Balanced Cayley graphs and balanced planar graphs

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Abstract

A balanced graph is a bipartite graph with no induced circuit of length $2 \pmod 4$. These graphs arise in linear programming. We focus on graph-algebraic properties of balanced graphs to prove a complete classification of balanced Cayley graphs on abelian groups. Moreover, in Section 5 of this paper, we prove that there is no cubic balanced planar graph. Finally, some remarkable conjectures for balanced regular graphs are also presented.

Key words: Cayley graph, balanced graph

1 Introduction

A $\{0, 1\}$-matrix is *balanced* if the sum of the entries of every submatrix that is minimal with respect to the property of containing 2 nonzero entries per row and per column, is congruent to $0 \pmod 4$. Balanced matrices were introduced by Berge [3] in the context of hypergraphs, and they arise naturally in linear programming [10].

There has been considerable study of balanced matrices; the reader might check [5] or [6] for a survey on the main results and horizons on balanced matrices.

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Every $\{0,1\}$-matrix is also the bipartite adjacency matrix of a bipartite graph. Specifically, if X is a bipartite graph with vertex bipartition (U, V), then the bipartite adjacency matrix for X is the $\{0,1\}$-matrix A with $A_{u,v} = 1$ if and only if $u \in U, v \in V$ and $\{u, v\} \in E(X)$. So, it is natural to consider which bipartite graphs have balanced adjacency matrices. Equivalently, which bipartite graphs have no induced circuits of length $2 \pmod 4$. We refer to such graphs as *balanced graphs*. Most results on balanced graphs are restricted to some subclass. For instance, balanced graphs in which every induced circuit has length 4 have been characterised in [9]. It is our aim in this paper to provide some insight into the structure of some additional classes of balanced graphs.

All graphs in this paper are connected, so, by abuse of terminology we use the term "graph" to mean "connected graph".

We will present three main results in this paper, together with a number of conjectures. In two of our results, we restrict our attention to a significant family of balanced graphs, and characterise all of the graphs in that family. First, we characterise balanced Cayley graphs on abelian groups. Then we prove that there are no cubic balanced planar graphs. In the remaining result, we provide a condition on the number of vertices of a k-regular balanced graph.

Before we can state our characterisation of balanced Cayley graphs on abelian groups, a number of definitions will be required.

Let G be a group and S a subset of G such that S is closed under taking inverses, i.e. $S = S^{-1}$. The *Cayley graph* of G with connection set S, denoted $\mathrm{Cay}(G, S)$, is the graph with vertex set G and edge set $\{\{g, h\} \mid gh^{-1} \in S\}$. If X and Y are graphs, the lexicographic product of X with Y is the graph with vertices $V(X) \times V(Y)$, where (x, y) and (x', y') are adjacent if and only if either $\{x, x'\} \in E(X)$, or $x = x'$ and $\{y, y'\} \in E(Y)$. We denote by $\overline{K_t}$ the complement of the complete graph K_t, i.e. $\overline{K_t}$ has t vertices and no edges. Further, we denote by C_l the cycle of length l and $C_2 = K_2$ for the degenerate case. The following terminology will be used in our characterisation.

Definition 1.1 *Let l, t be in $\mathbb{Z}^+$, with $l = 2$ or $l \equiv 0 \pmod 4$ and $l \geq 8$. The lexicographic product of C_l with $\overline{K_t}$ is called an* (l, t)-cycle.

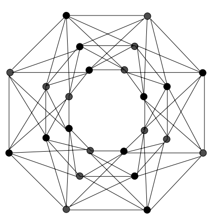


Fig. 1.1. $(8, 3)$-cycle

Figure 1.1 shows the example of an $(8, 3)$-cycle.

We recall that the circuits of a Cayley graph X are well studied and the problem of "understanding" the lengths of the circuits of X has attracted considerable attention, see [1] and [11]. For example, it is known that if X is a bipartite Cayley graph on an abelian group, then X has circuits of any even length, i.e. $4, 6, \ldots, |X|$, see [4]. Unfortunately, not much is known about the *induced* circuits of X. So, in this paper, we focus only on induced circuits. Since it seems hard to get general results on induced circuits of Cayley graphs, we restrict our interest to balanced Cayley graphs. Specifically, in Section 2 and Section 3, we study balanced Cayley graphs on abelian groups. We will show that this class of graphs is very restricted. Indeed, we get the following result.

Theorem 1.2 *If G is an abelian group and $S \subseteq G$, then the graph* $\mathrm{Cay}(G, S)$ *is balanced if and only if it is isomorphic to an (l, t)-cycle.*

In Section 4, we use a theorem of Berge [3] and Fulkerson, Hoffman, and Oppenheim [7] to prove the following result.

Theorem 1.3 *If X is a k-regular balanced graph, then the number of vertices of X is divisible by $2k$.*

In Section 5, we narrow our focus to cubic balanced planar graphs and we prove the following theorem.

Theorem 1.4 *There is no cubic balanced planar graph.*

Finally, two conjectures for balanced regular graphs are presented in Section 6. These conjectures are based on the results for balanced Cayley graphs on abelian groups, on Theorem 1.4 and on some exhaustive computer computations.

2 Balanced circulant graphs

It seems to the authors that the statements and the proofs in Section 2 and 3 are neater using a multiplicative notation. So, all groups in the following two sections will be written using multiplicative notation. As usual, if G is a group and $a \in G$, the symbol $\langle a \rangle$ denotes the subgroup of G generated by a, and 1_G denotes the identity element in G.

Let A_n denote a cyclic group of order n, generated by a, and let $S \subseteq A_n \backslash \{1_{A_n}\}$ be closed under taking inverses. The graph $\mathrm{Cay}(A_n, S)$ is said to be a *circulant graph* of order n. Recall that $\mathrm{Cay}(A_n, S)$ is bipartite if and only if n is even and $S \subseteq \{a^i \mid i \text{ odd}\}$.

Lemma 2.1 *If $X = \mathrm{Cay}(A_n, S)$ is balanced and $a, a^3 \in S$, then X is $K_{n/2,n/2}$.*

Proof. We prove, by induction on l, that a^{2l+1} lies in S for every l. If $l = 0$ or 1, then there is nothing to prove, by hypothesis a, a^3 lie in S. Now, assume that the claim is true for every index i such that $i \leq l-1$, and so, $\{a, a^3, \ldots, a^{2l-1}\} \subseteq S$. Since $S = S^{-1}$, if $S = \{a, a^3, \ldots, a^{2l-1}\}$, then $S = \{a^i \mid i \text{ odd}\}$ and there is nothing to prove. Assume $\{a, a^3, \ldots, a^{2l-1}\} \subsetneq S$. Let $m = 2l-1$ and let $t > 0$ be minimum with $a^{m+t} \in S$. Set $t = qm + r$ where q and r are non-negative integers with $0 \leq r < m$. We show that $q = 0$ and $r = 2$ and thus $a^{2l+1} \in S$. Note that since $S^{-1} = S$, we may assume that $m < n/2$.

Suppose instead that $q \geq 1$. The following table lists the vertices of an induced circuit C in X with length 2 (mod 4). In reading the table, the following remarks may be useful. Since t is even and m is odd, q and r have the same parity. Each cycle ends with the vertex $a^{(q+1)m+r} = a^{m+t}$. Now, we leave to the reader the straightforward work of checking that every given circuit is induced, since all differences between non-consecutive vertices are either even, or strictly between m and $m+t$. Since X is balanced, it follows that $q = 0$.

$q \pmod 4$	r	C
0	0	$(1, a^m, a^{2m}, \ldots, a^{(q+1)m})$
0	2	$(1, a, a^2, a^{m+2}, a^{m+3}, a^{m+4}, a^{2m+4}, a^{3m+4}, \ldots, a^{qm+4}, a^{(q+1)m+r})$
0	≥ 4	$(1, a, a^2, a^{m+2}, a^{2m+2}, \ldots, a^{(q+1)m+2}, a^{(q+1)m+r-1}, a^{(q+1)m+r})$
1	≥ 1	$(1, a, a^2, a^{m+2}, a^{2m+2}, a^{3m+2}, \ldots, a^{qm+2}, a^{(q+1)m}, a^{(q+1)m+r})$
2	0	$(1, a, a^2, a^{m+2}, a^{2m+2}, \ldots, a^{qm+2}, a^{(q+1)m})$
2	≥ 2	$(1, a^m, a^{2m}, \ldots, a^{(q+1)m}, a^{(q+1)m+r-1}, a^{(q+1)m+r})$
3	≥ 1	$(1, a^m, a^{2m}, \ldots, a^{(q+1)m}, a^{(q+1)m+r})$

Since $q = 0$, we have r is even. If $r \geq 4$, then $(1, a, a^2, a^{m+2}, a^{m+3}, a^{m+r})$ is an induced circuit of length 6. Thus $r = 2$ and the induction is complete.

We have proved that $S = A_n \backslash \langle a^2 \rangle$ and so X is a complete bipartite graph. □

Next we consider the case that $a^3 \notin S$.

Lemma 2.2 *Let $X = \mathrm{Cay}(A_n, S)$ be balanced with $a \in S$, $|S| > 1$ and $a^3 \notin S$. If $l > 2$ is minimum such that $a^{l-1} \in S$, then $l \equiv 0 \pmod 4$, $l \geq 8$, l divides n and $S = \{a^{il\pm 1} \mid 0 \leq i \leq (n/l) - 1\}$.*

Proof. The circuit $(1, a, a^2, \ldots, a^{l-1})$ is induced in X, therefore $l \equiv 0 \pmod 4$. Moreover, since $a^3 \notin S$, we have $l \geq 8$. We use induction on k to prove the following claim, from which Lemma 2.2 would follow.

Claim 2.2.1 *If $k \in \mathbb{Z}^+$, then $S \cap \{a^j \mid -1 \le j \le kl+1\} = \{a^{il\pm 1} \mid 0 \le i \le k\}$.*

Let $k = 1$. In this case, by hypothesis on a and on l, we have only to show that a^{l+1} lies in S. The circuit $C = (1, a, a^2, a^{l+1}, a^{l+2}, a^{l+3}, a^4, a^5, a^6, \ldots, a^{l-1})$ has length $l+2$ and thus is not induced. Using the hypothesis that $l > 2$ is minimum with $a^{l-1} \in S$, it is easy to check that $\{1, a^{l+1}\}$, $\{1, a^{l+3}\}$, $\{a, a^{l+2}\}$, $\{a^2, a^{l+3}\}$ are the only possible chords for C; hence $a^{l+1} \in S$ or $a^{l+3} \in S$. If $a^{l+3} \in S$ and $a^{l+1} \notin S$, then $(1, a^{l-1}, a^l, a^{l+1}, a^{l+2}, a^{l+3})$ is an induced circuit of length 6 in X, a contradiction. Thus $a^{l+1} \in S$ and the claim holds when $k = 1$.

Assume that $S \cap \{a^j \mid -1 \le j \le kl+1\} = \{a^{il\pm 1} \mid 0 \le i \le k\}$ for some $k \ge 1$. Let $t > 0$ be minimal such that $a^{kl+t+1} \in S$. We have to prove that $t = l-2$. If $t \equiv 0 \pmod 4$ and $t \le l-3$, then $(1, a^{kl+1}, a^{kl+2}, \ldots, a^{kl+t+1})$ is an induced circuit of length $t+2$ in X, a contradiction. Similarly, if $t \equiv 2 \pmod 4$ and $t \le l-3$, then $(1, a^{kl+t+1}, a^{t+2}, a^{t+3}, \ldots, a^{l-1})$ is an induced circuit of length $l-t$, a contradiction. This yields $t \ge l-2$.

Consider the circuit $C = (1, a^{kl-1}, a^{(k+1)l}, a^{(k+1)l+1}, a^{l+2}, a)$ of X. Clearly, C must have a chord, so, $a^{(k-1)l-3} \in S$, or $a^{(k+1)l-1} \in S$ or $a^{(k+1)l+1} \in S$. Since $(k-1)l-3 \le kl+1$ and $(k-1)l-3 \not\equiv \pm 1 \pmod l$, by induction hypothesis, we have $a^{(k-1)l-3} \notin S$. This says that either $a^{(k+1)l-1} \in S$ or $a^{(k+1)l+1} \in S$. If $a^{(k+1)l+1} \in S$ and $a^{(k+1)l-1} \notin S$, then the circuit $(1, a, a^2, \ldots, a^{l-2}, a^{(k+1)l-1}, a^{(k+1)l}, a^{(k+1)l+1)})$ is induced of length $l+2$, a contradiction. Thus $a^{(k+1)l-1} \in S$.

Now, to conclude the inductive argument, it remains to prove that $a^{(k+1)l+1}$ is in S. Let $t > 0$ be minimal such that $a^{(k+1)l-1+t} \in S$. If $t > 4$, then

$$(1, a, a^2, a^{(k+1)l+1}, a^{(k+1)l+2}, a^{(k+1)l+3}, a^{kl+4}, a^{kl+5}, a^{kl+6}, \ldots, a^{(k+1)l-1})$$

is an induced circuit of length $l+2$. Similarly, if $t = 4$, then the circuit $(1, a^{(k+1)l-1}, a^{(k+1)l}, a^{(k+1)l+1}, a^{(k+1)l+2}, a^{(k+1)l+3})$ is induced of length 6. This yields $t = 2$, and thus $a^{(k+1)l+1} \in S$. Claim 2.2.1 follows. ■

Now, we are ready to conclude the proof of Lemma 2.2. Since S is closed under taking inverses, Claim 2.2.1 and $l \ne 4$ imply that $n - (l-1) = kl+1$ for some integer k. Thus l divides n and the lemma follows. □

If X is a graph and $u \in V(X)$, then the *neighbours* of u, denoted $N(u)$, are the elements of $N(u) = \{v \in V(X) \mid \{u, v\} \in E(X)\}$. Vertices u, v of X are *twins* if $N(u) = N(v)$. Being twins is an equivalence relation on $V(X)$. We say that two vertices u, v of X are non-trivial twins if $u \ne v$ and u, v are twins. If $u \in V(X)$ then we denote by $X \backslash u$ the induced subgraph of X on $V(X) \setminus \{u\}$.

Lemma 2.3 *If X is a bipartite graph with non-trivial twins u and v, then X*

is balanced if and only if $X\backslash u$ is balanced.

PROOF. Induced subgraphs of balanced graphs are balanced, and thus $X\backslash u$ is balanced whenever X is balanced. Conversely, suppose $X\backslash u$ is balanced. In particular, as $N(u) = N(v)$, we have that $X\backslash v$ is also balanced. Let C be an induced circuit in X. If $u \notin V(C)$, or if $v \notin V(C)$, then C is isomorphic to an induced circuit in $X\backslash u$, or $X\backslash v$ (respectively), and thus $|V(C)| \equiv 0 \pmod 4$. If $u, v \in V(C)$ then, since C is induced and $N(u) = N(v)$, we have $|V(C)| = 4$. In either case every induced circuit of X has length $0 \pmod 4$, therefore X is balanced. □

Note that the complete bipartite graph $K_{t,t}$ is a $(2,t)$-cycle. In the following lemma, we prove that (l,t)-cycles are circulant graphs.

Lemma 2.4 *Let l, t be in $\mathbb{Z}^+$ with $l = 2$ or $l \equiv 0 \pmod 4$ and $l \geq 8$, A_{lt} be a cyclic group of order lt with generator a and $S = \{a^{il\pm1} \mid 0 \leq i \leq t-1\}$. The circulant graph $\mathrm{Cay}(A_{lt}, S)$ is isomorphic to an (l,t)-cycle.*

PROOF. We leave it to the reader to check that, for each $j \in \{0, \ldots, l-1\}$, the set $\{a^{il+j} \mid 0 \leq i \leq t-1\} = \langle a^l \rangle a^j$ is the twin class of a^j in $\mathrm{Cay}(A_{lt}, S)$. Furthermore, the quotient graph through the twin equivalence relation is a cycle of length l. Now, the result is straightforward. □

As a corollary of the results we have proved, we get the following theorem.

Theorem 2.5 *A circulant bipartite graph X with a generator in its connection set is balanced if and only if X is an (l,t)-cycle.*

PROOF. By Lemmas 2.1, 2.2 and 2.4, we have that every balanced circulant graph X with a generator in its connection set is an (l,t)-cycle. Conversely, with Lemma 2.3, it is easily verified that (l,t)-cycles are balanced. □

3 Balanced Cayley graphs on abelian groups

In this section Theorem 2.5 is used to prove Theorem 1.2.

Let G be an abelian group with $S \subseteq G$ such that $X = \mathrm{Cay}(G, S)$ is balanced. Choose $a \in S$. Recall that if $U \subseteq V(X)$, then $X[U]$ denotes the subgraph of X induced by U. In particular, we have $X[\langle a \rangle] = \mathrm{Cay}(\langle a \rangle, S \cap \langle a \rangle)$. Therefore $X[\langle a \rangle]$ is a circulant graph with a generator in its connection set. By Theorem 2.5, $X[\langle a \rangle]$ is an (l,t)-cycle. Moreover, by Lemmas 2.1 and 2.2, we have that either $l = 2$ or $l \equiv 0 \pmod 4$, $l \geq 8$, l divides $|a|$ and

$$S \cap \langle a \rangle = \{a^{il\pm1} \mid 0 \leq i \leq (|a|/l) - 1\}. \tag{1}$$

If $G = \langle a \rangle$, then Theorem 1.2 follows. Thus we assume $G \neq \langle a \rangle$.

For the remainder of Section 3, G, S, X, a and l are fixed, with the meaning defined in the preceding paragraph. Before going into the proof of Theorem 1.2, we would like to point out explicitly where the group G being abelian is used. We recall that $\mathrm{Aut}(X)$ contains the right regular representation of G, so, if $e \in E(X)$ and $g \in G$, then $eg = ge$ is an edge of X. Consequently, whenever H, Hb are two cosets of G and $b \in S \backslash H$, then there are "plenty" of edges between H and Hb, indeed $\{h, hb\}$ is an edge for any $h \in H$. In other words, any edge e from a vertex of H to a vertex of Hb determines a matching $\{he \mid h \in H\}$. Without the hypothesis of G being abelian we could only say that $\{eh \mid h \in H\}$ is a family of edges of X and no further "structure" on this family of edges or where these edges lie could be assumed.

Lemma 3.1 *If $b \in S \backslash \langle a \rangle$, then the subgraph of X induced by $\langle a \rangle \cup b\langle a \rangle$ is an $(l, 2t)$-cycle.*

Proof. We prove three claims, from which the lemma follows.

Claim 3.1.1 *If $i \equiv j \pmod{l}$ and $ba^i \in S$, then $ba^j \in S$.*

Assume towards a contradiction that $i \equiv j \pmod{l}$, $ba^i \in S$ and $ba^j \notin S$. Consider the subgraph of X induced by the union of $\{a^k \mid k \equiv 0 \pmod{l}\}$ and $\{ba^k \mid k \equiv i \pmod{l}\}$. Note that each vertex in this subgraph has degree equal to $|S \cap \{ba^k \mid k \equiv i \pmod{l}\}|$. In particular, since 1_G and a^{i-j} have the same degree and $\{1_G, ba^i\}$ is an edge but $\{a^{i-j}, ba^i\}$ is not, there exists $k \equiv i \pmod{l}$ such that $\{a^{i-j}, ba^k\}$ is an edge but $\{1_G, ba^k\}$ is not. Now, $j+1-k \equiv 1 \pmod{l}$, so, by Eq. 1, we have $a^{j+1-k} \in S$. Therefore, $(1_G, ba^i, ba^{j+1}, ba^k, a^{i-j}, a)$ is an induced circuit in X and it has length 6, a contradiction. Claim 3.1.1 follows. ■

Suppose $l = 2$, i.e. $X[\langle a \rangle]$ is a complete bipartite graph. Then $X[\langle a \rangle]$ is a $(2, t)$-cycle where $2t = |a|$, and $a^i \in S$ for all odd i. Since $b \in S$, Claim 3.1.1 yields $ba^j \in S$ for all even j. This yields

$$S \cap (\langle a \rangle \cup b\langle a \rangle) = \{a^i, ba^j \mid i \equiv 1 \pmod{2}, j \equiv 0 \pmod{2}\}. \tag{2}$$

We leave it to the reader to prove that Eq. 2 yields $X[\langle a \rangle \cup b\langle a \rangle]$ is a $(2, 2t)$-cycle.

Hence we assume that $l \geq 8$, i.e. $X[\langle a \rangle]$ is not a complete bipartite graph.

Claim 3.1.2 *If $ba^i \in S$, then $ba^{i+2} \in S$ or $ba^{i-2} \in S$.*

We argue by contradiction, so, assume $ba^{i-2}, ba^{i+2} \notin S$. The circuit $C = (1_G, a, a^2, ba^{i+2}, ba^{i+3}, \ldots, ba^{i+l-1}, ba^i)$ has length $l + 2$. So C is not induced. Using Eq. 1 to study the possible chords in C and applying Claim 3.1.1, we get that either $ba^{i-2} \in S$ or $ba^{i+m} \in S$ for some $2 \leq m \leq l-4$. Since $ba^{i-2}, ba^{i+2} \notin S$, we have $ba^{i+m} \in S$ for some $4 \leq m \leq l-4$. Let $m > 0$ be minimum with $ba^{i+m} \in S$. The circuit $(1_G, ba^i, ba^{i+1}, \ldots, ba^{i+m})$ is induced in X of length $m+2$. Therefore $m \equiv 2 \pmod 4$. Hence $6 \leq m \leq l-6$. The 10 - cycle $(1_G, ba^i, ba^{i+1}, ba^{i+2}, ba^{i+3}, a^3, ba^{i+m+3}, ba^{i+m+2}, ba^{i+m+1}, ba^{i+m})$ has only one possible chord, namely $\{1_G, ba^{i+m+2}\}$. Since X is balanced, we have $ba^{i+m+2} \in S$. Now, the circuit $(1, ba^{i+m+2}, a^2, ba^{i+2}, ba^{i+1}, ba^i)$ is induced in X and has length 6, a contradiction. ■

Since $b \in S$, Claim 3.1.2 says that either $ba^2 \in S$ or $ba^{-2} \in S$. Since the roles of a and a^{-1} are interchangeable in our arguments, we assume, throughout the rest of Lemma 3.1, that $ba^2 \in S$.

Claim 3.1.3 *If $ba^i \in S$, then $i \equiv 0 \pmod l$ or $i \equiv 2 \pmod l$.*

We argue by contradiction. So, let $i \geq 0$ such that $ba^i \in S$ and $i \not\equiv 0, 2 \pmod l$, in particular, pick such an i as small as possible. Claim 3.1.1 yields $4 \leq i \leq l-2$. Write $i = m+2$, so, $2 \leq m \leq l-4$. Since the circuit $(1_G, ba^2, ba^3, \ldots, ba^{2+m})$ is induced, we have $m \equiv 2 \pmod 4$, and thus $2 \leq m \leq l-6$.

The circuit $C = (1_G, ba^{2+m}, ba^{3+m}, \ldots, ba^{l-1}, b)$ of X has length $l - m \equiv 2 \pmod 4$. Therefore, C has a chord. It follows that there exists m' with $ba^{2+m+m'} \in S$ and $4 + m \leq 2 + m + m' \leq l-2$. Pick m' minimal with these properties.

By Claim 3.1.2, either $ba^{m+m'} \in S$ or $ba^{4+m+m'} \in S$. If $ba^{m+m'} \in S$, then $m' = 2$ by minimality of m'. However, when $m' = 2$, the circuit $(1_G, ba^2, ba^3, a^3, ba^{5+m}, ba^{4+m})$ is induced. This contradiction implies that $ba^{4+m+m'} \in S$ and $m' > 2$.

Claim 3.1.2 with $i = 2+m$ implies that either $ba^m \in S$ or $ba^{4+m} \in S$. Since $m' > 2$, the minimality of m' implies that $ba^{4+m} \notin S$. Thus $ba^m \in S$. If $m > 2$, this contradicts the minimality of i, so $m = 2$. In particular, $ba^{6+m'} \in S$.

Consider the circuit $C = (1_G, ba^2, ba^3, a^3, ba^{7+m'}, ba^{6+m'})$. Since $m = 2$, we have $2 \leq m' \leq l-6$. By Eq. 1, the only possible chord in C is $\{ba^2, ba^{7+m'}\}$ and thus $a^{m'+5} \in S$. Since $m'+5 \leq l-1$, Eq. 1 implies that $m'+5 = l-1$. Since $ba^{2+m+m'} \in S$, we have $ba^{l-2} \in S$. By Claim 3.1.1, $ba^{-2} \in S$ and thus the circuit $(1, ba^2, ba, a^3, ba^5, ba^4)$ is induced in X. This contradiction finally implies Claim 3.1.3. ■

Since $b, ba^2 \in S$, Claims 3.1.1 and 3.1.3 show that $ba^i \in S$ if and only if $i \equiv 0, 2 \pmod l$. Thus

$$S \cap (\langle a \rangle \cup b\langle a \rangle) = \{a^i, ba^j \mid i \equiv \pm 1 \pmod l, j \equiv 0, 2 \pmod l\}. \quad (3)$$

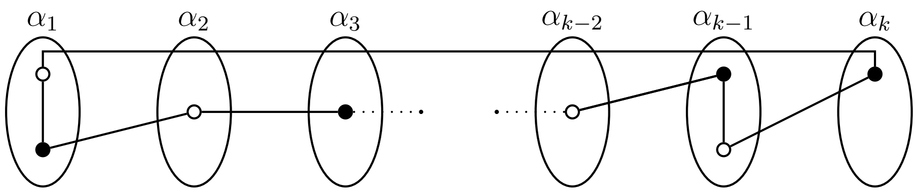
We leave it to the reader to check that Eq. 3 yields that $X[\langle a \rangle \cup b\langle a \rangle]$ is an

$(l, 2t)$-cycle (show that a^i and ba^j are twins in $X[\langle a\rangle \cup b\langle a\rangle]$ if and only if $j \equiv i+1 \pmod{l}$). The proof of Lemma 3.1 is complete. □

Proof of Theorem 1.2. Let $\widehat{X}$ denote the quotient graph $X/\langle a\rangle$. That is, $\widehat{X}$ has a vertex for each coset of $\langle a\rangle$, and the vertices $b_1\langle a\rangle$ and $b_2\langle a\rangle$ are adjacent in $\widehat{X}$ if and only if $b_1\langle a\rangle \cap b_2\langle a\rangle = \emptyset$ and there exists $u_1 \in b_1\langle a\rangle$ and $u_2 \in b_2\langle a\rangle$ with $\{u_1, u_2\} \in E(X)$.

Case 1: $l = 2$, i.e. $X[\langle a\rangle]$ is a complete bipartite graph.

Let $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ be an induced circuit in $\widehat{X}$. Note that by Lemma 3.1, $X[\alpha_i \cup \alpha_{i+1}]$ is a complete bipartite graph (indices $\bmod k$). If $k \geq 4$ then it is straightforward to exhibit an induced circuit of X of length $2 \pmod{4}$. For instance, if $k \equiv 0 \pmod{4}$ then pick a path of length 2 in $X[\alpha_1]$ and a path of length 2 in $X[\alpha_{k-1}]$ and extend it to an induced circuit of X using a unique vertex from each $X[\alpha_i]$, where $i \neq 1, k-1$.



All other cases are fairly similar. This proves that every induced circuit in $\widehat{X}$ has length 3, and thus $\widehat{X}$ is complete. Now, it is easy to check, using Lemma 3.1 and the fact that $\widehat{X}$ is complete, that X is isomorphic to a $(2, t|G|/|a|)$-cycle.

Case 2: $l \geq 8$.

We first prove a claim.

Claim 3.1.4 *The graph $\widehat{X}$ is complete. Further, let $\alpha_1, \alpha_2, \alpha_3$ be three distinct vertices of $\widehat{X}$. If $x_2 \in V(X[\alpha_2])$ has twins $x_1 \in V(X[\alpha_1])$ in $X[\alpha_1 \cup \alpha_2]$ and $x_3 \in V(X[\alpha_3])$ in $X[\alpha_2 \cup \alpha_3]$, then x_1 and x_3 are twins in $X[\alpha_1 \cup \alpha_3]$.*

Let $(\alpha_1, \alpha_2, \alpha_3)$ be a path of length 3 in $\widehat{X}$. For $i \in \{1, 2, 3\}$, choose $x_i \in V(X[\alpha_i])$ such that x_1 and x_2 are twins in $X[\alpha_1 \cup \alpha_2]$ and x_2 and x_3 are twins in $X[\alpha_2 \cup \alpha_3]$ (this is feasible because, by Lemma 3.1, the graphs $X[\alpha_1 \cup \alpha_2]$ and $X[\alpha_2 \cup \alpha_3]$ are $(l, 2t)$-cycles). Since the multiplication on the right by an element of $\langle a\rangle$ is an automorphism of $X[\alpha_1 \cup \alpha_2]$, we have that x_1a^j and x_2a^j are twins in $X[\alpha_1 \cup \alpha_2]$ for any j. Similarly, x_2a^j and x_3a^j are twins in $X[\alpha_2 \cup \alpha_3]$. This yields that $(x_1, x_1a, x_2a^2, x_3a, x_3, x_2a^{-1})$ is a circuit of length 6 in X, therefore it has a chord. Since $l \geq 8$, we have that $\{x_2a^2, x_2a^{-1}\} \notin E(X)$, therefore either $\{x_1, x_3a\}$ or $\{x_1a, x_3\}$ is an edge of X. Without loss of generality, we may assume that $\{x_1, x_3a\}$ is an edge (the role of a and a^{-1} is interchangeable). In particular, $\{\alpha_1, \alpha_3\} \in E(\widehat{X})$, and thus $\widehat{X}$ is complete.

Now, it remains to prove that x_1 and x_3 are twins in $X[\alpha_1 \cup \alpha_3]$. Since $\widehat{X}$ is complete, Lemma 3.1 yields that $X[\alpha_1 \cup \alpha_3]$ is an $(l, 2t)$-cycle. Moreover, since $\{x_1, x_3a\} \in E(X)$, we have that either x_1 is a twin with x_3 in $X[\alpha_1 \cup \alpha_3]$ or x_1 is a twin with x_3a^2 in $X[\alpha_1 \cup \alpha_3]$. In the latter case, $(x_1, x_1a, x_1a^2, \ldots, x_1a^{l-5}, x_3a^{l-2}, x_2a^{l-1})$ is an induced circuit of length $l-2$, and by this contradiction, x_1 and x_3 are twins in $X[\alpha_1 \cup \alpha_3]$. The claim is proved. ■

Now, we leave it to the reader to check that Lemma 3.1 and Claim 3.1.4 yield that X is a $(l, t|G|/|a|)$-cycle. The proof of Theorem 1.2 is complete. □

4 The number of vertices in a regular balanced graph

The graph $X = \mathrm{Cay}(G, S)$ is regular with degree $|S|$. Theorem 1.2 implies that if X is balanced and G is abelian, then $|G|$ is divisible by $2|S|$. We prove Theorem 1.3, which states that this divisibility criterion holds for all regular balanced graphs. To do this we need first some terminology and a well-known result in linear programming on balanced matrices.

We recall that if A is an $n \times m$ $\{0,1\}$-matrix, then the set partitioning polytope defined by A is the set $R(A) = \{x \in \mathbb{R}^m \mid Ax = \mathbf{1}_n, \mathbf{0}_m \leq x \leq \mathbf{1}_m\}$, where $\mathbf{1}_n$, respectively $\mathbf{1}_m$, denotes a column vector of length n, respectively m, whose entries are all equal to 1 and $\mathbf{0}_m$ denotes the zero vector of length m. Note that $R(A)$ is a convex polytope. A set partitioning polytope is said to be *integral* if all its vertices (i.e. extremal points) have only integer-valued components.

The following characterization of balanced matrices is due to Berge [3] and Fulkerson, Hoffman, and Oppenheim [7].

Theorem 4.1 (Berge [3], Fulkerson, Hoffman, and Oppenheim [7]) *If A is a balanced matrix, then $R(A)$ is integral.*

We note that a proof of Theorem 4.1 is also in [5] (see Theorem 2.1), where a more general result is proved. Now, Theorem 1.3 is a corollary of Theorem 4.1.

PROOF OF THEOREM 1.3. Let X be a k-regular balanced graph and A be the bipartite adjacency matrix of X. Since X is regular, A is a square matrix of size $n = \frac{1}{2}|V(X)|$. Since the vector $\frac{1}{k}\mathbf{1}_n$ lies in the set partitioning polytope $R(A)$, we have $R(A) \neq \emptyset$. Thus, by Theorem 4.1, there exists a vertex x of $R(A)$ with integer-valued components. As x lies in $R(A)$, we have $\mathbf{0}_n \leq x \leq \mathbf{1}_n$, so, x has components in $\{0,1\}$. Set $t = \sum_{i=1}^{n} x_i$. The equation $Ax = \mathbf{1}_n$ and the fact that X is k-regular yield $tk = n = 1/2|V(X)|$. The theorem follows. □

5 Cubic balanced planar graphs

In this section we deal with cubic planar graphs and we prove Theorem 1.4.

Batagelj [2] proved that all 3-connected cubic bipartite planar graphs can be obtained from the cube by a succession of two elementary operations. The first operation is called *diamond inflation* of a vertex and it replaces a vertex with a "diamond", see Figure 5.1(a). The second operation is called A_1 *subdivision* and it applies to a pair of non-adjacent edges $\{u, v\}$, $\{w, z\}$, see Figure 5.1(b).

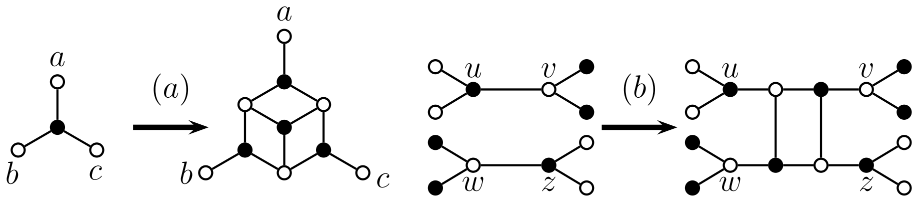


Fig. 5.1. (a) diamond inflation, (b) A_1 subdivision

We state Batagelj's theorem precisely.

Theorem 5.1 (Batagelj [2]) *Every 3-connected cubic bipartite planar graph can be obtained from the cube by a succession of diamond inflations and A_1 subdivisions. The operations can be chosen such that the intermediate graphs are planar and bipartite.*

Lemma 5.2 *Let X be a cubic bipartite planar graph with connectivity $\kappa(X) = 2$. There exists a 2-vertex cut $\{u, v\}$ such that $X \backslash \{u, v\}$ has a component Y with the following property: there exist two nonadjacent vertices a, b in Y such that $\overline{Y} = (V(Y), E(Y) \cup \{ab\})$ is a cubic 3-connected bipartite planar graph.*

Proof. Choose a 2-vertex cut-set $\{a, b\}$ so as to minimise the order of one of the resulting components, C. If a has a single neighbour a' in C, then $\{a', b\}$ is a 2-vertex cut-set that contradicts the minimality of C; similarly for b. So each of a and b has two neighbours in C. A counting argument, using the regularity and the bipartition of the graph X, forces a and b to have opposite colours, so the neighbours of a and b are all distinct. Further, a and b are not adjacent, since there must be another component to which at least one of them is joined by an edge, and each has only three neighbours.

We claim that the choice of Y to be the induced subgraph on $V(C) \cup \{a, b\}$ (with the neighbours of a and b that are not in C as the 2-vertex cut-set) satisfies the claims of this lemma. The minimality of C is sufficient to ensure that $\overline{Y}$ is 3-connected, and it is clearly cubic, bipartite and planar. $\square$

Let X be a 3-connected cubic bipartite planar graph, v be a vertex of X and w_1, w_2, w_3 be the three neighbours of v. Let $p_i = (w_i, w_{i1}, \ldots, w_{in_i}, w_{i+1})$ be the path in X from w_i to w_{i+1} (indices $\mod 3$) such that the circuit (v, p_i) is the boundary of a face of X, see Fig. 5.2. Let us denote by S_v the subgraph of X with vertex set $\{v, w_1, w_2, w_3, w_{ij} \mid 1 \le i \le 3, 1 \le j \le n_i\}$ and edge set $\{vw_1, vw_2, vw_3, w_iw_{i1}, w_{in_i}w_{i+1}, w_{ij}w_{i(j+1)} \mid 1 \le i \le 3, 1 \le j \le n_i - 1\}$, i.e. S_v is the graph in Fig. 5.2(a). For instance, the bold edges in Fig. 5.2(b) show the subgraph S_v for the cube.

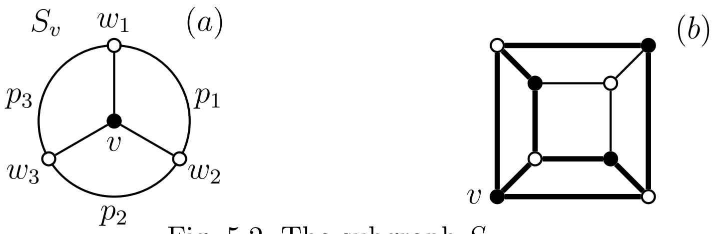


Fig. 5.2. The subgraph S_v

Claim 5.2.1 *S_v is an induced subgraph of X.*

Proof. This is true for the cube. Now use Theorem 5.1 and induction. ∎

Claim 5.2.2 $\cap_{v \in V(X)} E(S_v) = \emptyset$.

Proof. This is readily true for the cube. Now use Theorem 5.1 and induction to conclude. ∎

If $X = (V, E)$ is a graph and $e \in E$, then we denote by $X \backslash e$ the graph $(V, E \backslash \{e\})$.

Lemma 5.3 *Let X be a 3-connected cubic bipartite planar graph. Then X is unbalanced. If e is an edge of X, then $X \backslash e$ is unbalanced.*

Proof. Let v be a vertex of a 3-connected cubic bipartite planar graph X. We claim that S_v is unbalanced. If the path p_i has length $0 \pmod 4$, for some $i \in \{1, 2, 3\}$, then the circuit (v, p_i) is induced and has length $2 \pmod 4$, so S_v is unbalanced. Finally, if p_i has length $2 \pmod 4$, for every i, then the circuit (p_1, p_2, p_3) is induced in S_v and has length $2 \pmod 4$, so S_v is unbalanced. Thus our claim is proved.

Now, by Claim 5.2.1, S_v is an induced subgraph of X. Therefore, X is unbalanced.

Let e be an edge of X. By Claim 5.2.2, there exists $v \in V(X)$ such that $e \notin E(S_v)$. Now, by Claim 5.2.1, S_v is an induced subgraph of $X \backslash e$. Therefore, $X \backslash e$ is unbalanced. □

Proof of Theorem 1.4. Let X be a cubic bipartite planar graph. We have to prove that X is unbalanced. By Lemma 5.3, we may assume that

$\kappa(X) = 2$. By Lemma 5.2, there exist nonadjacent a, b in $V(X)$ and Y an induced subgraph of X such that $\overline{Y} = (V(Y), E(Y) \cup \{e\})$ is a 3-connected bipartite planar graph, where $e = ab$. Now, by Lemma 5.3, $Y = \overline{Y} \backslash e$ is unbalanced. So, X is unbalanced. The proof of Theorem 1.4 is complete. $\square$

6 Conjectures

The conjectures presented here are supported by computer searches performed with the invaluable help of GAP [8], including an exhaustive analysis of the graphs from Gordon Royle's web page.

Conjecture 6.1 *If X is a connected vertex-transitive balanced graph, then X is an (l, t)-cycle.*

The truth of Conjecture 6.1 would imply that the only connected vertex-transitive balanced graphs of odd degree are the complete bipartite graphs.

Every cubic balanced graph known to the authors has non-trivial twins. So, we present the following conjecture which has been verified for all graphs with fewer than 54 vertices.

Conjecture 6.2 *If X is a cubic balanced graph, then X has non-trivial twins.*

The truth of Conjecture 6.2 might shed some new light on the graph structure of a cubic balanced graph. For example the following conjectures, interesting in their own right, are implied by the validity of Conjecture 6.2.

(i) $K_{3,3}$ is the only connected vertex-transitive cubic balanced graph;
(ii) every cubic balanced graph has girth four;
(iii) Conforti-Rao conjecture is true for cubic balanced graphs, see [6] page 54.

We leave it as an exercise to the reader to prove that Conjecture 6.2 yields $(i), (ii)$ and (iii). As a matter of curiosity we point out that the truth of Conjecture 6.2 would also yield Theorem 1.4. Indeed, it is easy to show that cubic bipartite graphs with non-trivial twins are not planar graphs.

We conclude by reporting the number of "small" cubic balanced graphs; $f(d)$ denotes the number of connected cubic balanced graphs on d vertices. [3]

d	6	12	18	24	30	36
$f(d)$	1	1	4	13	74	527

[3] for the graphs corresponding to the values of d contact the second author

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