Connectivity of Cayley Graphs: A Special Family

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1 Introduction

Taking any finite group G, let $H \subset G$ be such that $1 \notin H$ (where 1 represents the identity element of G) and $h \in H$ implies $h^{-1} \in H$. The *Cayley graph* $X(G; H)$ is the graph whose vertices are labelled with the elements of G, in which there is an edge between two vertices g and gh if and only if $h \in H$. The exclusion of 1 from H eliminates the possibility of loops in the graph. The inclusion of the inverse of any element which is itself in H means that an edge is in the graph regardless of which endvertex is considered.

It has been suggested that Cayley graphs should form good networks and several papers have been written about their fault tolerance. In particular, B. Alspach [2] proves certain results about the fault tolerance of a particular class of Cayley graphs, and isolates one family of graphs with interesting properties. The purpose of this paper is to exhibit some characteristics of graphs in this family.

The *fault tolerance* of a graph is defined to be the largest number of vertices whose deletion cannot disconnect the graph. The *connectivity* of a graph is the smallest number of vertices whose deletion disconnects the graph. For a graph X, $\kappa(X)$ denotes its connectivity. It is easy to see that the connectivity of a non-complete graph is always one more than its fault tolerance. The term connectivity will be used in this paper.

An *automorphism* of a graph X is a permutation σ of the vertices of X with the property that if u and v are vertices of X, then there is an edge

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from u to v if and only if there is an edge from $\sigma(u)$ to $\sigma(v)$. In the case of a Cayley graph $X(G;H)$, left-multiplication by any element of G is clearly an automorphism. A graph is said to be *vertex-transitive* if for all vertices u and v, there is an automorphism which sends u to v. Clearly, Cayley graphs have this property. Furthermore, any vertex-transitive graph X must be regular, of degree k, and since removing all vertices adjacent with a particular vertex disconnects a graph, then $\kappa(X) \leq k$ and if $\kappa(X) = k$ we say X has *optimal connectivity*. It is of interest to determine which graphs are optimal in this respect.

A *minimum cutset* of a graph X is a set C, $|C| = \kappa(X)$, of vertices whose deletion disconnects the graph. If $c(Y)$ is the size of a smallest component in the graph Y, then $A \subset X$ is an *atom* of X if $|A| = \min\{c(X - C) : C$ is a minimum cutset of $X\}$ and there exists a $C \subset V(X)$ such that A is a component of $X - C$.

M. Watkins obtained the following results regarding cutsets and atoms in vertex-transitive graphs.

Theorem 1.1 (Watkins, 1968 [5]). *In a connected vertex-transitive graph, distinct atoms are vertex-disjoint.*

Theorem 1.2 (Watkins, 1968 [5]). *If A_1 and A_2 are sets of vertices forming distinct atoms in a vertex-transitive graph, and A_1 has associated cutset C_1, then either $A_2 \subset C_1$, or A_2 and C_1 are disjoint.*

He also proves the following simple corollary to these theorems.

Corollary 1.3 (Watkins, 1968 [5]). *The size of the atoms of a vertex-transitive graph must divide the connectivity of the graph.*

2 Quasi-Minimal Generating Sets

Again, let us take G to be a finite group, and $H \subset G$. We take $\langle H \rangle$ to be the subgroup of G generated by H. We say H is a *minimal generating set* for G if $\langle H \rangle = G$ and every proper subset of H generates a proper subgroup of G. If H is a minimal generating set for G, then $h \in H$ implies $h^{-1} \notin H$. The definition is therefore modified to accommodate Cayley graphs. A Cayley set H of the group G is *minimal* if it generates G, and $H \setminus \{h, h^{-1}\}$ is a proper subgroup of G for all $h \in H$. The following theorem was proved in 1981.

Theorem 2.1 (Godsil, 1981 [4]). *If H is a minimal Cayley set for the finite group G, then the Cayley graph $X(G;H)$ has optimal connectivity.*

The question became, could this result be extended to embrace other classes of Cayley graphs? As may be deduced from the name, quasi-minimal generating sets (first defined by Babai [3]) bear some resemblance to minimal generating sets, and so were a logical class to try.

The set H is said to be a *quasi-minimal Cayley set* if it generates G, and if its elements can be ordered in such a way that the following hold.

1. If $h_i \in H$, then h_i^{-1} is either h_{i-1}, h_i, or h_{i+1}.
2. If H_i denotes the set $\{h_1, h_2, \ldots, h_i\}$, then for each i such that h_i has order 2, $\langle H_{i-1} \rangle$ is a proper subgroup of $\langle H_i \rangle$. For each i such that h_i has order greater than 2, and $h_i^{-1} = h_{i-1}$, $\langle H_{i-2} \rangle$ is a proper subgroup of $\langle H_i \rangle$.

Using results achieved by M. Watkins [5], B. Alspach was able to prove the following theorem regarding Cayley graphs with quasi-minimal generating sets.

Theorem 2.2 (Alspach, 1992 [2]). *If H is a quasi-minimal Cayley generating set of the finite group G, then the Cayley graph $X = X(G;H)$ has connectivity $|H|$, or connectivity $|H|-1$ and atoms of size 2.*

Let $\mathcal{F}$ be the family of all Cayley graphs $X = X(G;H)$ where H is a quasi-minimal generating set for G, $\kappa(X) = |H|-1$, and X has atoms of size 2. From Theorem 2.2 and Corollary 1.3, it follows clearly that any graph $X(G;H) \in \mathcal{F}$ has odd degree; that is, $|H| = 2n+1$ for some natural number n. The proof of Theorem 2.2, which is by induction, further shows that any graph in this family of degree $2n+1$ must be composed of copies of a graph $X'(G';H') \in \mathcal{F}$, which has degree $2n-1$, together with edges connecting them.

It was also shown in [2] that graphs in $\mathcal{F}$ must have degree at least 5, and that the only graphs of degree 5 in $\mathcal{F}$ are of the form $C_{2k} \wr K_2$, where C_{2k} denotes the cycle of length $2k$, and $\wr$ denotes the wreath (or lexicographic) product, as defined in [2]. In fact, we may generalize the result as follows.

Theorem 2.3 *If $X \in \mathcal{F}$ has degree $2n+1$, $n \geq 2$, then X is isomorphic to $Y \wr K_2$ where Y is vertex-transitive and the degree of Y is n.*

PROOF. From Theorem 2.2, every atom contains 2 vertices, both of which have degree $2n+1$ and the connectivity of the graph is $2n$. Thus for each atom there are precisely $2n$ vertices adjacent to both vertices of the atom, and the atom is K_2. Since X is vertex-transitive, every vertex must be part of some atom, and from Theorems 1.1 and 1.2, the $2n$ vertices which form the cutset for an atom must themselves fully comprise n disjoint atoms. This shows that X is isomorphic to $Y \wr K_2$ for some graph Y with degree n. Since any atom must be sent to another atom under any automorphism of the graph X and X is vertex-transitive, the result follows.■

It was further shown in [2] that the generating sets for graphs in $\mathcal{F}$ must be of the form $H = \{h_1, h_2, h_2^{-1}, \ldots, h_{n+1}, h_{n+1}^{-1}\}$, where h_1 has order 2, and $h^2 = h_1$ for all $h \in H \setminus \{h_1\}$ The edges of the atom arise from h_1, and h_j and h_j^{-1} result in edges joining pairs of atoms (and thus yielding K_4s). Thus each new pair of elements joins together at least two disjoint subgraphs of degree $2i-1$ to form subgraphs of degree $2i+1$. Graphs in $\mathcal{F}$ are in fact precisely those Cayley graphs with a quasi-minimal generating set of this sort.

What remained to be determined, after the work done in [2], was whether or not the form in which edges connect the graphs in $\mathcal{F}$ of degree $2n-1$ referred to previously, could be determined. The object of this paper is to exhibit some results along these lines which indicate that such a characterization is by no means a simple one. Even for graphs in $\mathcal{F}$ which have degree 7, no simple characterization exists.

3 The Quotient Graph

There is a slightly simpler way of looking at graphs in $\mathcal{F}$. In Theorem 2.3, it was shown that every graph $X(G; H)$ in $\mathcal{F}$ is of the form $Y \wr K_2$, for some vertex-transitive graph Y with degree $\frac{|H|-1}{2}$. We define this Y to be the *quotient graph* of X.

Notice that the element h_1 commutes with every element in the group G. Therefore, $\{1, h_1\}$ is a normal subgroup of G. Now consider the group $G' = G/\{1, h_1\}$, together with the subset $H' = \{h_i\{1, h_1\}\}, 2 \leq i \leq \frac{|H|+1}{2}$. Construct the graph $X(G'; H')$ which is easily seen to be Y. The elements of H' with ordering inherited in the obvious way from H are all of order 2, and must form a quasi-minimal (and perhaps even a minimal) generating set for G'. Thus, the quotient graph for any graph in $\mathcal{F}$ is itself a Cayley graph with a minimal or quasi-minimal generating set, composed entirely of elements of order 2.

Theorem 3.1 *Given a Cayley graph $Y = X(G'; H')$ of degree d, where H' is a minimal or quasi-minimal generating set for G', and $h^2 = 1$ for all $h \in H'$, Y is the quotient graph for some graph $X(G; H) \in \mathcal{F}$. In other words, $Y \wr K_2 \in \mathcal{F}$.*

Proof. If $H' = \{h_2, h_3, \ldots, h_{d+1}\}$, then take an element $h_1 \notin G'$, define $h_1^2 = 1$, and let $H = \{h_1, h_2, h_1h_2, \ldots, h_{d+1}, h_1h_{d+1}\}$, where $h_i^2 = h_1$, $2 \leq i \leq d+1$ and all other relations are inherited from G'. Define $G = \langle H \rangle$. One clearly sees that G is the union of G' and h_1G'. Then H as ordered is a quasi-minimal Cayley generating set for G (note that $h_i^{-1} = h_1h_i$). Also, $X(G; H) \in \mathcal{F}$ since H has the required form as mentioned earlier. Finally, as $\{1, h_1\}$ is normal in G, $G/\{1, h_1\}$ is isomorphic to G' and hence Y is the quotient graph of $X(G; H)$.$\blacksquare$

In the case of a graph X in $\mathcal{F}$ with degree 5, finding an appropriate quotient graph from which to build X means finding a group with a 2-element quasi-minimal generating set in which both elements have order 2. The corresponding Cayley graphs are precisely the cycles of even length and the group is dihedral. In the case of a graph X in $\mathcal{F}$ with degree 7, the problem of finding an appropriate quotient graph is that of finding a group with a 3-element quasi-minimal generating set in which all three elements have order 2.

These conditions are both necessary and sufficient to find a graph in $\mathcal{F}$ (every graph in $\mathcal{F}$ has a quotient graph based on such a group, and every such group forms the quotient graph for a graph in $\mathcal{F}$).

4 Graphs in $\mathcal{F}$ with Degree 7

Certain necessary conditions for a graph $X(G; H)$ of degree 7 to belong to $\mathcal{F}$ are obvious. We know that atoms have size 2 and the local subgraph around any atom must be as shown in Figure 1.

The *underlying graph*, Y, of $X(G; H) \in \mathcal{F}$ is defined as follows: Y is the unique graph obtained from $X(G; H)$ by replacing each of the disjoint $C_{2k} \wr K_2$s isomorphic to $X(G; H \setminus \{h_4, h_4^{-1}\})$, with a single vertex, and joining two vertices of Y if and only if the corresponding $C_{2k} \wr K_2$s had edges connecting them in X. Note that Y may be an underlying graph for infinitely many distinct graphs in $\mathcal{F}$, simply by varying k. Further, left-multiplication by any element of G clearly moves these $C_{2k} \wr K_2$s as blocks. Since left-multiplication by some element of G will send any one of these blocks to any other, Y is vertex-transitive.

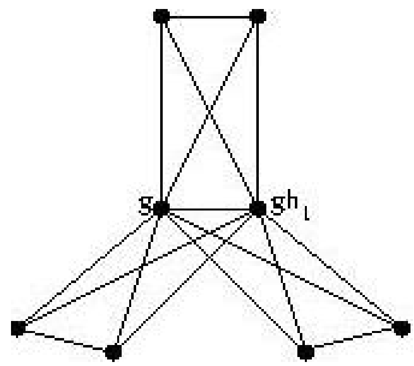

Figure 1: The local subgraph around an atom

Note that Y need not be a Cayley graph. The following example has the Petersen graph as its underlying graph, yet has a quasi-minimal generating set of the required form. Let $H = \{h_1, h_2, h_2^{-1}, h_3, h_3^{-1}, h_4, h_4^{-1}\}$, where $h_1^2 = h_2^4 = h_3^4 = h_4^4 = 1$, and $h_2^2 = h_3^2 = h_4^2 = h_1$. Let $G = \langle H \rangle$ as usual. Also, assume $(h_2h_3)^3 = 1$, $(h_2h_4)^5 = (h_3h_4)^5 = 1$, $(h_2h_3h_2h_4)^2 = 1$, and $(h_ih_jh_k)^5 = 1$, $\{i,j,k\} = \{2,3,4\}$. Factoring out h_1 for simplicity, and feeding these relations into the "grelgroup" function on Maple, yields confirmation that the factor group has order 60, so G has order 120. The graph $X(G;H)$ defined by this group is in $\mathcal{F}$, yet the Petersen graph, which underlies X, has been proven to be non-Cayley. The corresponding quotient graph is shown in Figure 2. Vertex-transitivity is not a sufficient condition for a graph Y to be the underlying graph of a graph of degree 7 in $\mathcal{F}$. For example, the graph corresponding to the edges of the prism (see Figure 3), underlies no graph of degree 7 in $\mathcal{F}$. A stronger necessary property, which this graph lacks, will now be established.

Theorem 4.1 *Let Y be the underlying graph of $X(G;H) \in \mathcal{F}$, where $|H| = 7$. Then Y has the property that, for any vertex v with neighbours u and w, there is an automorphism $\sigma \in Aut(Y)$ such that $\sigma(v) = v$, and $\sigma(u) = w$.*

Proof. Let $H = \{h_1, h_2, h_2^{-1}, h_3, h_3^{-1}, h_4, h_4^{-1}\}$. Since the $C_{2k} \wr K_2$s in $X(G, H \setminus \{h_4, h_4^{-1}\})$ are blocks of imprimitivity of G acting on X by left-multiplication, every such automorphism of X induces an automorphism of Y. Consider vertices u, v and w in Y, where v is adjacent to both u and w, and their corresponding blocks are B_u, B_v and B_w in X. Let A_1 be an atom in B_v which is connected to B_u. We may assume that one of the vertices in this atom is the identity element 1 since X is vertex-transitive.

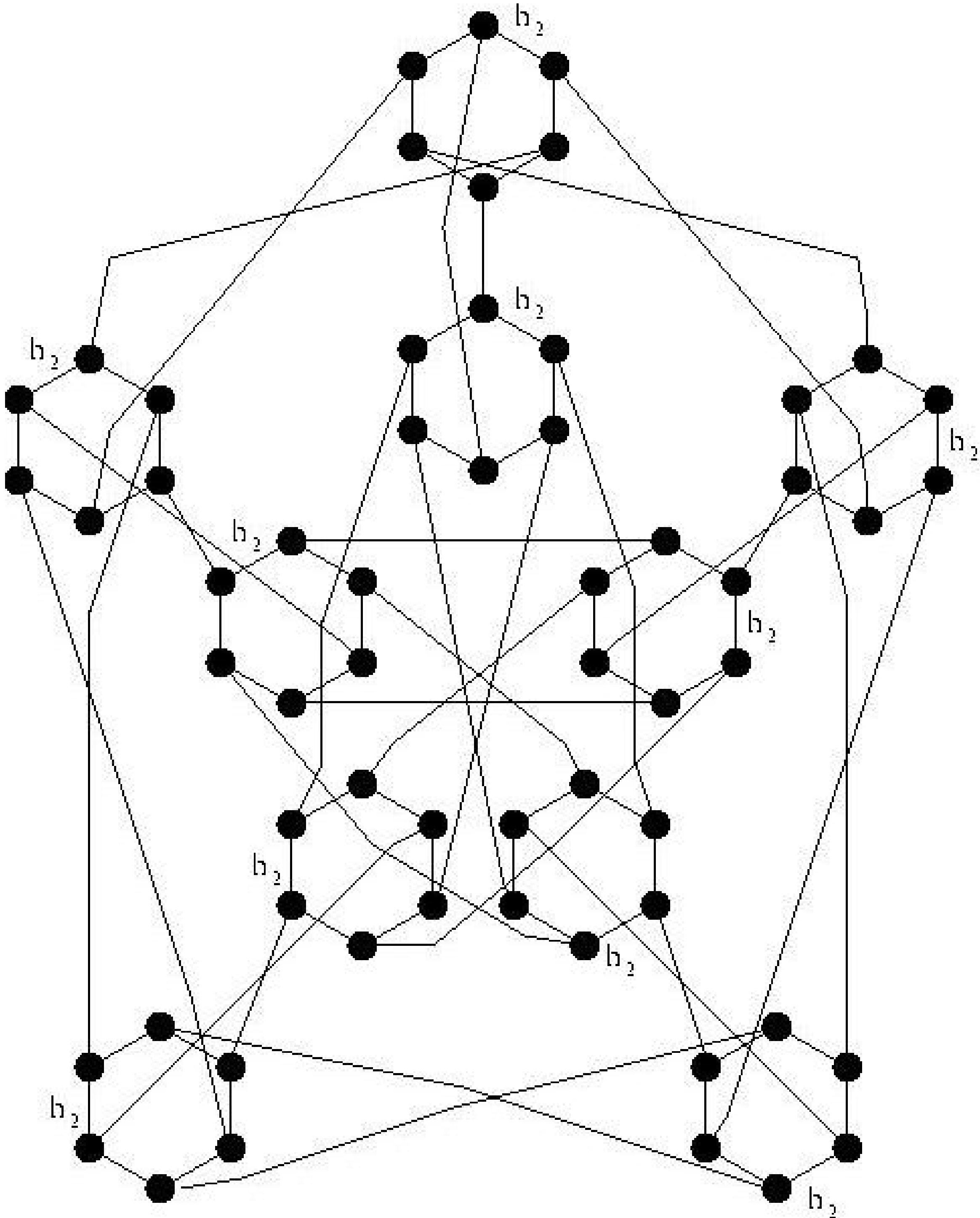

Figure 2: The quotient graph of a graph in $\mathcal{F}$ with the Petersen graph underlying

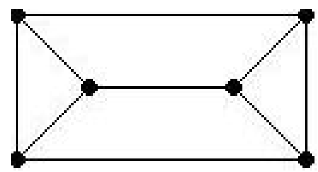

Figure 3: A vertex-transitive graph which is not arc-transitive

The other element must then be h_1. Now consider an atom A_2 in B_v which is connected to B_w. Let one of the vertices in A_2 be labelled g, $g \in G$. Then left-multiplication by g is an automorphism sending A_1 to A_2. Under this automorphism B_v is fixed and B_u is mapped to B_w. This induces an automorphism σ on Y which fulfils the required conditions. ∎

This leads immediately to

Corollary 4.2 *The underlying graph Y of a graph in $\mathcal{F}$ is arc-transitive.*

Proof. Simply use Theorem 4.1 and the fact that Y is vertex-transitive. ∎

The fact that a graph is arc-transitive, however, is still not sufficient to show that it underlies a graph of degree 7 in $\mathcal{F}$. We will now demonstrate that K_5 underlies no graph of degree 7 in $\mathcal{F}$.

Suppose it did, and let $X = X(G; H)$ be the quotient graph of some such graph in $\mathcal{F}$. Since K_5 is the underlying graph, there are five $C_{2k} \wr K_2$s in the original graph, corresponding to five C_{2k}s in X. Call these cycles $B_1, \ldots, B_5$. Now, B_i is connected to each of the other cycles by an edge since the underlying graph is K_5. By the argument of Theorem 4.1, any pair of these cycles can be moved via an automorphism to any other pair, so B_i must have some constant number j edges connecting it to each of the other cycles, meaning that B_i is in fact a C_{4j} for some $j \geq 1$.

We first deal with the case $j = 1$. Now, $H = \{h_2, h_3, h_4\}$, $h_2^2 = h_3^2 = h_4^2 = 1$ and G is generated by H. The elements h_2 and h_3 together generate a 4-cycle since $j = 1$, and the element h_4 interconnects the 4-cycles. Since the underlying graph is K_5, there are five of the 4-cycles, which we may denote B_1, B_2, B_3, B_4, B_5 where B_1 contains the identity element. Again, since the underlying graph is K_5, we may assume without loss of generality that B_2 contains h_4, B_3 contains h_2h_4, B_4 contains $h_2h_3h_4 = h_3h_2h_4$, and B_5 contains h_3h_4. Now we consider the order of h_2h_4. Since G has order 20, h_2h_4 must have order 2, 4, 5 or 10 (since X has only 10 edges corresponding to right-multiplication by h_4, the longest cycle in X consisting of edges corresponding to h_2 and h_4 alternately, cannot have more than 20 edges, meaning the order of h_2h_4 is at most 10). If h_2h_4 has order 2, then $h_2h_4h_2 = h_4$ must be in both B_2 and B_3 since h_2h_4 is in B_3, a contradiction. Suppose then that the order of h_2h_4 is 4. Consider the 8-cycles generated by right-multiplying alternately by h_2 and h_4 from any starting point. These are clearly disjoint and include all edges corresponding to h_4, but there are 10 such edges and 4 are in each such cycle, a contradiction. Now suppose h_2h_4 has order 10. Then the edges corresponding to h_2 and h_4 must form a single cycle of length 20. Since each B_i contains two edges corresponding to h_2, if we follow this cycle from any element, there is a first point at which we return to some B_i which we have already visited, and this first point must occur after no more than 5 steps (by h_2h_4) since there are only 5 blocks. Due to transitivity, we may assume without loss of generality that the first repeated block is B_1 and that we first visited the identity, so we obtain $(h_2h_4)^k = h_3$ or $(h_2h_4)^k = h_2h_3$ for some $k \leq 5$. Now multiply by h_2h_4 again. We end up in either B_4 or B_5 by definition; however, beginning from h_2h_4 in B_3 and multiplying by $(h_2h_4)^k$, we get that the same element, $(h_2h_4)^{k+1}$ must be either $h_2h_4h_3$ or $h_2h_4h_2h_3$, each of which is in B_3, a contradiction. Precisely the same arguments can be carried out interchanging the roles of h_2 and h_3, so we are left with the only remaining case, that h_2h_4 and h_3h_4 both have order 5. It can be quickly verified that there is no way of connecting the blocks so that this is the case.

We may now suppose $j > 1$ and there is a graph of degree 7 in $\mathcal{F}$ with K_5 as its underlying graph. Again we consider only the quotient graph, $X(G; H)$. Again H is as above, and G is generated by the three involutions which comprise H. Let G' denote the group generated by h_2 and h_3, which forms the cycles of length $4j$ in X. Since the underlying graph is K_5, the index of G' in G is 5. If G' were normal in G, we would have the order of G/G' being 5, but h_4G' would be an element of order 2, a contradiction. So G' is not normal in G. We now mod out the largest normal subgroup of G which is contained in G'. In the graph, we may have identified some vertices and so caused multiple edges, but we still have the 5 blocks which were originally cycles, and since each block still has an edge leading to each other block, these blocks each have at least 4 distinct elements. Due to this method of reduction, we may assume without loss of generality that in the original groups, G' contains no nontrivial normal subgroups of G. We split the proof into two cases.

Case 1. G is not solvable.

The action of G on the cosets of G' determines a nontrivial homomorphism from G to S_5. Because G' contains no normal subgroup of G and the kernel of the homomorphism is contained in G' the homomorphism is faithful, meaning the image of the homomorphism is isomorphic to G. So the image of the homomorphism must be either S_5 or A_5 since any group of order less than 60 is solvable. But neither S_5 nor A_5 has a dihedral subgroup of index 5, as G has (G'), so this case cannot occur.

Case 2. G is solvable.

Let N be a minimal normal subgroup of G. Since G is solvable, we know that N is an elementary abelian p-group for some prime p. Since $N \not\subset G'$, we have $G'N$ is a group which strictly contains G', but since G' has index 5 in G, $G'N$ must be G itself. Hence we must have $p = 5$. Also, the intersection of G' with N is normal in both G' (clearly) and N (since N is abelian). Thus this intersection is normal in $G'N = G$, and being contained in G' must be the identity. This forces N to be the cyclic group on five elements. Consider the homomorphism ψ from G to the group of automorphisms of N defined by $\psi(g)(n) = gng^{-1}$. Its kernel is $\{g \in G : gn = ng \text{ for all } n \in N\}$. The group of automorphisms of N is easily seen to be abelian now that we know what N is, so if we take any element of the form $g_1^{-1}g_2^{-1}g_1g_2$ where g_1 and g_2 are in G', we find that its image under ψ is the identity since ψ is a homomorphism and its image is abelian. The group of all such elements in G' (the commutator subgroup of G') is hence normal in G since it is normal in G' and commutes with every element of N. But this means that it must be the identity, and so $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G'$. Notice that if h_2 were in any normal subgroup G'' of G which was contained in G' and hence reduced to the identity of the quotient group we considered, G'/G'' would consist of G'' and the single coset h_3G'', a contradiction since we must be left with a group of order at least 4. Thus, G' is still generated by two involutions, and G' is abelian. The only possibility is that G' has order 4, but this is precisely the case we dealt with initially.

Thus, K_5 underlies no graph of degree 7 in $\mathcal{F}$, showing that edge-transitivity is not a sufficient condition for an arbitrary graph to underlie a graph of degree 7 from $\mathcal{F}$.

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