Non-Vanishing of Modular L-Functions with Large Level

by

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ABSTRACT

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This thesis studies the non-vanishing of the twisted modular L-function $L_f(s, \chi)$ for a fixed weight k, varying level N and a fixed Dirichlet character $\chi \pmod q$ where $(q, N) = 1$. Here f is a newform of level N. Let $\mathcal{F}_N$ be the set of newforms of weight k and level N.

(1) It is proved that

$$C_k \frac{N}{(\log N)^2} \leq \sharp\{f \in \mathcal{F}_N \colon\ L_f(\frac{k}{2}, \chi) \neq 0\}$$

for prime N large enough. Here, C_k is a constant depending only on k.

(2) It is proved that for real Dirichlet characters χ_1 and χ_2 with $\chi_1\chi_2(-N) = 1$ and $k > 2$,

$$C'_k \frac{N}{(\log N)^6} \leq \sharp\{f \in \mathcal{F}_N \colon\ L_f(\frac{k}{2}, \chi_1) L_f(\frac{k}{2}, \chi_2) \neq 0\}$$

for prime N large enough. Here, C'_k is a constant depending only on k.

(3) In the case $k = 2$, it is proved that under the assumption of the Generalized Riemann Hypothesis for $L_f(s)$ and the assumption of $L_{sym^2(f)}(\frac{3}{2} + it) \ll N^{\frac{1}{2}-\eta}$, for some $\eta > 0$

$$cN \leq \sharp\{f \in \mathcal{F}_N \colon\ L_f(1) = 0 \textit{ and } L'_f(1) \neq 0\}$$

for prime N large enough. Here $L'_f(s)$ is the derivative of $L_f(s)$ and c $(0 < c < 1)$ is an absolute constant.

During the course of the proof of (3), a "semi-orthogonality" relation between the Fourier coefficients of $\mathcal{F}_N^-$ (newforms with root number -1) is given. Using this

relation and the symmetric square L-function properties, upper bounds for

$$\sum_{f\in\mathcal{F}_N^-}\frac{r_f}{4\pi < f, f >}, \ and \sum_{f\in\mathcal{F}_N^-} r_f^{\frac{1}{2}}$$

and asymptotic formula for

$$\sum_{f\in\mathcal{F}_N^-} < f, f >$$

are obtained, where r_f is the vanishing order of $L_f(s)$ at $s = 1$ and $< \ldots >$ denotes the Petersson inner product.

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NOTATIONS

$f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

$f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a constant C such that $|f(x)| \leq Cg(x)$

$\mathcal{H}$: the upper half-plane

$\mathcal{H}^*$: $\mathcal{H} \bigcup \mathbb{Q} \bigcup \{\infty\}$

$GL_2{}^+(\mathbb{R})$: the group of 2×2 matrices with real entries and positive determinant

$SL_2(\mathbb{Z})$: the group of 2×2 matrices with integer entries and determinant equal to 1

$\overline{\Gamma_0(N)}$: the subgroup of $SL_2(\mathbb{Z})$ consist of matrices $(a_{ij})_{2\times 2}$ which a_{21} is divisible by N

$\Gamma_0(N)$: $\overline{\Gamma_0(N)}$ mod its center

$S_k(N)$: the space of cusp forms of weight k and level N

$< f, g >_N$: the Petersson inner product of f and g in $S_k(N)$

$a_f(n)$: the n-th Fourier coefficient of the cusp form f

$L_f(N)$: the L-function associated to the cusp form f

W_N: the Atkin-Lehner involution

$S_k{}^+(N)$: the $(-1)^{\frac{k}{2}}$-eigenspace of W_N in $S_k(N)$

$S_k{}^-(N)$: the $(-1)^{\frac{k}{2}+1}$-eigenspace of W_N in $S_k(N)$

T_p $(p \nmid N)$, U_q $(q|N)$: the Hecke operators

$\mathcal{F}_N$: the set of normalized newforms of weight k and level N

χ: a Dirichlet character

$L_f(s, \chi)$: the twisted L-function associated to f and χ

ϵ_f: the root number of $L_f(s)$

$\tau(x)$: the Gauss sum

ϵ_χ: the root number of $L_f(s, \chi)$

$L_{sym^2(f)}(s)$: the symmetric square L-function associated to f

r_f: the vanishing order of $L_f(s)$ at $s = \frac{k}{2}$

$P_n(z, k, N)$: the Poincaré series of weight k and level N for $S_k(N)$

$\hat{P}_n(m, k, N)$: the m-th coefficient of the Fourier expansion of $P_n(z, k, N)$

$P_n^-(z, k, N)$: the Poincaré series of weight k and level N for $S_k^-(N)$

Γ_∞: the stabilizer of ∞ in $\Gamma_0(N)$

$\Gamma(s)$: the Gamma function

$S(m, n; c)$: the Kloosterman sum

$J_{k-1}(t)$: the Bessel function of order $k-1$

δ_{mn}: the Kronecker delta

T_p $(p \nmid N)$, C_q $(q|N)$: the Pizer operators

$\mathcal{P}_N$: the Pizer basis

$Tr(f)$: the trace function

$tr(T_{e^2})$: the trace of the e^2-th Hecke operator

$\zeta(s)$: the Riemann zeta function

$\zeta_N(s)$: the Riemann zeta function with the Euler factors corresponding to $p|N$ removed

$\mathbf{d}(n)$: the number of positive divisors of n

$\phi(n)$: the Euler phi function

$\Lambda(n)$: the Von Mangoldt function

$\mu(n)$: the Möbius function

$q^r \parallel N$: $q^r \mid N$ but $q^{r+1} \nmid N$

(m, n): the greatest common divisor of m and n

Contents

Chapter 1

Introduction and Statement of Results

We recall some basic facts about modular forms (see [12] and [15] for details).

1.1 Modular forms

Let $\mathcal{H}$ denote the upper half-plane

$$\mathcal{H} = \{z = x + iy : \quad x \in \mathbb{R},\ y > 0\}.$$

Let $GL_2{}^{+}(\mathbb{R})$ be the group of 2×2 matrices with real entries and positive determinant. Then $GL_2{}^{+}(\mathbb{R})$ acts on $\mathcal{H}$ as a group of holomorphic automorphisms

$$\gamma \ : \ z \mapsto \frac{az+b}{cz+d},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2{}^{+}(\mathbb{R}).$$

Let $\mathcal{H}^{*}$ denote the union of $\mathcal{H}$ and the rational numbers $\mathbb{Q}$ together with a symbol ∞ (or $i\infty$). The rational numbers together with ∞ are called *cusps*.

Let f be a holomorphic function on $\mathcal{H}$ and k a positive integer. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2{}^+(\mathbb{R}).$$

define the *stroke operator* "$|_k$" as

$$(f|_k\gamma)(z) = (det\gamma)^{\frac{k}{2}}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Sometimes, we simply write $f|\gamma$ for $f|_k\gamma$. Note that $(f|\gamma)|\sigma = f|\gamma\sigma$.

Let $SL_2(\mathbb{Z})$ be the group of 2×2 matrices with integer entries and determinant 1 and let Γ be a subgroup of finite index of it. Suppose f is a holomorphic function on $\mathcal{H}$ such that $f|\gamma = f$ for all $\gamma \in \Gamma$. Since Γ has finite index,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^M = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \in \Gamma$$

for some positive integer M. Hence $f(z+M) = f(z)$ for all $z \in \mathcal{H}$. So f has a "*Fourier expansion at infinity*" in the form of

$$f(z) = \sum_{n=-\infty}^{\infty} a_f(n) q_M{}^n, \qquad q_M = e^{\frac{2\pi i z}{M}}.$$

We say that f is *holomorphic at infinity* if $a_n = 0$ for all $n < 0$. We say it *vanishes at infinity* if $a_n = 0$ for all $n \leq 0$.

Let $\sigma \in SL_2(\mathbb{Z})$. Then $\sigma^{-1}\Gamma\sigma$ also has finite index and $(f|\sigma)|\gamma = f|\sigma$ for all $\gamma \in \sigma^{-1}\Gamma\sigma$. So $f|\sigma$ also has a Fourier expansion at infinity. We say that f is *holomorphic at the cusps* if $f|\sigma$ is holomorphic at infinity for all $\sigma \in SL_2(\mathbb{Z})$. We say that f *vanishes at the cusps* if $f|\sigma$ vanishes at infinity for all $\sigma \in SL_2(\mathbb{Z})$.

Now for $N \geq 1$ let

$$\overline{\Gamma_0(N)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \ c \equiv 0 \ (mod\ N) \right\}$$

and $\Gamma_0(N) = \overline{\Gamma_0(N)}/\{\pm 1\}$. Note that $\overline{\Gamma_0(N)}$ is of finite index in $SL_2(\mathbb{Z})$ (Here, we follow the unconventional notation of [4] to be consistent with the results of [5]).

A *modular form of weight k and level N* is a holomorphic function f on $\mathcal{H}$ such that

(i) $f|\gamma = f$ for all $\gamma \in \Gamma_0(N)$,

(ii) f is holomorphic at the cusps.

Such a modular form is called a *cusp form* if it vanishes at the cusps.

The modular forms of weight k and level N form a finite dimensional vector space $M_k(N)$ and this has a subspace $S_k(N)$ consisting of cusp forms. Note that since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the same as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in $\Gamma_0(N)$, (i) shows that $S_k(N) = \{0\}$ if k is odd. From now on we assume that k is even.

Moreover, one can define an inner product called *Petersson inner product* on $S_k(N)$ by

$$< f, g >= \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}.$$

Note that if $N_1 \mid N_2$ then $S_k(N_1) \subset S_k(N_2)$. However, the value of the Petersson inner product depends on N. To emphasize this dependency sometimes we write $< f, g >_N$.

1.2 L-function of a cusp form

Now if $f \in S_k(N)$ since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ its Fourier expansion at $i\infty$ is of the form

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz), \quad e(z) = e^{2\pi i z}.$$

Attached to f, we define the *the L-function associated to f* by the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}.$$

We can show that $L_f(s)$ represents an analytic function for $Re(s) > \frac{k+1}{2}$. This is a consequence of the fact that $a_f(n) = O(n^{\frac{k-1}{2}})$ (see formula (1.1)).

Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. It is not an element of $SL_2(\mathbb{Z})$ unless $N = 1$. However.

$$W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N).$$

Moreover. $f|W_N^2 = f$. W_N is called the *Atkin-Lehner involution.*

More generally for any prime q dividing N with $q^r \parallel N$ (i.e. $q^r \mid N$ but $q^{r+1} \nmid N$). let

$$W_q = \begin{pmatrix} q^r x & y \\ Nz & q^r w \end{pmatrix}$$

where x, y, z and w are any integers satisfying $det(W_q) = q^r$. W_q is called the "*W_q operator*" of Atkin and Lehner.

Since W_N is a linear transformation of the vector space $S_k(N)$ and $W_N^2 = 1$, it decomposes the space of cusp forms (modular forms) to complementary subspaces corresponding to the eigenvalues ± 1. Set

$$S_k^+(N) = \left\{ f \in S_k(N): \quad f|W_N = (-1)^{\frac{k}{2}} f \right\}.$$

$$S_k^-(N) = \left\{ f \in S_k(N): \quad f|W_N = (-1)^{\frac{k}{2}+1} f \right\}.$$

and so $S_k(N) = S_k^+(N) \dotplus S_k^-(N)$.

Then the following Theorem of Hecke guarantees the analytic continuation of $f \in S_k(N)$.

Theorem (Hecke) *Let $f \in S_k^{\pm}(N)$. Then $L_f(s)$ extends to an entire function and $\Lambda_f(s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L_f(s)$ satisfies the functional equation $\Lambda_f(s) = \pm\Lambda_f(k-s)$.*

Corollary *Let $f \in S_k(N)$. Then $L_f(s)$ extends to an entire function.*

Note Our definition of $S_k^+(N)$ and $S_k^-(N)$ is slightly different from the conventional ones which denote them as subspaces corresponding to the eigenvalues $+1$ and -1 for operator W_N, so for $\frac{k}{2}$ odd our $S_k^{\pm}(N)$ is their $S_k^{\mp}(N)$. The *root number* of $L_f(s)$

is the sign appearing in the functional equation of $L_f(s)$. In our notation $S_k^{\pm}(N)$ is the set of cusp forms whose L-functions have root number ± 1, respectively.

1.3 Hecke operators

Let $f \in M_k(N)$. Let p and q be primes such that $p \nmid N$ and $q \mid N$. The *Hecke operators* T_p and U_q are defined by

$$f \mid T_p = p^{\frac{k}{2}-1}\left[f \mid \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{e=0}^{p-1} f \mid \begin{pmatrix} 1 & e \\ 0 & p \end{pmatrix} \right].$$

$$f \mid U_q = q^{\frac{k}{2}-1}\left[\sum_{e=0}^{q-1} f \mid \begin{pmatrix} 1 & e \\ 0 & q \end{pmatrix} \right].$$

We can show that $f \mid T_p$, $f \mid U_q$ are also modular forms of weight k and level N, and furthermore they are cusp forms if f is a cusp form.

Let $f \in S_k(N)$. We will say that f is an *eigenform* if f is an eigenfunction for all the Hecke operators $\{T_p\ (p \nmid N),\ U_q\ (q|N)\}$. The following theorem gives the main property of eigenforms.

Theorem *The following conditions are equivalent:*

(i) f is an eigenform and $a_f(1) = 1$.

(ii) $L_f(s)$ has a product of the form

$$L_f(s) = \prod_{q|N}\left(1 - \frac{a_f(q)}{q^s}\right)^{-1} \prod_{p\nmid N}\left(1 - \frac{a_f(p)}{p^s} + \frac{1}{p^{2s+1-k}}\right)^{-1}$$

which converges absolutely for $Re(s) > \frac{k+1}{2}$.

We call the product given in part (ii) of the above theorem an *Euler Product.* Inspired by the above theorems we may think of finding a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{W_N,\ T_p\ (p \nmid N),\ U_q\ (q|N)\}$. We can show that there exists a basis for $S_k(N)$ consisting of eigenforms for all the operators $\{T_p\ (p \nmid N)\}$ and the operator W_N (see [1] Lemma 27). The existence of such a basis is the consequence of the fact that $\{T_p\ (p \nmid N),\ W_N\}$ form a commuting family of Hermitian linear operators (with respect to the Petersson inner product) and therefore from a theorem of linear algebra (see [10] p. 207, Theorem 8) the space of cusp forms is diagonalizable under these operators. Unfortunately the operators $\{U_q\ (q|N)\}$ are not Hermitian for $S_k(N)$ and we can not diagonalize $S_k(N)$ with respect to the operators $\{T_p\ (p \nmid N),\ U_q\ (q|N),\ W_N\}$. However, we may find such a basis for a certain subspace of $S_k(N)$ (For a proof of the fact that the operator W_N is Hermitian, see Lemma 14, Chapter 3).

1.4 Oldforms and newforms

In [1] Atkin and Lehner construct a subspace of $S_k(N)$ which is diagonizable under the operators $\{T_p\ (p \nmid N),\ U_q\ (q|N),\ W_N\}$. More precisely they showed the existence of a subspace of $S_k(N)$ whose $\{T_p\ (p \nmid N)\}$ eigenspaces are one dimensional. We call such a property "*multiplicity one*". Now since the $\{U_q\ (q|N),\ W_N\}$ commute with the $\{T_p\ (p \nmid N)\}$, the eigenform for the $\{T_p\ (p \nmid N)\}$ are eigenform for the $\{U_q\ (q|N),\ W_N\}$ too.

Let $N' \mid N\ (N' \neq N)$ and suppose that the $\{g_i\}$ is a basis of eigenforms for the $\{T_p\ (p \nmid N')\}$. Now if d is any divisor of $\frac{N}{N'}$ then $g_i(dz) \in S_k(N)$. Set

$$S_k{}^{old}(N) = span\{g_i(dz);\quad for\ any\ \ N' \mid N\ (N' \neq N),\ \ d \mid \frac{N}{N'}\}$$

We call $S_k{}^{old}(N)$ the space of *oldforms*. Its orthogonal complement under the Petersson inner product is denoted $S_k{}^{new}(N)$ and the eigenforms in this space are called *newforms*. So we have

$$S_k(N) = S_k{}^{old}(N) \bigoplus S_k{}^{new}(N).$$

If f is a newform then we can prove that $a_f(1) \neq 0$ and therefore we can normalize a newform such that $a_f(1) = 1$. Since the space of newforms has multiplicity one the set of *normalized newforms of weight k and level N* is uniquely determined. We denote it by $\mathcal{F}_N$. From the above discussion it is clear that if $f \in \mathcal{F}_N$. $L_f(s)$ is given by an absolutely convergent series on the half plane $Re(s) > \frac{k+1}{2}$, it has an analytic continuation to the whole plane. Moreover, it satisfies a functional equation and has an Euler product on the half-plane $Re(s) > \frac{k+1}{2}$. For the Fourier coefficients of a newform f we have the *Deligne bound*

$$|a_f(n)| \leq \mathbf{d}(n) n^{\frac{k-1}{2}} \tag{1.1}$$

where $\mathbf{d}(n)$ is the divisor function.

Now let $f \in \mathcal{F}_N$ and χ be a primitive Dirichlet character mod q with $(q, N) = 1$. The *twisted L-function* associated to f and χ is defined by

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) a_f(n)}{n^s}.$$

The twisted L-function is given by an absolutely convergent series on the half-plane $Re(s) > \frac{k+1}{2}$ and has an Euler product valid there. Also it has an analytic continuation which satisfies the following functional equation

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s, \chi) = \epsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s) L_f(k-s, \bar{\chi}) \tag{1.2}$$

where $\epsilon_\chi = \epsilon_f \chi(N) \tau(\chi)^2 q^{-1}$ with $\epsilon_f = \pm 1$ (the root number of f) which depends only on f and $\tau(\chi)$ is the Gauss sum (see [20] p. 93).

Now let $f \in \mathcal{F}_N$; then the *symmetric square L-function* associated to f is defined by

$$L_{sym^2(f)}(s) = \frac{\zeta_N(2s+2-2k)}{\zeta_N(s+1-k)} \sum_{n=1}^{\infty} \frac{(a_f(n))^2}{n^s} \tag{1.3}$$

where $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p|N$ removed.

Since f is a newform, we have

$$a_f(m) a_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d^{k-1} a_f\left(\frac{mn}{d^2}\right)$$

(see [12] p. 163 for a proof). Using this identity for $m = n$ we have

$$\frac{1}{\zeta_N(s+1-k)} \sum_{n=1}^{\infty} \frac{(a_f(n))^2}{n^s} = \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}.$$

Substituting this in (1.3) yields

$$L_{sym^2(f)}(s) = \zeta_N(2s+2-2k) \sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}. \tag{1.4}$$

Since $\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}$ is absolutely convergent for $Re(s) > k$, $L_{sym^2(f)}(s)$ is also absolutely convergent for $Re(s) > k$. By [8] we know that $L_{sym^2(f)}(s)$ extends to an entire function and satisfies a functional equation of the form

$$R(s) = A^{s-1}\Gamma(\frac{s+k-2}{2})\Gamma(\frac{s+k-1}{2})\Gamma(\frac{s}{2})L_{sym^2(f)}(s) = \omega R(3-s)$$

where $|\omega| = 1$, and A is a constant with $\log A = O(\log N)$ (see [13] p. 337).

1.5 Problems

The L-function of a cusp form is one of the many L-functions which one studies in number theory. Specifically, the investigation of the non-vanishing of L-functions has been one of the main themes in modern number theory. For example, the distribution of prime numbers in arithmetic progressions is intimately connected with non-vanishing properties of various L-functions.

In this thesis we study the non-vanishing of the L-function associated to a cusp form of weight k and level N. Specifically we consider the following problems.

Problem 1 *Find a lower bound for*

$$\sharp\{f \in \mathcal{F}_N; L_f(\frac{k}{2}, \chi) \neq 0\}$$

(here χ is a primitive Dirichlet character).

Problem 2 *Find a lower bound for*

$$\sharp\{f \in \mathcal{F}_N; L_f(\frac{k}{2}, \chi_1)L_f(\frac{k}{2}, \chi_2) \neq 0\}$$

(here χ_1 and χ_2 are distinct primitive Dirichlet characters).

Problem 3 *Find a lower bound for*

$$\sharp\{f \in \mathcal{F}_N; L'_f(\frac{k}{2}, \chi) \neq 0\}$$

(here L'_f is the derivative of L_f).

1.6 Statement of results

In Problem 1, we expect that for a positive proportion of the newforms $f \in \mathcal{F}_N$, $L_f(\frac{k}{2}, \chi) \neq 0$, however, it seems that we are still far from being able to prove this fact. The only known result concerning Problem 1 is one by W. Duke [5] for the case $k = 2$.

By comparing mean and mean square estimate for the twisted L-function $L_f(s, \chi)$ attached to a newform f of weight 2, Duke proved that there is a positive absolute constant C and a constant C_q depending only on q such that for any prime $N > C_q$ there are at least $CN(\log N)^{-2}$ newforms $f \in \mathcal{F}_N$ for which $L_f(1, \chi) \neq 0$.

Although this result does not give us a positive proportion of $\mathcal{F}_N$ for which $L_f(1, \chi) \neq 0$, it is an important result and has certain applications. For example, if A is the factor of the Jacobian of the modular curve $X_0(N)$ determined by $f \in \mathcal{F}_N$, then $L_f(1)$ is conjectured not to vanish if and only if the rank of the Mordell-Weil group of A over the set of rational numbers is zero . Thus, Duke's result gives a lower bound for the frequency of this occurrence for a prime level N.

The main difficulty in the generalization of the above result to the cusp forms of weight k is the contribution coming from oldforms of weight k. In chapter 2, by using a special construction of a basis for the space of cusp forms of weight k and level N, introduced by A. Pizer [17], we show that the contribution of oldforms is negligible.

and therefore we obtain a generalization of Duke's result to newforms of weight k and level N. More precisely, we prove the following result.

Theorem *Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants C_k (depending only on k) and constant $C_{q,k}$ (depending only on q and k) such that for prime $N > C_{q,k}$ there exist at least $C_k N(\log N)^{-2}$ newforms f of weight k and level N for which $L_f(\frac{k}{2}, \chi) \neq 0$.*

By using similar techniques and an estimation of sums of Fourier coefficients due to W. Duke, J.B. Friedlander and H. Iwaniec [6], we have been able to prove the following theorem about the non-vanishing of the product of two distinct twist of a modular L-function.

Theorem *Let $k > 2$ and χ_1 (mod q_1) and χ_2 (mod q_2) be fixed distinct real primitive Dirichlet characters such that $\chi_1\chi_2(-N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{q_1,q_2,k}$ (depending only on q_1, q_2 and k) such that for prime $N > C_{q_1,q_2,k}$ there exist at least $C_k N(\log N)^{-6}$ newforms f of weight k and level N for which $L_f(\frac{k}{2}, \chi_1)L_f(\frac{k}{2}, \chi_2) \neq 0$.*

If we set $r_f = ord_{s=\frac{k}{2}} L_f(s)$ and consider

$$\sum_{f\in\mathcal{F}_N} r_f \tag{1.5}$$

we may find a solution for problem 1 (in the case that χ is trivial) if we can find a good upper bound for (1.5). For example if we could prove that $\sum_{f\in\mathcal{F}_N} r_f \leq c\ dim\ S_k{}^{new}(N) + o(N)$ for some $c < 1$.

In [16] R. Murty by applying the machinery of the Weil explicit formula to newforms of weight 2 and prime level N, showed that under the assumptions of the Generalized Riemann Hypothesis for the L-functions of the newforms f and the Lindelöf hypothesis for the symmetric square L-function of f

$$\sum_{f\in\mathcal{F}_N} r_f \leq (\frac{11}{6} + \epsilon) dim\ S_2(N) + o(N)$$

as $N \to \infty$ for any $\epsilon > 0$. Since $\frac{11}{6} > 1$ this result does not help us with Problem 1.

The main technical tool in the proof of Duke and R. Murty's results is the "semi-orthogonality" of the Fourier coefficient of an orthonormal basis of $S_k(N)$ which is a consequence of the Petersson formulae about Poincaré series.

In chapter 3, we develop a new technical tool. We define a similar Poincaré series for $S_k^-(N)$, the complex vector space of cusp forms with root number -1 as defined in section 1.2, and then by analogy with the classical case, we get a "semi-orthogonality" relation for $S_k^-(N)$. As a consequence of this, by applying the methods developed in [16], and under the assumption of the Riemann hypothesis, we obtain an upper bound for

$$\sum_{f \in \mathcal{F}_N} \omega_f r_f$$

where $\omega_f = \frac{1}{4\pi < f,f >}$. Also under certain assumption on $L_{sym^2(f)}(s)$ on the line $\frac{3}{2} + it$, we obtain an asymptotic formula for

$$\sum_{f \in \mathcal{F}_N} < f, f > .$$

Finally, as a direct consequence of these two facts, we have

Corollary: *Assume the Riemann hypothesis for $L_f(s)$ and suppose that $L_{sym^2(f)}(\frac{3}{2} + it) << N^{\frac{1}{2}-\eta}$, for some $\eta > 0$, then for prime N large enough, a positive proportion of elements of $\mathcal{F}_N^-$ (and therefore $\mathcal{F}_N$) have order 1 at $s = 1$.*

Chapter 2

Non-Vanishing of Weight k Modular L-functions

2.1 A semi-orthogonality relation

We start by recalling some basic facts about Poincaré series (see [18] chapter 5 for more explanation).

We can show that $S_k(N)$ equipped with the Petersson inner product is a finite dimensional inner product space spanned by the Poincaré series

$$P_n(z, k, N) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{e(n\gamma z)}{(cz+d)^k}, \ n \geq 1$$

where $e(z) = e^{2\pi i z}$, $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, and $\Gamma_\infty$is the stabilizer of $i\infty$ in $\Gamma_0(N)$. We know that for $k > 2$ the above series is absolutely convergent.

If $f \in S_k(N)$, we write the Fourier expansion of f as

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

at $i\infty$.

Petersson proved (see [11] p. 206)

$$< P_n(.,k,N), f >= \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_f(n). \tag{2.1}$$

Now if $\{f_1, ..., f_r\}$ is an orthonormal basis for $S_k(N)$, and

$$P_n(.,k,N) = \sum_i c_i f_i$$

we have

$$c_i =< P_n(.,k,N), f_i > .$$

Therefore from (2.1)

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} P_n(.,k,N) = \sum_i a_{f_i}(n) f_i.$$

Now if $\hat{P}_n(m,k,N)$ is the m-th coefficient of the Fourier expansion of $P_n(z,k,N)$, by comparing the m-th coefficients on both sides we have

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \hat{P}_n(m,k,N) = \sum_i a_{f_i}(n) a_{f_i}(m). \tag{2.2}$$

But by a formula of Petersson we have the following explicit representation (see [11] p. 206)

$$\hat{P}_n(m,k,N) = (\frac{m}{n})^{\frac{k-1}{2}} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0 (mod\ N)} c^{-1} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) S(m,n;c) \right\} \tag{2.3}$$

where δ_{mn} is the Kronecker delta, $J_{k-1}(x)$ is the Bessel function of order $k-1$ which is defined by the following integral

$$J_{k-1}(t) = \frac{1}{2\pi i} \int_{z\bar{z}=1} \frac{\exp(\frac{t}{2}(z - z^{-1}))}{z^k} dz$$

and $S(m,n;c)$ is the Kloosterman sum

$$S(m,n;c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} e(\frac{ma+n\bar{a}}{c})$$

where $a\bar{a} \equiv 1 (\mathrm{mod}\ c)$.

From (2.2) and (2.3), one can get the following "semi-orthogonality" of the Fourier coefficients of an orthonormal basis of $S_k(N)$

$$\sum_i \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0 (mod N)} c^{-1} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) S(m,n;c) \right\}. \tag{2.4}$$

Now for $0 \neq f \in S_k(N)$ set

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} < f,f >}.$$

Then we have the following estimate.

Proposition 1 *If $\{f_1, \ldots, f_r\}$ is an orthogonal basis for $S_k(N)$, for m and n positive integers we have the inequality*

$$|\sum_i \omega_{f_i} \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}} \frac{a_{f_i}(n)}{\sqrt{n^{k-1}}} - \delta_{mn}| \leq M \mathbf{d}(N) N^{\frac{1}{2}-k} (m,n)^{\frac{1}{2}} \sqrt{(mn)^{k-1}}$$

where M is a constant depending only on k and $\mathbf{d}(N)$ is the number of divisor of N.

Proof: The following expression for the Bessel function of order $k-1$ is known (see [23] p. 60)

$$J_{k-1}(z) = \frac{z^{k-1}}{2^{k-1}\Gamma(k-\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} \cos(z\cos\theta) \sin^{2k-2}\theta \, d\theta.$$

From this we get the following bound for $z \geq 0$;

$$|J_{k-1}(z)| \leq \frac{\sqrt{\pi} z^{k-1}}{2^{k-1}\Gamma(k-\frac{1}{2})}. \tag{2.5}$$

Also we have Weil's bound for the Kloosterman sum (see [7]).

$$|S(m,n;c)| \leq (m,n,c)^{\frac{1}{2}}\mathbf{d}(c)c^{\frac{1}{2}}. \tag{2.6}$$

Now the Proposition follows by applying (2.5) and (2.6) in (2.4). □

Note Although in the case $k = 2$, one does not have the absolute convergence of the Poincaré series, nevertheless, Proposition 1 is valid in this case as well.

2.2 A basis for $S_k(N)$

We are going to generalize Duke's result to cusp forms of weight k and prime level N (see [5], Theorem 1, and also 1.6 of the Introduction).

The first difficulty that we encounter is that $\mathcal{F}_N$ is not a basis for $S_k(N)$ when k is large (more precisely if $k > 12$ and $k \neq 14$). So we must find a basis for $S_k(N)$ with good analytic properties. A theorem of Pizer guarantees the existence of such basis for $S_k(N)$.

In 1983 A. Pizer introduced the operators C_q on $S_k(N)$ for $q|N$, such that the action of C_q on the new part of $S_k(N)$ is the same as the action of the classical U_q operators. More precisely he defined C_q as

$$C_q = U_q + W_qU_qW_q + q^{\frac{k}{2}-1}W_q \quad if \quad q\|N$$

$$C_q = U_q + W_qU_qW_q \quad if \quad q^2|N.$$

Then he showed that T_p ($p \nmid N$), C_q ($q|N$) form a commuting family of Hermitian operators. Using this, he proved ([17] Theorem 3.10) the following result.

Theorem *There exists a basis $f_i(z)$ ($1 \leq i \leq dim\ S_k(N)$) of $S_k(N)$ such that each $f_i(z)$ is an eigenform for all the T_p and C_q operators with $p \nmid N$ and $q|N$. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e(z)$ be an element of this basis. Then $a_f(1) \neq 0$ and assuming $f(z)$ is normalized so that $a_f(1) = 1$, we have $f|T_p = a_f(p)f$ for all $p \nmid N$, $f|C_q = a_f(q)f$ for all $q|N$, and $a_f(nm) = a_f(n)a_f(m)$ whenever $(n,m) = 1$. Furthermore $f(z)$ is an* *eigenform for all W_q operators, $q|N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the T_p and C_q operators with $p \nmid N$ and $q|N$, then $g(z) = cf_i(z)$ for some $c \in \mathbb{C}^*$ and some unique i, $1 \leq i \leq dim\ S_k(N)$.*

Now let $\mathcal{P}_N$ be the basis of $S_k(N)$ given by the above theorem. The elements of $\mathcal{P}_N$ form an orthogonal basis for $S_k(N)$ and their L-functions have analytic continuation and satisfy certain functional equations. We can show that the action of C_q on $S_k(N)^{new}$ is the same as the action of U_q (see [17] Remark 2.9). This shows that $\mathcal{F}_N \subset \mathcal{P}_N$.

In the sequel we need an estimation for the Fourier coefficient of an oldform in $\mathcal{P}_N$. Suppose that N is prime and $f \in \mathcal{P}_N - \mathcal{F}_N$, then we can show the existence of $A \in \mathbb{C}$ such that

$$f(z) = h(z) + Ah(Nz)$$

where h is the normalized newform of weight k and level 1 associated to f (see [17], Proposition 3.6). From this we can get the following lemma.

Lemma 1 *With the above notations, $A = \pm N^{\frac{k}{2}}$.*

Proof: Since $f \in \mathcal{P}_N - \mathcal{F}_N$, we have

$$f(z) = h(z) + Ah(Nz).$$

Therefore, f is in the space generated by h and $h(Nz)$. From [17], Proposition 3.4, we know that this space is invariant under C_N. We can show that

$$h|C_N = c_h(N)h + N^k h(Nz)$$

$$h(Nz)|C_N = h + c_h(N)h(Nz)$$

where $c_h(N)$ is the N-th Fourier coefficient of h. We know that $a_f(N)$ is the eigenvalue of C_N operator. The above identities show that the C_N operator on the space generated by h and $h(Nz)$ can be represented by the following matrix

$$\begin{pmatrix} c_h(N) & 1 \\ N^k & c_h(N) \end{pmatrix}.$$

Therefore its characteristic polynomial is

$$x^2 - 2c_h(N)x + (c_h(N)^2 - N^k) = 0.$$

This shows that $a_f(N) = c_h(N) \pm N^{\frac{k}{2}}$, and so $A = a_f(N) - c_h(N) = \pm N^{\frac{k}{2}}$. □

Now by using Lemma 1, we give an estimation for the Fourier coefficient $a_f(n)$.

Lemma 2 *Suppose N is a prime and $f \in \mathcal{P}_N$. Then*

$$|a_f(n)| \leq c_0 n^{\frac{k}{2}}$$

where c_0 is an absolute constant independent of f.

Proof: If $f \in \mathcal{F}_N$ we know that $|a_f(n)| \leq \mathbf{d}(n)n^{\frac{k-1}{2}}$ (Deligne's bound) and therefore the result is clear.

If $f \in \mathcal{P}_N - \mathcal{F}_N$ then from [17] Proposition 3.6 follows that there exists an $A \in \mathbb{C}$ such that

$$f(z) = h(z) + Ah(Nz)$$

where h is the normalized newform of weight k and level 1 associated to f. Since $A = a_f(N) - c_h(N)$, where $c_h(N)$ is the N-th Fourier coefficient of h, Lemma 1 follows that $|A| = N^{\frac{k}{2}}$.

Now if $(n, N) = 1$ then $a_f(n) = c_h(n)$ and therefore the Deligne bound implies the result, and if $(n, N) \neq 1$ then $n = mN$ and we can write

$$a_f(Nm) = c_h(Nm) + Ac_h(m).$$

By using the Deligne bound for the Fourier coefficients of h we get

$$|a_f(Nm)| \leq \mathbf{d}(Nm)(Nm)^{\frac{k-1}{2}} + N^{\frac{k}{2}}\mathbf{d}(m)m^{\frac{k-1}{2}}$$

$$= (\frac{\mathbf{d}(Nm)}{(Nm)^{\frac{1}{2}}} + \frac{\mathbf{d}(m)}{m^{\frac{1}{2}}})(Nm)^{\frac{k}{2}}.$$

The result follows from the fact that $\mathbf{d}(n) = O(n^{\frac{1}{2}})$ with an absolute constant. □

2.3 Critical values on average

Now we give a representation of $L_f(\frac{k}{2}, \chi)$ as a sum of two convergent series for $f \in \mathcal{P}_N$.

Lemma 3 *For any $x > 0$, let*

$$\mathcal{A}(x) = \sum_{n \geq 1} \chi(n) a_f(n) n^{-\frac{k}{2}} \{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (\frac{2\pi n}{x})^j \} e^{-\frac{2\pi n}{x}}.$$

Where χ is a fixed primitive Dirichlet character mod q with $(q, N) = 1$. Then we have

$$L_f(\frac{k}{2}, \chi) = \mathcal{A}(x) + \epsilon_\chi \bar{\mathcal{A}}(Nq^2/x)$$

where ϵ_χ is the root number of $L_f(s, \chi)$ and $\bar{\mathcal{A}}$ is the conjugate of $\mathcal{A}$.

Proof: Define the function $\mathcal{E}(x)$ by

$$\mathcal{E}(x) = \frac{1}{2\pi i} \int_{(\frac{3}{4})} (-\frac{1}{x})^s \Gamma(s + \frac{k}{2}) \frac{ds}{s}$$

then

$$\frac{1}{\Gamma(\frac{k}{2})} \mathcal{E}(-\frac{1}{x}) = (\sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (\frac{1}{x})^j) e^{-\frac{1}{x}}. \qquad (2.7)$$

This is true because

$$\frac{1}{\Gamma(\frac{k}{2})} \mathcal{E}(-\frac{1}{x}) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^\infty e^{-t} t^{\frac{k}{2}} (\frac{1}{2\pi i} \int_{(\frac{3}{4})} (xt)^s \frac{ds}{s}) \frac{dt}{t}.$$

But we know that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1 & if\ x > 1 \\ \frac{1}{2} & if x = 1 \\ 0 & if x < 1 \end{cases}.$$

Therefore

$$\frac{1}{\Gamma(\frac{k}{2})}\mathcal{E}(-\frac{1}{x}) = \frac{1}{\Gamma(\frac{k}{2})}\int_{\frac{1}{x}}^{\infty} e^{-t}t^{\frac{k}{2}-1}dt = (\sum_{j=o}^{\frac{k}{2}-1}\frac{1}{j!}(\frac{1}{x})^{j})e^{-\frac{1}{x}}.$$

Now by definition of $\mathcal{E}(x)$, it is clear that

$$\mathcal{A}(x) = \frac{1}{2\pi i}\int_{(\frac{3}{4})} L_f(s+\frac{k}{2},\chi)(\frac{x}{2\pi})^{s}\frac{\Gamma(s+\frac{k}{2})}{\Gamma(\frac{k}{2})}s^{-1}ds$$

moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, and using the functional equation (1.2) for $L_f(s)$ yields

$$\mathcal{A}(x) = L_f(\frac{k}{2},\chi) + \epsilon_\chi\int_{(-\frac{3}{4})}(\frac{2\pi x}{q^2 N})^{s}\frac{\Gamma(-s+\frac{k}{2})}{\Gamma(\frac{k}{2})}L_f(-s+\frac{k}{2},\bar{\chi})s^{-1}ds$$

Now changing variables $s \mapsto -s$ gives the result. □

Lemma 4 *We have* $\sum_{n=1}^{\infty}\frac{n^k}{(N^{2\pi})^n} = O(N^{-2\pi})$ *and* $\sum_{n=1}^{\infty} n^k e^{-\frac{n}{b}} = O(b^{k+1})$, *where* $b > 1$.

Proof: We know that the geometric series $\sum_{n=1}^{\infty} x^n$ is uniformly convergent to $\frac{1}{1-x}$ on any closed sub-interval of $(-1,1)$. Now by using induction and term by term differentiation of the geometric series, we can show that

$$\sum_{n=1}^{\infty} n^k x^n = \frac{(-1)^{k-1}xP(x)}{(x-1)^{k+1}} \qquad (*)$$

where $P(x) = x^{k-1} + a_{k-2}x^{k-2} + \ldots + 1$ is a polynomial of degree $k-1$.

Now the result easily follows by substituting $x = N^{-2\pi}$ and $x = e^{-\frac{1}{b}}$ in $(*)$. □

From Proposition 1 and Lemma 3, we can get the following asymptotic formula.

Proposition 2 *Let χ be a fixed primitive character modulo q. Then we have*

$$\sum_{f\in\mathcal{P}_N}\omega_f L_f(\frac{k}{2},\chi) = 1 + O(N^{-\frac{1}{2}}(\log N)^{k-1})$$

for N prime. The implied constant depends on q and k.

Proof: Choosing $x = q^2 N \log N$ in Lemma 3 gives

$$\bar{A}(\frac{Nq^2}{x}) = \sum_{n\geq 1} \overline{\chi(n)} a_f(n) n^{-\frac{k}{2}} \left\{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!}(2\pi n \log N)^j \right\} (N^{-2\pi})^n.$$

Using Lemma 2 we get

$$|\bar{A}(\frac{Nq^2}{x})| \leq \sum_{n\geq 1} |a_f(n)| n^{-\frac{k}{2}} \frac{k}{2}(2\pi n \log N)^{\frac{k}{2}-1}(N^{-2\pi})^n$$

$$\leq c_0 \frac{k}{2}(2\pi)^{\frac{k}{2}-1}(\log N)^{\frac{k}{2}-1} \sum_{n\geq 1} \frac{n^{\frac{k}{2}-1}}{(N^{2\pi})^n}.$$

$$\leq c_0 \frac{k}{2}(2\pi)^{\frac{k}{2}-1}(\log N)^{\frac{k}{2}-1} O(N^{-2\pi}).$$

Therefore from Lemma 3 we get

$$L_f(\frac{k}{2}, \chi) = \sum_{n\geq 1} \chi(n) a_f(n) \left\{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!}(\frac{2\pi n}{q^2 N \log N})^j \right\} e^{-\frac{2\pi n}{q^2 N \log N}} n^{-\frac{k}{2}} + O(N^{-6}(\log N)^{\frac{k}{2}-1}).$$

From this, we get

$$\sum_{f\in \mathcal{P}_N} \omega_f L_f(\frac{k}{2}, \chi) - 1 = \sum_{n\geq 1} \chi(n) (\sum_{f\in\mathcal{P}_N} \omega_f \frac{a_f(n)}{\sqrt{n^{k-1}}} - \delta_{1n}) \left\{ \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!}(\frac{2\pi n}{q^2 N \log N})^j \right\} \frac{1}{\sqrt{n}} e^{-\frac{2\pi n}{q^2 N \log N}}$$

$$+ (\sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!}(\frac{2\pi}{q^2 N \log N})^j) e^{-\frac{2\pi}{q^2 N \log N}} - 1 + (\sum_{f\in\mathcal{P}_N} \omega_f) O(N^{-6}(\log N)^{\frac{k}{2}-1}).$$

Note that

$$(\sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!}(\frac{2\pi}{q^2 N \log N})^j) e^{-\frac{2\pi}{q^2 N \log N}} - 1 = \left(\sum_{j=\frac{k}{2}}^{\infty} \frac{1}{j!}(\frac{2\pi}{q^2 N \log N})^j \right) e^{-\frac{2\pi}{q^2 N \log N}}$$

From Proposition 1, with $m = n = 1$ we get

$$\sum_{f\in\mathcal{P}_N} \omega_f = 1 + O(N^{\frac{1}{2}-k}).$$

Also, by applying $m = 1$ in Proposition 1 and using the above identities and Lemma 4, we have

$$|\sum_{f\in\mathcal{P}_N} \omega_f L_f(\frac{k}{2},\chi)-1| \leq M_1 N^{\frac{1}{2}-k} \sum_{n\geq 1} n^{k-2} e^{-\frac{2\pi n}{q^2 N \log N}} + \left(\sum_{j=\frac{k}{2}}^{\infty} \frac{1}{j!} (\frac{2\pi}{q^2 N \log N})^j \right) e^{-\frac{2\pi}{q^2 N \log N}}$$

$$+M_2 N^{-6} (\log N)^{\frac{k}{2}-1} \leq M_3 N^{-\frac{1}{2}} (\log N)^{k-1} + M_4 (N \log N)^{-\frac{k}{2}} + M_2 N^{-6} (\log N)^{\frac{k}{2}-1}$$

where M_1, M_2, M_3, M_4 are constants. This completes the proof. □

Now let $P_f(s) = L_f(s,\chi_1) L_f(s,\chi_2)$ where χ_1 and χ_2 are fixed primitive Dirichlet characters mod q_1 and q_2. Then we have $P_f(s) = \sum_{l\geq 1} b_f(l) l^{-s}$, where

$$b_f(l) = \sum_{mn=l} \chi_1(m)\chi_2(n) a_f(m) a_f(n).$$

Define for $x > 0$

$$g(x) = \frac{1}{2\pi i} \int_{(\frac{3}{4})} (2\pi)^{-2s} \frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} x^{-s} \frac{ds}{s} \tag{2.8}$$

and set $\mathcal{B}(x) = \sum_{l\geq 1} b_f(l) l^{-\frac{k}{2}} g(\frac{l}{x})$. Then we have

Lemma 5 *Let $f \in \mathcal{P}_N$ for $N \geq 1$ and suppose that χ_1 and χ_2 are primitive Dirichlet characters mod q_1, q_2 with $(q_1 q_2, N) = 1$. For any $x > 0$, we have*

$$P_f(\frac{k}{2}) = \mathcal{B}(x) + \hat{\epsilon}_{\chi_1\chi_2} \bar{\mathcal{B}}(\frac{(Nq_1q_2)^2}{x})$$

where $\hat{\epsilon}_{\chi_1\chi_2} = \chi_1\chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2 (q_1q_2)^{-1}$ is the root number of $P_f(s)$ and $\bar{\mathcal{B}}$ is the conjugate of $\mathcal{B}$.

Proof: By the definition of $g(x)$, it is clear that

$$\mathcal{B}(x) = \frac{1}{2\pi i} \int_{(\frac{3}{4})} x^s (2\pi)^{-2s} \frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2} P_f(s+\frac{k}{2}) s^{-1} ds.$$

Now by moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, and using the functional equation for $P_f(s)$ which is a direct consequence of (1.2), we get

$$\mathcal{B}(x) = P_f(\frac{k}{2}) + \frac{\hat{\epsilon}_{\chi_1\chi_2}}{2\pi i}\int_{(-\frac{3}{4})}\left(\frac{(Nq_1q_2)^2}{x}\right)^{-s}(2\pi)^{2s}\frac{\Gamma(-s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}\bar{P}_f(-s+\frac{k}{2})s^{-1}ds.$$

Now changing variables $s \mapsto -s$ yields the result. □

We come now to the following important proposition.

Proposition 3 *Let χ be a primitive Dirichlet character. Then*

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(\frac{k}{2},\chi)|^2 = \sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = \prod_{p|q}(1-p^{-1})\log N + c + O(N^{-\frac{1}{2}}\log N)$$

for N prime with $(q,N)=1$, where c and the implied constant depend on q and k.

Proof: In Lemma 5, set $\chi_1=\chi$, $\chi_2=\bar{\chi}$, we have $\mathcal{B}=\bar{\mathcal{B}}$ and $\hat{\epsilon}_{\chi\bar{\chi}}=1$. By Lemma 5 with $x=Nq^2$, we have

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2\sum_f\omega_f\sum_{l\geq 1}b_f(l)l^{-\frac{k}{2}}g(\frac{l}{Nq^2})$$

$$= 2\sum_{m,n\geq 1}\chi(m)\bar{\chi}(n)g(\frac{mn}{Nq^2})\frac{1}{(mn)^{\frac{1}{2}}}\sum_f\omega_f\frac{a_f(m)}{\sqrt{m^{k-1}}}\frac{a_f(n)}{\sqrt{n^{k-1}}}. \tag{2.9}$$

By Proposition 1, it is clear that

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2\sum_{n\geq 1}|\chi(n)|^2 g(\frac{n^2}{Nq^2})n^{-1} + R \tag{2.10}$$

where

$$R \ll N^{\frac{1}{2}-k}\sum_{m,n\geq 1}g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1}. \tag{2.11}$$

Now the first term on the right hand side of (2.10) is evaluated using the definition of g as

$$\frac{1}{\pi i}\int_{(\frac{3}{4})}L(2s+1,\chi_0)(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}(Nq^2)^s\frac{ds}{s}$$

where χ_0 is the principal character mod q and $L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-\frac{1}{p^s})$. Since the integrand has a double pole at $s=0$, by moving the line of integration from $\frac{3}{4}$ to $-\frac{1}{2}$, we see that the above integral is equal to

$$\prod_{p|q}(1-p^{-1})\log N + c + O(N^{-\frac{1}{2}}). \tag{2.12}$$

Now in (2.11) we calculate $\sum_{m,n\geq 1} g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1}$. It is

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2})}(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}\left(\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)}\right)(Nq^2)^s\frac{ds}{s}$$

because the integrand does not have any poles in the strip $\frac{3}{4} < Re(s) < \frac{k+1}{2}$ and $\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)}$ is absolutely convergent. Next we use the following identity

$$\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s-\frac{k}{2}+1)} = \frac{\zeta(2s-k+\frac{3}{2})\zeta(s-\frac{k}{2}+1)^2}{\zeta(2s-k+2)} \tag{2.13}$$

(See [5] Lemma 4). By moving the line of integration from $\frac{k+1}{2}$ to $\frac{k}{2}-\epsilon$ ($\epsilon > 0$) we get

$$\sum_{m,n\geq 1} g(\frac{mn}{Nq^2})(m,n)^{\frac{1}{2}}(mn)^{\frac{k}{2}-1} \sim c_1 N^{\frac{k}{2}}\log N \tag{2.14}$$

and by (2.11), $R \ll N^{\frac{1}{2}-\frac{k}{2}}\log N$. This and (2.12) prove the Proposition. □

2.4 A lower bound for the Petersson inner product

To obtain a lower bound for $< f, f >_N$ when $f \in \mathcal{P}_N$, we need to introduce the *trace function* which maps $S_k(N)$ to $S_k(1)$. More precisely suppose N is a prime, we can show that the elements

$$\left\{\gamma_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}, 0 \leq j < N\right\}$$

are right coset representatives for $\Gamma = \Gamma_0(N)\backslash\Gamma_0(1)$. Then for $f \in S_k(N)$ we define

$$Tr(f) = \sum_{j=-1}^{N} f|\gamma_j.$$

It is clear that $Tr(f) \in S_k(1)$. Let W_N be the usual Atkin-Lehner involution. Since $W_N{}^{-1}\Gamma_0(N)W_N = \Gamma_0(N)$, it is clear that $f|W_N \in S_k(N)$. We have the following lemma regarding the calculation of $Tr(f|W_N)$.

Lemma 6 *If h is a normalized newform of weight k and level 1, then*

$$Tr(h|W_N) = N^{1-\frac{k}{2}}c_h(N)h$$

where $c_h(N)$ is the N-th Fourier coefficient of h.

Proof: From the definition of trace we have

$$Tr(h|W_N) = \sum_{j=-1}^{N}(h|W_N)|\gamma_j = h|W_N + \sum_{j=0}^{N-1}\left(h\left|\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\right.\right)\begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix}$$

But $W_N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, since $h \in S_k(1)$

$$Tr(h|W_N) = N^{\frac{k}{2}}h(Nz) + N^{-\frac{k}{2}}\sum_{j=0}^{N-1} h\left(\frac{z+j}{N}\right) = N^{1-\frac{k}{2}}T_N(h)$$

where T_N is the N-th Hecke operator, since h is a normalized newform the result is clear. □

Now we use the above lemma to evaluate $< h, h(Nz) >_N$.

Lemma 7 *If h is a normalized newform of level 1, then*

$$< h, h(Nz) >_N \;= N^{1-k}c_h(N)< h, h >_1.$$

Proof: Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, and $h \in S_k(1)$, and the operator W_N is Hermitian, we have

$$< h, h(Nz) >_N \quad = N^{-\frac{k}{2}} < h|W_N, h(z) >_N.$$

Now let F be a fundamental domain of $\Gamma_0(1)\backslash\mathcal{H}$. Then since $\Gamma_0(1) = \bigcup_{i=-1}^{N-1} \Gamma_0(N)\gamma_i$,

$$F' = \bigcup_{i=-1}^{N-1} \gamma_i F$$

is a fundamental domain of $\Gamma_0(N)\backslash\mathcal{H}$. This is because if $z \in \mathcal{H}$ there exist $z' \in F$ and $\gamma \in \Gamma_0(1)$ such that $z = \gamma z'$. We can write $\gamma = \gamma'\gamma_i$ where $\gamma' \in \Gamma_0(N)$, this shows that $z = \gamma'(\gamma_i z')$. Therefore any $z \in \mathcal{H}$ is equivalent to an element in $\gamma_i F$ for some i.

Now suppose that for $z_1, z_2 \in intF'$ there exist $\gamma' \in \Gamma_0(N)$ such that $\gamma' z_1 = z_2$, where $z_1 \in \gamma_i F$ and $z_2 \in \gamma_j F$. This shows that $\gamma_j{}^{-1}\gamma'\gamma_i z_1' = z_2'$ where $z_1', z_2' \in F$ and $\gamma_j{}^{-1}\gamma'\gamma_i \in \Gamma_0(1)$ which is impossible. this shows that $F' = \bigcup_i \gamma_i F$ is a fundamental domain for $\Gamma_0(N)\backslash\mathcal{H}$.

So we have

$$< h, h(Nz) >_N = N^{-\frac{k}{2}} \sum_{j=-1}^{N-1} \int_{\gamma_j F} (h|W_N)(z)\overline{h(z)} y^k \frac{dxdy}{y^2}.$$

Using the change of variable $z = \gamma_j w$, where $w = u + iv$ we find that this is

$$= N^{-\frac{k}{2}} \sum_{j=-1}^{N-1} \int_F ((h|W_N)|\gamma_j)(w)\overline{(h|\gamma_j)(w)} v^k \frac{dudv}{v^2}$$

But $h \in S_k(1)$ and so $h|\gamma_j = h$. Hence, by using Lemma 6 the above expression is

$$N^{-\frac{k}{2}} < Tr(h|W_N), h >_1 = N^{1-k} c_h(N) < h, h >_1. \ \Box$$

Now we use the above Lemma to get a lower bound for $< f, f >_N$.

Lemma 8 *If $f \in \mathcal{P}_N - \mathcal{F}_N$ and N is a prime then*

$$< f, f >_N \;\; \geq \; (N - 4N^{\frac{1}{2}} + 1) < h, h >_1.$$

Proof: By applying Lemma 7 we have

$$< f, f >_N \;\; = \;\; < h + Ah(Nz), h + Ah(Nz) >_N \;\; \geq \;\; \left(N + 1 + 2AN^{1-k}c_h(N)\right) < h, h >_1. \tag{2.15}$$

By Lemma 1, $|A| = N^{\frac{k}{2}}$ and therefore

$$|2AN^{1-k}c_h(N)| \leq 4N^{\frac{1}{2}} \tag{2.16}$$

Applying (2.16) to (2.15) gives the desired result. □

Now we are in a situation that we can establish an upper bound for

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1} < f, f >_N}.$$

Proposition 4 *If $f \in \mathcal{P}_N - \mathcal{F}_N$, for N prime large enough*

$$\omega_f \ll_k \frac{1}{N}$$

with implied constant depending on k.

Proof: This is clear from Lemma 8. □

Proposition 5 *If $f \in \mathcal{F}_N$ for N prime large enough*

$$\omega_f \ll_k \frac{\log N}{N}$$

with implied constant depending on k.

Proof: Set

$$q(s) = \zeta_N(2s + 2 - 2k) \sum_{n=1}^{\infty} \frac{(a_f(n))^2}{n^s}.$$

The Rankin-Selberg method shows that $q(s)$ has a pole at $s = k$ of residue

$$\frac{\pi(4\pi)^k \phi(N)}{2\Gamma(k)N^2} < f, f >_N$$

(see [21] p. 90). So from the definition of $L_{sym^2(f)}(s)$ (see the Introduction) it is clear that for N prime

$$L_{sym^2(f)}(k) = \frac{\pi(4\pi)^k}{2\Gamma(k)N} < f, f >_N. \tag{2.17}$$

But the extension of the Main theorem of [9] to holomorphic cusp forms, together with the fact that for prime N no $f \in \mathcal{F}_N$ is a lift from $GL(1)$, implies that

$$L_{sym^2(f)}(k) \gg_k \frac{1}{\log N} \tag{2.18}$$

(see [9] p. 178, remark and paragraph following the Main Theorem). Now the result follows from (2.17) and (2.18). □

We are in the situation that we can prove the main theorem of this chapter.

Theorem 1 *Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{q,k}$ (depending only on q and k) such that for prime $N > C_{q,k}$ there exist at least $C_k N(\log N)^{-2}$ newforms f of weight k and level N for which $L_f(\frac{k}{2}, \chi) \neq 0$.*

Proof: By the Cauchy-Schwarz inequality and Proposition 4, we have

$$|\sum_{f\in\mathcal{P}_N} \omega_f L_f(\frac{k}{2}, \chi)|^2 \leq \left(\sum_{f\in\mathcal{F}_N; L_f(\frac{k}{2},\chi)\neq 0} \omega_f + \sum_{f\in\mathcal{P}_N - \mathcal{F}_N; L_f(\frac{k}{2},\chi)\neq 0} \omega_f \right) \sum_{f\in\mathcal{P}_N} \omega_f |L_f(\frac{k}{2}, \chi)|^2$$

$$\ll \left(\sharp\{f \in \mathcal{F}_N;\ L_f(\frac{k}{2}, \chi) \neq 0\} \frac{\log N}{N} + 2 dim S_k(1) \frac{1}{N} \right) \sum_{f\in\mathcal{P}_N} \omega_f |L_f(\frac{k}{2}, \chi)|^2$$

Now theorem follows from Propositions 2, 3 and 5. □

2.5 Non-vanishing of product of twisted modular L-functions

We may try to use the above trick to find a lower bound for the number of newforms f for which $P_f(s) = L_f(s,\chi_1)L_f(s,\chi_2)$ is non-zero at the center of the critical strip. Here we assume that χ_1 and χ_2 are real and distinct such that $\chi_1\chi_2(-N) = 1$. To do this we need to derive asymptotic formulae for $\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2})$ and $\sum_{f\in\mathcal{P}_N}\omega_f|P_f(\frac{k}{2})|^2$.

Proposition 6 *Let χ_1 (mod q_1) and χ_2 (mod q_2) be distinct real primitive Dirichlet characters such that $\chi_1\chi_2(-N) = 1$, then for N prime we have*

$$\sum_{f\in\mathcal{P}_N}\omega_f P_f(\frac{k}{2}) = 2L(1,\chi_1\chi_2) + O(N^{-\frac{1}{2}}\log N)$$

where the implied constant depend on q_1, q_2 and k.

Proof: In Lemma 5 we have $\hat{\epsilon}_{\chi_1\chi_2}$. This is because $(\tau(\chi_1))^2 = \chi_1(-1)q_1$ and $(\tau(\chi_2))^2 = \chi_2(-1)q_1$ (see [20] p.91), and therefore $\hat{\epsilon}_{\chi_1\chi_2} = \chi_1\chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2(q_1q_2)^{-1} = \chi_1\chi_2(-N) = 1$. So we may repeat the proof of Proposition 3 line by line. The result follows with the observation that

$$\frac{1}{\pi i}\int_{(\frac{3}{4})} L(2s+1,\chi_1\chi_2)(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}(Nq_1q_2)^s\frac{ds}{s}$$

is equal to

$$2L(1,\chi_1\chi_2) + O(N^{-\frac{1}{2}}). \ \square$$

We recall from (2.8) the definition of $g(x)$ as

$$g(x) = \frac{1}{2\pi i}\int_{(\frac{3}{4})}(2\pi)^{-2s}\frac{\Gamma(s+\frac{k}{2})^2}{\Gamma(\frac{k}{2})^2}x^{-s}\frac{ds}{s}$$

Let for $x > 0$ and a non-negative integer v

$$K_v(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(u+\frac{1}{u})} u^{-(v+1)} du.$$

be the K_v-Bessel function.

In the next lemma we give a representation of $g(x)$ as a sum of the K-Bessel functions.

Lemma 9 $g(x) = \frac{2}{\Gamma(\frac{k}{2})} \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (2\pi\sqrt{x})^{\frac{k}{2}+j} K_{\frac{k}{2}-j}(4\pi\sqrt{x})$

Proof: From definition of $g(x)$ and Γ function we have

$$I = \Gamma(\frac{k}{2})^2 g(x) = \frac{1}{2\pi i}\int_{(\frac{3}{4})} (\int_0^\infty \int_0^\infty t_1^{s+\frac{k}{2}-1} t_2^{s+\frac{k}{2}-1} e^{-(t_1+t_2)} dt_1 dt_2)(4\pi^2 x)^{-s} \frac{ds}{s}.$$

By interchanging the order of integration we get

$$I = \int_0^\infty t_1^{\frac{k}{2}-1} e^{-t_1} (\int_{\frac{4\pi^2 x}{t_1}}^\infty e^{-t_2} t_2^{\frac{k}{2}-1} dt_2) dt_1.$$

Now by integration by parts we have

$$I = \Gamma(\frac{k}{2}) \sum_{j=0}^{\frac{k}{2}-1} \frac{1}{j!} (4\pi^2 x)^j \int_0^\infty t^{\frac{k}{2}-1-j} e^{-(t+\frac{4\pi^2 x}{t})} dt. \tag{2.19}$$

But we know that

$$\int_0^\infty t^{\frac{k}{2}-1-j} e^{-(t+\frac{4\pi^2 x}{t})} dt = 2(4\pi^2 x)^{\frac{k}{4}-\frac{j}{2}} K_{\frac{k}{2}-j}(4\pi\sqrt{x}) \tag{2.20}$$

(see [22] p. 235, Formula 9.42).

Substituting (2.20) in (2.19) yields the result. □

Lemma 10 $g(x) \ll \begin{cases} 1 & for \quad x \le 1 \\ x^{\frac{k}{2}-\frac{3}{4}} e^{-4\pi\sqrt{x}} & for \quad x > 1 \end{cases}$.

Proof: By moving the line of integration from $\frac{3}{4}$ to $-\frac{3}{4}$, we have

$$g(x) = 1 + O(x^{\frac{3}{4}})$$

which proves the Lemma if $x \leq 1$.

If $x > 1$, we know

$$K_v(x) = (\frac{\pi}{2x})^{\frac{1}{2}} e^{-x}[1 + O(\frac{1}{x})]$$

(see [24] p. 202). Now applying this identity to Lemma 9, yields the result. □

Lemma 11 *Let $f \in \mathcal{P}_N$ then*

$$a_f(m)a_f(n) = \sum_{d|(m,n)} d^{k-1} a_f(\frac{mn}{d^2})$$

if $(m, N) = 1$.

Proof: We consider the collection of operators $\{T_n, (n, N) = 1, n \in \mathbb{N}\}$ such that $\{T_p, (p, N) = 1, p\ prime\}$ is the collection of the classical Hecke operators as defined in section (1.3). Also we assume that T_n, $(n, N) = 1$ satisfies the following identities

(i) $T_m T_n = T_{mn}$ if $(m, n) = 1$,

(ii) $T_p T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$ if $(p, N) = 1$.

From here it is clear that if $(m, N) = 1$, T_m is the classical Hecke operator. Now if $f \in \mathcal{P}_N$, f is an eigenform for T_m and $T_m(f) = a_f(m)f$. But we know that if $f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz)$ then

$$a_f(m)f(z) = T_m(f)(z) = \sum_{n=1}^{\infty} (\sum_{\substack{d|(m,n) \\ (d,N)=1}} d^{k-1} a_f(\frac{mn}{d^2}))e(nz) \tag{2.21}$$

(see [12] p. 163). Equating the n-th Fourier coefficient of the two sides of (2.21) and noting that $(m, N) = 1$ yields the result. □

Lemma 12 *Under the assumption of Proposition 6, for $f \in \mathcal{P}_N$ and $X = Nq_1q_2(\log N)^2$, we have*

$$P_f(\frac{k}{2}) = \sum_{l \leq X} c_l a_f(l) + O(N^{-11})$$

where $c_l \ll \frac{d(j)}{l^{\frac{k}{2}}} \log N$ and the implied constants depends on q_1q_2 and k.

Proof: In Lemma 5 set $x = Nq_1q_2$, then we have

$$P_f(\frac{k}{2}) = 2\sum_{l=1}^{\infty} b_f(l)l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2}).$$

Now by using Lemma 10 and the fact that $b_f(l)\leq c_0{}^2\mathbf{d}(l)l^{\frac{k}{2}}$, we can estimate

$$2\sum_{l>X} b_f(l)l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2})$$

using the integral

$$\int_{Nq_1q_2(\log N)^2}^{\infty} \frac{1}{(Nq_1q_2)^{\frac{k}{2}-\frac{3}{4}}} t^{\frac{k}{2}-\frac{3}{4}} e^{\frac{-4\pi\sqrt{t}}{\sqrt{Nq_1q_2}}} dt$$

which is $O(N^{-11})$. Therefore

$$P_f(\frac{k}{2}) = 2\sum_{l\leq X} b_f(l)l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2}) + O(N^{-11}). \tag{2.22}$$

In (2.22) the sum can be written as

$$\sum_{l\leq X} 2l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2}) \sum_{mn=l} \chi_1(m)\chi_2(n)a_f(m)a_f(n) = (*) + (\dagger) \tag{2.23}$$

where $(*)$ is the sum over the terms with $(m, N) = 1$, and $(\dagger)$ is the sum over the terms with $N|m$.

Using Lemma 11 in (2.23) yields

$$(*) = \sum_{l\leq X} 2l^{-\frac{k}{2}}g(\frac{l}{Nq_1q_2}) \sum_{mn=l,(m,N)=1} \chi_1(m)\chi_2(n) \sum_{d|(m,n)} d^{k-1}a_f(\frac{l}{d^2}).$$

Now by setting $j = \frac{l}{d^2}$ and rearranging the above sum, we have

$$(*) = \sum_{j\leq X} \left(\sum_{d\leq\sqrt{\frac{X}{j}}} \frac{2}{j^{\frac{k}{2}}d}\, g(\frac{jd^2}{Nq_1q_2}) \sum_{\substack{mn=jd^2\\ d|(m,n)}} \chi_1(m)\chi_2(n) \right) a_f(j) = \sum_{j\leq X} \alpha_j a_f(j) \tag{2.24}$$

where $\alpha_j \ll \frac{\mathbf{d}(j)}{j^{\frac{k}{2}}} \log N$ by using Lemma 10.

Now suppose that $N|m$. Since $m \leq X = Nq_1q_2(\log N)^2$, for N large enough we can assume that $m = m_0N$ where $(m_0, N) = 1$. Using the multiplicative property of $a_f(n)$'s, we have

$$(\dagger) = \sum_{l \leq X} 2l^{-\frac{k}{2}} g(\frac{l}{Nq_1q_2}) \sum_{mn=l, m=m_0N} \chi_1(m)\chi_2(n)a_f(N) \sum_{d|(m_0,n)} d^{k-1}a_f(\frac{l}{Nd^2}).$$

Now set $\frac{l}{Nd^2} = j$. Rearranging $(\dagger)$ yields

$$(\dagger) = \sum_{j \leq \frac{X}{N}} \left(\sum_{d \leq \sqrt{\frac{X}{Nj}}} \frac{2N^{-\frac{k}{2}}a_f(N)}{j^{\frac{k}{2}}d} g(\frac{jd^2}{q_1q_2}) \sum_{\substack{mn=Njd^2, m=m_0N \\ d|(m_0,n)}} \chi_1(m)\chi_2(n) \right) a_f(j) = \sum_{j \leq \frac{X}{N}} \beta_j a_f(j) \tag{2.25}$$

where $\beta_j \ll \frac{\mathbf{d}(j)}{j^{\frac{k}{2}}} \log N$, here again we are using Lemma 10 and the fact that $|a_f(N)| \leq c_0N^{\frac{k}{2}}$.

Now the result follows from (2.22), (2.24) and (2.25). □

We now employ the following mean value result.

Lemma 13 *For N prime and complex numbers c_n we have*

$$\sum_{f \in \mathcal{P}_N} \omega_f |\sum_{l \leq X} c_l a_f(l)|^2 = (1 + O(N^{-1}X \log X)) \sum_{l \leq X} l|c_l|^2$$

with an absolute implied constant.

Proof: See [6] Theorem 1. □

Proposition 7 *Under the assumption of Proposition 6 we have*

$$\sum_{f \in \mathcal{P}_N} \omega_f |P_f(\frac{k}{2})|^2 \ll (\log N)^5$$

for $k > 2$.

Proof: Apply Lemma 13 to Lemma 12. □

Note: In the case $k = 2$, since $\sum_{l \le X} \mathbf{d}^2(l) l^{-1} \ll (\log N)^4$, we have

$$\sum_{f \in \mathcal{P}_N = \mathcal{F}_N} \omega_f |P_f(\frac{k}{2})|^2 \ll (\log N)^9.$$

We can now state and prove the following theorem.

Theorem 2 *Let $k > 2$ and $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ be fixed real distinct primitive Dirichlet characters such that $\chi_1\chi_2(-N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{q_1 q_2, k}$ (depending only on q_1, q_2 and k) such that for prime $N > C_{q_1, q_2, k}$ there exist at least $C_k N (\log N)^{-6}$ newforms f of weight k and level N for which $P_f(\frac{k}{2}) = L_f(\frac{k}{2}, \chi_1) L_f(\frac{k}{2}, \chi_2) \neq 0$.*

Proof: By the Cauchy-Schwarz inequality and Proposition 4, we have

$$|\sum_{f \in \mathcal{P}_N} \omega_f P_f(\frac{k}{2})|^2 \le \left(\sum_{f \in \mathcal{F}_N; P_f(\frac{k}{2}) \neq 0} \omega_f + \sum_{f \in \mathcal{P}_N - \mathcal{F}_N; P_f(\frac{k}{2}) \neq 0} \omega_f \right) \sum_{f \in \mathcal{P}_N} \omega_f |P_f(\frac{k}{2})|^2$$

$$\ll \left(\sharp\{f \in \mathcal{F}_N : \ P_f(\frac{k}{2}) \neq 0\} \frac{\log N}{N} + 2 dim S_k(1) \frac{1}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f |P_f(\frac{k}{2})|^2$$

Now theorem follows from Propositions 6, 7, 4 and 5. □

Note: In the case $k = 2$ we get the lower bound $C_2 N (\log N)^{-10}$ for the number of non-vanishing $P_f(\frac{k}{2})$ (see [5] Theorem 2).

Chapter 3

A "Semi-Orthogonality" Relation for $S_k^-(N)$ and Its Applications

3.1 Poincaré series for $S_k^-(N)$

We know that if k is odd, $S_k(N) = \{0\}$. So as we mentioned before we assume that k is even and consider the following subspace of $S_k(N)$.

$$S_k^-(N) = \left\{ f \in S_k(N) : \ f|W_N = (-1)^{\frac{k}{2}+1} f \right\}$$

where W_N is the Atkin-Lehner involution. We know that for every element f of $S_k^-(N)$ if we set

$$L_f(s) = \sum_{n \geq 1} a_f(n) n^{-s}$$

then $L_f(s)$ has an analytic continuation to the whole plane and satisfies the functional equation $\Lambda(s) = -\Lambda(k-s)$ where

$$\Lambda(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L_f(s)$$

(see section 1.2 or [12] p. 140). In other words $S_k^-(N)$ is the subspace of cusp forms with root number -1.

In Chapter 2 we mentioned that $S_k(N)$ equipped with the Petersson inner product is a finite dimensional inner product space spanned by the Poincaré series

$$P_n(z,k,N) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \frac{e(n\gamma z)}{(cz+d)^k},\ n \geq 1$$

where $e(z) = e^{2\pi i z}$, $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and Γ_∞ is the stabilizer of $i\infty$ in $\Gamma_0(N)$. We know that for $k > 2$ the above series is absolutely convergent. From now on we assume that k is even and $k > 2$. We define

$$P_n^-(z,k,N) = P_n(z,k,N) + (-1)^{\frac{k}{2}+1} P_n(z,k,N)|W_N.$$

Note that since $(W_N)^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}$, then

$$P_n^-(z,k,N)|W_N = P_n(z,k,N)|W_N + (-1)^{\frac{k}{2}+1} P_n(z,k,N)$$

$$= (-1)^{\frac{k}{2}+1}\left(P_n(z,k,N) + (-1)^{\frac{k}{2}+1} P_n(z,k,N)|W_N\right) = (-1)^{\frac{k}{2}+1} P_n^-(z,k,N).$$

So, $P_n^-(z,k,N) \in S_k^-(N)$.

As we mentioned before, the operator W_N is Hermitian with respect to the Petersson inner product. We continue with giving a proof of this fact.

Lemma 14 *For $f, g \in S_k(N)$, $< f|W_N, g > = < f, g|W_N >$.*

Proof: Let F be a fundamental domain for $\Gamma_0(N)$. Since $W_N^{-1}\Gamma_0(N)W_N = \Gamma_0(N)$, we have

$$< f, g|W_N > = \int_F f(z)\overline{(g|W_N)(z)}y^k \frac{dxdy}{y^2}$$

$$= \int_{W_N^{-1}F} (f|W_N)(z)\overline{((g|W_N)|W_N)(z)}y^k \frac{dxdy}{y^2}$$

$$= \int_{W_N^{-1}F} (f|W_N)(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

$$= \int_{W_N^{-1}\Gamma_0(N)W_N\backslash H} (f|W_N)(z)\overline{g(z)}y^k \frac{dxdy}{y^2} = < f|W_N, g > . \square$$

Lemma 15 *If $\{f_1, \ldots, f_s\}$ is an orthonormal basis for $S_k^-(N)$,*

$$< P_n^-(z,k,N), f_i > \; = \frac{2\Gamma(k-1)}{(4\pi n)^{k-1}} a_{f_i}(n).$$

Proof: Let $P_n^-(z,k,N) = \sum_i c_i f_i$. Then

$$c_i = \; < P_n^-(z,k,N), f_i > \; = \; < P_n(z,k,N), f_i > + < (-1)^{\frac{k}{2}+1} P_n(z,k,N)|W_N, f_i > .$$

By Lemma 14,

$$c_i = \; < P_n(z,k,N), f_i > + < (-1)^{\frac{k}{2}+1} P_n(z,k,N), f_i|W_N >= 2 < P_n(z,k,N), f_i > .$$

From (2.1) we know $< P_n(z,k,N), f_i > \; = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} a_{f_i}(n)$. This completes the proof.
□

From Lemma 15, we deduce that if $\{f_1, \ldots, f_s\}$ is an orthonormal basis for $S_k^-(N)$, then

$$\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} P_n^-(z,k,N) = \sum_i a_{f_i}(n) f_i.$$

Let $\hat{P}_n^-(m,k,N)$ denote the m-th Fourier coefficient of $P_n^-(.,k,N)$. By comparing the m-th Fourier coefficients on both sides, we set

$$\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} \hat{P}_n^-(m,k,N) = \sum_i a_{f_i}(n) a_{f_i}(m). \tag{3.1}$$

This shows that to get a "semi-orthogonality" relation for $S_k^-(N)$, we need to compute $\hat{P}_n^-(m,k,N)$.

3.2 A "semi-orthogonality" relation for $S_k^-(N)$

Now we calculate $\hat{P}_n(m,k,N)|W_N$. To do this, we set

$$\Gamma_\infty = \left\{ U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N);\; l \in \mathbb{Z} \right\}$$

and recall the definition of the Poincaré series

$$P_n(z,k,N) = \sum_{\gamma'\in\Gamma_\infty\backslash\Gamma_0(N)} \frac{e(n\gamma' z)}{(c'z+d')^k}$$

where $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

If we set $S = \Gamma_\infty\backslash\Gamma_0(N)$ then we can decompose S with respect to $W_N\Gamma_\infty {W_N}^{-1}$ in the following way. Let $R = \Gamma_\infty\backslash\Gamma_0(N)/W_N\Gamma_\infty {W_N}^{-1}$, and suppose (R) is a set of representatives for R in $\Gamma_0(N)$. For any $\gamma' \in \Gamma_\infty\backslash\Gamma_0(N)$ there exist $\gamma \in (R)$ and $l \in \mathbb{Z}$ such that

$$\gamma' = \gamma W_N U^l {W_N}^{-1} = \begin{pmatrix} a-bNl & b \\ c-dNl & d \end{pmatrix}.$$

Here $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$. So we have

$$P_n(z,k,N) = \sum_{\gamma\in(R)} \sum_{l=-\infty}^{+\infty} \frac{e(n\gamma W_N U^l {W_N}^{-1} z)}{((c-dNl)z+d)^k}.$$

Now we apply the W_N operator to $P_n(z,k,N)$ to get

$$P_n(z,k,N)|W_N = N^{\frac{k}{2}}(-Nz)^{-k} P_n(W_N z,k,N)$$

$$= N^{\frac{k}{2}} \sum_{\gamma\in(R)} \sum_{l=-\infty}^{+\infty} \frac{e(n\frac{bN(z+l)-a}{dN(z+l)-c})}{(dN(z+l)-c)^k} = N^{\frac{k}{2}} \sum_{\gamma\in(R)} h_{n,\gamma}(z). \qquad (3.2)$$

Since the function $h_{n,\gamma}$ is periodic with period 1, it has a Fourier expansion

$$h_{n,\gamma}(z) = \sum_m b_{n,\gamma}(m) e(mz), \quad Im z \geq \alpha > 0.$$

Now we follow the method of [19] to calculate $b_{n,\gamma}(m)$.

Considering the uniform convergence of the series defining $h_{n,\gamma}(z)$, we see that

$$b_{n,\gamma}(m) = \int_0^1 h_{n,\gamma}(z)e(-mz)dx = \int_{-\infty}^{+\infty} \frac{e(n\frac{bNz-a}{dNz-c})}{(dNz-c)^k} e(-mz)dx.$$

If $w = dNz - c = u + iv$, then $v > 0$. This is because we can assume that $d > 0$. Note that $d \neq 0$ because $\gamma \in \Gamma_0(N)$. Therefore we have

$$b_{n,\gamma}(m) = \frac{1}{dN} e(\frac{nb}{d} + \frac{m(\frac{-c}{N})}{d}) \int_{-\infty}^{+\infty} \frac{e(-(\frac{nw^{-1}}{d} + \frac{mw}{dN}))}{w^k} du.$$

Since $d > 0$ and $v = A$ (a fixed positive number), $b_{n,\gamma}(m)$ is defined by the above convergent integral, because

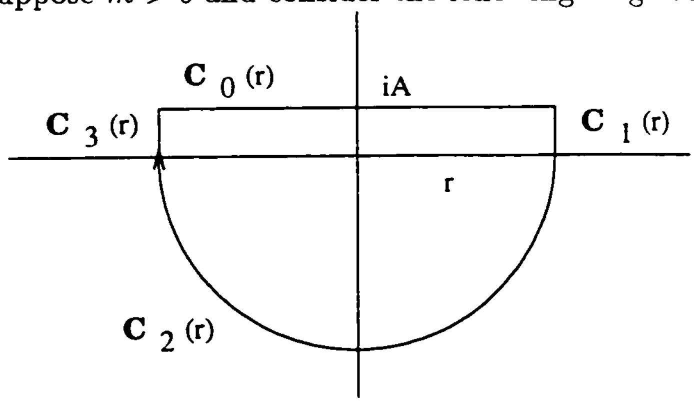
$$|b_{n,\gamma}(m)| \leq \frac{1}{dN} \int_{-\infty}^{+\infty} \frac{\exp(\frac{2\pi v}{d}(\frac{m}{N} - \frac{n}{u^2+v^2}))}{(u^2+v^2)^{\frac{k}{2}}} du < \infty.$$

The following lemma gives an exact expression for $b_{n,\gamma}(m)$.

Lemma 16 *We have,* $b_{n,\gamma}(m) = \begin{cases} M_{n,\gamma}(m) & m > 0 \\ 0 & m \leq 0 \end{cases}$, *where*

$$M_{n,\gamma}(m) = \frac{1}{dN} e(\frac{nb}{d} + \frac{m(-\frac{c}{N})}{d}) \int_{w\bar{w}=\frac{nN}{m}} \frac{e(-(\frac{mw}{dN} + \frac{nw^{-1}}{d}))}{w^k} dw.$$

Proof: First of all suppose $m > 0$ and consider the following diagram.



For $s = 1, 2, 3$, set

$$I_s(r) = \int_{\mathbf{C}_s(r)} \frac{e(-\frac{\frac{m}{N}w + nw^{-1}}{d})}{w^k} dw.$$

We will show that for $s = 1, 2, 3$

$$\lim_{r\to\infty} |I_s(r)| = \lim_{r\to\infty} \left| \int_{\mathbf{C}_s(r)} \frac{e(-\frac{\frac{m}{N}w+nw^{-1}}{d})}{w^k} dw \right| = 0.$$

If $r > max(1, \sqrt{\frac{nN}{m}})$, then

$$\frac{m}{N}r - \frac{n}{r} > 0$$

and on the $\mathbf{C}_1(r)$ and $\mathbf{C}_2(r)$,

$$0 < \frac{\frac{nN}{m}}{r^2+v^2} < 1.$$

So

$$|I_2(r)| \leq \frac{1}{r^{k-1}} \int_0^{\pi} \exp(-\frac{2\pi}{d}(\frac{m}{N}r - \frac{n}{r})\sin\theta)d\theta \leq \frac{\pi}{r^{k-1}}$$

and for $s = 1, 3$

$$|I_s(r)| \leq \int_0^{A} \frac{\exp(\frac{2\pi m v}{dN}(1 - \frac{\frac{nN}{m}}{r^2+v^2}))}{(r^2+v^2)^{\frac{k}{2}}} dv \leq \frac{1}{r^k} \int_0^{A} \exp(\frac{2\pi m}{dN}v)dv.$$

Therefore, we can evaluate $b_{n,\gamma}(m)$ by integrating clockwise around a circle with center at origin, for example the circle $w\bar{w} = \frac{nN}{m}$. This shows that $b_{n,\gamma}(m) = M_{n,\gamma}(m)$.

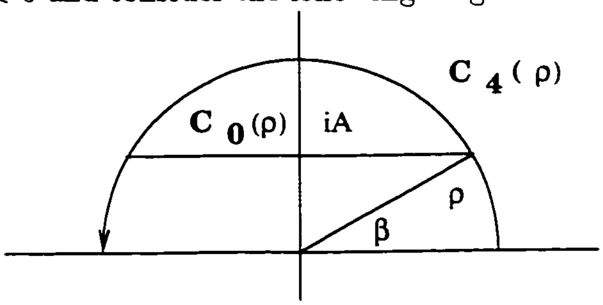
If $m = 0$, again we can show that $\lim_{r\to\infty} |I_s(r)| = 0$ for $s = 1, 2, 3$, and therefore

$$b_{n,\gamma}(0) = \frac{1}{dN} e(\frac{nb}{d}) \int_{\mathbf{C}} \frac{e(-\frac{n}{dw})}{w^k} dw$$

where $\mathbf{C}$ is the unit circle. Calculation of the residue of

$\frac{e(-\frac{n}{dw})}{w^k}$ at $w = 0$ shows that $b_{n,\gamma}(m) = 0$ if $m = 0$.

Now suppose $m < 0$ and consider the following diagram.



For $m = -\mu (\mu > 0)$, we have

$$|I_4(\rho)| \le \frac{1}{\rho^{k-1}} \int_{\beta}^{\pi-\beta} \exp(-\frac{2\pi}{d}(\frac{\mu}{N}\rho + \frac{n}{\rho})\sin\theta)d\theta \le \frac{\pi}{\rho^{k-1}}.$$

Since the integrand is analytic in the region enclosed by $\mathbf{C}_0(\rho)$ and $\mathbf{C}_4(\rho)$, we deduce that $b_{n,\gamma}(m) = 0$ if $m < 0$. □

From Lemma 16 and (3.2), we have

$$P_n(z, k, N)|W_N = N^{\frac{k}{2}} \sum_{\gamma\in(R)} \left\{ \sum_{m>0} M_{n,\gamma}(m) e(mz) \right\}$$

By using the integral representation

$$J_{k-1}(t) = \frac{1}{2\pi i} \int_{z\bar{z}=1} \frac{\exp(\frac{t}{2}(z - z^{-1}))}{z^k} dz$$

of the Bessel function of order $k-1$, and substituting $t = \frac{4\pi\sqrt{mn}}{\sqrt{N}d}$ and $z = -i\sqrt{\frac{m}{nN}}w$, we get

$P_n(z, k, N)|W_N$

$$= N^{\frac{k}{2}} \sum_{m>0} \left\{ \sum_{\gamma\in(R)} \frac{1}{dN} e(\frac{nb}{d} + \frac{m(-\frac{c}{N})}{d})\ (2\pi)i^{-k}(\frac{m}{n})^{\frac{k-1}{2}} \frac{1}{N^{\frac{k-1}{2}}} J_{k-1}(\frac{4\pi\sqrt{mn}}{d\sqrt{N}}) \right\} e(mz). \tag{3.3}$$

Now let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (R)$. Since (R) is a set of representatives for $\Gamma_\infty \backslash \Gamma_0(N)/W_N \Gamma_\infty W_N^{-1}$, we can assume that

$$0 \le b < d,\quad 0 \le (-c) < dN,\quad (d, N) = 1,\quad N|(-c),\quad ad - bc = 1.$$

So in (3.3) we can express the inner sum in terms of d. For $d \ge 1$ with $(d, N) = 1$, we are dealing with the following exponential sum

$$\sum_{0\le b<d,(b,d)=1,N|(-c)} e(\frac{nb + m(-\frac{c}{N})}{d}) \tag{3.4}$$

Since $ad - bc = 1$ and the exponential sum (3.3) depends on b and $-\frac{c}{N}$ only mod d, we can substitute $N^{\phi(d)-1}\bar{b}$ for $-\frac{c}{N}$, where ϕ is the Euler function and $\bar{b}$ is the inverse of b mod d. Note that

$$-\frac{c}{N} \equiv N^{\phi(d)-1}\bar{b} \quad (mod\ d). \tag{3.5}$$

Since $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, for given d and b $(0 \leq b < d,\ (b,d) = 1)$ there exists c such that $N|c$ and $ad - bc = 1$. If we assume that $0 \leq (-c) < dN$ then (3.5) shows that such c is unique.

Substituting (3.5) in (3.4) shows that (3.4) is actually a Kloosterman sum in the following way

$$\sum_{0 \leq b < d, b\bar{b} \equiv 1\ (mod\ d)} e(\frac{nb + (mN^{\phi(d)-1})\bar{b}}{d}) = S(n, mN^{\phi(d)-1}; d). \tag{3.6}$$

Therefore (3.3) and (3.6) yields

$P_n(z,k,N)|W_N$

$$= \sum_{m>0} \left\{ N^{-\frac{1}{2}}(2\pi)i^{-k}(\frac{m}{n})^{\frac{k-1}{2}} \sum_{d,\ (d,N)=1} d^{-1}S(n, mN^{\phi(d)-1}; d)J_{k-1}(\frac{4\pi\sqrt{mn}}{d\sqrt{N}}) \right\} e(mz). \tag{3.7}$$

Now we are in the situation that we can derive a "semi-orthogonality" relation for the Fourier coefficients of an orthonormal basis of $S_k^-(N)$.

Let $\{f_1, ..., f_s\}$ be an orthonormal basis for $S_k^-(N)$, and let $P_n^-(z,k,N) = \sum_i c_i f_i$. From Lemma 15 we get

$$\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} P_n^-(.,k,N) = \sum_i a_{f_i}(n) f_i.$$

Now if $\hat{P}_n^-(m,k,N)$ is the m-th coefficient of the Fourier expansion of $P_n^-(z,k,N)$, we have

$$\frac{(4\pi n)^{k-1}}{2\Gamma(k-1)} \hat{P}_n^-(m,k,N) = \sum_i a_{f_i}(n) a_{f_i}(m)$$

and by definition of the Poincaré series for $S_k^-(N)$ we have

$$\hat{P}_n^-(m,k,N) = \hat{P}_n(m,k,N) + (-1)^{\frac{k}{2}+1}\hat{P}_n(m,k,N)|W_N \tag{3.8}$$

so by applying (2.3) and (3.7) in (3.8) we get

Theorem 3 *Let $\{f_1,...,f_s\}$ be an orthonormal basis for $S_k^-(N)$. Then*

$$\sum_i \frac{a_{f_i}(m)}{\sqrt{m^{k-1}}}\frac{\overline{a_{f_i}(n)}}{\sqrt{n^{k-1}}} = \frac{(4\pi)^{k-1}}{2\Gamma(k-1)}\left\{\delta_{mn} + 2\pi i^{-k}\sum_{c\equiv 0(mod\ N)} c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right.$$

$$\left. -2\pi N^{-\frac{1}{2}}\sum_{d,\ (d,N)=1} d^{-1}S(n,mN^{\phi(d)-1};d)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{d\sqrt{N}}\right)\right\}.$$

As a consequence of the above theorem we have

Proposition 8 *If $\{f_1,...,f_s\}$ is an orthogonal basis for $S_k^-(N)$ and m,n are positive integers, then we have the inequality*

$$\left|\sum_i \omega_{f_i}\frac{a_{f_i}(m)}{\sqrt{m^{k-1}}}\frac{\overline{a_{f_i}(n)}}{\sqrt{n^{k-1}}} - \frac{1}{2}\delta_{mn}\right| \le M\mathbf{d}(N)N^{-\frac{k}{2}}(m,n)^{\frac{1}{2}}\sqrt{(mn)^{k-1}}$$

where $\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}<f,f>}$, M is a constant depending only on k, and $\mathbf{d}(N)$ is the number of divisors of N.

Proof: Similar to Proposition 1, the result follows easily from the following bound for the Bessel function $J_{k-1}(z)$ for $z \ge 0$

$$|J_{k-1}(z)| \le \frac{\sqrt{\pi}z^{k-1}}{2^{k-1}\Gamma(k-\frac{1}{2})}$$

(see Proposition 1) and the Weil bound for the Kloosterman sum (see [7]), i.e.

$$|S(m,n;c)| \le (m,n,c)^{\frac{1}{2}}\mathbf{d}(c)c^{\frac{1}{2}}. \ \square$$

Note In the case of $k=2$, one does not have absolute convergence of the Poincaré series. Nevertheless, Theorem 3 and Proposition 8 are valid in this case as well. To see this, we use a method of Hecke and define

$$P_n(z, 2+2s, N) = \sum_{\Gamma_\infty \backslash \Gamma_0(N)} \frac{e(n\gamma z)}{(cz+d)^2 |cz+d|^{2s}}$$

for a positive real number s.

We can show that as $s \to 0^+$ the above series tends to a cusp form of weight 2 (see [18] pp. 183-191 for details). So we define

$$P_n(z, 2, N) = \lim_{s \to 0^+} P_n(z, 2+2s, N).$$

It can be shown that

$$\hat{P}_n(m, 2, N) = (\frac{m}{n})^{\frac{1}{2}} \left\{ \delta_{mn} - 2\pi \sum_{c \equiv 0 \ (mod\ N)} c^{-1} S(m, n; c) J_1(\frac{4\pi\sqrt{mn}}{c}) \right\}$$

(see [18] p. 188). To obtain a semi-orthogonality relation in the case $k = 2$, we need to calculate $\hat{P}_n(z, 2, N)|W_N$, the m-th Fourier coefficient of $P_n(m, 2, N)|W_N$. Since

$$P_n(z, 2, N)|W_N = \lim_{s \to 0^+} (P_n(z, 2+2s, N)|W_N)$$

we start by finding the Fourier expansion of $P_n(z, 2+2s, N)|W_N$.

Following the notation of the beginning of this section, let (R) be a set of representative for $R = \Gamma_\infty \backslash \Gamma_0(N) / W_N \Gamma_\infty W_N{}^{-1}$ in $\Gamma_0(N)$. Then we have

$$P_n(z, 2+2s, N) = \sum_{\gamma \in (R)} \sum_{l=-\infty}^{+\infty} \frac{e(n\gamma W_N U^l W_N{}^{-1} z)}{((c - dNl)z + d)^2 |(c - dNl)z + d|^{2s}}$$

where $U^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Now we apply the W_N operator on $P_n(z, 2+2s, N)$ to get

$$P_n(z, 2+2s, N)|W_N = N^{1+2s} |z|^{2s} \sum_{\gamma \in (R)} \sum_{l=-\infty}^{+\infty} \frac{e(n\frac{bN(z+l)-a}{dN(z+l)-c})}{(dN(z+l) - c)^2 |dN(z+l) - c|^{2s}}. \qquad (*)$$

Now in $(*)$ set $z = \zeta + \frac{c}{dN}$, then we have

$$P_n(\zeta + \frac{c}{dN}, 2+2s, N)|W_N = N^{1+2s}|\zeta + \frac{c}{dN}|^{2s} \sum_{\gamma \in (R)} \frac{e(\frac{nb}{d})}{(dN)^{2+2s}} \sum_{l=-\infty}^{+\infty} \frac{e(-\frac{n}{d^2 N(\zeta+l)})}{(\zeta+l)^2 |\zeta+l|^{2s}}. \quad (\dagger)$$

Let

$$F_s(\zeta) = \sum_{l=-\infty}^{+\infty} \frac{e(-\frac{n}{d^2 N(\zeta+l)})}{(\zeta+l)^2 |\zeta+l|^{2s}}.$$

To sum this series we make use of the Poisson summation formula

$$\sum_{l=-\infty}^{+\infty} f(l) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(w) e(-mw) dw.$$

This is valid for any function f defined on $\mathbb{R}$ for which f'' exists and is continuous, $f(x)$ and $f'(x)$ tend to zero as $|x| \to \infty$ and $|f|$, $|f'|$ and $|f''|$ are integrable over $\mathbb{R}$. This is true for

$$f(w) = \frac{e(-\frac{n}{d^2 N(\zeta+w)})}{(\zeta+w)^2 |\zeta+w|^{2s}}$$

for real w and $s > -\frac{1}{2}$. Therefore by applying the Poisson summation formula to $F_s(\zeta)$ we get

$$F_s(\zeta) = \sum_{m=-\infty}^{+\infty} I_m(\zeta, s) e(m\zeta)$$

where

$$I_m(\zeta, s) = \int_{-\infty}^{+\infty} \frac{e(-\frac{n}{d^2 N(\zeta+w)}) e(-m(\zeta+w))}{(\zeta+w)^2 |\zeta+w|^{2s}} dw.$$

In this integral $s > -\frac{1}{2}$ and ζ is any point of $\mathcal{H}$.

Using $I_m(\zeta, s)$, $(\dagger)$ can be written as

$$P_n(\zeta + \frac{c}{dN}, 2+2s, N)|W_N = N^{1+2s}|\zeta + \frac{c}{dN}|^{2s} \sum_{\gamma \in (R)} \frac{e(\frac{nb}{d})}{(dN)^{2+2s}} \sum_{m=-\infty}^{+\infty} I_m(\zeta, s) e(m\zeta).$$

Substituting $\zeta = z - \frac{c}{dN}$ in $I_m(\zeta, s)$ and noticing that $I_m(\zeta, s) = I_m(z + \frac{c}{dN}, s) = I_m(z, s)$ in the above identity yields to

$$P_n(z, 2+2s, N)|W_N = N^{1+2s}|z|^{2s} \sum_{\gamma \in (R)} \frac{e(\frac{nb+m(-\frac{c}{N})}{d})}{(dN)^{2+2s}} \sum_{m=-\infty}^{+\infty} I_m(z, s) e(mz).$$

Similar to the case $k > 2$, the above identity can be written as

$$P_n(z, 2+2s, N)|W_N = N^{1+2s}|z|^{2s} \sum_{d,\ (d,N)=1} \sum_{m=-\infty}^{+\infty} \frac{S(n, mN^{\phi(d)-1}; d)}{(dN)^{2+2s}} I_m(z, s)e(mz). \quad (**)$$

Now suppose c_0 is any positive number and let $Im\ z \geq 2c_0$ then we can prove that

$$|I_m(z,s)e(mz)| \leq \frac{2\Gamma(\frac{1}{2})\Gamma(s+\frac{1}{2})}{c_0{}^{2s+1}\Gamma(s+1)} e^{-2\pi(|m|-\frac{n}{d^2N})c_0}$$

(see [18] Theorem 5.7.1 for a proof).

By applying this upper bound and also the Weil bound for the Kloosterman sum, it is clear that the double series on the right of $(**)$ is uniformly convergent for $s \geq 0$ and any $\eta = Im\ z \geq 2c_0 > 0$. Therefore

$$P_n(z, 2, N)|W_N = \lim_{s\to 0^+} (P_n(z, 2+2s, N)|W_N)$$

$$= \sum_{d,(d,N)=1} \sum_{m=-\infty}^{+\infty} \frac{S(n, mN^{\phi(d)-1}; d)}{Nd^2} (\lim_{s\to 0^+} I_m(z,s))e(mz). \quad (***)$$

But it can be proved that

$$\lim_{s\to 0^+} I_m(z,s) = I_m(z,0) = \begin{cases} 0 & m \leq 0 \\ -2\pi d N^{\frac{1}{2}} (\frac{m}{n})^{\frac{1}{2}} J_1(\frac{4\pi\sqrt{nm}}{d\sqrt{N}}) & m > 0 \end{cases}$$

(see [18] Theorem 5.7.1 for a proof).

Now by substituting the above expression for $\lim_{s\to 0^+} I_m(z,s)$ in $(***)$ we derive (3.7) in the case $k = 2$. Now it is clear that Theorem 3 and Proposition 8 are valid in the case $k = 2$ too.

Note From now on for simplicity we assume that $k = 2$ and N is a prime.

3.3 Non-vanishing of the derivative of modular L-functions

Here we follow [16] and apply the explicit formula method to get an upper bound for

$$\sum_{f \in \mathcal{F}_N^-} r_f^{\frac{1}{2}}$$

where $r_f = ord_{s=1} L_f(s)$. To start, let us write

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{q|N} (1 - \frac{a_f(q)}{q^s})^{-1} \prod_{p \nmid N} (1 - \frac{a_f(p)}{p^s} + \frac{1}{p^{2s-1}})^{-1}$$

$$= \prod_{q|N} (1 - \frac{a_f(q)}{q^s})^{-1} \prod_{p \nmid N} (1 - \frac{\alpha_p}{p^s})^{-1} (1 - \frac{\bar{\alpha_p}}{p^s})^{-1}.$$

This is true because f is a newform and therefore it has an Euler product (see 1.3). Now set

$$c_f(n) = \begin{cases} {\alpha_p}^a + \bar{\alpha}^a & \text{if } n = p^a \text{ and } p \nmid N \\ (a_f(q))^a & \text{if } n = q^a \text{ and } q|N \\ 0 & \text{otherwise} \end{cases}.$$

Here $\alpha_p + \bar{\alpha}_p = a_f(p)$, with $|\alpha_p| = \sqrt{p}$.

Lemma 17 *(Weil's explicit formula) Let $F : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(a) there is an $\epsilon > 0$ such that $F(x) \exp((1+\epsilon)x)$ is integrable and of bounded variation,

(b) the function $\frac{F(x)-F(0)}{x}$ is of bounded variation.

Define

$$\Phi(\gamma) = \int_{-\infty}^{\infty} F(x) e^{i\gamma x} dx.$$

Then,

$$\sum_{L_f(1+i\gamma)=0} \Phi(\gamma) = 2F(0) \log \frac{\sqrt{N}}{2\pi} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1+it)\Phi(t)dt - 2\sum_{n \geq 1} \frac{c_f(n)}{n} \Lambda(n) F(\log n)$$

where $\Lambda(n)$ *is Von Mangoldt function and the sum on the left hand side is over* γ *such that* $L_f(1+i\gamma)=0$, $1 \le Re(1+i\gamma) \le \frac{3}{2}$.

Proof : See [14]. □

We choose $T>0$ and define

$$F(x) = \begin{cases} 2T - |x| & if\ |x| \le 2T \\ 0 & otherwise \end{cases}.$$

Then F satisfies the conditions of Lemma 17 and

$$\Phi(\gamma) = (\frac{2\sin\gamma T}{\gamma})^2.$$

Moreover, the corresponding integral involving the logarithmic derivative of the gamma function is easily estimated to be $O(T)$. To see this, let $T>1$ and consider the following integral

$$I = \int_0^\infty \frac{\Gamma'}{\Gamma}(1+it)(\frac{2\sin tT}{t})^2\,dt = 4\int_0^{\frac{1}{T}} + 4\int_{\frac{1}{T}}^\infty = I_1 + I_2.$$

Since $\sin x \le x$ if $x>0$, I_1 is $O(T)$ as the gamma function is bounded in this range. Also we know that

$$\frac{\Gamma'}{\Gamma}(1+it) = O(\log(|t|+2))$$

(see [3] p. 73). Therefore

$$I_2 \ll 4\int_{\frac{1}{T}}^\infty \frac{\log(t+2)}{t^2}dt = 4T\log(2+\frac{1}{T}) + 4\int_{\frac{1}{T}}^\infty \frac{dt}{t(t+2)} \ll T.$$

This shows that $I=O(T)$. Also, we have

$$\Phi(0) = T^2\lim_{\gamma\to 0}(\frac{2\sin\gamma T}{\gamma T})^2 = 4T^2$$

and therefore choosing $T=\frac{(\log x)}{2}$, we will have $\Phi(0)=(\log x)^2$. On the assumption of the GRH (Generalized Riemann Hypothesis) γ is real and so $\Phi(\gamma)\ge 0$. Thus from the explicit formula we get

$$r_f(\log x)^2 \le 2(\log x)\log\frac{\sqrt{N}}{2\pi} - 2\sum_{n\le x}\frac{c_f(n)}{n}\Lambda(n)\log\frac{x}{n} + O(\log x). \tag{3.9}$$

Before continuing we state some analytic estimations which will be used in future.

Lemma 18 *1.* $\sum_{p\le x}\log p\log\frac{x}{p} \ll x$.

2. $\sum_{p^2\le x}\frac{\log p}{p}(\log\frac{x}{p^2}) \sim \frac{1}{4}(\log x)^2$.

Proof: Set

$$b(n) = \begin{cases} 1 & n \textit{ prime} \\ 0 & \textit{otherwise} \end{cases}.$$

Then by using partial summation we get

$$\sum_{p\le x}\log p = \sum_{n\le x} b(n)\log n = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt \sim x - \int_2^x \frac{dt}{\log t} \ll x \tag{3.10}$$

where $\pi(x) \sim \frac{x}{\log x}$ is the number of prime less than x. Now using (3.10) and partial summation yields

$$\sum_{p\le x}\log p\log\frac{x}{p} = \sum_{n\le x} b(n)\log n\log\frac{x}{n} \ll x.$$

The second formula is derived in a similar fashion by using the partial summation formula. □

Now we have the following theorem

Theorem 4 *Let N be prime. Suppose that for each newform $f \in \mathcal{F}_N^-$. $L_f(s)$ satisfies the analogue of the Riemann hypothesis. Then*

$$\sum_{f\in\mathcal{F}_N^-} \omega_f r_f \le \frac{3}{4} + O((\log N)^{-2}).$$

Proof: From (3.9), we get

$$(\log x)^2 \sum_{f\in\mathcal{F}_N^-} \omega_f r_f \le 2(\log x)(\log\frac{\sqrt{N}}{2\pi}) \sum_{f\in\mathcal{F}_N^-} \omega_f$$

$$-2\sum_{n\le x}\frac{\Lambda(n)}{n}\log\frac{x}{n}\Big(\sum_{f\in\mathcal{F}_N^-}\omega_f c_f(n)\Big)+O\Big((\log x)\sum_{f\in\mathcal{F}_N^-}\omega_f\Big).$$

Note that Proposition 8 with $m = n = 1$ gives

$$\sum_{f\in\mathcal{F}_N^-}\omega_f=\frac{1}{2}+O(N^{-1}).$$

Therefore,

$$(\log x)^2\sum_{f\in\mathcal{F}_N^-}\omega_f r_f\le(\log x)\log\frac{\sqrt{N}}{2\pi}-2\sum_{n\le x}\frac{\Lambda(n)}{n}\log\frac{x}{n}\Big(\sum_{f\in\mathcal{F}_N^-}\omega_f c_f(n)\Big)+O(\log x). \tag{3.11}$$

We now study $\sum_{f\in\mathcal{F}_N^-}\omega_f c_f(n)$.

In the case $n = p$ prime, $c_f(p) = a_f(p)$ and therefore by Proposition 8

$$\sum_{f\in\mathcal{F}_N^-}\omega_f a_f(p)=O(N^{-1}p)$$

which by Lemma 18 contributes

$$O\Big(\sum_{p\le x}\log p\log\frac{x}{p}N^{-1}\Big)=O(N^{-1}x)$$

to the second sum in (3.11). The contribution from $n = p^a$ with $a \ge 3$ is at most

$$\sum_{n=p^a\le x,a\ge3}\frac{\Lambda(n)}{\sqrt{n}}\log\frac{x}{n}\ll(\log x)\sum_{a\ge3,p}\frac{\log p}{p^{\frac{a}{2}}}\ll\log x.$$

We still have to deal with $n = p^2$. Note that

$$c_f(p^2)=\alpha_p{}^2+\bar{\alpha}_p^2=(\alpha_p+\bar{\alpha}_p)^2-2\alpha_p\bar{\alpha}_p=a_f(p)^2-2p.$$

But $a_f(p)^2 = a_f(p^2) + p$. By the Rankin-Selberg method,

$$\sum_{p\le x}\frac{\log p}{p}a_f(p)^2\sim x.$$

Therefore, by partial summation, we deduce that

$$\sum_{p\leq x} \frac{\log p}{p^2} a_f(p)^2 \sim \log x$$

and so,

$$\sum_{p^2\leq x} \frac{\log p}{p^2} a_f(p)^2 \sim \frac{1}{2}\log x.$$

Again, by partial summation,

$$\sum_{p^2\leq x} \frac{\log p}{p^2}(\log \frac{x}{p^2}) a_f(p)^2 \sim \frac{1}{4}(\log x)^2.$$

In addition by Lemma 18,

$$-2\sum_{p^2\leq x} \frac{\log p}{p}(\log \frac{x}{p^2}) \sim -\frac{1}{2}(\log x)^2$$

so that

$$\sum_{p^2\leq x} \frac{\log p}{p^2}(\log \frac{x}{p^2})(a_f(p)^2 - 2p) \sim -\frac{1}{4}(\log x)^2$$

as $x \to \infty$. Summing over f with weights ω_f, we obtain a contribution of

$$\frac{1}{4}(\log x)^2 + O(N^{-1}(\log x)^2).$$

At last, we have

$$\sum_{f\in\mathcal{F}_N^-} \omega_f r_f \leq \frac{\log \frac{\sqrt{N}}{2\pi}}{\log x} + \frac{1}{4} + O(N^{-1} + N^{-1}x(\log x)^{-2} + (\log x)^{-1})$$

which simplifies to

$$\frac{1}{4} + \frac{\log N}{2\log x} + O(N^{-1}x(\log x)^{-2}).$$

Choosing $x = N$, we get

$$\sum_{f\in\mathcal{F}_N^-} \omega_f r_f \leq (\frac{3}{4} + O((\log N)^{-2}))$$

which completes the proof. □

Note In the Proof of Theorem 4 we estimated $\sum_{f\in\mathcal{F}_N^-}\omega_f c_f(n)$ for $n = p^a (p \nmid N)$. We can show that the above estimations are valid in the case $n = q^a (q|N)$ too. Note that $c_f(q) = \pm 1$ (see [1] p. 147).

Now we are going to give an asymptotic formula for the Petersson inner product on average. To do this we start with reviewing some fact about the symmetric square L-function, which we denote by $L_{sym^2(f)}(s)$. The value of $L_{sym^2(f)}(s)$ at $s = 2$ and the Petersson inner product are related with each other as follows

$$L_{sym^2(f)}(2) = \frac{8\pi^3}{N} < f, f > \tag{3.12}$$

(see [21] p. 90). So to find an asymptotic formula for $\sum_{f\in\mathcal{F}_N^-} < f, f >$ it is enough to find one for $\sum_{f\in\mathcal{F}_N^-} L_{sym^2(f)}(2)$.

We start by recalling the following identity

$$L_{sym^2(f)}(s) = \zeta_N(2s-2)\sum_{n=1}^{\infty}\frac{a_f(n^2)}{n^s} = \sum_{n=1}^{\infty}\frac{g_f(n)}{n^s} \tag{3.13}$$

where $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p \mid N$ removed (see 1.4 for details).

Consider the integral

$$\frac{1}{2\pi i}\int_{(2)} L_{sym^2(f)}(2+s)T^s\Gamma(s)ds = \sum_{n=1}^{\infty}\frac{g_f(n)}{n^2}\exp\left(-\frac{n}{T}\right)$$

and this is

$$= L_{sym^2(f)}(2) + \frac{1}{2\pi i}\int_{(-\frac{1}{2})} L_{sym^2(f)}(2+s)T^s\Gamma(s)ds. \tag{3.14}$$

In (3.14) the integral is easily estimated as

$$O(N^\theta T^{-\frac{1}{2}})$$

on the assumption that $L_{sym^2(f)}(\frac{3}{2}+it) \ll N^\theta$.

From the Phragmén-Lindelöf theorem it follows that

$$L_{sym^2(f)}(\frac{3}{2}+it) \ll N^{\frac{1}{2}}(\log N)^3$$

(see [13] p. 336 for details). Also assuming the Lindelöf hypothesis (which is a consequence of the generalized Riemann hypothesis for $L_{sym^2(f)}(s)$), θ can be any positive number. Therefore from (3.14)

$$L_{sym^2(f)}(2) = \sum_{n=1}^{\infty} \frac{g_f(n)}{n^2} \exp(-\frac{n}{T}) + O(N^{\theta} T^{-\frac{1}{2}}). \qquad (3.15)$$

Now we derive an expression for $g_f(n)$. From (3.13) we have

$$\sum_{n=1}^{\infty} \frac{g_f(n)}{n^s} = (\sum_{\substack{n=1\\(n,N)=1}}^{\infty} \frac{n^2}{(n^2)^s})(\sum_{n=1}^{\infty} \frac{a_f(n^2)}{n^s}) = (\sum_{u=1}^{\infty} \frac{a_u}{u^s})(\sum_{v=1}^{\infty} \frac{a_f(v^2)}{v^s}).$$

Here

$$a_u = \begin{cases} d^2 & if\ u = d^2,\ (d,N)=1 \\ 0 & otherwise \end{cases}$$

and

$$g_f(n) = \sum_{uv=n} a_u a_f(v^2) = \sum_{\substack{(d,N)=1\\ d^2e=n}} d^2 a_f(e^2).$$

Substituting the expression for $g_f(n)$ in (3.15) gives

Proposition 9 *$L_{sym^2(f)}(2) = \sum_{d,e,(d,N)=1} \frac{a_f(e^2)}{d^2e^2} \exp(-\frac{d^2e}{T}) + O(N^{\theta}T^{-\frac{1}{2}})$ where θ is the positive number satisfying $L_{sym^2(f)}(\frac{3}{2}+it) \ll N^{\theta}$.*

From Proposition 9, it is clear that to find an asymptotic formula for $\sum_{f\in\mathcal{F}_N^-} L_{sym^2(f)}(2)$, we need an estimation for $\sum_{f\in\mathcal{F}_N^-} a_f(e^2)$. By using the Selberg trace formula we have the following proposition.

Proposition 10 *For N prime and e prime to N*

$$\sum_{f\in\mathcal{F}_N^-} a_f(e^2) = \frac{N-1}{24} + O\left(\left(e^2\mathbf{d}(e^2) + N^{\frac{1}{2}}e\mathbf{d}(e^2)\right)(\log eN)^2\right).$$

Proof: It is clear that

$$\sum_{f\in\mathcal{F}_N^-} a_f(e^2) = \frac{tr(T_{e^2}) + tr(T_{e^2}|W_N)}{2}.$$

Here $tr(T_{e^2})$ is the trace of the e^2-th Hecke operator on $S_2(N)$ and estimation of the trace of the Hecke operators as given in Proposition 2.8 of [2]. □

We need the following two lemmas in the proof of our asymptotic formula for $\sum_{f\in\mathcal{F}_N} L_{sym^2(f)}(2)$.

Lemma 19 *If* $d(n)$ *is the number of divisors of* n*, then*

(i) $\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta(s)^3\eta(s)}{\lambda(s)}$, *where* $\zeta(s)$ *is the Riemann zeta function and* $\eta(s)$ *and* $\lambda(s)$ *are Dirichlet series which are absolutely convergent for* $Re(s) > \frac{1}{2}$.

(ii) $\sum_{n=1}^{\infty} d(n^2)\exp(-\frac{n}{T}) \ll T(\log T)^2$.

(iii) $\sum_{n=1}^{\infty} \frac{d(n^2)}{n}\exp(-\frac{n}{T}) \ll (\log T)^3$.

Proof: Since $\mathrm{d}(n)$ is a multiplicative function, we have

$$\sum_{n=1}^{\infty}\frac{\mathrm{d}(n^2)}{n^s} = \prod_{p\ prime}(\sum_{j=0}^{\infty}\frac{2j+1}{p^{js}}) = \prod_p(1+\frac{3}{p^s}+\frac{5}{p^{2s}}+\ldots)$$

$$= \prod_p(1+\frac{3}{p^s})(1+\frac{*}{p^{2s}}+\frac{**}{p^{3s}}+\ldots) = \prod_p(1-\frac{3}{p^s})^{-1}(1-\frac{9}{p^{2s}})(1+\frac{*}{p^{2s}}+\frac{**}{p^{3s}}+\ldots)$$

$$= \prod_p(1-\frac{3}{p^s})^{-1}\sum_{n=1}^{\infty}\frac{a_n}{n^s}. \quad (\dagger)$$

Let $\eta(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$, then $a_n = 0$ if n has a prime factor with multiplicity one. This shows that $\eta(s)$ is absolutely convergent for $Re(s) > \frac{1}{2}$. Now we have

$$(\zeta(s))^3 = \prod_{p\ prime}(1-\frac{1}{p^s})^{-3} = \prod_p(1-\frac{3}{p^s}+\frac{3}{p^{2s}}-\frac{1}{p^{3s}})^{-1}$$

$$= \prod_p(1-\frac{3}{p^s})^{-1}(1+\frac{\times}{p^{2s}}+\frac{\times\times}{p^{3s}}+\ldots)^{-1} = \prod_p(1-\frac{3}{p^s})^{-1}\sum_{n=1}^{\infty}\frac{b_n}{n^s}.$$

Let $\lambda(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$, again similar to $\eta(s)$, $\lambda(s)$ is absolutely convergent for $Re(s) > \frac{1}{2}$. Substituting

$$\prod_p (1 - \frac{3}{p^s})^{-1} = \frac{(\zeta(s))^3}{\lambda(s)}$$

in (†) yields (i).

(ii) Since

$$e^{-\frac{1}{T}} = \frac{1}{2\pi i}\int_{(2)} T^s\Gamma(s)ds$$

we have

$$\sum_{n=1}^{\infty} \mathbf{d}(n^2)e^{-\frac{n}{T}} = \frac{1}{2\pi i}\int_{(2)} (\sum_{n=1}^{\infty} \frac{\mathbf{d}(n^2)}{n^s})T^s\Gamma(s)ds.$$

By part (i) this integral is

$$= \frac{1}{2\pi i}\int_{(2)} \frac{(\zeta(s))^3\eta(s)}{\lambda(s)}T^s\Gamma(s)ds.$$

By moving the line of integration from 2 to $\frac{3}{4}$ we get

$$\sum_{n=1}^{\infty} \mathbf{d}(n^2)e^{-\frac{n}{T}} = O(T(\log T)^2) + \frac{1}{2\pi i}\int_{(\frac{3}{4})} \frac{(\zeta(s))^3\eta(s)}{\lambda(s)}T^s\Gamma(s)ds \ll T(\log T)^2.$$

This completes the proof of (ii).

(iii) Similar to part (ii), we set

$$\sum_{n=1}^{\infty} \frac{\mathbf{d}(n^2)}{n}e^{-\frac{n}{T}} = \frac{1}{2\pi i}\int_{(2)} (\sum_{n=1}^{\infty} \frac{\mathbf{d}(n^2)}{n^{s+1}})T^s\Gamma(s)ds$$

$$= \frac{1}{2\pi i}\int_{(2)} \frac{(\zeta(s+1))^3\eta(s+1)}{\lambda(s+1)}T^s\Gamma(s)ds.$$

The result follows by moving the line of integration from 2 to $-\frac{1}{4}$ and the calculation of the residue at $s = 0$. □

Lemma 20 $dim S_2^-(N) = \frac{N}{24} + O(\sqrt{N})$.

Proof: It is known that for $N > 3$ a prime the exact number of forms in $\mathcal{F}_N$ is given by $\sharp\mathcal{F}_N = \frac{1}{12}(N + \alpha(N))$, where $\alpha(N) = -13, \ -5, \ -7$, or 1 according to whether

$N \equiv 1, 5, 7, or\ 11 (mod\ 12)$. Now the result follows from the fact that

$$dim\ S_2^-(N) = \frac{1}{2} dim\ S_2(N) + O(\sqrt{N})$$

(see [16] p. 276). □

Now we can give an asymptotic formula for the Petersson inner product on average.

Theorem 5 *If we assume* $L_{sym^2(f)}(\frac{3}{2} + it) \ll N^{\frac{1}{2}-\eta}$, *for some* $\eta > 0$, *then*

$$\sum_{f \in \mathcal{F}_N^-} < f, f > = \frac{\pi}{12} (dim S_2^-(N))^2 + O(N^{2-\frac{\eta}{2}}).$$

Proof: By Proposition 9 we have

$$\sum_{f \in \mathcal{F}_N^-} L_{sym^2(f)}(2) = \sum_{d,e,(d,N)=1} \frac{\exp\left(-\frac{d^2 e}{T}\right)}{d^2 e^2} \sum_{f \in \mathcal{F}_N^-} a_f(e^2) + O(N^{\theta+1} T^{-\frac{1}{2}}).$$

Since f is a newform with root number -1, we have $a_f(N) = -1$ and for $(e_0, N) = 1$, $a_f(N^{2m} e_0^2) = (a_f(N))^{2m} a_f(e_0^2) = a_f(e_0^2)$ (see [1] p. 147, Theorem 3). Therefore the above identity can be written as

$\sum_{f \in \mathcal{F}_N^-} L_{sym^2(f)}(2)$

$$= \left(\sum_{\substack{d,e \\ (d,N)=1,(e,N)=1}} \frac{\exp\left(-\frac{d^2 e}{T}\right)}{d^2 e^2} + \sum_{m=1}^{\infty} \frac{1}{N^{2m}} \sum_{\substack{d,e \\ (d,N)=1,(e,N)=1}} \frac{\exp\left(-\frac{d^2 e N}{T}\right)}{d^2 e^2} \right) \sum_{f \in \mathcal{F}_N^-} a_f(e^2) + O(N^{\theta+1} T^{-\frac{1}{2}}).$$

By Proposition 10

$\sum_{f \in \mathcal{F}_N^-} L_{sym^2(f)}(2)$

$$= \frac{N}{24} \left(\sum_{\substack{d,e \\ (d,N)=1,(e,N)=1}} \frac{\exp\left(-\frac{d^2 e}{T}\right)}{d^2 e^2} + \sum_{m=1}^{\infty} \frac{1}{N^{2m}} \sum_{\substack{d,e \\ (d,N)=1,(e,N)=1}} \frac{\exp\left(-\frac{d^2 e N}{T}\right)}{d^2 e^2} \right) + \vartheta + O(N^{\theta+1} T^{-\frac{1}{2}})$$

where

$$\vartheta \ll \left(\sum_{d,e} \frac{\exp\left(-\frac{d^2 e}{T}\right)}{d^2 e^2} \right.$$

$$+\sum_{m=1}^{\infty}\frac{1}{N^{2m}}\sum_{d,e}\frac{\exp\left(-\frac{d^2eN}{T}\right)}{d^2e^2}\Bigg)\left(\mathbf{d}(e^2)e^2(\log eN)^2+e\mathbf{d}(e^2)N^{\frac{1}{2}}(\log eN)^2\right).$$

From Lemma 19 we know that

$$\sum_e \mathbf{d}(e^2)\exp\left(-\frac{d^2e}{T}\right)\ll\frac{T}{d^2}(\log\frac{T}{d^2})^2$$

and

$$\sum_e \frac{\mathbf{d}(e^2)}{e}\exp\left(-\frac{d^2e}{T}\right)\ll(\log\frac{T}{d^2})^3$$

Therefore by using the partial summation formula. we deduce that

$$\vartheta\ll T(\log T)^4+T(\log T)^2(\log N)^2+N^{\frac{1}{2}}(\log T)^5+N^{\frac{1}{2}}(\log T)^3(\log N)^2.$$

The main term is deduced by an estimation of

$$\sum_{(d,N)=1}\frac{\exp\left(-\frac{d^2e}{T}\right)}{d^2}.$$

From the integral formula we have

$$\sum_{(d,N)=1}\frac{\exp\left(-\frac{d^2}{T}\right)}{d^2}=\frac{1}{2\pi i}\int_{(2)}\zeta_N(2+2s)T^s\Gamma(s)ds$$

$$=\zeta_N(2)+\frac{1}{2\pi i}\int_{(-\frac{1}{2}+\epsilon)}\zeta_N(2+2s)T^s\Gamma(s)ds$$

which gives us

$$\sum_{(d,N)=1}\frac{\exp\left(-\frac{d^2}{T}\right)}{d^2}=\zeta_N(2)+O(T^{-\frac{1}{2}+\epsilon}). \tag{3.16}$$

Therefore

$$\frac{N}{24}\sum_{\substack{d,e\\(d,N)=1,(e,N)=1}}\frac{\exp\left(-\frac{d^2e}{T}\right)}{d^2e^2}=\frac{N}{24}\sum_{\substack{e\\(e,N)=1}}\frac{1}{e^2}\left(\zeta_N(2)+O((\frac{T}{e})^{-\frac{1}{2}+\epsilon})\right).$$

This is easily seen to be equal to

$$\frac{N}{24}\left(\frac{\pi^2}{6}\right)^2 + O(NT^{-\frac{1}{2}+\epsilon}).$$

Hence from (3.12), for any $T > 0$, we have

$$\frac{8\pi^3}{N}\sum_{f\in\mathcal{F}_N^-} < f, f >= \frac{N}{24}\left(\frac{\pi^2}{6}\right)^2 + O(T(\log T)^4 + T(\log T)^2(\log N)^2$$

$$+N^{\frac{1}{2}}(\log T)^5 + N^{\frac{1}{2}}(\log T)^3(\log N)^2) + O(N^{\theta+1}T^{-\frac{1}{2}}).$$

Now by the assumption of theorem, $\theta = \frac{1}{2} - \eta$. Therefore setting $T = N^{1-\eta}$ we have

$$\sum_{f\in\mathcal{F}_N^-} < f, f >= \frac{\pi}{12}(dim S_2^-(N))^2 + O(N^{2-\frac{\eta}{2}}). \ \square$$

Now from Theorems 4 and 5 we can deduce the following upper bound for the $\sum_{f\in\mathcal{F}_N^-} r_f^{\frac{1}{2}}$.

Corollary 1 *Let N be prime. Assume the Riemann hypothesis for $L_f(s)$ and suppose that $L_{sym^2(f)}(\frac{3}{2} + it) \ll N^{\frac{1}{2}-\eta}$, for some $\eta > 0$, then*

$$\sum_{f\in\mathcal{F}_N^-} r_f^{\frac{1}{2}} \le \frac{\pi}{2} dim S_2^-(N) + o(N)$$

as $N \to \infty$.

Proof: By the Cauchy-Schwarz inequality we have

$$\sum_{f\in\mathcal{F}_N^-} r_f^{\frac{1}{2}} \le \Big(\sum_{f\in\mathcal{F}_N^-} \omega_f r_f\Big)^{\frac{1}{2}}\Big(\sum_{f\in\mathcal{F}_N^-} \frac{1}{\omega_f}\Big)^{\frac{1}{2}}$$

$$= \Big(\sum_{f\in\mathcal{F}_N^-} \frac{r_f}{4\pi < f, f >}\Big)^{\frac{1}{2}}\Big(\sum_{f\in\mathcal{F}_N^-} 4\pi < f, f >\Big)^{\frac{1}{2}}.$$

By Theorems 4 and 5, this is

$$2\sqrt{\pi}\left(\frac{3}{4}+o(1)\right)^{\frac{1}{2}}\left(\frac{\pi}{12}(dimS_2{}^-(N))^2+O(N^{2-\frac{n}{2}})\right)^{\frac{1}{2}}$$

which is

$$\leq \frac{\pi}{2}dimS_2{}^-(N)+o(N)$$

as $N\to\infty$. □

Note The upper bound given in Corollary 1 is actually weaker than what we can deduce from a result of Brumer. In [2] Brumer proved that under the assumption of the Riemann hypothesis for $L_f(s)$,

$$\sum_{f\in\mathcal{F}_N{}^-} r_f \leq (\frac{3}{2}+\epsilon)\ dimS_2{}^-(N)$$

for any $\epsilon>0$ and N sufficiently large (see [2] Theorem 3.15). By using the Cauchy-Schwarz inequality, this yields

$$\sum_{f\in\mathcal{F}_N{}^-} r_f{}^{\frac{1}{2}} \leq (\frac{3}{2}+\epsilon)^{\frac{1}{2}}\ dimS_2{}^-(N).$$

The following non-vanishing result is a direct consequence of Corollary 1.

Corollary 2 *Under the assumptions of Corollary 1 for $L_{sym^2(f)}(s)$ for any $f\in\mathcal{F}_N{}^-$, and for prime N large enough a positive proportion of elements of $\mathcal{F}_N{}^-$ (and therefore $\mathcal{F}_N$) have order 1 at $s=1$.*

3.4 An approximate trace formula for $S_2{}^-(N)$

In this section as another application of "semi-orthogonality" relation, we use Theorem 3 to derive a formula for

$$\sum_{\mathcal{F}_N{}^-} a_f(n)$$

where $\mathcal{F}_N^-$ is the set of newforms in $S_2^-(N)$. We follow [16] closely.

Proposition 11 *Suppose that $L_{sym^2(f)}(s) \ll N^\theta$, for some $\theta > 0$. If n is not a square, we have for any $T > 0$*

$$\sum_{f\in\mathcal{F}_N^-} a_f(n) = O(nT\mathbf{d}(n) + \sqrt{n}\mathbf{d}(n)N^{1+\theta}T^{-\frac{1}{2}})$$

where $\mathbf{d}(n)$ is the number of divisors of n. If n is a square, we have

$$\sum_{f\in\mathcal{F}_N^-} a_f(n) = \frac{N}{4\pi^2}\left\{\zeta_N(2) + O(T^{-\frac{1}{2}+\epsilon}n^{\frac{1}{4}-\frac{\epsilon}{2}})\right\}$$

$$+O(nT\mathbf{d}(n) + \sqrt{n}\mathbf{d}(n)N^{1+\theta}T^{-\frac{1}{2}}).$$

Proof: From (3.12) we have

$$\sum_{f\in\mathcal{F}_N^-} a_f(n) = N\sum_{f\in\mathcal{F}_N^-}\frac{a_f(n)}{8\pi^3 < f, f >}L_{sym^2(f)}(2)$$

Now from Proposition 9 and definition of $\omega_f = \frac{1}{4\pi<f,f>}$ we get

$$\frac{1}{N}(2\pi^2)\sum_{f\in\mathcal{F}_N^-} a_f(n) = \sum_{d,e,(d,N)=1}\frac{\exp(-\frac{d^2e}{T})}{d^2e^2}\left(\sum_{f\in\mathcal{F}_N^-}\omega_f a_f(n)a_f(e^2)\right)$$

$$+\sum_{f\in\mathcal{F}_N^-}\omega_f a_f(n)O(N^\theta T^{-\frac{1}{2}}).$$

Now by applying Proposition 8 and the Deligne bound for $a_f(n)$, we get

$$\frac{1}{N}(2\pi^2)\sum_{f\in\mathcal{F}_N^-} a_f(n) = \sum_{d,e,(d,N)=1}\frac{\exp(-\frac{d^2e}{T})}{d^2e^2}\Big(\frac{\delta_{n,e^2}\sqrt{n}e}{2}$$

$$+O(N^{-1}ne^2(n,e^2)^{\frac{1}{2}}) + O(\sqrt{n}d(n)N^\theta T^{-\frac{1}{2}}))$$

Here we are using the fact that $\sum_{f\in\mathcal{F}_N^-}\omega_f = \frac{1}{2}+O(N^{-1})$, and also $\#\mathcal{F}_N^- = O(N)$ (see Lemma 20). The error term arising from the sum is

$$g(n)=\sum_{d|n}\mu(d)\sqrt{\frac{n}{d}}$$

where $\mu(d)$ is the Möbius function. From the Möbius inversion formula we know that

$$m^{\frac{1}{2}}=\sum_{d|m}g(d).$$

Now using the definition of $g(n)$, we can rewrite the above sum in the error as

$$N^{-1}n\sum_{\delta|n}g(\delta)\sum_{d,e,\delta|e^2}\frac{\exp(-\frac{d^2e}{T})}{d^2}=N^{-1}n\sum_{\delta|n}g(\delta)\sum_{d}\frac{1}{d^2}\sum_{e,\delta|e^2}\exp(-\frac{d^2e}{T})$$

It is easily seen that the above expression

$$\ll N^{-1}nT\sum_{\delta|n}\frac{g(\delta)}{\sqrt{\delta}}=N^{-1}nT\sum_{\delta|n}\sum_{d|\delta}\frac{\mu(d)}{\sqrt{d}}=N^{-1}nT\sum_{d|n}\frac{\mu(d)}{\sqrt{d}}$$

which is

$$\ll N^{-1}nT\mathbf{d}(n).$$

This proves the Proposition when n is not a square.

When $n=e^2$, substituting (3.16) into the main term gives the desired result. □

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