has a simple pole at $s = 1$ and it is holomorphic everywhere else. See also [12, Theorem 5.10]. \square

Corollary 2.8. *Let π be a self-dual cuspidal representation (not necessarily normalized) of $\mathrm{GL}_m(\mathbb{A}_F)$. Then $L(s, \pi \times \chi)$ satisfies a classical zero-free region for any given ray class character χ modulo \mathfrak{q} .*

Proof. We consider two cases.

Case 1: Assume that $\pi \otimes \chi$ is self-dual. Then by Proposition 2.7 a classical zero-free region is furnished for $L(s, \pi \times \chi)$.

Case 2: Assume that $\pi \otimes \chi$ is not self-dual. We claim that in such case $\pi \otimes \chi \not\cong (\pi \otimes \chi)^\vee \otimes |\cdot|^{i\tau}$ for any $\tau \in \mathbb{R}$. Suppose, on the contrary, that $\pi \otimes \chi \simeq (\pi \otimes \chi)^\vee \otimes |\cdot|^{i\tau}$ for some $\tau \in \mathbb{R}$. As π is self-dual, we conclude that

$$\pi \otimes \chi^2 \simeq \pi \otimes |\cdot|^{i\tau}.$$

From here, by employing (2.7), we have

$$\omega_\pi \chi^{2m} = \omega_\pi |\cdot|^{im\tau}.$$

As π is unitary, the last identity implies that

$$\chi^{2m} = |\cdot|^{im\tau}.$$

Now, since χ is of finite order, we conclude that $\tau = 0$ and thus $\pi \otimes \chi$ is self-dual, a contradiction.

Thus, $\pi \otimes \chi \not\cong (\pi \otimes \chi)^\vee \otimes |\cdot|^{i\tau}$ for any $\tau \in \mathbb{R}$, which implies the holomorphy of $L(s, (\pi \otimes \chi) \times (\pi \otimes \chi))$ everywhere. Therefore, by Proposition 2.7 and (2.8), $L(s, \pi \times \chi)$ has a classical zero-free region. \square

The following result shows that, for certain GL_2 representations, we can dispensed with the self-duality condition of π' in Theorem 2.4.

Theorem 2.9. *Let π and π' be non-dihedral cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Assume that π' is not twist-equivalent to π . Then there is an effective absolute constant $c > 0$ such that $L(s, \pi \times \pi')$ has no zero in the region*

$$\sigma \geq 1 - \frac{c}{\log(A_{\pi \times \pi'}(|t| + 2 + \lambda)^{4n_F})},$$

where λ is the maximum of the absolute value of the infinite parameters of π and π' .

Proof. This is Theorem 4.12(b) in [24]. \square

Remark 2.10. In Theorem 4.12(b) in [24], it is further assumed that π and $\tilde{\pi}'$ are not twist-equivalent by a product of a quadratic character and $|\cdot|^{i\tau}$. However, since π' is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$, $\tilde{\pi}' \simeq \pi' \otimes \omega_{\pi'}^{-1}$, where $\omega_{\pi'}$ is the central character of π' . Thus, if π' is not twist-equivalent to π , then π and $\tilde{\pi}'$ are not twist-equivalent by a product of a quadratic character and $|\cdot|^{i\tau}$.

Corollary 2.11. *Let π and π' be non-dihedral cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$. Assume that π' is not twist-equivalent to π . Then $L(s, \pi \times \pi' \times \chi)$ satisfies a classical zero-free region for any given ray class character χ modulo \mathfrak{q} .*

Proof. Under the given assumptions, $\pi \otimes \chi$ and π' are non-dihedral, and moreover π' is not twist-equivalent to $\pi \otimes \chi$. In addition, from the theory of Rankin-Selberg L -functions (see, e.g., [12, p. 97] for $F = \mathbb{Q}$), we have the relation

$$|\kappa_{\pi \otimes \chi}(i, j, v)| \leq |\kappa_\pi(i, v)| + |\kappa_{\chi}(j, v)| \leq |\kappa_\pi(i, v)| + 1$$

between the infinite parameters of $\pi \otimes \chi$ and the infinite parameters of π . Thus the claimed assertion is a direct corollary of Theorem 2.9 and (2.2). \square

We next review some results on the existence and the locations of the exceptional zeros of L -functions. We start by a Siegel-type bound on the location of the exceptional zeros of the ray class L -functions.

Theorem 2.12. *Let χ be a ray class character modulo \mathfrak{q} . Let β_χ be the possible exceptional zero of $L(s, \chi)$. Then, given $\epsilon > 0$, there is a constant $\kappa(\epsilon)$, depending only on ϵ , such that*

$$\beta_\chi \leq 1 - \frac{\kappa(\epsilon)}{N\mathfrak{q}^\epsilon}.$$

Proof. See [18, Section 1, Lemma 11]. \square

The following result summarizes some cases for which the non-existence of exceptional zeros is known.

Theorem 2.13. (i) *Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and assume that either π is not self-dual or $n = 2, 3$. Then $L(s, \pi)$ does not admit an exceptional zero.*

(ii) *Let π and π' be non-dihedral cuspidal representations of $\mathrm{GL}_2(\mathbb{A}_F)$ that are not twist-equivalent. Then $L(s, \pi \times \pi')$ admits no exceptional zero.*

Proof. Part (i) is a consequence of [8, Corollary 3.2], [8, Theorem C(3)], and [3, Theorem 1]. Part (ii) follows from [24, Theorem A]. \square

Finally, we deduce a log-free zero-density estimate for an automorphic representation twisted by a ray class character.

Theorem 2.14. *Let F be a number field of degree n_F . Let $\Pi = \boxplus_i \pi_i$ be an automorphic representation for $\mathrm{GL}_m(\mathbb{A}_F)$, where each π_i is a cuspidal representation for $\mathrm{GL}_{m_i}(\mathbb{A}_F)$. Set*

$$N(\sigma, T, \Pi) = \#\{\rho = \Re(\rho) + i\Im(\rho) \mid L(\rho, \Pi) = 0, \Re(\rho) \geq \sigma, |\Im(\rho)| \leq T\}.$$

Then there is an absolute constant $c_1 > 0$ such that for $T \geq 1$ and $0 \leq \sigma \leq 1$, one has

$$N(\sigma, T, \Pi) \ll_{\Pi} \sum_i m_i^2(\mathfrak{q}(\pi_i)T^{n_F})^{c_1 m_i^2(1-\sigma)}.$$

Consequently, given a ray class character χ modulo \mathfrak{q} , for $T \geq 1$ and $0 \leq \sigma \leq 1$, there is a positive constant d_{Π} such that

$$N(\sigma, T, \Pi \times \chi) \ll_{\Pi} ((N\mathfrak{q})T)^{d_{\Pi}(1-\sigma)}.$$

Proof. For each i , [14, Corollary 1.2] asserts that

$$N(\sigma, T, \pi_i) \ll m_i^2(\mathfrak{q}(\pi_i)T^{n_F})^{c_1 m_i^2(1-\sigma)}$$

for $1/2 \leq \sigma \leq 1$ and $T \geq n_F$. Now the first part of the theorem follows immediately from the fact that $N(\sigma, T, \Pi) = \sum_i N(\sigma, T, \pi_i)$. Finally, by employing the bound (2.8) for $\mathfrak{q}(\pi_i \times \chi)$, we conclude the proof. (Note that the above bound trivially extends to $0 \leq \sigma \leq 1$.) \square

3. THEOREM 1.3 IMPLIES THEOREM 1.1

We need to show that the conditions (i), (ii), (iii), and (iv) of Theorem 1.3 hold for pairs π and π' associated with Π satisfying either (a), (b), or (c). Note that $\pi' = 1$ in (a) and $\pi' = \tilde{\pi}$ in (b). We observe that, by Propositions 2.1 and 2.2, (i) holds for π and π' associated with Π in (a), (b), or (c). We now establish (ii), (iii), and (iv), for corresponding π and π' in (a), (b), or (c).

(a) The condition (ii) is true, since for any character χ , $\pi \otimes \chi$ is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A}_F)$, with $m > 1$, and thus $L(s, \pi \times 1 \times \chi)$ is holomorphic. Also, by Corollary 2.5, (iii) holds.

To verify the condition (iv), we first note that, by Theorem 2.13(i), for $m = 2$ and 3 none of the $L(s, \pi \times 1 \times \chi)$'s admit an exceptional zero in their classical zero-free region. If π is not essentially self-dual, then each $\pi \otimes \chi$ is not self-dual. (Suppose, on the contrary, that for some character χ the contragredient representation of $\pi \otimes \chi$ is equivalent to $\pi \otimes \chi$. A direct calculation shows that $\pi \simeq \tilde{\pi} \otimes \bar{\chi}^2$, a contradiction.) Therefore, by Corollary 2.5, $L(s, \pi \times 1 \times \chi)$ admits no exceptional zero if π is not essentially self-dual. Thus, (iv) holds trivially.

(b) Let $\Pi \simeq \pi \boxtimes \tilde{\pi}$. Then, by Theorem 2.3(ii) we have

$$(3.1) \quad L(s, \pi \times \tilde{\pi} \times \chi) = L(s, \mathrm{Ad}(\pi) \otimes \chi)L(s, \chi)$$

for any ray class character χ . Since $\mathrm{Ad}(\pi)$ is cuspidal, $\mathrm{Ad}(\pi) \otimes \chi$ is also cuspidal, and so $L(s, \mathrm{Ad}(\pi) \otimes \chi)$ is holomorphic. Hence, if $L(s, \pi \times \tilde{\pi} \times \chi)$ admits a pole, then it is contributed by $L(s, \chi)$. This happens only if χ is the principal character χ_0 . Thus (ii) holds.

To verify (iii), we note that since $\mathrm{Ad}(\pi)$ is self-dual, by (3.1), Corollary 2.8, and the classical zero-free region for $L(s, \chi)$, we deduce that $L(s, \pi \times \tilde{\pi} \times \chi)$ has either no zeros or possibly only one simple real zero β_{χ} in the region

$$(3.2) \quad \sigma \geq 1 - \frac{c_{\pi}}{\log((N\mathfrak{q})(|t| + 3))}$$

for some $c_{\pi} > 0$ only depending on π . More precisely, this region is obtained by the intersection of the zero-free region for $L(s, \mathrm{Ad}(\pi) \otimes \chi)$ given by Corollary 2.8 with the classical zero-free region for $L(s, \chi)$ given in Corollary 2.5. Note that since χ has degree one, then the constant c in the

classical zero-free region for $L(s, \chi)$ is absolute, so c_π in (3.2) is independent of χ . Thus, (iii) holds.

Now, as $\text{Ad}(\pi) \otimes \chi$ is cuspidal and of degree 3, the first part of Theorem 2.13 yields the non-existence of the exceptional zero for $L(s, \text{Ad}(\pi) \otimes \chi)$. Thus, if the exceptional zero β_χ of $L(s, \pi \times \tilde{\pi} \times \chi)$ exists, it has to come from $L(s, \chi)$. Therefore, β_χ depends only on χ . In this case, by Theorem 2.12, there is a constant $\kappa(\epsilon)$, depending only on ϵ , such that

$$\beta_\chi \leq 1 - \frac{\kappa(\epsilon)}{N\mathfrak{q}^\epsilon},$$

where \mathfrak{q} is the modulus of the ray class character χ . Thus, (iv) holds trivially.

(c) Under the assumptions, we know that $\pi \otimes \chi$ and π' are non-dihedral cuspidal representations of $\text{GL}_2(\mathbb{A}_F)$ that are not twist-equivalent. Thus, it follows from Theorem 2.3(i) that $L(s, \pi \times \pi' \times \chi) = L(s, (\pi \otimes \chi) \boxtimes \pi')$ for the cuspidal representation $(\pi \otimes \chi) \boxtimes \pi'$ of $\text{GL}_4(\mathbb{A}_F)$. So, $L(s, \pi \times \pi' \times \chi)$ is entire. This settles (ii).

The condition (iii) is a direct consequence of Corollary 2.11.

Finally, by Theorem 2.13(ii), we know that under the given conditions $L(s, \pi \times \pi' \times \chi)$ has no exceptional zero. Thus, (iv) holds.

Hence, by Theorem 1.3, for Π satisfying either (a), (b), or (c), we have that for any $\gamma > 0$, there exists $c = c(\Pi) > 0$ such that for any ideal \mathfrak{q} , with $N\mathfrak{q} \leq (\log x)^\gamma$, and any ideal \mathfrak{a} relatively prime to \mathfrak{q} , (1.8) holds. Now (1.3) follows from (1.8) and Proposition 2.2.

4. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. We start by collecting some analytic properties of $L(s, \pi \times \pi' \times \chi)$ in the following lemma.

Lemma 4.1. (i) For $\delta > 0$, let $\mathbb{C}(\delta)$ denote the set

$$\mathbb{C} \setminus \{s \in \mathbb{C} : |s + \kappa_{\pi \times \pi' \times \chi}(i, j, v) + 2k| \leq \delta, \text{ for } v \in S_\infty, 1 \leq i \leq m, 1 \leq j \leq m', \text{ and integers } k \geq 0\}.$$

Let $\sigma \leq -1/2$. Then for all $s = \sigma + it \in \mathbb{C}(\delta)$,

$$\frac{L'(s, \pi \times \pi' \times \chi)}{L(s, \pi \times \pi' \times \chi)} \ll_{\pi, \pi', \delta} \log((N\mathfrak{q})|s|).$$

(ii) For any integer $m \geq 2$, there is T_m , with $m \leq T_m \leq m + 1$, such that

$$\frac{L'(\sigma \pm iT_m, \pi \times \pi' \times \chi)}{L(\sigma \pm iT_m, \pi \times \pi' \times \chi)} \ll_{\pi, \pi'} \log^2((N\mathfrak{q})T_m)$$

uniformly for $-2 \leq \sigma \leq 2$.

(iii) Let $N(t, \pi \times \pi' \times \chi)$ be the number of the zeros ρ of $L(s, \pi \times \pi' \times \chi)$ in the region $0 \leq \Re(s) \leq 1$, where $t - 1 \leq \Im(\rho) \leq t + 1$. Then

$$N(t, \pi \times \pi' \times \chi) \ll_{\pi, \pi'} \log((N\mathfrak{q})(|t| + 3)).$$

(iv) Let

$$b_{\pi, \pi'}(\chi) = \lim_{s \rightarrow 0} \left(\frac{L'(s, \pi \times \pi' \times \chi)}{L(s, \pi \times \pi' \times \chi)} - \frac{r}{s} \right),$$

where the integer $r \geq 0$ is the order of vanishing of $L(s, \pi \times \pi' \times \chi)$ at $s = 0$. Then

$$b_{\pi, \pi'}(\chi) = O_{\pi, \pi'}(\log N\mathfrak{q}) - \sum_{\substack{0 < \Re(\rho) \leq 1 \\ |\Im(\rho)| \leq 1}} \frac{1}{\rho},$$

where $\rho \neq 0$ ranges over the non-trivial zeros of $L(s, \pi \times \pi' \times \chi)$.

Proof. For (i) see [20, p. 177] for a single GL_2 automorphic L -function over $F = \mathbb{Q}$. The general case is similar. See [15, Lemma 4.3(a)(d)] for (ii) and (iii) for $F = \mathbb{Q}$, again the proof for general F is similar. The assertion (iv) is a consequence [12, Proposition 5.7(2)], (2.2), and (2.4). \square

The following lemma evaluates a contour integral that will appear in the proof.

Lemma 4.2. *Let $b > 1$, $2 \leq T \leq x$, and $\mathrm{Nq} \leq x$. We have*

$$(4.1) \quad \frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{L'}{L}(s, \pi \times \pi' \times \chi) \frac{x^s}{s} ds = \delta(\pi \times \pi' \times \chi)x - \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} \\ - b_{\pi, \pi'}(\chi) + O_{\pi, \pi'}\left(\frac{x \log x}{T}\right) + O_{\pi, \pi'}\left(x^{1-\frac{1}{m^2+1}-\frac{1}{(m')^2+1}}\right),$$

where $\delta(\pi \times \pi' \times \chi)$ is 1 if $L(s, \pi \times \pi' \times \chi)$ has a simple pole at $s = 1$ and is zero otherwise, $\rho \neq 0$ ranges over the non-trivial zeros of $L(s, \pi \times \pi' \times \chi)$, and $b_{\pi, \pi'}(\chi)$ is the expression defined in part (iv) of Lemma 4.1.

Proof. The proof is standard and follows closely the arguments given in [5, Chapter 19] for the case $\pi = \pi' = 1$ over $F = \mathbb{Q}$ and the arguments given in Proposition 4.2 of [1] for a single automorphic L -function π over $F = \mathbb{Q}$. In fact (4.1) is a consequence of computing the residues of the integrand upon moving the line of integration to the left and employing (2.3) and parts (i), (ii), and (iii) of Lemma 4.1. Note that the errors terms

$$O(\log x) + O_{\pi, \pi'}\left(\frac{x \log^2((\mathrm{Nq})T)}{T \log x}\right) + O_{\pi, \pi'}\left(\frac{x \log((\mathrm{Nq})T)}{T}\right)$$

appearing in the process can be combined as the first error term in (4.1) under the assumptions $T \leq x$ and $\mathrm{Nq} \leq x$. See [1, Proposition 4.2] for details. \square

We also need a version of the truncated Perron's formula due to Liu and Ye [16, Theorem 2.1].

Lemma 4.3. *Let $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be an absolutely convergent series in the half-plane $\sigma > \sigma_a$. Let $B(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}$ for $\sigma > \sigma_a$. Then for $b > \sigma_a$, $x \geq 2$, $T \geq 2$, and $H \geq 2$,*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\sum_{x-x/H < n \leq x+x/H} |a_n|\right) + O\left(\frac{Hx^b B(b)}{T}\right).$$

We now have all the necessary tools for the proof in our disposal.

Proof of Theorem 1.3. We only describe the proof that implies (1.8). The argument can be adjusted to obtain (1.6). Let $\epsilon_0 > 0$ be as given in the assumption (i) in the statement of the theorem. Assume that $x \geq 2$, $T \geq 4$, $T \leq x^{\epsilon_0}$, and $\mathrm{Nq} \leq x$. In Lemma 4.3, set $H = T^{1/2}$, $b = 1 + 1/\log x$, and $f(s) = -\frac{L'}{L}(s, \pi \times \pi' \times \chi)$. Then employing Proposition 2.1 for $u = x/T^{1/2}$ and the bound (2.1) for $\mathfrak{p} \mid (A_\pi, A_{\pi'})$ yield

$$(4.2) \quad \sum_{\mathrm{Nn} \leq x}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) \chi(\mathfrak{n}) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{L'}{L}(s, \pi \times \pi' \times \chi) \frac{x^s}{s} ds \\ + O_{\pi, \pi'}\left(\frac{x \log x}{T^{1/2}}\right) + O_{\pi, \pi'}\left(x^{1-\frac{1}{m^2+1}-\frac{1}{(m')^2+1}} (\log x)^2\right).$$

For the integral in (4.2), by employing Lemma 4.2 and part (iv) of Lemma 4.1, we deduce

$$(4.3) \quad \frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{L'}{L}(s, \pi \times \pi' \times \chi) \frac{x^s}{s} ds = \delta(\pi \times \pi' \times \chi)x - \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} \\ + \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq 1 \\ \rho \neq \beta_\chi, 1-\beta_\chi}} \frac{1}{\rho} + \frac{1}{\beta_\chi} + \frac{1}{1-\beta_\chi} + O_{\pi, \pi'} \left(\frac{x \log x}{T} \right) + O_{\pi, \pi'} \left(x^{1-\frac{1}{m^2+1} - \frac{1}{(m')^2+1}} \right),$$

where β_χ is the possible exceptional zero of $L(s, \pi \times \pi' \times \chi)$, and, as later, all terms contributed by β_χ should be omitted if β_χ does not exist.

Next we focus on the sums in (4.3) involving the non-trivial zeros ρ . First of all, by the assumption (iii) and the symmetry of the non-trivial zeros respect to line $\Re(s) = 1/2$, for any low-lying zero $\rho \neq \beta_\chi, 1 - \beta_\chi$, we have $\rho^{-1} = O(\log N\mathfrak{q})$. Thus, by Lemma 4.1(iii), we deduce that

$$(4.4) \quad \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq 1 \\ \rho \neq \beta_\chi, 1-\beta_\chi}} \frac{1}{\rho} = O_{\pi, \pi'}(\log^2 N\mathfrak{q}),$$

so this term can be absorbed in the second error term in the right-hand side of (4.3). Secondly, by Lemma 4.1(iii), we have

$$\sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ 3 < |\Im(\rho)| \leq T}} \frac{1}{|\rho|} \ll \sum_{3 \leq t < T} \frac{N(t, \pi \times \pi' \times \chi)}{t} \ll_{\pi, \pi'} (\log T) \log((N\mathfrak{q})T).$$

This together with the assumption (iii), the classical zero-free region of $L(s, \pi \times \pi' \times \chi)$, gives

$$(4.5) \quad \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ 3 < |\Im(\rho)| \leq T}} \left| \frac{x^\rho}{\rho} \right| \ll_{\pi, \pi'} (\log T) (\log((N\mathfrak{q})T)) x^{1-c_{\pi, \pi'}(\log((N\mathfrak{q})(T+3)))^{-1}}.$$

Similarly, for non-exceptional zeros ρ with $|\Im(\rho)| \leq 3$, the assumption (iii) yields

$$(4.6) \quad \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq 3 \\ \rho \neq \beta_\chi, 1-\beta_\chi}} \left| \frac{x^\rho}{\rho} \right| \ll_{\pi, \pi'} (\log^2 N\mathfrak{q}) x^{1-c_{\pi, \pi'}(\log(6N\mathfrak{q}))^{-1}},$$

where β_χ is the possible exceptional zero of $L(s, \pi \times \pi' \times \chi)$. Now let $T = \exp((\log x)^{1/2})$. Then, for $N\mathfrak{q} \leq \exp((\log x)^{1/2})$, (4.5) and (4.6) yield

$$(4.7) \quad \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq T \\ \rho \neq \beta_\chi, 1-\beta_\chi}} \left| \frac{x^\rho}{\rho} \right| \ll_{\pi, \pi'} x \exp(-\hat{c}_{\pi, \pi'}(\log x)^{1/2})$$

for a constant $\hat{c}_{\pi, \pi'}$ depending only on π and π' .

Recall that χ is of modulus \mathfrak{q} and that by the assumption (iv), for any $\epsilon > 0$, there is a constant $\kappa(\epsilon) := \kappa(\epsilon, \pi, \pi')$, so that the exceptional zero β_χ of $L(s, \pi \times \pi' \times \chi)$ satisfies $\beta_\chi \leq 1 - \kappa(\epsilon)N\mathfrak{q}^{-\epsilon}$. Thus, for $N\mathfrak{q} \leq (\log x)^\gamma$, we have

$$(4.8) \quad \frac{x^{\beta_\chi} - 1}{\beta_\chi} + \frac{x^{1-\beta_\chi} - 1}{1-\beta_\chi} \leq 2 \frac{x^{\beta_\chi} - 1}{1-\beta_\chi} \ll \frac{x^{1-\kappa(\epsilon)(\log x)^{-\epsilon\gamma}}}{\kappa(\epsilon)(\log x)^{-\epsilon\gamma}}.$$

For $\gamma > 0$, let $\epsilon = 1/3\gamma$. Then, for x satisfying $(\log x)^\gamma \leq \exp((\log x)^{1/2})$, $T = \exp((\log x)^{1/2})$, and $N\mathfrak{q} \leq (\log x)^\gamma$, from (4.7) and (4.8), we get

(4.9)

$$\sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq T \\ \rho \neq \beta_\chi, 1 - \beta_\chi}} \left| \frac{x^\rho}{\rho} \right| + \left| \frac{x^{\beta_\chi} - 1}{\beta_\chi} + \frac{x^{1 - \beta_\chi} - 1}{1 - \beta_\chi} \right| \\ \ll_{\pi, \pi'} x \left(\exp(-\hat{c}_{\pi, \pi'}(\log x)^{1/2}) + \kappa(1/3\gamma)^{-1}(\log x)^{1/3} \exp(-\kappa(1/3\gamma)(\log x)^{2/3}) \right).$$

Let $C := C(\pi, \pi', \gamma)$ be a positive constant such that for $x > C$,

(4.10)
$$\max\{4, (\log x)^\gamma\} \leq T = \exp((\log x)^{1/2}) \leq x^{\epsilon_0}.$$

Thus, for $x > C$, T as in (4.10), and $N\mathfrak{q} \leq (\log x)^\gamma$, by employing (4.9) in (4.3), the asymptotic formula (4.2) can be written as

(4.11)
$$\sum_{N\mathfrak{n} \leq x}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) \chi(\mathfrak{n}) = \delta(\pi \times \pi' \times \chi)x + O_{\pi, \pi', \gamma} \left(x \exp(-c(\log x)^{1/2}) \right)$$

for some $c := c(\pi, \pi') > 0$.

The final result follows from (4.11), the orthogonality property of ray class characters, i.e.,

(4.12)
$$\sum_{\substack{N\mathfrak{n} \leq x \\ \mathfrak{n} \sim \mathfrak{a} \pmod{\mathfrak{q}}}}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) = \frac{1}{h(\mathfrak{q})} \sum_{\chi \pmod{\mathfrak{q}}} \bar{\chi}(\mathfrak{a}) \sum_{N\mathfrak{n} \leq x}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) \chi(\mathfrak{n}),$$

and the assumption (ii) in the statement of the theorem. \square

5. THEOREM 1.6 IMPLIES THEOREM 1.5

It is enough to show that the conditions (i), (ii), (iii), (iv), and (v) of Theorem 1.6 hold for pairs π and π' associated with Π in Theorem 1.5. Then (1.11) together with Proposition 2.2 imply (1.9). Note that (i), (ii), (iii) and (v) of Theorem 1.6 are the same as (i), (ii), (iii) and (iv) in Theorem 1.3. So following the arguments of Section 3, the conditions (i), (ii), (iii), and (v) hold for π and π' associated with Π in Theorem 1.5. Since Π in (a) is cuspidal and, by Theorem 2.3(i), in (b) (resp., (c)) is automorphic (resp., cuspidal), then, by Theorem 2.14, the condition (iv) of Theorem 1.6 also holds for the corresponding π and π' .

6. PROOF OF THEOREM 1.6

Proof of Theorem 1.6. We only describe the proof that implies (1.11) since the argument can be adjusted to establish (1.10). The proof closely follows the proof of Theorem 1 in [21]. Let $\epsilon_0 > 0$ be as given in the assumption (i) of the theorem and without loss of generality assume that $0 < \epsilon_0 \leq 4/5$. Let x , T , $N\mathfrak{q}$, H , b , and $f(s)$ be as in the beginning of the proof of Theorem 1.3, so by following the initial steps of the proof of Theorem 1.3 we get (4.2) and (4.3). Let $0 < \theta \leq \epsilon_0/4$, $T = x^{4\theta}$, $y = x^{1-\theta}$, and $N\mathfrak{q} \leq x^\theta$. Then, from (4.2), (4.3), and (4.4), we deduce

(6.1)
$$\sum_{x-y < N\mathfrak{n} \leq x}^* \Lambda(\mathfrak{n}) a_{\pi \times \pi'}(\mathfrak{n}) \chi(\mathfrak{n}) = \delta(\pi \times \pi' \times \chi)y - \sum_{\substack{0 < |\Re(\rho)| \leq 1 \\ |\Im(\rho)| \leq T}} \hat{g}(\rho) + O_{\pi, \pi'}(y x^{-\theta} \log x),$$

where \hat{g} is the Mellin transform of $g = \mathbf{1}_{(x-y, x]}$ (the indicator function of the interval $(x - y, x]$). Next, we note that for $T \geq 4$ we can find a constant $\tilde{c}_{\pi, \pi'}$ such that

$$\frac{c_{\pi, \pi'}}{\log((N\mathfrak{q})(T + 3))} \geq \frac{\tilde{c}_{\pi, \pi'}}{\log((N\mathfrak{q})T)},$$

where $c_{\pi, \pi'}$ is the constant given in the assumption (iii).

To control the zero-sum in (6.1), we shall apply the assumptions (iii) and (iv) of the theorem, and [12, Theorem 5.8] together with (2.2), to deduce

$$\begin{aligned} \sum_{\rho \neq \beta_\chi} |\hat{g}(\rho)| &\leq \int_{x-y}^x (\log t) \left(\int_0^{1 - \frac{\tilde{c}_{\pi, \pi'}}{\log((Nq)T)}} N(\sigma, T, \pi \times \pi' \times \chi) t^{\sigma-1} d\sigma \right) dt \\ &\quad + N(0, T, \pi \times \pi' \times \chi) \int_{x-y}^x \frac{dt}{t} \\ &\ll_{\pi, \pi'} y(\log x) \int_0^{1 - \frac{\tilde{c}_{\pi, \pi'}}{\log((Nq)T)}} ((Nq)T)^{d_{\pi \times \pi'}(1-\sigma)} x^{\sigma-1} d\sigma + (y/x)T \log((Nq)T), \end{aligned}$$

where β_χ is the possible exceptional zero of $L(s, \pi \times \pi' \times \chi)$. Now let $\theta \leq 1/(10d_{\pi, \pi'})$ so that $((Nq)T)^{d_{\pi, \pi'}} \leq x^{1/2}$. (Recall that $Nq \leq x^\theta$ and $T = x^{4\theta}$.) With these choices, we get

$$(6.2) \quad \sum_{\rho \neq \beta_\chi} |\hat{g}(\rho)| \ll_{\pi, \pi'} y \exp\left(-\frac{c_0}{\theta}\right) + yx^{4\theta-1} \log x$$

for a constant c_0 depending on π and π' . Now, by choosing x such that $(\log x)^{-1/2} \leq \theta$ we deduce $x^{-\theta} \log x < \exp(-1/2\theta)$. So, by adjusting $\tilde{c}_{\pi, \pi'}$, we see that the error terms $O_{\pi, \pi'}(yx^{-\theta} \log x)$ in (6.1) and $O_{\pi, \pi'}(yx^{4\theta-1} \log x)$ in (6.2) can be absorbed in the term $y \exp(-\frac{c_0}{\theta})$ in (6.2). (Note that $x^{4\theta-1} \leq x^{-\theta}$ since $\theta \leq \epsilon_0/4 \leq 1/5$.) Thus, inserting the derived bound for $\sum_{\rho \neq \beta_\chi} |\hat{g}(\rho)|$ in (6.1) yields

$$(6.3) \quad \sum_{x-y < Nn \leq x}^* \Lambda(n) a_{\pi \times \pi'}(n) \chi(n) = \delta(\pi \times \pi' \times \chi) y + O(|\hat{g}(\beta_\chi)|) + O\left(y \exp\left(-\frac{c_0}{\theta}\right)\right).$$

To treat the term involving $|\hat{g}(\beta_\chi)|$, observe that the mean value theorem implies that

$$|\hat{g}(\beta_\chi)| = \int_{x-y}^x t^{\beta_\chi-1} dt = \frac{x^{\beta_\chi} - (x-y)^{\beta_\chi}}{\beta_\chi} = y\xi^{\beta_\chi-1},$$

for some $\xi \in (x-y, x)$. We conclude that

$$|\hat{g}(\beta_\chi)| = O(yx^{\beta_\chi-1}).$$

Now inserting this bound in (6.3) and employing the orthogonality relation (4.12), together with the assumptions (ii) and (v) in the statement of the theorem, imply the result. \square

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