Descending Rational Points on Elliptic Curves to Smaller Fields

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Abstract

In this paper, we study the Mordell-Weil group of an elliptic curve as a Galois module. We consider an elliptic curve E defined over a number field K whose Mordell-Weil rank over a Galois extension F is 1, 2 or 3. We show that E acquires a point (points) of infinite order over a field whose Galois group is one of $C_n \times C_m$ $(n = 1, 2, 3, 4, 6, m = 1, 2)$, $D_n \times C_m$ $(n = 2, 3, 4, 6, m = 1, 2)$, $A_4 \times C_m$ $(m = 1, 2)$, $S_4 \times C_m$ $(m = 1, 2)$. Next, we consider the case where E has complex multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field $\mathfrak{K}$ contained in K. Suppose that the $\mathcal{O}$-rank over a Galois extension F is 1 or 2. If $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ and $h_{\mathfrak{K}}$ (class number of $\mathfrak{K}$) is odd, we show that E acquires positive $\mathcal{O}$-rank over a cyclic extension of K or over a field whose Galois group is one of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, an extension of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$, or a central extension by the dihedral group. Finally, we discuss the relation of the above results to the vanishing of L-functions.

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1 Introduction

Let E be an elliptic curve defined over a number field K. By the Mordell-Weil theorem, the group $E(K)$ of points of E with coordinates in K is finitely generated. We write $rank(E(K))$ for the rank of $E(K)$ modulo torsion. Let F be a finite Galois extension

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of K with group G. In this paper, we consider the Mordell-Weil group $E(F)$ as a $\mathbb{Z}[G]$-module. Since the torsion subgroup $E(F)_{tors}$ has been extensively studied (see for example, Serre [20]), we shall restrict ourselves to the free part of $E(F)$. The question of studying this as a Galois module was raised in the works of Mazur [9], Mazur and Swinnerton-Dyer [10], Coates and Wiles [3] Rohrlich [16], and [17], to name a few.

Philosophically, it is of interest to note one basic difference between the free part and the torsion part as Galois modules. For example, consider the Galois module of ℓ-torsion points $E[\ell]$. The field $K(E[\ell])$ obtained by adjoining the coordinates of points in $E[\ell]$ has Galois group contained in $\mathrm{Aut}(E[\ell]) \simeq \mathrm{GL}_2(\mathbb{Z}/\ell)$. Serre's theorem tells us that if E is without complex multiplication, then for large ℓ, it is in fact equal to $\mathrm{GL}_2(\mathbb{Z}/\ell)$. On the other hand, let $K(E(F)_{free})$ be the field generated by adjoining the coordinates of any free $\mathbb{Z}[\mathrm{Gal}(F/K)]$-submodule of $E(F) \otimes \mathbb{Q}$ to K and suppose that $rank(E(F)) = r$, then $\mathrm{Gal}(K(E(F)_{free})/K)$ is conjugate to a subgroup of $\mathrm{GL}_r(\mathbb{Z})$. This imposes two restrictions on this Galois group. Firstly, by Jordan's theorem, a finite subgroup of $\mathrm{GL}_r(\mathbb{C})$ has a normal Abelian subgroup of index bounded by a function of r alone. Secondly, this is an integral representation. By the work of Nori [14], there are many restrictions on the finite subgroups of $\mathrm{GL}_r(\mathbb{Z})$. Another restriction imposed on these Galois groups arises from the fact that the height pairing on the Mordell-Weil group is respected by the action of Galois.

In another direction, there is the connection with the L function of the elliptic curve. A well known theorem of Coates and Wiles [3] for CM elliptic curves asserts that if $E(K)$ is infinite, then the L-function $L(E/K, s)$ vanishes at $s = 1$. From the work of Kolyvagin [7], a similar result is known for (modular) elliptic curves over $\mathbb{Q}$. This is in accordance with the general conjecture of Birch and Swinnerton-Dyer. Here, we shall discuss the following:

Problem 1: Let F/K be a finite *Galois* extension. If $E(F)$ is infinite, does $L(E/F, s)$ vanish at $s = 1$?

Since the extensions of Coates-Wiles and Kolyvagin theorems to Abelian extensions are known (due respectively to Arthaud [1], and Rubin [18] in the CM-case and Kato (unpublished) in the modular case), we will show that the existence of an Abelian subextension M of F/K with $E(M)$ infinite implies a positive answer to Problem 1 (see Theorem 4). So we shall consider the following related problem.

Problem 2: Let F/K be a finite *Galois* extension. If $E(F)$ is infinite, then under what conditions can we produce an Abelian subextension M of K ($K \subseteq M \subseteq F$) such that $E(M)$ is infinite?

We wish to draw the analogy of this question with a result of Stark [22] for Artin L-functions. He shows that if F/K is Galois and the Dedekind zeta function $\zeta_F(s)$ has a *simple* zero at a point $s = s_0$, then there is a subextension $K \subseteq M \subseteq F$ with the property that $\zeta_M(s_0) = 0$ and M/K is Abelian (in fact, cyclic). Moreover, if N is any other subfield satisfying $\zeta_N(s_0) = 0$, we must have $M \subseteq N$.

In section 4, we consider an elliptic curve E defined over K whose Mordell-Weil rank over a Galois extension F is 1 or 2. If the rank of $E(F)$ is one, we observe (Theorem 1, (i)) that a Stark type result holds here. If the rank of $E(F)$ is two, we show that E acquires two points of infinite order over a cyclic extension of K with Galois group C_n ($n = 1, 2, 3, 4, 6$) contained in F or over a dihedral extension with Galois group D_n ($n = 2, 3, 4, 6$). Then we establish a similar result in the rank three case (Theorem 1, (iii)). In the case that E has complex multiplication, we can also study the Mordell-Weil group $E(F)$ as an $\mathcal{O}[G]$-module. Here E has complex multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field $\mathfrak{K}$ contained in K. We are able to establish the analogues of the above results in the case that $E(F)$ has $\mathcal{O}$-rank 1 or 2 (Theorems 2 and 3).

In the final section, by considering the order of vanishing of the L-function of E at a point $s = \omega$, we investigate some analytic analogues of our results in section 4. In the case of a simple zero, we prove an analogue of Stark's theorem for a certain class of elliptic curves (Corollary 1). Also, by analogy with [13], we formulate a statement for higher order zeros but it would depend on the holomorphy of the L-functions obtained by twisting the L-function of E with certain Artin characters (see Proposition 6).

It is clear that much work remains to be done to elucidate the Galois module structure of the Mordell-Weil group. We hope that the explicit results of this paper may help in this effort.

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2 The minimal subfield

Definition: Let E be an elliptic curve defined over K and let F/K be an extension (not necessarily Galois) of number fields. Suppose that $rank(E(F)) = r$, then the *minimal subfield* F_r is a subfield with $K \subseteq F_r \subseteq F$, such that
(i) $rank(E(F_r)) = rank(E(F))$.
(ii) If $K \subseteq M \subseteq F$ and $rank(E(M)) = rank(E(F))$, then $F_r \subseteq M$.

Proposition 1 *For any finite extension F/K and elliptic curve E defined over K with $rank(E(F)) = r$, F_r exists and is unique. Also, if F/K is Galois then F_r/K is Galois.*

Proof: We need only prove that if $K \subseteq M_1, M_2 \subseteq F$ are subfields such that

$$rank(E(M_1)) = rank(E(M_2)) = r$$

then

$$rank(E(M_1 \cap M_2)) = r.$$

Indeed, $E(M_1) \otimes \mathbb{Q} = E(M_2) \otimes \mathbb{Q}$. Hence, there is a lattice L contained in $E(M_1) \cap E(M_2)$ which is of finite index in both $E(M_1)$ and $E(M_2)$. But then L is fixed by $\mathrm{Gal}(\tilde{F}/M_1)$ and by $\mathrm{Gal}(\tilde{F}/M_2)$ where $\tilde{F}$ is the normal closure of F/K. Thus, it is fixed by $\mathrm{Gal}(\tilde{F}/(M_1 \cap M_2))$ and so it is contained in $E(M_1 \cap M_2)$. Thus the rank of $E(M_1 \cap M_2)$ is r as claimed.

If F/K is Galois, we can apply this argument to M and a conjugate of M, and from this, we see that the minimal subfield is necessarily Galois over K. □

Now we give another description of the minimal subfield. Let F/K be a finite Galois extension, then since $\mathrm{Gal}(F/K)$ acts on $E(F) \otimes \mathbb{Q}$, we have a representation

$$\rho : \mathrm{Gal}(F/K) \to \mathrm{Aut}(E(F) \otimes \mathbb{Q}) \simeq \mathrm{GL}_r(\mathbb{Q})$$

where $rank(E(F)) = r$. Then, there exists a free submodule of $E(F) \otimes \mathbb{Q}$ of rank r on which $\mathrm{Gal}(F/K)$ acts. For example, if $m = |E(F)_{tors}|$, then we can take $mE(F)$. Each such submodule X (say) gives a representation

$$\rho_X : \mathrm{Gal}(F/K) \to \mathrm{Aut}(X) \simeq \mathrm{GL}_r(\mathbb{Z}).$$

Moreover, different choices of X yield representations isomorphic over $\mathbb{Q}$. In particular, $Ker(\rho_X)$ is equal to $Ker(\rho)$ and is independent of X. Thus, the field $K(X)$ obtained by adjoining the coordinates of points in X to K is independent of the choice of X. We denote this field by $K(E(F)_{free})$.

Proposition 2 *Let F/K be a finite Galois extension. If $rank(E(F)) = r \geq 1$, then*
(i) there is a subextension M, Galois over K such that $E(M) \otimes \mathbb{Q} = E(F) \otimes \mathbb{Q}$ and the representation

$$\rho_f : \mathrm{Gal}(M/K) \to \mathrm{Aut}(E(M) \otimes \mathbb{Q})$$

is faithful. Moreover, $Im(\rho_f)$ is conjugate to a finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$.
(ii) $M = K(E(F)_{free})$.
(iii) M is the minimal subfield defined in the beginning of the section.

Proof: (i) Suppose that ρ is the representation of $\mathrm{Gal}(F/K)$ in $E(F) \otimes \mathbb{Q}$. Let M be the fixed field of $ker\rho$. Since

$$(E(F) \otimes \mathbb{Q})^{Ker\rho} = (E(F) \otimes \mathbb{Q})^{\mathrm{Gal}(F/M)} = E(M) \otimes \mathbb{Q}$$

(see [16], p. 126) and since M is the fixed field of $ker\rho$, $\mathrm{Gal}(F/M)$ acts trivially on $E(F) \otimes \mathbb{Q}$. This shows that $E(F) \otimes \mathbb{Q} = E(M) \otimes \mathbb{Q}$ and ρ_f is faithful. The argument before the proposition shows that $Im(\rho_f)$ is conjugate to a finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$.

(ii) This is clear from the argument before the proposition.

(iii) Let $K \subseteq L \subseteq F$ and $rank(E(L)) = rank(E(F))$, then from the proof of Proposition 1, we know that $rank(E(L \cap M)) = rank(E(M))$ and $E(M) \otimes \mathbb{Q} = E(L \cap M) \otimes \mathbb{Q}$. This shows that $\mathrm{Gal}(M/(L \cap M))$ acts trivially on $E(M) \otimes \mathbb{Q}$ and therefore it is contained in the kernel of the representation ρ_f. But $ker\rho_f = \{id\}$, which implies that $\mathrm{Gal}(M/(L \cap M)) = \{id\}$. Thus $L \cap M = M$ and therefore $M \subseteq L$. This proves that M is the minimal subfield. □

Proposition 3 *Let F/K be a finite Galois extension, then the degree of the minimal subfield F_r over K is bounded as a function of r alone.*

Proof: By Proposition 2, we can consider $\mathrm{Gal}(F_r/K)$ as a finite subgroup of $\mathrm{GL}_r(\mathbb{Z})$ (and therefore $\mathrm{GL}_r(\mathbb{C})$). By Jordan's theorem a finite subgroup of $\mathrm{GL}_r(\mathbb{C})$ has a normal Abelian subgroup G_1 whose index is bounded by a function of r alone. So it is enough to prove that the order of G_1 is bounded by a function of r alone.

Now, let L be the fixed field of G_1 in F_r/K, and let ρ_1 be the restriction of the representation ρ_f (defined in Proposition 2) to $G_1 = \mathrm{Gal}(F_r/L)$. Then

$$\rho_1 = \psi_1 \oplus \psi_2 \oplus ... \oplus \psi_r$$

where ψ_i's are one dimensional characters of G_1. Since the values of the ψ_i satisfy a degree r polynomial over $\mathbb{Q}$, if ψ_i takes values in $\mathbb{Q}(\zeta_{m_i})$, we must have $\phi(m_i) \leq r$. Since ρ_1 is faithful, this implies that the order of G_1 is bounded by a function of r alone. □

3 Group theoretic lemmas

In this section, we collect some group theoretic results which will be needed in the sequel.

Lemma 1 *Let the representation $\rho : G \to \mathrm{GL}_2(\mathbb{Z})$ be faithful, then*
(i) if ρ is reducible, G is cyclic C_n ($n =$ 1, 2, 3, 4, 6) or $G \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \simeq D_2$.
(ii) if ρ is irreducible, G is dihedral D_n ($n =$ 3, 4, 6).

Proof: (i) Suppose that ρ is reducible. Let χ be the character of ρ. Then $\chi = \psi_1 + \psi_2$ over $\mathbb{C}$, where ψ_1 and ψ_2 are one dimensional characters of G. As the characteristic polynomial of ρ has coefficients in $\mathbb{Z}$, we must have $\psi_1 = \overline{\psi_2}$ or ψ_1 and ψ_2 characters of order 2. Since ρ is faithful, in the latter case, $G \simeq \mathbb{Z}/2$ or $G \simeq \mathbb{Z}/2 \oplus \mathbb{Z}/2 \simeq D_2$ and in the former case, G is cyclic.

Now if r is a generator of the cyclic group G and $ord(r) = n$, then $\rho(r)$ is conjugate to a diagonal matrix over $\mathbb{C}$ like

$$\begin{pmatrix} e^{\frac{2\pi i h}{n}} & 0 \\ 0 & e^{-\frac{2\pi i h}{n}} \end{pmatrix}$$

where $0 \leq h < n$ and $(h, n) = 1$. Here, $e^{\frac{2\pi i h}{n}}$ is a primitive n-th root of unity which is also a root of a quadratic polynomial over $\mathbb{Z}$ (i.e. the characteristic polynomial of the above matrix). Therefore $\phi(n) = [\mathbb{Q}(e^{\frac{2\pi i h}{n}}) : \mathbb{Q}] \leq 2$ and so $n = 1,\ 2,\ 3,\ 4,\ 6$.

(ii) Since ρ is faithful, we can consider G as a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$. We know that a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$ is conjugate to a subgroup of $\mathrm{O}_2(\mathbb{R})$ and is therefore cyclic or dihedral (see [15], p. 22, Theorem 9). As ρ is irreducible, $G \simeq D_n =$

$\langle r,\ s;\ r^n = 1,\ s^2 = 1,\ srs = r^{-1}\rangle$. Let $H = \langle r\rangle$, then $\chi|_H = \psi_1 + \psi_2$ over $\mathbb{C}$, where $\psi_1(r) = e^{\frac{2\pi ih}{n}}$ and $\psi_2(r) = e^{-\frac{2\pi ih}{n}}$ (see [19], p. 37), so by reasoning similar to part (i), $ord(H) = n = 1,\ 2,\ 3,\ 4,\ 6$. Moreover, $n \neq 1,\ 2$ since in these cases D_n is Abelian. □

Let H_1 and H_2 be subgroups of a group G and let $x \in G$. Set

$$J(H_1, H_2, x) = H_2 \cup \{xg|\ g \in H_1,\ g \notin H_2\}.$$

Lemma 2 *Let H_1 and H_2 be subgroups of a group G such that $H_2 \subset H_1$ and $[H_1 : H_2] = 2$. Let $x \in G - H_2$ be an element of order 2 which commutes with all elements of H_1. Then*
(i) $J(H_1, H_2, x)$ is a subgroup of G.
(ii) $H_1 \simeq H_2 \times C_2$ if $x \in H_1$.
(iii) $H_1 \simeq J(H_1, H_2, x)$ if $x \notin H_1$.

Proof: It is straightforward. □

Lemma 3 *Let the representation $\rho : G \to \mathrm{GL}_3(\mathbb{Z})$ be faithful, then G is isomorphic to one of the following:*

$$C_n \times C_m, \quad D_p \times C_m, \quad A_4 \times C_m, \quad S_4 \times C_m$$

where $n =\ 1,\ 2,\ 3,\ 4,\ 6$, $p =\ 2,\ 3,\ 4,\ 6$ and $m =\ 1,\ 2$.

Proof: Since ρ is faithful we consider G as a finite subgroup of $\mathrm{O}_3(\mathbb{R})$. First suppose that $G \subset \mathrm{SO}_3(\mathbb{R})$. Then it is known that G is either cyclic, dihedral, A_4, S_4 or A_5 (see [15], p. 35, Theorem 11). Note that in this case if $A \in G$, then there is an orthonormal matrix P such that

$$P^{-1}AP = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(see [15], p. 35, corollary 1), with $tr(P^{-1}AP) \in \mathbb{Z}$. Therefore $2\cos\alpha \in \mathbb{Z}$. It is easily seen from here that if $G \subset \mathrm{SO}_3(\mathbb{R})$, the order of any element of G must be 2, 3, 4 or 6, and therefore G must be one of the following

$$C_n\ (n =\ 1,\ 2,\ 3,\ 4,\ 6), \quad D_p\ (p =\ 2,\ 3,\ 4,\ 6), \quad A_4, \quad S_4. \qquad (*)$$

Now suppose that $G \not\subset \mathrm{SO}_3(\mathbb{R})$. Let $G_s = G \cap \mathrm{SO}_3(\mathbb{R})$ and note that $-I$ (I is the identity matrix) is an element of order 2 in $\mathrm{O}_3(\mathbb{R})$ which is not in G_s and it commutes with all elements of G. Therefore, by Lemma 2, either $G \simeq G_s \times C_2$ or $G \simeq J(G, G_s, -I)$. G_s and $J(G, G_s, -I)$ are finite subgroups of $\mathrm{SO}_3(\mathbb{R})$ and therefore they are in the list given in $(*)$. This completes the proof. □

Now let $\mathcal{O}$ denote the ring of integers of an imaginary quadratic field $\mathfrak{K}$. We fix an embedding $\mathfrak{K} \hookrightarrow \mathbb{C}$.

Notation. We denote the center of a group G by $Cent(G)$.

Lemma 4 *Let G be a group with a normal subgroup H of prime index. Let $\rho : G \to \mathrm{GL}_2(\mathcal{O})$ be a faithful and irreducible representation of G, and let χ be the character of ρ. Then*
(i) either $\chi = Ind_H^G \psi$, $\psi(1) = 1$ or $\chi|_H$ is irreducible.
In the case that $\chi = Ind_H^G \psi$, $\psi(1) = 1$, let us set $N = Ker\ \psi$.
(ii) If $N = \{id\}$, then $H \simeq C_n$ ($n =$ 2, 3, 4, 6, 8, 12).
(iii) If $N \neq \{id\}$ and $[G : H] = 2$ then for all $\sigma \in G - H$ we have $N \cap \sigma^{-1} N \sigma = \{id\}$.

Proof: (i) By Proposition 24 of [19] (p. 61), there exists a subgroup J of G, unequal to G and containing H such that either $\chi = Ind_J^G \psi$, $\psi(1) = 1$ or $\chi|_J$ is isotypic. Since H has prime index in G then $J = H$.

If $\chi|_H$ is isotypic and reducible then $H \subset Cent(G)$. But G/H is cyclic and therefore $G/Cent(G)$ is also cyclic. This implies that G is Abelian which is a contradiction since G has a two dimensional irreducible representation. The only other possibility is that $\chi|_H$ is irreducible.

(ii) Since ψ is faithful, H is isomorphic to a finite subgroup of $\mathbb{C}^\times$ and therefore is cyclic. A characteristic polynomial argument similar to the one in Lemma 1 shows that the order n, say, of this group can only be 2, 3, 4, 5, 6, 8, 10 or 12 ($n \neq 1$, since G cannot be Abelian). Since H is cyclic, $\chi|_H = \psi + \psi'$.

Now if $n = 5$, ψ and ψ' take values in the group of 5-th roots of unity, and therefore $\chi|_H$ takes values in $\mathbb{Q}(\zeta_5) \cap \mathfrak{K} = \mathbb{Q}$. The characteristic polynomial of $\rho|_H$ has real coefficients and so either ψ and ψ' are both real or ψ' is the complex conjugate of ψ. Since ψ has order 5, the first case cannot occur. Hence, we are in the second case, and this implies that the character $\chi|_H$ takes values in $\mathbb{Q}(\zeta_5)^+$ which is not $\mathbb{Q}$ and this is a contradiction. Therefore, $n \neq 5$. In a similar way, we can show that $n \neq 10$.

(iii) If $N \neq \{id\}$ then N cannot be normal in G. Indeed, if $N \lhd G$ then $N \subset Ker\ \chi$ and this is not possible as ρ is faithful. Now $[G : H] = 2$ and therefore there exists exactly one conjugate of N, say $N' = \sigma^{-1} N \sigma$. Then $N \cap N' = \{id\}$ because $N \cap N' \subset Ker\ \chi$, $N \cap N' \lhd G$ and ρ is faithful. □

Remark 1. If $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, in part (ii) of Lemma 2, we can prove that n is not equal to 8 and 12. This is true since in these cases $\chi|_H$ takes values in $\mathbb{Q}(\zeta_8)^+$ or $\mathbb{Q}(\zeta_{12})^+$ which are not $\mathbb{Q}$.

Lemma 5 *Let $5 \not| d_{\mathfrak{K}}$ (discriminant of $\mathfrak{K}$). Then, the order of any finite subgroup of $\mathrm{GL}_2(\mathcal{O})$ is not divisible by 5.*

Proof: Let G be a finite subgroup of $\mathrm{GL}_2(\mathcal{O})$. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes $q \equiv 2 \pmod{5}$ such that q splits completely in $\mathcal{O}$. Let $q = \mathfrak{q}_1 \mathfrak{q}_2$ in $\mathcal{O}$. We choose q large enough such that the restriction of the reduction map

$$\mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{q}_1 \mathcal{O})$$

to G is injective. But $Card(\mathrm{GL}_2(\mathcal{O}/\mathfrak{q}_1\mathcal{O})) = Card(\mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z})) = (q^2-1)(q^2-q) \equiv 1 \pmod 5$. This proves the lemma. □

Lemma 6 *Let G be a subgroup of $\mathrm{GL}_2(\mathcal{O})$, then either G is Abelian or $Cent(G) \simeq \{id\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/6\mathbb{Z}$.*

Proof: We consider G as a subgroup of $\mathrm{GL}_2(\mathfrak{K})$. Let

$$C(G) = \{\alpha \in \mathrm{GL}_2(\mathfrak{K}) :\ \alpha\gamma = \gamma\alpha \text{ for all } \gamma \in G\}\,.$$

Then, G is either Abelian or

$$C(G) = \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \ : c \in \mathfrak{K}^* \right\}$$

(see [21], p. 179, problem 2.6. (a)). Now the lemma follows from the facts that

$$Cent(G) = C(G) \cap G$$

and $\mathcal{O}^* \simeq \{id\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/6\mathbb{Z}$. □

4 $E(F)$ of $\mathbb{Z}$-rank 1, 2, 3 or $\mathcal{O}$-rank 1 or 2

In this section, we assume that $E(F)$ is infinite of either $\mathbb{Z}$-rank ≤ 3 or $\mathcal{O}$-rank ≤ 2. We apply the results of the previous section to determine the minimal subfield in the case that $E(F)$ has $\mathbb{Z}$-rank 1, 2 or 3. We also consider the case that E has multiplication by the ring of integers $\mathcal{O}$ of an imaginary quadratic field $\mathfrak{K}$ and $E(F)$ has $\mathcal{O}$-rank 1 or 2. In the latter situation, we are able to determine the minimal subfield in all cases but one.

Theorem 1 *Let E be an elliptic curve defined over K and let F be a finite Galois extension of K. Let M be the minimal subfield.*
(i) If $rank(E(F)) = 1$, then M is a cyclic subextension of K and $[M : K] = 1$ or 2.
(ii) If $rank(E(F)) = 2$, then M is either a cyclic subextension of K and $[M : K] = 1,\ 2,\ 3,\ 4,\ 6$ or a dihedral subextension of K and $[M : K] = 4,\ 6,\ 8,\ 12$.
(iii) If $rank(E(F)) = 3$, then $\mathrm{Gal}(M/K)$ is one of the following:

$$C_n \times C_m, \quad D_p \times C_m, \quad A_4 \times C_m, \quad S_4 \times C_m$$

where $n =\ 1,\ 2,\ 3,\ 4,\ 6$, $p =\ 2,\ 3,\ 4,\ 6$ and $m =\ 1,\ 2$.

Proof: (i) M/K is the subextension given in Proposition 2. It is clear that since ρ_f is faithful, $\mathrm{Gal}(M/K)$ is isomorphic to a subgroup of $\mathrm{GL}_1(\mathbb{Z}) \simeq \mathbb{Z}^* = \{\pm 1\}$ which is cyclic and has order 1 or 2.

(ii), (iii) Let ρ_f be the faithful representation given in Proposition 2. Applying Lemmas 1 and 3 on ρ_f imply the results. □

Now we show that in part (ii) of Theorem 1, M cannot be a dihedral extension of degree 12 of K, if we assume the Birch and Swinnerton-Dyer conjecture and some other assumptions.

Let M be a dihedral extension of $\mathbb{Q}$ and let C be the fixed field of the cyclic subgroup H of the dihedral Galois group in $M/\mathbb{Q}$. So $[C:\mathbb{Q}]=2$ and $[M:C]=n$ (say) ($n \geq 3$). We have

$$L(E/M,s) = L(E/C,s)\prod_i L(E/\mathbb{Q}\otimes Ind_H^G\psi_i,s)^2$$

where ψ_i are characters of $H=\mathrm{Gal}(M/C)$. Since G is dihedral, the twisted L-function $L(E/\mathbb{Q}\otimes Ind_H^G\psi_i,s)$ has root number ± 1, depending on the parity of the order of vanishing of the twisted L-function at $s=1$.

Now assume that the Birch and Swinnerton-Dyer conjecture is true. Then the assumption that $rank(E(M))=2$, and the above factorization of L-functions implies that we have the following possibilities:
(i) $L(E/C,1)=0$
(ii) exactly one of the factors $L(E/\mathbb{Q}\otimes Ind_H^G\psi_i,s)$ has a simple zero at $s=1$. In the first case, we must have $L(E/C,s)$ vanishing to order 2 at $s=1$ and none of the two-dimensional twists vanishes. In particular, all the root numbers must satisfy

$$w(E/\mathbb{Q}\otimes Ind_H^G\psi_i)=1$$

for all i. In the second case, $L(E/C,1)\neq 0$ and there is a unique i such that $L(E/\mathbb{Q}\otimes Ind_H^G\psi_i,1)=0$. Since this zero is simple

$$w(E/\mathbb{Q}\otimes Ind_H^G\psi_i)=-1.$$

Moreover, as none of the others vanish, all of the other root numbers are equal to +1.
Now it is clear that if M is the minimal subfield then (i) cannot be true and thus we are in the situation (ii).

Proposition 4 *Let E be a modular elliptic curve of conductor N defined over $\mathbb{Q}$ and suppose that the Birch and Swinnerton-Dyer conjecture is true. Also with the above notation assume that N and conductor of $Ind_H^G\psi_i$'s are relatively prime and for all i, $\chi_i=\det(Ind_H^G\psi_i)$ is even. Then, in part (ii) of Theorem 1 (for $K=\mathbb{Q}$) the minimal subfield M cannot be a dihedral extension of degree 12.*

Proof: Let M be the minimal subfield in Theorem 1 and follow the notations before the proposition. By a result of Rohrlich (see [16], p. 125, Proposition 1), the root number can be calculated as follows. Let χ_i be the determinant of $Ind_H^G\psi_i$. If χ_i is even, then

$$w(E/\mathbb{Q}\otimes Ind_H^G\psi_i)=\chi_i(N).$$

Now, χ_i is a quadratic character which can be computed by the following formula:

$$\chi_i \;=\; \epsilon\psi_i \circ \mathrm{Ver}$$

where ϵ is the character of $C/\mathbb{Q}$ and Ver is the transfer map (Verlagerung) given by

$$\mathrm{Ver}(g) \;=\; \begin{cases} g^2 & \text{if } g \notin H \\ g.\delta g \delta^{-1} & \text{if } g \in H. \end{cases}$$

Here, δ is a fixed element of $G - H$ of order 2. Now, $\psi(\delta g \delta^{-1}) = \overline{\psi(g)}$ and so $\psi \circ \mathrm{Ver}$ is trivial on H. Moreover, $\mathrm{Ver}(\delta) = 1$. Hence, $\psi_i \circ \mathrm{Ver} = 1$ and $\chi_i = \epsilon$ is a quadratic character independent of ψ_i. Thus, the root numbers $w(E/\mathbb{Q} \otimes Ind_H^G \psi_i)$'s are all equal. But from the argument before the proposition, we know that there is a unique i such that $w(E/\mathbb{Q} \otimes Ind_H^G \psi_i) = -1$ and all of the others are $+1$. Now since the number of irreducible two dimensional characters of D_n is $\frac{n-1}{2}$ if n is odd and $\frac{n-2}{2}$ if n is even, we have $\epsilon(N) = -1$ and $n = 3$ or 4. □

Now let E be an elliptic curve defined over a number field K which has complex multiplication by $\mathcal{O}$, the ring of integers of an imaginary quadratic number field $\mathfrak{K}$ contained in K ($\mathfrak{K} \subseteq K$), and let F be a finite Galois extension of K. (We fix once and for all an embedding $\mathfrak{K} \hookrightarrow \mathbb{C}$.) Since E has complex multiplication by $\mathcal{O}$ and E is defined over K, we can fix an isomorphism between the ring of endomorphisms of E and $\mathcal{O}$ and equip $E(F)$ with an $\mathcal{O}$ action. (Note that all the endomorphisms of E are defined over K.)

We consider the submodule $mE(F)$ of the $\mathcal{O}$-module $E(F)$, where m is the order of the $\mathcal{O}$-torsion submodule of $E(F)$, then $mE(F)$ is a finitely generated torsion free module over $\mathcal{O}$ which is projective since $\mathcal{O}$ is a Dedekind domain. Moreover, there exist free $\mathcal{O}$-modules M_1 and M_2, such that

$$M_1 \subset mE(F) \subset M_2$$

and M_1 and M_2 have the same rank. We call this common rank, the $\mathcal{O}$-rank of $E(F)$. (For the above algebraic facts, see [8], p. 168, Problems 11 and 13.) Note that $2\ rank_{\mathcal{O}}(E(F)) = rank(E(F))$.

Remark 2. If the field of complex multiplication $\mathfrak{K}$ is not contained in K, still we can consider $E(F)$ as an $\mathcal{O}$-module if we assume that $\mathfrak{K}K \subset F$. Also, we want to mention that the upcoming results in this section are also valid for elliptic curves with complex multiplication by a non-maximal order in $\mathfrak{K}$.

Now we can consider the $\mathfrak{K}$-module $mE(F) \otimes_{\mathcal{O}} \mathfrak{K} = E(F) \otimes_{\mathcal{O}} \mathfrak{K}$ as a representation space for $\mathrm{Gal}(F/K)$ to get the following representation:

$$\rho : \mathrm{Gal}(F/K) \to \mathrm{Aut}(E(F) \otimes_{\mathcal{O}} \mathfrak{K}) \simeq \mathrm{GL}_r(\mathfrak{K})$$

where $r = rank_{\mathcal{O}}(E(F))$. It is clear that we can define an $\mathcal{O}$-analogue of the minimal subfield and establish an $\mathcal{O}$-analogue of Propositions 1, 2 and 3. Note that in the

$\mathcal{O}$-analogue of Proposition 2, we have to assume that r and $h_{\mathfrak{K}}$ (the class number of $\mathfrak{K}$) are relatively prime to make sure that $Im(\rho_f)$ is conjugate to a finite subgroup of $GL_r(\mathcal{O})$. (For more explanation about this condition see [4], Theorem 23.17, p. 530.) Also note that if $rank_{\mathcal{O}}(E(F)) = r$ then the $\mathcal{O}$-minimal subfield is the same as the minimal subfield F_{2r} defined in the beginning of Section 2.

Proposition 5 *If $rank_{\mathcal{O}}(E(F)) = 1$, then the minimal subfield is a cyclic subextension M of K and $[M : K] = 1,\ 2,\ 3,\ 4$ or 6.*

Proof: Since $(h_{\mathfrak{K}}, 1) = 1$, the argument before the proposition implies that $Im(\rho_f)$ can be considered as a subgroup of $\mathrm{GL}_1(\mathcal{O})$. Now the proof is exactly the $\mathcal{O}$-analogue of part (i) of Theorem 1. Note that $\mathrm{GL}_1(\mathcal{O}) \simeq \mathcal{O}^*$ which is cyclic and has order 1, 2, 4 or 6. □

If $rank_{\mathcal{O}}(E(F)) = 2$ and $h_{\mathfrak{K}}$ is odd, then $\rho(\mathrm{Gal}(F/K))$ is isomorphic to a finite subgroup of $\mathrm{GL}_2(\mathcal{O})$. We apply the group theoretic lemmas of the previous section to obtain some useful information about the representation ρ and the group $\mathrm{Gal}(F/K)$.

Theorem 2 *Suppose that $h_{\mathfrak{K}}$ is odd and $rank_{\mathcal{O}}(E(F)) = 2$. Then there is a Galois subextension $K \subseteq S \subseteq F$ with $rank_{\mathcal{O}}(E(S)) > 0$ such that $G = \mathrm{Gal}(S/K)$ is one of the following:*
(i) G is cyclic of order $1,\ 2,\ 3,\ 4\ ,6\ ,8\ ,$ or 12.
(ii) $G/Cent(G) \simeq D_n$. More precisely G satisfies one of the following:
(a) $G \simeq D_3$.
(b) $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$ and $G/Cent(G) \simeq D_n$ $(n =\ 2,\ 3,\ 4,\ 6,\ 8)$.
(c) $Cent(G) \simeq \mathbb{Z}/3\mathbb{Z}$ and $G/Cent(G) \simeq D_n$ $(n =\ 2,\ 3,\ 4,\ 6)$.
(d) $Cent(G) \simeq \mathbb{Z}/4\mathbb{Z}$ and $G/Cent(G) \simeq D_n$ $(n =\ 2,\ 3,\ 4)$.
(e) $Cent(G) \simeq \mathbb{Z}/6\mathbb{Z}$ and $G/Cent(G) \simeq D_n$ $(n =\ 2,\ 3,\ 6)$.
(iii) $Cent(G) \neq \{id\}$ and $G/Cent(G) \simeq A_4$ or S_4.
In (ii) and (iii), $rank_{\mathcal{O}}(E(S)) = 2$. In fact, S is the minimal subfield in these cases.

Proof: Let $\rho : \mathrm{Gal}(F/K) \to GL_2(\mathcal{O})$ be the representation of $\mathrm{Gal}(F/K)$ in $E(F) \otimes_{\mathcal{O}} \mathfrak{K}$ and χ be its character. By the $\mathcal{O}$-analogue of Proposition 2, we can assume that ρ is faithful. Also we know that $G/Cent(G)$ is isomorphic to a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$ and therefore (see [20]) is isomorphic to C_n, D_n, A_4, S_4 or A_5. By Lemma 5, $G/Cent(G)$ cannot be isomorphic to A_5. Note that since $h_{\mathfrak{K}}$ is odd, $\mathfrak{K} = \mathbb{Q}(\sqrt{-p})$ for prime p with $-p \equiv 1 \pmod 4$ or $\mathfrak{K} = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$, and therefore $5 \not| d_{\mathfrak{K}}$.

If ρ is reducible, let χ be the character of ρ. We have $\chi = \psi_1 + \psi_2$ over $\mathbb{C}$, where ψ_1 and ψ_2 are one dimensional characters of G. Let S be the fixed field of $Ker\ \psi_1$ in F/K. Then ψ_1 is a faithful and irreducible character of $\mathrm{Gal}(S/K)$, which implies that $\mathrm{Gal}(S/K)$ is cyclic and $rank_{\mathcal{O}}(E(S)) \neq 0$. Indeed, (see [16], p. 126)

$$(E(F) \otimes_{\mathcal{O}} \mathbb{C})^{\mathrm{Gal}(F/S)} = E(S) \otimes_{\mathcal{O}} \mathbb{C}.$$

Now a characteristic polynomial argument similar to the one in Lemma 1 implies that $[S : K] = 1$, 2, 3, 4, 6, 8 or 12.

Thus, we may suppose that ρ is irreducible. Then, since G is not Abelian $G/Cent(G)$ cannot be cyclic. Suppose that $G/Cent(G)$ is isomorphic to A_4 or S_4. In this case, we must have $Cent(G) \neq \{1\}$. Indeed, G is not isomorphic to A_4, since A_4 does not have any 2-dimensional irreducible representation. This also implies that if $G \simeq S_4$, and χ is the character of ρ then $\chi = Ind_{A_4}^{S_4}\psi$, $\psi(1) = 1$ (see part (i) of Lemma 4). But it is known that any 1-dimensional representation of A_4 is trivial on the Klein 4-group V_4 (see [19], p. 42). Since $V_4 \lhd S_4$, we have

$$V_4 \subset Ker(Ind_{A_4}^{S_4}\psi) = Ker\ \chi.$$

However, χ is the character of the faithful representation ρ. This is a contradiction. Therefore, G is not isomorphic to S_4.

It remains to analyze the possibility $G/Cent(G) \simeq D_n$. Let A be the cyclic subgroup of order n in D_n. Let L be the fixed field of $Cent(G)$ in F/K and M be the fixed field of A in L/K. If $H = \mathrm{Gal}(F/M)$ then $H/Cent(G) \simeq A$ is cyclic and therefore H is Abelian. Clearly H has index 2 in G, thus by part (i) of Lemma 4, $\chi = Ind_H^G\psi$, $\psi(1) = 1$. Let $N = Ker\ \psi$.

By part (ii) of Lemma 4 if $N = \{id\}$, then $H \simeq C_n$ ($n = 2,\ 3,\ 4,\ 6,\ 8,\ 12$). By Lemma 6, $Cent(G) \simeq \{id\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. As $Cent(G) \subseteq H$, we have the following possibilities. If $Cent(G) \simeq \{id\}$ then $G \simeq D_n$. In this case n must be odd, since $Cent(D_n) \neq \{id\}$ for n even. This proves that $G \simeq D_3$. If $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$ then $G/Cent(G) \simeq D_n$ ($n = 1,\ 2,\ 3,\ 4,\ 6$). But $n \neq 1$ since in that case G is Abelian. Similarly, if $Cent(G) \simeq \mathbb{Z}/3\mathbb{Z}$ then $G/Cent(G) \simeq D_n$ ($n = 2,\ 4$), if $Cent(G) \simeq \mathbb{Z}/4\mathbb{Z}$ then $G/Cent(G) \simeq D_n$ ($n = 2,\ 3$) and if $Cent(G) \simeq \mathbb{Z}/6\mathbb{Z}$ then $G/Cent(G) \simeq D_n$ ($n = 2$).

Now suppose that $N \neq \{id\}$. First note that since $\chi = Ind_H^G\psi$, $\psi(1) = 1$, then $\chi|_H = \psi + \psi^\sigma$ where $\sigma \in G - H$ and $\psi^\sigma(x) = \psi(\sigma^{-1}x\sigma)$ for $x \in H$ (See [19], Proposition 22, p. 58). This shows that $Ker\ \psi^\sigma = \sigma^{-1}N\sigma \neq \{id\}$. Let R be the fixed field of N in F/M, since F is the minimal subfield and $K \subset R \subsetneq F$, it is clear that $rank_{\mathcal{O}}(E(R)) = 1$. In a similar way, we can show that $rank_{\mathcal{O}}(E(R^\sigma)) = 1$ (R^σ is the fixed field of $\sigma^{-1}N\sigma$ in F/M).

Now since $rank_{\mathcal{O}}(E(R)) = 1$, the action of $Gal(R/M)$ on $E(R) \otimes_{\mathcal{O}} \mathfrak{K}$ is given by ψ. This shows that R is the minimal subfield and therefore it is cyclic of degree 1, 2, 3, 4, 6 (Proposition 5). A similar statement holds for R^σ.

By part (iii) of Lemma 4,

$$Ker\ \psi \cap Ker\ \psi^\sigma = N \cap \sigma^{-1}N\sigma = \{id\}.$$

This implies that $F = RR^\sigma$. Hence,

$$|H| = [F : M] = \frac{[R : M][R^\sigma : M]}{[R \cap R^\sigma : M]} = \frac{[R : M]^2}{[R \cap R^\sigma : M]}.$$

An easy calculation implies that $[F:M]=4,\ 8,\ 9,\ 12,\ 16,\ 18,\ 36$, which can be checked from the following table:

Table 1

$[R:M]$	$[R\cap R^\sigma:M]$	$[F:M]$
2	1	4
3	1	9
4	1, 2	8, 16
6	1, 2, 3	12, 18, 36.

Note that $[R:M]\neq 1$, since otherwise $R=R^\sigma=M$.

By Lemma 6, $Cent(G)\simeq\{id\},\ \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$. If $|Cent(G)|=1$ then $G\simeq D_n$, this implies that $N\lhd G$ and therefore $N\subset Ker\ \chi$ which is a contradiction since $N\neq\{id\}$ and χ is faithful. If $|Cent(G)|=4$ and $N\neq\{id\}$, then the proof of Lemma 6 shows that $\mathfrak{K}=\mathbb{Q}(\sqrt{-1})$ and therefore $[R:M]=2,\ 4$, thus $[F:M]=\ 8,\ 16$ and so $G/Cent(G)\simeq D_n$ $(n=\ 2,\ 4)$. If $|Cent(G)|=6$ and $N\neq\{id\}$, then $[F:M]=12,\ 18,\ 36$ and so $G/Cent(G)\simeq D_n$ $(n=\ 4,\ 6,\ 12)$.

If $|Cent(G)|=2$, we can refine the above argument to show that $[F:M]$ cannot be 9, 18 or 36. Since $H=\mathrm{Gal}(F/M)$ contains $Cent(G)$, the order of H is even and so $[F:M]\neq 9$. To show that $[F:M]\ \neq\ 18$ or 36, recall that $N\neq\{id\}$ and $|Cent(G)|=2$. We first claim that N is a 2-group (in fact, it is a cyclic [1] 2-group). This is true, because as N and H are Abelian, they can be written as a direct sum of their Sylow subgroups

$$N=N_2\oplus N_{odd},\quad H=H_2\oplus H_{odd}$$

where N_2 (respectively H_2) is the 2-primary part of N (respectively H). Since $H/Cent(G)$ is cyclic and $|Cent(G)|=2$, it follows that H_{odd} is cyclic. Moreover, $H_{odd}\lhd G$, and since $N_{odd}\subset H_{odd}$ and H_{odd} is cyclic, $N_{odd}\lhd G$. This shows that for $\sigma\in G-H$

$$N_{odd}\subset N\cap\sigma^{-1}N\sigma=\{id\}$$

and therefore $N=N_2$.

Now let M_2 be the fixed field of H_2 in F/M. Since N is a subgroup of H_2, it is clear that R (the fixed field of N in F/M) is a Galois extension of M_2, and since R/M is cyclic with $[R:M]=\ 1,\ 2,\ 3,\ 4,\ 6$, R is a cyclic extension of M_2 and $[R:M_2]=\ 1,\ 2,\ 4$. A similar statement holds for R^σ/M_2. We have

$$|H|=[F:M_2][M_2:M]=\frac{[R:M_2]^2}{[R\cap R^\sigma:M_2]}[M_2:M].$$

The following table summarizes the possibilities for $[F:M]$ in this case.

[1] Note that $N\cap Cent(G)=\{id\}$ and $N\simeq N/N\cap Cent(G)\simeq NCent(G)/Cent(G)\subset H/Cent(G)\simeq A$, where A is the cyclic subgroup of order n in D_n.

Table 2

$[R : M_2]$	$[R \cap R^\sigma : M_2]$	$[M_2 : M]$	$[F : M]$
2	1	1, 3	4, 12
4	1, 2	1	8, 16.

So if $|Cent(G)| = 2$ and $N \neq \{id\}$, then $[F : M] = 4,\ 8,\ 12,\ 16$ and so $G/Cent(G) \simeq D_n$ ($n =\ 2,\ 4,\ 6,\ 8$). Similarly, if $|Cent(G)| = 3$ and $N \neq \{id\}$, we can prove that $N = N_{odd}$, and $[F : M] = 9,\ 18$ and so $G/Cent(G) \simeq D_n$ ($n =\ 3,\ 6$).

Now it is easy to verify the list given in part (ii) of the statement of the theorem. This completes the proof. □

Remark 3. It might be of interest to note that a group G with cyclic center $Cent(G)$ having the property that $G/Cent(G) \simeq D_n$ is necessarily a product HK with H and K Abelian, with $H \cap K = Cent(G)$. Moreover, if $Cent(G)$ has order m, then H has order mn and K has order $2m$. In some cases, we can say more. For example, if $n = 3$ and $m = 2, 3, 4$, then $G \simeq Cent(G) \times D_n$.

Definition: The *generalized quaternion group* Q_{4n} is defined with the following presentations:

$$Q_{4n} = \langle x,\ y :\ x^{2n} = 1,\ x^n = y^2,\ yxy^{-1} = x^{-1} \rangle$$

Theorem 3 *Suppose that* $h_{\mathfrak{K}}$ *is odd and* $rank_{\mathcal{O}}(E(F)) = 2$ *and* $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$. *Then there is a Galois subextension* S *with* $K \subseteq S \subseteq F$ *and* $rank_{\mathcal{O}}(E(S)) > 0$ *such that* $G = \mathrm{Gal}(S/K)$ *is one of the following:*
(i) G *is cyclic of order* $1,\ 2,\ 3,\ 4$ *or* 6.
(ii) G *is isomorphic to* D_n ($n =\ 3,\ 4,\ 6$) *or* Q_{4n} ($n =\ 2,\ 3$).
(iii) $G \simeq \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ *or* G *is isomorphic to an extension of* $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ *by* $\mathbb{Z}/2\mathbb{Z}$ *with* $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$. *This can occur only if* $d_{\mathfrak{K}} \not\equiv 1$ *(mod 8).*
In (ii) and (iii), $rank_{\mathcal{O}}(E(S)) = 2$. *In fact,* S *is the minimal subfield in these cases.*

Proof: First note that since $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$ in part (ii) of Lemma 4, $n \neq 8,\ 12$ (see Remark 1). Applying this fact in the proof of Theorem 2 implies (i) if ρ (defined in the proof of Theorem 2) is reducible. In the case that ρ is irreducible and $G/Cent(G) \simeq D_n$, from the assumptions of the theorem, we conclude that $G \simeq D_3$ or $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$ and $G/Cent(G) \simeq D_n(n = 2,\ 3)$[2]. Now it is easy to verify the list given in part (ii) of the statement of the theorem, by referring to the list of non-Abelian groups of order 8 and 12 (see for example [5], Appendix B, p. 238).

So, we may suppose that ρ is irreducible and $G/Cent(G)$ is isomorphic to either A_4 or S_4 and that $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$.

[2] Note that $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$, however, $n = 4,\ 6,\ 8$ never occur. This is true since $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$ and therefore in the proof of Theorem 2 if $N = \{id\}$, then $H \simeq C_n(n =\ 2,\ 3,\ 4,\ 6)$ and if $N \neq \{id\}$, then in Table 1, $[R : M] = 2$.

Let $G/Cent(G) \simeq A_4$. Suppose that L is the fixed field of $Cent(G)$ in F/K and M is the fixed field of V_4 (Klein's 4-group) in L/K. Set $H \simeq Gal(F/M)$. Since $H/Cent(G) \cong V_4$ and $V_4 \lhd A_4$, it follows that $H \lhd G$, also it is clear that $[G : H] = 3$. Suppose that $\chi|_H$ is reducible. Then, by part (i) of Lemma 4, $\chi = Ind_H^G \psi$, $\psi(1) = 1$. This can never happen because $[G : H] = 3$ and χ is 2 dimensional.

Thus $\chi|_H$ is irreducible. Note that H is the 2-Sylow subgroup of G and it is of order 8. As it is necessarily non-Abelian, it is isomorphic to either D_4 or Q_8 (the quaternion group of order 8). In either case, G is the semidirect product of H and $\mathbb{Z}/3\mathbb{Z}$.

If $H \simeq Q_8$, then $G \simeq \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$. This group has three 2-dimensional irreducible representations. For two of these, the character takes values in $\mathbb{Q}(\sqrt{-3})$ (see for example [12], p. 61) and hence we can exclude these. The remaining representation has character values in $\mathbb{Z}$. If the restriction of this representation to Q_8 is irreducible (as we are assuming), it is a representation of Schur index 2 (see [19], p. 94, Exercise 12.3) and it is realizable over $\mathfrak{K}$ if and only if $\mathfrak{K}$ can be embedded in the quaternion algebra $\mathbb{D}$ over $\mathbb{Q}$ which is ramified at 2 and ∞. But if $d_{\mathfrak{K}} \equiv 1 \pmod 8$, then $\mathfrak{K}$ cannot be embedded in $\mathbb{D}$ as the prime 2 splits in this field. Thus, if $d_{\mathfrak{K}} \equiv 1 \pmod 8$ this case cannot occur.

If $H \simeq D_4$, then let J be the cyclic subgroup of order 4. Let A be a 3-Sylow subgroup of G. It acts by conjugation on J (as J contains all elements of order 4 in D_4). Moreover, it must act trivially as $\mathrm{Aut}(J)$ is cyclic of order 2. Hence, AJ is cyclic of order 12. Let P be the quadratic extension of K which is fixed by AJ. Restricting our representation ρ to AJ, we find it is reducible and given by two characters ψ_1 and ψ_2 (say). ψ_1 and ψ_2 take values in the group of 12-th roots of unity. The character of ρ on H thus takes values in $\mathbb{Q}(\zeta_{12}) \cap \mathfrak{K} = \mathbb{Q}$ (as $\mathfrak{K} \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$). In particular, it is real and so either ψ_1 and ψ_2 are both real or ψ_2 is the complex conjugate of ψ_1. Since $\rho|_H$ is faithful, the first case cannot occur as it would imply that H has order at most 4. Hence, we are in the second case, and this implies that ψ_1 is of order 12. But then, the character takes values in $\mathbb{Q}(\zeta_{12})^+$ which is not $\mathbb{Q}$ and this is a contradiction. Thus, this case also cannot occur.

Let $G/Cent(G) \simeq S_4$. Again let L be the fixed field of $Cent(G)$ in F/K, M be the fixed field of A_4 in L/K and $H = \mathrm{Gal}(F/M)$. Suppose first that $\chi|_H$ is reducible. Then by part (i) of Lemma 4, $\chi = Ind_H^G \psi$, $\psi(1) = 1$. Let $N = Ker\ \psi$. It is clear that $N \neq \{id\}$, since otherwise by part (ii) of Lemma 4, H is cyclic which is impossible. Let R be the fixed field of N, then $rank_{\mathcal{O}}(E(R)) > 0$. Since ρ is faithful, we must have $rank_{\mathcal{O}}(E(R)) = 1$. This implies that R is the minimal subfield and therefore it is cyclic of order 1 or 2 (Proposition 5). Since $N \cap \sigma^{-1} N \sigma = \{id\}$, we have $F = RR^\sigma$ and then by a calculation similar to one used in the proof of Theorem 2, we deduce $[F : M] = 4$ and hence $[F : K] = 8$, contradicting our assumption that $G/Cent(G) \simeq S_4$.

Now consider the case $\chi_1 = \chi|_H$ is irreducible. We argue as in the A_4 case. Let us set H_1 to be the 2-Sylow subgroup of H. Note that it is a normal subgroup. Now, if we have $\chi_1|_{H_1}$ reducible, this would force ρ_1 to be the induction of a character from

H_1 to H (by part (i) of Lemma 4) contradicting the fact that ρ_1 is a 2-dimensional representation. On the other hand, if $\chi_1|_{H_1}$ is irreducible, then H_1 is either the quaternion group of order 8 or the dihedral group of order 8 and both of these cases are dealt with as in the A_4 case using the fact that our representation has to be realizable over $\mathfrak{K}$. This shows that if $d_{\mathfrak{K}} \not\equiv 1 \pmod 8$, then $H \simeq \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ and therefore G is an extension of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$. This completes the proof of the theorem. □

5 Vanishing of L-functions

5.1 Non-CM case

Let E be an elliptic curve defined over $\mathbb{Q}$ and let $L(E/\mathbb{Q}, s)$ be the L-function of E over $\mathbb{Q}$. Kolyvagin [7] proved that for a (modular) elliptic curve E if $rank(E(\mathbb{Q})) \geq 1$ then $L(E/\mathbb{Q}, 1) = 0$ (see [6], p. 356, Theorem 20.5.2. (b)). This result is generalized to any finite Abelian extension of $\mathbb{Q}$ by Kato (unpublished).

Theorem 4 *Let E be a modular elliptic curve defined over $\mathbb{Q}$ and let F be a finite solvable extension of $\mathbb{Q}$. Suppose that $rank(E(F)) \geq 1$.*
(i) If $E(F) \otimes \mathbb{Q}$ is an Abelian $\mathrm{Gal}(F/\mathbb{Q})$ module then $L(E/F, 1) = 0$.
(ii) If $rank(E(F)) = 1$ then $L(E/F, 1) = 0$.
(iii) If $rank(E(F)) = 2$ then either $L(E/F, 1) = 0$ or the minimal subfield is a dihedral extension of $\mathbb{Q}$ of degree 6, 8 or 12.
(iv) If $rank(E(F)) = 3$ then either $L(E/F, 1) = 0$ or $\mathrm{Gal}(M/K)$ (M is the minimal subfield) is one of the following:

$$A_4, \quad S_4, \quad A_4 \times C_2, \quad S_4 \times C_2.$$

Proof: (i) Since $E(F) \otimes \mathbb{Q}$ is an Abelian Galois module, by Proposition 2, there is an Abelian subextension M of $\mathbb{Q}$ such that $rank(E(M)) \geq 1$. Now Kato's generalization of Kolyvagin's theorem implies the vanishing of $L(E/M, s)$ at $s = 1$. By Theorem 2 of [11], $L(E/F, s)$ is divisible by $L(E/M, s)$. Hence, $L(E/F, s)$ also vanishes at $s = 1$. This completes the proof.

(ii) By part (i) of Theorem 1, $E(F) \otimes \mathbb{Q}$ is a cyclic Galois module, and the result follows from part (i).

(iii) It follows from part (ii) of Theorem 1 and (i).

(iv) Let $\rho_f : \mathrm{Gal}(M/K) \to \mathrm{GL}_3(\mathbb{Z})$ be the faithful representation given in Proposition 2. We prove that if ρ_f is reducible then $L(E/F, 1) = 0$. Let ρ_f be reducible, then since its degree is 3, ρ_f has a one dimensional representation ψ of $\mathrm{Gal}(M/K)$ as a direct summand. Let M_1 be the fixed field of $ker\ \psi$ in M/K. It is clear that E has a point of infinite order on M_1 and M_1 is at most quadratic over $\mathbb{Q}$. As in (i), we conclude that $L(E/M_1) = 0$ which implies $L(E/F, 1) = 0$.

Now note that in part (iii) of Theorem 1, the only groups with a possible three dimensional irreducible representation, are those given in the statement of the theorem. This completes the proof. □

Remark 4. If $M/\mathbb{Q}$ is a dihedral extension of degree $2n$ such that the fixed field C of the cyclic subgroup of order n of $\mathrm{Gal}(M/\mathbb{Q})$ is imaginary quadratic and of discriminant prime to the conductor of E, and $(E(M) \otimes \mathbb{C})^{\chi} \neq 0$ is infinite (χ is a two dimensional character of $\mathrm{Gal}(M/\mathbb{Q})$), then by recent work of Bertolini and Darmon [2], $L(E/\mathbb{Q} \otimes \chi, 1) = 0$. Applying this with the factorization of the L-function of E over M (see the paragraph before Proposition 4) and part (ii) of Theorem 1, we deduce that if F is a finite solvable extension of $\mathbb{Q}$ such that any quadratic subfield is imaginary and of discriminant prime to the conductor of E, and $rank(E(F)) = 2$ then $L(E/F, 1) = 0$.

5.2 CM case

Let E be an elliptic curve defined over an imaginary quadratic field K and having complex multiplication by $\mathcal{O}$, the ring of integers of K. Let $L(E/K, s)$ be the L-function of E over K. It is known that $L(E/K, s)$ is the product of two Hecke L-series of K (see [21], p. 175, Theorem 10.5) and therefore it is defined on the whole complex plane. Coates and Wiles [3] proved that if $\mathrm{rank}(E(K)) \geq 1$ then $L(E/K, 1) = 0$. Arthaud [1] generalized this result to any finite Abelian extension of K. She proved that if F is a finite Abelian extension of K such that $\mathrm{rank}(E(F)) \geq 1$ then $L(E/F, 1) = 0$. The work of Rubin [18] established this under some conditions even if E is not defined over K.

Theorem 5 *Let E be an elliptic curve defined over an imaginary quadratic field K and having complex multiplication by $\mathcal{O}$, the ring of integers of K. Let F/K be a finite Galois extension and let $rank_{\mathcal{O}}(E(F)) \geq 1$.*
(i) If $E(F) \otimes_{\mathcal{O}} K$ is an Abelian $K[G]$-module then $L(E/F, 1) = 0$.
(ii) If $rank_{\mathcal{O}}(E(F)) = 1$ then $L(E/F, 1) = 0$.
(iii) If $rank_{\mathcal{O}}(E(F)) = 2$ and $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then either $L(E/F, 1) = 0$ or the Galois group of the minimal subfield over K is isomorphic to one of the following:
a) D_n ($n =$ 3, 4, 6), Q_{4n} ($n =$ 2, 3).
b) $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ or an extension of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ by $\mathbb{Z}/2\mathbb{Z}$ with $Cent(G) \simeq \mathbb{Z}/2\mathbb{Z}$. This can occur only if $K \neq \mathbb{Q}(\sqrt{-7})$.

Proof: (i) By the $\mathcal{O}$-analogue of Proposition 2, there is an Abelian subextension M of K such that $rank_{\mathcal{O}}(E(M)) \geq 1$. Now by Arthaud's theorem [1], $L(E/M, 1) = 0$. By Theorem 1 of [11], $L(E/F, s)$ is divisible by $L(E/M, s)$. Hence $L(E/F, 1) = 0$.

(ii) By Proposition 5, $E(F) \otimes_{\mathcal{O}} K$ is a cyclic $K[G]$-module, and the result follows from part (i).

(iii) It follows from Theorem 3 and (i). Note that since the j-invariant $j(E) \in K$ then $h_K = 1$, and $K = \mathbb{Q}(\sqrt{-7})$ is the only imaginary quadratic number field with $h_K = 1$ that for it $d_K \equiv 1 \pmod 8$. □

6 Elliptic analogue of Stark's theorem

In this section, we investigate the analytic analogue of the minimal subfield. In this, we are guided by the results of Stark [22] about simple zeros of Dedekind zeta functions.

Definition: Let E be an elliptic curve defined over K and let F be an extension of K. For each zero ω of $L(E/F, s)$, the *analytic minimal subfield* F_ω is a subfield over K with $K \subseteq F_\omega \subseteq F$ such that
(i) $ord_{s=\omega} L(E/F_\omega, s) = ord_{s=\omega} L(E/F, s)$.
(ii) If $K \subseteq M \subseteq F$ and $ord_{s=\omega} L(E/M, s) = ord_{s=\omega} L(E/F, s)$, then $F_\omega \subseteq M$.

Proposition 6 *Let F/K be a Galois extension with Galois group G, and suppose that $L(E/K \otimes \chi, s)$ is holomorphic at $s = \omega$ for any irreducible character χ of G. Then the analytic minimal subfield F_ω exists and it is Galois over K.*

Proof: We have the factorization

$$L(E/F, s) = \prod_{\chi \in Irr(G)} L(E/K \otimes \chi, s)^{\chi(1)}$$

where $Irr(G)$ is the set of irreducible characters of G. Consider the set

$$Z_\omega = \{\chi| \ \ L(E/K \otimes \chi, \omega) = 0\}.$$

Define

$$H_\omega = \bigcap_{\chi \,\in\, Z_\omega} Ker\ \chi.$$

Then H_ω is a normal subgroup of G and we let F_ω denote its fixed field, which is Galois over K. Using the holomorphy of $L(E/K \otimes \chi, s)$, it is easy to see that $ord_{s=\omega} L(E/F, s) = ord_{s=\omega} L(E/F_\omega, s)$.

Now let M be any field between F and K. Put $H = \mathrm{Gal}(F/M)$ and let 1_H be the identity character of H. We have

$$Ind_H^G 1_H = \sum_{\chi \in Irr(G)} a_\chi \chi, \quad 0 \le a_\chi \le \chi(1), \quad a_\chi \in \mathbb{Z}.$$

Thus,

$$L(E/M, s) = L(E/K \otimes Ind_H^G 1_H, s) = \prod_{\chi \in Irr(G)} L(E/K \otimes \chi, s)^{a_\chi}.$$

This shows that if $ord_{s=\omega} L(E/M, s) = ord_{s=\omega} L(E/F, s)$, then

$$\sum a_\chi n_\chi \ = \ \sum \chi(1) n_\chi$$

where n_χ denotes the order of $L(E/K \otimes \chi, s)$ at $s = \omega$. Hence, $a_\chi = \chi(1)$ for all $\chi \in Z_\omega$. We have

$$a_\chi = \langle Ind_H^G 1_H, \chi\rangle_G \; = \; \langle 1_H, \chi|_H\rangle_H \; = \; \frac{1}{|H|}\sum_{g\in H}\chi(g).$$

Now if $a_\chi = \chi(1)$, then as $|\chi(g)| \leq \chi(1)$, we must have $\chi(g) = \chi(1)$ for all $g \in H$ and therefore $H \subset Ker\ \chi$ and this holds for all $\chi \in Z_\omega$. In other words $H \subset H_\omega$. This proves that $F_\omega \subseteq M$. □

Definition. We say that E satisfies the Taniyama conjecture over a field K if the L-function $L(E/K, s)$ is the L-function $L(\pi, s)$ of an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_K)$, where $\mathbb{A}_K$ is the adèle ring of K.

Proposition 7 *Suppose that E satisfies the Taniyama conjecture over K. Let F be a solvable extension of K and let χ be a character of $G = \mathrm{Gal}(F/K)$. Then, $L(E/K \otimes \chi, s)$ is holomorphic at $s = \omega$ if ω is a simple zero of $L(E/F, s)$.*

Proof: Let H be a subgroup of G and let χ and ψ denote irreducible characters of G and H. Set

$$\theta_G = \sum_\chi n_\chi \chi, \qquad \theta_H = \sum_\psi n_\psi \psi$$

where n_χ and n_ψ denote the orders of zeros of $L(E/K \otimes \chi, s)$ and $L(E/F^H \otimes \psi, s)$ at $s = \omega$ respectively (F^H is the fixed field of H in F/K). By Proposition 1 of [11]

$$\theta_G|_H = \theta_H. \qquad (*)$$

Suppose g is an element of G and let $H = \langle g\rangle$ be the cyclic group generated by g. Then, $L(E/F^H \otimes \psi, s)$ is analytic (see [11], p. 492, Proof of Theorem 2) and since

$$L(E/F, s) = \prod_\psi L(E/F^H \otimes \psi, s)^{\psi(1)}$$

and $ord_{s=\omega} L(E/F, s) = 1$, then $\theta_H = \psi$ for some irreducible character ψ of H. From $(*)$, $\theta_G(g)$ is a root of unity and therefore

$$\sum_\chi {n_\chi}^2 = \left\langle \sum_\chi n_\chi \chi, \sum_\chi n_\chi \chi \right\rangle$$

$$= \frac{1}{|G|}\sum_{g\in G} |\theta_G(g)|^2 = 1.$$

This shows that all n_χ's except one are 0. By taking $H = \langle 1\rangle$, we have $\theta_G(1) = 1$ and thus the remaining n_χ is 1. This proves that $L(E/K \otimes \chi, s)$ is analytic at $s = \omega$. □

Corollary 1 *Under the assumptions of the above proposition F_ω exists. Moreover, F_ω is a cyclic extension of K. If ω is real, $[F_\omega : K] \leq 2$.*

Proof: By the previous proposition $L(E/K \otimes \chi, s)$ is holomorphic at $s = \omega$, thus if $ord_{s=\omega} L(E/F, s) = 1$ then there is a $\chi \in Irr(G)$ such that $ord_{s=\omega} L(E/K \otimes \chi, s) = 1$ and $\chi(1) = 1$. Now by Proposition 6, F_ω is the fixed field of $Ker\ \chi$. Since χ is one dimensional F_ω is a cyclic extension of K. Moreover, if ω is real

$$ord_{s=\omega} L(E/K \otimes \overline{\chi}, s) \ = \ ord_{s=\omega} L(E/K \otimes \chi, s).$$

Hence, $\chi = \overline{\chi}$. □

Remark 5. Let F be a Galois extension of K, then Corollary 1 is still true if E is an elliptic curve with complex multiplication. Note that in this case, we can remove the hypothesis that F/K is solvable, as E satisfies the Taniyama conjecture over any Galois extension of K (see [11], p. 488, Lemma 2).

Corollary 2 *Let E be an elliptic curve defined over a number field K. Suppose that E has complex multiplication by an order in an imaginary quadratic field contained in K. Let F be a Galois extension of K and let χ be a character of $G = \mathrm{Gal}(F/K)$. Then, $L(E/K \otimes \chi, s)$ is holomorphic at $s = \omega$ if ω is a double zero of $L(E/F, s)$, and ω is real. Moreover, F_ω exists and F_ω is a cyclic extension of K.*

Proof: We have the factorization

$$L(E/K, s) \ = \ L(\psi_K, s) L(\overline{\psi}_K, s)$$

where ψ_K is a Hecke character of K. Over F,

$$L(E/F, s) \ = \ L(\psi_F, s) L(\overline{\psi_F}, s)$$

where ψ_F denotes the restriction of ψ_K to $\mathrm{Gal}(\overline{F}/F)$. As ω is real, both factors on the right vanish at $s = \omega$. As $ord_{s=\omega} L(E/F, s) = 2$, it follows that

$$ord_{s=\omega} L(\psi_F, s) \ = \ ord_{s=\omega} L(\overline{\psi_F}, s) \ = \ 1.$$

Now the argument of Proposition 7 implies that all $L(\psi_K \otimes \chi, s)$ are holomorphic at $s = \omega$ and that F_ω exists and is a cyclic extension of K. □

Finally, we show that we can replace the assumption of holomorphy in the statement of Proposition 6, with a milder condition if we assume that E has complex multiplication and F is contained in a solvable extension of K (F/K is not necessarily Galois).

Proposition 8 *Suppose that F/K has solvable normal closure, and let E be an elliptic curve defined over K which has complex multiplication. Suppose that for any two subfields M_1 and M_2 with the property that*

$$ord_{s=\omega} L(E/M_1, s) = ord_{s=\omega} L(E/M_2, s) = ord_{s=\omega} L(E/F, s)$$

the quotient

$$\frac{L(E/M_1M_2,s)L(E/M_1\cap M_2,s)}{L(E/M_1,s)L(E/M_2,s)}$$

is holomorphic at $s=\omega$. Then the analytic minimal subfield F_ω exists.

Proof: Let $\mathcal{S}$ be the set of subfields M of F with

$$ord_{s=\omega}L(E/M,s)=ord_{s=\omega}L(E/F,s).$$

We prove that $\mathcal{S}$ is closed under intersections and thus has a minimal element. Let M_1 and M_2 be in $\mathcal{S}$, then by the hypothesis

$$\frac{L(E/M_1M_2,s)L(E/M_1\cap M_2,s)}{L(E/M_1,s)L(E/M_2,s)}$$

is holomorphic at ω. Moreover, by the main result of [11] (see Theorem 1), $L(E/M_1,s)$ divides $L(E/M_1M_2,s)$ and $L(E/M_1M_2,s)$ divides $L(E/F,s)$. Thus,

$$ord_{s=\omega}L(E/M_1,s)\leq ord_{s=\omega}L(E/M_1M_2,s)\leq ord_{s=\omega}L(E/F,s)$$

and therefore we have equality throughout. Hence,

$$ord_{s=\omega}L(E/M_1\cap M_2,s)\geq ord_{s=\omega}L(E/F,s).$$

The reverse inequality also holds (as $L(E/M_1\cap M_2,s)$ divides $L(E/F,s)$). This proves that $\mathcal{S}$ has a minimal element F_ω. □

Remark 6. Note that the assumption of holomorphy in the previous proposition is implied by the holomorphy of $L(E/K\otimes\chi,s)$ at $s=\omega$ (see [22], p. 151, Lemma 12).

Remark 7. Proposition 8 is also true, in the case that E satisfies the Taniyama conjecture over K and F is a solvable extension of K.

References

[1] N. Arthaud, On Birch and Swinnerton-Dyer's conjecture for elliptic curves with complex multiplication. I., *Comp. Math.* **37** (1978), 209-232.

[2] M. Bertolini and H. Darmon, The p-adic Birch and Swinnerton-Dyer conjecture, *in preparation.*

[3] J. Coates and A. Wiles, On the conjecture of Birch and Swinnerton-Dyer, *Invent. Math.* **39** (1977), 223-251.

[4] C.W. Curtis and I. Reiner, *Methods of representation theory, Volume I*, Wiley Interscience, New York, 1981.

[5] J. F. Humphreys, *A course in group theory*, Oxford University Press, 1996.

[6] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, Second Edition, Springer-Verlag, 1990.

[7] V. A. Kolyvagin, Finiteness of $E(\mathbb{Q})$ and III($\mathbb{Q}$) for a class of Weil curves, *Math. USSR Izv.* **32** (1989), 523-542.

[8] S. Lang, *Algebra*, Third Edition, Addison-Wesley, 1993.

[9] B. Mazur, Rational points on Abelian varieties in towers of number fields, *Invent. Math.*, **18**(1972), 183-266.

[10] B. Mazur and H.P.F. Swinnerton-Dyer, Arithmetic of Weil curves, *Invent. Math.*, **25**(1974), 1-61.

[11] M. R. Murty and V. K. Murty, Base change and the Birch- Swinnerton-Dyer conjecture, *Contemp. Math.* **143** (1993), 481-494.

[12] V. K. Murty, Holomorphy of Artin L-functions, in: Proc. Ramanujan Centennial Intl. Conf. ed. R. Balakrishnan et. al., pp. 55-66, Ramanujan Math. Society, Madras, 1988.

[13] V. K. Murty, Class numbers of CM-fields with solvable normal closure, *Compositio Math.*, to appear.

[14] M. V. Nori, On subgroups of $\mathrm{GL}_n(\mathbf{F}_p)$, *Invent. Math.* **88** (1987), 257-275.

[15] E. Rees, *Notes on Geometry*, Springer-Verlag, 1983.

[16] D. Rohrlich, The vanishing of certain Rankin-Selberg convolutions, in: *Automorphic forms and analytic number theory*, ed. R. Murty, pp. 123-133, CRM, Montréal, 1990.

[17] D. Rohrlich, Galois theory, elliptic curves, and root numbers, *Comp. Math.* **100** (1996) 311-349.

[18] K. Rubin, Elliptic curves with complex multiplication and the conjecture of Birch and Swinnerton-Dyer, *Invent. Math.*, **64**(1981), 455-470.

[19] J.-P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, 1977.

[20] J.-P. Serre, Propriétés galoisiennes des points d'ordre fini des courbes elliptiques, *Invent. Math.* **15** (1972), 259-331.

[21] J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.

[22] H. Stark, Some effective cases of the Brauer-Siegel theorem, *Invent. Math.* **23** (1974), 135-152.

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