Explicit upper bounds for $\prod_{p \leq p_{\omega(n)}} \frac{p}{p-1}$

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Abstract

Let $\omega(n)$ be the number of distinct prime divisors of n, $\phi(n)$ be the Euler totient function, $\sigma(n)$ be the sum of divisors of n, p_n be the n-th prime and γ be the Euler constant. We consider the arithmetic function

$$f(n) = \prod_{\substack{p \leq p_{\omega(n)} \\ p \text{ prime}}} \frac{p}{p-1}$$

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and show that

$$\overline{\lim} \frac{f(n)}{e^\gamma \log\log n} = 1.$$

Next we describe an algorithm that for a given $0 < \epsilon < 1$ determines all the exceptions to the inequality

$$f(n) < e^\gamma (1+\epsilon) \log\log n.$$

Finally by employing this algorithm we establish some explicit upper bounds for $n/\phi(n)$ and $\sigma(n)/n$. More specifically, we prove that

$$\frac{\sigma(n)}{n \log\log n} \leq \frac{\sigma(180)}{180 \log\log 180} = (1.0338...)e^\gamma, \quad \text{for} \quad n \geq 121.$$

1 Introduction

Let $\sigma(n)$ denote the sum of divisors function and $\phi(n)$ the Euler totient function, so that $\sigma(n)\phi(n) < n^2$. Nicolas [4] proved that $n/\phi(n) > e^\gamma \log\log n$ infinitely often. Also for the smaller quantity $\sigma(n)/n$, Robin showed that $\sigma(n)/n > e^\gamma \log\log n$ infinitely often provided the Riemann Hypothesis is false. More precisely, let $g(n) = \sigma(n)/n \log\log n$, then in [5], Robin proved the following.

Theorem 1.1 (**Robin**) *The Riemann Hypothesis is true if and only if*

$$g(n) < e^\gamma, \quad \text{for} \quad n \geq 5041.$$

Here, γ is the Euler constant.

As a consequence of this theorem one can show that under the assumption of the Riemann Hypothesis the only values of n that fail $g(n) < e^\gamma$ are $n = 2$, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240,360, 720, 840, 2520 and 5040 ([5], p. 204). The following table shows the values of $g(n)$ in decreasing order for the above exceptions (other than $n = 2$) to the inequality $g(n) < e^\gamma$.

n	3	4	6	12	8	5	24
$g(n)$	14.177...	5.357...	3.429...	2.563...	2.561...	2.521...	2.162...

10	18	60	36	30	120	20	48
2.158...	2.041...	1.986...	1.980...	1.960...	1.915...	1.913...	1.908...

16	72	180	9	360	240	2520	840
1.899...	1.863...	1.841...	1.834...	1.833...	1.822...	1.804...	1.797...

84	5040	720
1.791...	1.790...	1.782...

The following assertion is a direct corollary of Robin's theorem together with the values recorded in the above table.

Corollary 1.2 *Under the assumption of the Riemann Hypothesis, we have the following inequalities:*
(i) $g(n) \leq g(3) = (7.959914266...)e^\gamma$, *for* $n \geq 3$.
(ii) $g(n) \leq g(4) = (3.008117079...)e^\gamma$, *for* $n \geq 4$.
(iii) $g(n) \leq g(6) = (1.925450381...)e^\gamma$, *for* $n \geq 5$.
(iv) $g(n) \leq g(12) = (1.439267874...)e^\gamma$, *for* $n \geq 7$.
(v) $g(n) \leq g(24) = (1.213946496...)e^\gamma$, *for* $n \geq 13$.
(vi) $g(n) \leq g(60) = (1.115266133...)e^\gamma$, *for* $n \geq 25$.
(vii) $g(n) \leq g(120) = (1.075588326...)e^\gamma$, *for* $n \geq 61$.
(viii) $g(n) \leq g(180) = (1.033867784...)e^\gamma$, *for* $n \geq 121$.
(ix) $g(n) \leq g(360) = (1.029419589...)e^\gamma$, *for* $n \geq 181$.
(x) $g(n) \leq g(2520) = (1.013215898...)e^\gamma$, *for* $n \geq 361$.
(xi) $g(n) \leq g(5040) = (1.005558981...)e^\gamma$, *for* $n \geq 2521$.

One of our goal in this paper is to prove some of the above inequalities unconditionally. Note that establishing one of these inequalities in one line will also establish all the inequalities in the previous lines. The inequality (iv) is proved by Robin ([5], Proposition 2). This inequality gives an improvement of a previous result of Ivić [3]. Here we will prove the following.

Theorem 1.3 $g(n) \leq g(180) = (1.0338...)e^\gamma$, *for* $n \geq 121$. *More precisely,* $g(n) < (1.03)e^\gamma$, *for* $n \geq 121$ *except for* $n = 180$.

The methodology of the proof is as follows. Let

$$f(n) = \prod_{\substack{p \leq p_{\omega(n)} \\ p \text{ prime}}} \frac{p}{p-1},$$

where $\omega(n)$ is the number of distinct prime divisors of n and p_n is the n-th prime. We develop an algorithm that generates all the exceptions to the inequality

$$f(n) < (1.03)e^\gamma \log\log n.$$

Since $\frac{\sigma(n)}{n} < f(n)$, it is clear that the exceptions to

$$g(n) \leq g(180) = (1.0338...)e^\gamma$$

are among the exceptions to

$$f(n) < (1.03)e^{\gamma} \log\log n.$$

So numerically checking the exceptions generated by the algorithm against the inequality

$$g(n) < (1.03)e^{\gamma}$$

will establish the result.

The structure of the paper is as follows. In section 2, we study some properties of the arithmetic function $f(n)$. We describe the algorithm in section 3. In the last section we employ our algorithm to establish an explicit upper bound for $n/\phi(n)$ and prove Theorem 1.3.

2 The Arithmetic Function $f(n)$

Let

$$f(n) = \prod_{\substack{p \leq p_{\omega(n)} \\ p \text{ prime}}} \frac{p}{p-1}.$$

We have

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq f(n).$$

The right-hand side inequality is trivial and the left-hand side one is Theorem 329 of [1].

For given n let $p_1, \cdots, p_{\omega(n)-l}$ be the primes less than or equal to $\log n$ and $p_{\omega(n)-l+1}, \cdots, p_{\omega(n)}$ be those which exceed $\log n$. One can show that

$$l < \frac{\log n}{\log\log n}$$

and

$$f(n) < \left(1 - \frac{1}{\log n}\right)^{-\frac{\log n}{\log\log n}} \prod_{p \leq \log n} \frac{p}{p-1}. \tag{1}$$

Proposition 2.1 $\overline{\lim} \dfrac{f(n)}{e^{\gamma} \log\log n} = 1.$

Proof. First of all recall Mertens's theorem

$$\prod_{p \leq x} \frac{p}{p-1} \sim e^{\gamma} \log x$$

as $x \to \infty$ ([1], Theorem 429). Now an application of Mertens's theorem in (1) implies

$$\overline{\lim} \frac{f(n)}{e^\gamma \log\log n} \leq 1.$$

Let $n_x = \prod_{p \leq x} p$. By the Prime Number Theorem we know that $\log\log n_x \sim \log x$ as $x \to \infty$, so by Mertens's theorem

$$\lim_{x\to\infty} \frac{f(n_x)}{e^\gamma \log\log n_x} = 1.$$

This completes the proof. □

Corollary 2.2 *For any $\epsilon > 0$, there exists a number N_ϵ, such that*

$$f(n) < e^\gamma(1+\epsilon)\log\log n \tag{2}$$

for all $n > N_\epsilon$.

Our next goal is to establish explicit versions for inequality (2). More precisely, for a given $\epsilon > 0$, we like to find an algorithm that finds the smallest value for N_ϵ. In next section we describe an algorithm that for given $\epsilon > 0$ generates all the exceptions to the inequality (2).

3 Explicit Upper Bounds for $f(n)$

For $\epsilon > 0$, let $M_\epsilon = \exp\left(\exp\left(\sqrt{\frac{2.50637}{\epsilon e^\gamma}}\right)\right)$.

Lemma 3.1 (2) *holds for $n > M_\epsilon$ with $\omega(n) \geq 3$.*

Proof. Let $Q_{\omega(n)} = p_1 \cdots p_{\omega(n)}$. Then by Theorem 15 of [6], we have

$$f(n) = \prod_{p \mid Q_{\omega(n)}} \frac{p}{p-1} = \frac{Q_{\omega(n)}}{\phi(Q_{\omega(n)})} < e^\gamma \log\log Q_{\omega(n)} + \frac{2.50637}{\log\log Q_{\omega(n)}}$$

for n with $\omega(n) > 1$. Note that $Q_{\omega(n)} \leq n$ and the function $g(t) = e^\gamma t + \frac{2.50637}{t}$ is increasing for $t \geq 1.2$. So for any n with $\omega(n) \geq 3$, we have

$$f(n) < e^\gamma \log\log n + \frac{2.50637}{\log\log n}.$$

From here, it is clear that (2) holds for any $n > M_\epsilon$ with $\omega(n) \geq 3$. $\square$

For integer $\beta \geq 1$, let

$$n_\beta = \exp\left(\exp\left(\frac{1}{(1+\epsilon)e^\gamma}\prod_{p \leq p_\beta}\frac{p}{p-1}\right)\right).$$

Note that n_β depends on ϵ. For simplicity we use n_β instead of $n_{\beta,\epsilon}$.

Lemma 3.2 (2) *holds for all* $n > n_\beta$ *with* $\omega(n) \leq \beta$.

Proof. Let $n > n_\beta$ with $\omega(n) \leq \beta$, then

$$f(n) \leq \prod_{p \leq p_\beta}\frac{p}{p-1} < e^\gamma(1+\epsilon)\log\log n. \ \square$$

Since for $0 < \epsilon < 1$, $n_2 < M_\epsilon$ then the following is a direct corollary of Lemmas 3.1 and 3.2.

Corollary 3.3 *Let* $0 < \epsilon < 1$. *Then for* $n > M_\epsilon$ *the inequality* (2) *holds.*

Next we give a description for all exceptions to the inequality $f(n) < e^\gamma(1+\epsilon)\log\log n$ with $\omega(n) \leq \beta$.

Lemma 3.4 *Let* $\epsilon > 0$ *be real and* $\beta \geq 2$ *be an integer.*

(i) *If* $\prod_{p \leq p_\beta} p > n_\beta$ *then* (2) *holds for all* $n_{\beta-1} < n \leq n_\beta$.

(ii) *If* $\prod_{p \leq p_\beta} p \leq n_\beta$ *then integers with* β *distinct prime divisors not exceeding* n_β *fail* (2).

Proof. (i) If $\prod_{p \leq p_\beta} p > n_\beta$ then for any $n \leq n_\beta$ we have $\omega(n) \leq \beta - 1$, and so by Lemma 3.2, (2) holds for $n_{\beta-1} < n \leq n_\beta$.

(ii) If n is an integer with exactly β prime factors and $n \leq n_\beta$, then

$$f(n) = \prod_{p \leq p_\beta}\frac{p}{p-1} \geq e^\gamma(1+\epsilon)\log\log n. \ \square$$

We are ready to describe our algorithm.

Algorithm for finding exceptions to $f(n) < e^\gamma(1+\epsilon)\log\log n$

- Input: $0 < \epsilon < 1$.
- Calculate M_ϵ.
- Find the largest β such that
$$\prod_{p \le p_\beta} p \le M_\epsilon.$$
- While
$$\prod_{p \le p_\beta} p > n_\beta$$
then $\beta \leftarrow \beta - 1$.
- Calculate n_α for $1 \le \alpha \le \beta$.
- For any $1 \le \alpha \le \beta$, find all integers with α distinct prime divisors not exceeding n_α and write them in a file.
- Output: all the exceptions.

The correctness of this algorithm is a direct consequence of the previous lemmas. For simplicity from now on we call an exception to the inequality $f(n) < e^\gamma(1+\epsilon)\log\log n$ simply *an exception.*

Example: For $\epsilon = 0.07$ we have

$$[M_{0.07}] = 28865752845259709512271057170344353684 0.$$

Since 26 is the greatest number such that $\prod_{p \le p_\beta} p \le M_{0.07}$, we have 26 as initial value of β. Step 4 of the algorithm gives 18 as the final value of β. Then for any α $(1 \le \alpha \le \beta)$ step 6 of the algorithm construct all the exceptions. The following table records the number of exceptions for each α.

$\alpha = \omega(n)$	**Number of Exceptions for** $\epsilon = .07$
1	11
2	69
3	373
4	2319
5	7418
6	27134
7	66268
8	197450
9	454229
10	542533
11	740427
12	564065
13	329802
14	210907
15	106791
16	27963
17	3043
18	212

For example there are 212 exceptions with $\omega(n) = 18$. In total there are $3,281,014$ exceptions to the inequality $f(n) \leq (1.07)e^{\gamma} \log\log n$.

4 Applications

We recall that

$$\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq f(n).$$

Moreover,

$$\overline{\lim} \frac{\sigma(n)}{ne^{\gamma} \log\log n} = \overline{\lim} \frac{n}{\phi(n)e^{\gamma} \log\log n} = \overline{\lim} \frac{f(n)}{e^{\gamma} \log\log n} = 1$$

(see [1], p. 353 and Proposition 2.1). We can use the above inequality together with our algorithm to establish explicit upper bounds for $\sigma(n)/n$ and $n/\phi(n)$. For example by numerically checking the exceptions generated by the algorithm for $\epsilon = .07$ against the inequality $\frac{n}{\phi(n)} < (1.07)e^{\gamma} \log\log n$ we find out that there are 6569 exceptions to $\frac{n}{\phi(n)} < (1.07)e^{\gamma} \log\log n$, the largest being

$$23457676271881394196654 0.$$

So we have

Proposition 4.1

$$\frac{n}{\phi(n)} < (1.07)e^{\gamma}\log\log n, \quad \text{for} \quad n > 23457676271881394196654 0.$$

Using the same method we can deduce that there are only 14 exceptions to the inequality $\frac{\sigma(n)}{n} < (1.07)e^{\gamma}\log\log n$. The largest of these exceptions is 120. Hence

$$\frac{\sigma(n)}{n} < (1.07)e^{\gamma}\log\log n, \quad \text{for} \quad n \geq 121.$$

As it is expected, the number of exceptions increases dramatically as ϵ gets smaller. For example for $\epsilon = .06$ there are $32,707,736$ exceptions and for $\epsilon = .05$ there are $798,101,126$ exceptions. If we are only interested in upper bounds for $\sigma(n)$ we can reduce the number of possible exceptions to the inequality $\sigma(n) < e^{\gamma}(1+\epsilon)\log(\log n)$ significantly by the following two observations.

First of all by a result of Robin ([5], Theorem 2), for $n \geq 3$, we have

$$\frac{\sigma(n)}{n} \leq e^{\gamma}\log\log n + \frac{0.6483}{\log\log n}.$$

So by an argument similar to Lemma 3.1

$$\frac{\sigma(n)}{n} < e^{\gamma}(1+\epsilon)\log\log n \tag{3}$$

for $n > \tilde{M}_{\epsilon} = \exp\left(\exp\left(\sqrt{\frac{0.6483}{\epsilon e^{\gamma}}}\right)\right)$ with $\omega(n) \geq 2$. Note that $\tilde{M}_{\epsilon}$ is much smaller than M_{ϵ}. So we only need to check (3) for the exceptions not exceeding $\tilde{M}_{\epsilon}$.

Secondly, by a recent result of Choie, Lichiardopol, Moree and Solé [2] we know that if $n \geq 5041$ does not satisfy Robin's inequality $\sigma(n) < e^{\gamma}n\log\log n$ then n is even, is neither square free nor squarefull and is divisible by a fifth power > 1. In other words all $n \geq 5041$ that are odd, square free, squarefull and are not divisible by a fifth power > 1 satisfy (3).

Let S_1 be the set of natural numbers not exceeding $\tilde{M}_{\epsilon}$. Let S_2 be those numbers in S_1 that are even, are neither square free nor squarefull and are divisible by a fifth power > 1. The following table gives the number of exceptions, the number of exceptions in S_1 and the number of exceptions in S_2 for different values of ϵ.

ϵ	**# of exceptions**	**# of exceptions in S_1**	**# of exceptions in S_2**
.07	$3,281,014$	$10,190$	135
.06	$32,707,734$	$63,076$	850
.05	$798,101,116$	$418,627$	$5,672$
.04		$23,472,726$	$323,069$
.03		$4,420,980,851$	$55,258,878$
.029		$15,910,840,055$	$183,084,959$

So for proving Theorem 1.3 we need to check (3) in the case $\epsilon = .03$ for about 55,000,000 values of n.

Theorem 1.3 $g(n) \leq g(180) \simeq (1.033867784...)e^{\gamma}$, for $n \geq 121$. *More precisely,* $g(n) < (1.03)e^{\gamma}$ *for* $n \geq 121$ *except for* $n = 180$.

Proof. By running our algorithm for $\epsilon = .03$ we generate all the exceptions to the inequality $f(n) < (1.03)e^{\gamma} \log\log n$ in S_2. By checking these exceptions against the inequality $g(n) < (1.03)e^{\gamma}$ we find out that 180 is the largest n that does not satisfy $g(n) < (1.03)e^{\gamma}$. $\square$

Note on Computations The algorithm is implemented in C++. The checking of exceptions against the inequality $g(n) < (1 + \epsilon)e^{\gamma}$ is done by the Maple 10. The checking running time for $\epsilon = .03$ is 40 hours in a 3 GHz Intel Pentium 4 with 1 GB of memory.

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