AUTOMORPHISM GROUPS OF WREATH PRODUCT DIGRAPHS

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Abstract. We strengthen a classical result of Sabidussi giving a necessary and sufficient condition on two graphs, X and Y, for the automorphsim group of the wreath product of the graphs, $\mathrm{Aut}(X \wr Y)$ to be the wreath product of the automorphism groups $\mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$. We also generalize this to arrive at a similar condition on color digraphs.

The main purpose of this paper is to revisit a well-known and important result of Sabidussi [17] giving a necessary and sufficient condition for the wreath product $X \wr Y$ (defined below) of two graphs X and Y to have automorphism group $\mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$, the wreath product of the automorphism group of X and the automorphism group of Y (defined below). We will both strengthen Sabidussi's result and generalize it. First, Sabidussi only considered almost locally finite graphs X and finite graphs Y. (A graph is almost locally finite if the set of vertices of infinite degree is finite.) The condition that X be almost locally finite is needed for Sabidussi's proof, but is clearly not needed in general. Indeed, note that $\overline{X \wr Y}$, the complement of $X \wr Y$, has the same automorphism group as $X \wr Y$, $\overline{X \wr Y} = \bar{X} \wr \bar{Y}$, but $\bar{X}$ is not almost locally finite if X is infinite and almost locally finite. We will show that no restriction on X whatsoever is needed. We also weaken the requirement on Y: rather than requiring Y to be finite, we only require that Y not be isomorphic to a proper induced subgraph of itself.

Next, since Sabidussi published his original paper, the wreath product of digraphs and color digraphs have also been considered in various contexts. We will give a necessary and sufficient condition for $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$ for a color digraph X and a color digraph Y, provided that X does not contain a specific forbidden digraph (which is infinite), and that Y is not isomorphic to a proper induced color subdigraph of itself.

We then turn to the case where X is also finite and both X and Y are vertex-transitive graphs (this is a common context in which Sabidussi's result is applied), and show that if X and Y are not both complete or both edgeless, then there exist vertex-transitive graphs X' and Y' such that $X \wr Y = X' \wr Y'$ and $\mathrm{Aut}(X \wr Y) = \mathrm{Aut}(X') \wr \mathrm{Aut}(Y')$.

Finally, the wreath product of Cayley graphs arises naturally in the study of the Cayley Isomorphism problem (definitions are provided in the third section, where this work appears). We show that if X and Y are CI-graphs of abelian groups G_1 and G_2,

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respectively, then $X \wr Y$ need not be a CI-graph of $G_1 \times G_2$, and then give a necessary condition on G_1 and G_2 that will ensure that $X \wr Y$ is a CI-graph of $G_1 \times G_2$.

The *wreath product* of two color digraphs C and D is defined to be the graph whose vertices are the ordered pairs (v, w) for which v is a vertex of C, and w is a vertex of D. There is an arc of color k from (v, w) to (v', w') if either of the following holds:

- $v = v'$ and there is an arc of color k from w to w' in D; or
- there is an arc of color k from v to v' in C.

Another way of describing the wreath product of C and D is that each vertex of C is replaced by a copy of D, and we include all possible arcs of color k from the copy of D corresponding to the vertex v of C to the copy of D corresponding to the vertex v' of C, if and only if there is an arc of color k from v to v' in C. We denote the wreath product of C and D by $C \wr D$.

The name wreath product was chosen because of the close connection (mentioned earlier) to the wreath product of automorphism groups.

Suppose we have two permutation groups, Γ and Γ', acting on the sets Ω and Ω', respectively. The *wreath product* of Γ with Γ', denoted $\Gamma \wr \Gamma'$, is defined as follows. It is the group of all permutations δ acting on $\Omega \times \Omega'$ for which there exist $\gamma \in \Gamma$ and an element γ'_v of Γ' for each $v \in \Omega$, such that

$$\delta(v, w) = (\gamma(v), \gamma'_v(w)) \text{ for every } (v, w) \in \Omega \times \Omega'.$$

It is always the case that $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) \leq \mathrm{Aut}(C \wr D)$, for color digraphs C and D. This is mentioned as an observation in [17], for example, in the case of graphs, and color digraphs are equally straightforward.

In fact, it is very often the case that $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) = \mathrm{Aut}(C \wr D)$. Harary claimed that this was always the case in [12], but this was corrected by Sabidussi in [17], who provided a characterization for precisely when $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) = \mathrm{Aut}(C \wr D)$, where C is an almost locally finite graph and D is a finite graph.

The first section of this paper will give the strengthening of Sabidussi's result explained above. The second section will use results from the first section to consider the question of what $\mathrm{Aut}(C \wr D)$ can be, if it is not $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, and will give the result mentioned previously, on vertex-transitive graphs.

The final section will produce the results that relate to the Cayley Isomorphism problem for graphs that are wreath products of Cayley graphs on abelian groups.

1. Extending Sabidussi's Result

Before we can state and prove the appropriate extension of Sabidussi's characterisation of when $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, we need to introduce some additional notation and terminology.

In what follows, we state definitions and results in terms of color digraphs; color graphs can be modelled as color digraphs by replacing each edge of color k by a digon (arcs in both directions) of color k, for every color k, so all of the definitions and results also hold for color graphs.

Definition 1. *In any color digraph X, we say that the vertices x and y are a* neighbour-equivalent k-pair *of vertices, if $x \neq y$ and the following two conditions hold:*

(1) *there are arcs of color k from x to y, and from y to x; and*
(2) *for any color i, the open out-neighbourhood and in-neighbourhood, of color i of the vertex x, are equal to the open out-neighbourhood and in-neighbourhood (respectively), of color i of the vertex y.*

That is, a neighbour-equivalent k-pair of vertices is a pair of vertices that are mutually adjacent via two arcs of color k, and that, with the exception of this mutual adjacency, have exactly the same in-neighbours and out-neighbours of every color.

It is straightforward to verify that being a neighbour-equivalent k-pair is an equivalence relation, and therefore partitions the vertices of any graph into equivalence classes. We call these equivalence classes *neighbour-equivalent k-classes* of vertices.

Suppose that a color digraph X has arcs of r colors, 1 through r. In all that follows in this section, we assume that all non-arcs of every color digraph have been replaced by arcs of a new color, color 0. This serves to simplify our notation and some aspects of the proofs. Thus, for any ordered pair of vertices v_1 and v_2, there will be an arc of color k from v_1 to v_2 for some $0 \leq k \leq r$.

Definition 2. *For any color k $(0 \leq k \leq r)$, we say that the k-complement of X is disconnected if, upon removing all arcs of color k, the underlying graph is disconnected.*

That is, the k-complement of X is disconnected if X has a pair of vertices x and y, for which every path between x and y in the underlying graph of X must use an edge of color k.

Notice that saying that the 0-complement of X is disconnected is equivalent to saying that X is disconnected.

Notation 1. *For any wreath product $C \wr D$ of color digraphs C and D, and any vertex v of C, we use $D^{(v)}$ to denote the copy of D in $C \wr D$ that corresponds to the vertex v of C.*

With that final piece of notation, we are ready to begin our proofs. There are several lemmata that will be required along the way to our characterisation of when $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$.

Lemma 1. *Suppose that $X = C \wr D = C' \wr D'$, where C, D, C' and D' are color digraphs. For every vertex v of C, every vertex w of C' for which $D'^{(w)} \not\subset D^{(v)}$, and every vertex $v' \neq v$ of C for which $D'^{(w)} \cap D^{(v')} \neq \emptyset$, we conclude that*

- *there is some color k for which every arc from any vertex of $D^{(v)} \setminus D'^{(w)}$ to any vertex of $D^{(v)} \cap D'^{(w)}$ has color k. Furthermore, this color k is the color of the arcs from the vertices of $D^{(v)}$ to the vertices of $D^{(v')}$.*
- *Similarly, there is some color k' for which every arc to any vertex of $D^{(v)} \setminus D'^{(w)}$ from any vertex of $D^{(v)} \cap D'^{(w)}$ has color k'. Furthermore, k' is the color of the arcs from the vertices of $D^{(v')}$ to $D^{(v)}$.*

Proof. First, if either $D^{(v)} \cap D'^{(w)}$ or $D^{(v)} \setminus D'^{(w)}$ contains no vertices, the result is vacuously true. So we may assume that each of these sets contains at least one vertex. Let z_1 be any vertex in $D^{(v)} \cap D'^{(w)}$.

Since $D'^{(w)} \not\subset D^{(v)}$, there must be some vertex, z_2 say, in $D'^{(w)} \setminus D^{(v)}$. Let v' be the vertex of C for which z_2 is a vertex of $D^{(v')}$. Let k be the color of the arc from z_1 to z_2. Since $X = C \wr D$, all arcs from vertices in $D^{(v)}$ to vertices in $D^{(v')}$ have color k. Therefore, with this fixed choice of z_2, any choice for $z_1 \in D^{(v)} \cap D'^{(w)}$ will have an arc of color k to z_2.

Let z_3 be any vertex of X in $D^{(v)} \setminus D'^{(w)}$. Since $z_3 \in D^{(v)}$, there must be an arc of color k from z_3 to z_2. Now, z_3 is in some $D'^{(w')}$ for some vertex w' of C', and since there is an arc of color k from $z_3 \in D'^{(w')}$ to $z_2 \in D'^{(w)}$ and $X = C' \wr D'$, we must have every possible arc of color k from vertices in $D'^{(w')}$ to vertices in $D'^{(w)}$. Since $z_3 \in D'^{(w')}$ and $z_1 \in D'^{(w)}$, in particular there is an arc of color k from z_3 to z_1.

Since the color k is fixed for any choice of $z_1 \in D^{(v)} \cap D'^{(w)}$ and for any choice of $z_3 \in D^{(v)} \setminus D'^{(w)}$, this shows that there is an arc of color k from every vertex in $D^{(v)} \setminus D'^{(w)}$ to every vertex in $D^{(v)} \cap D'^{(w)}$.

Reversing the direction of each arc and replacing k with k' in the argument above, completes the proof of the lemma. □

Before stating and proving the next lemma, we must introduce the forbidden subdigraph, F.

Definition 3. *The color digraph F has a countably infinite number of vertices, and arcs of two distinct colors. It is characterised by the property that its vertices can be labelled by the integers in such a way that whenever $i < j$, the arc from i to j has the first color, and the arc from j to i has the second color.*

Lemma 2. *Let C, C', D and D' be color digraphs, where $D \cong D'$ and D is not isomporphic to a proper induced color subdigraph of itself. Suppose that $X = C \wr D = C' \wr D'$, and that no induced subdigraph of C is isomorphic to F (where the arcs of F may have any two distinct colors). Suppose that there is some vertex v of C for which $D^{(v)}$ is not a copy of D'. Then*

- *whenever $D^{(v')}$ meets some $D'^{(w)}$ that meets $D^{(v)}$, the arcs from $D^{(v')}$ to $D^{(v)}$ have the same color as the arcs from $D^{(v)}$ to $D^{(v')}$; moreover, if this color is k, then*
- *the k-complement of D is disconnected.*

Proof. Let v be a vertex of C for which $D^{(v)}$ is not a copy of D'. As $D \cong D'$ is not isomorphic to any proper induced color subdigraph of itself, there must be some vertex w of C' such that $D'^{(w)} \cap D^{(v)} \neq \emptyset$, $D'^{(w)} \setminus D^{(v)} \neq \emptyset$, and $D^{(v)} \setminus D'^{(w)} \neq \emptyset$. Let $v' \neq v$ be a vertex of C such that $D^{(v')} \cap D'^{(w)} \neq \emptyset$. Since we have replaced non-arcs by arcs of color 0, and since $X = C \wr D$, all arcs from $D^{(v)}$ to $D^{(v')}$ have the same color, k, say. Also, all arcs from $D^{(v')}$ to $D^{(v)}$ have the same color, k'.

If we can establish that $k' = k$ (the first claim of this lemma), then by Lemma 1, the arcs from $D^{(v)} \cap D'^{(w)}$ to $D^{(v)} \setminus D'^{(w)}$ and the arcs from $D^{(v)} \setminus D'^{(w)}$ to $D^{(v)} \cap D'^{(w)}$

must all have color k. Since both of these sets are nonempty, the k-complement of D is disconnected, which establishes the second claim of this lemma.

The remainder of this proof will establish that $k' = k$. Now, we can form a chain forwards and backwards from $v_1 = v$ to include $v_2 = v'$, $v_3, \ldots$; v_0, $v_{-1}, \ldots$ of vertices of C such that for every integer i, there exists some vertex w_i of C' for which $D'^{(w_i)}$ meets both $D^{(v_i)}$ and $D^{(v_{i+1})}$ in at least one vertex, and $v_i \neq v_{i+1}$, $w_i \neq w_{i+1}$ for any i. Since $D \cong D'$ is not isomorphic to any proper induced subdigraph of itself (note that this is the only part of the proof that uses this assumption on D), this chain can never end (in either direction) by having some $D^{(v_{i+1})}$ completely contained in $D'^{(w_i)}$, or by having some $D'^{(w_i)}$ completely contained in $D^{(v_i)}$.

We show that in any such chain, if $b > a$ and $v_b \neq v_a$, then all arcs from $D^{(v_a)}$ to $D^{(v_b)}$ have color k, and all arcs from $D^{(v_b)}$ to $D^{(v_a)}$ have color k'. Inductively, assume that all arcs from $D^{(v_a)}$ to $D^{(v_{a+1})}$ have color k (the base case of this, with $a = 1$, has been established, and this makes sense, since $v_a \neq v_{a+1}$). Then since $D'^{(w_a)}$ meets $D^{(v_a)}$ in at least one vertex, and $D'^{(w_{a+1})}$ meets $D^{(v_{a+1})}$ in at least one vertex, all arcs from $D'^{(w_a)}$ to $D'^{(w_{a+1})}$ must have color k (since these two copies of D' are distinct). Now, since $D^{(v_{a+1})}$ meets $D'^{(w_a)}$ in at least one vertex, and $D^{(v_{a+2})}$ meets $D'^{(w_{a+1})}$ in at least one vertex, all arcs from $D^{(v_{a+1})}$ to $D^{(v_{a+2})}$ must have color k. This establishes that all arcs from $D^{(v_a)}$ to $D^{(v_{a+1})}$ have color k for any a, which will form the base case for our next induction.

Fix a, and inductively suppose that either all arcs from $D^{(v_a)}$ to $D^{(v_b)}$ have color k, or $v_b = v_a$, where $b > a$. If $v_b = v_a$ then since all arcs from $D^{(v_b)}$ to $D^{(v_{b+1})}$ have color k (by our last inductive argument), so do all arcs from $D^{(v_a)}$ to $D^{(v_{b+1})}$, completing the induction. Otherwise, Since $D'^{(w_a)}$ meets $D^{(v_a)}$ in at least one vertex, and $D'^{(w_b)}$ meets $D^{(v_b)}$ in at least one vertex, all arcs from $D'^{(w_a)}$ to $D'^{(w_b)}$ must have color k if $w_a \neq w_b$. And since $D^{(v_a)}$ meets $D'^{(w_a)}$ in at least one vertex, and $D^{(v_{b+1})}$ meets $D'^{(w_b)}$ in at least one vertex, all arcs from $D^{(v_a)}$ to $D^{(v_{b+1})}$ must have color k. On the other hand, if $w_a = w_b$, then $w_a \neq w_{b+1}$, but $D^{(v_b)}$ meets $D'^{(w_a)} = D'^{(w_b)}$ in at least one vertex and $D^{(v_{b+1})}$ meets $D'^{(w_{b+1})}$ in at least one vertex, so since all arcs from $D^{(v_b)}$ to $D^{(v_{b+1})}$ have color k (by our last inductive argument), so must all arcs from $D'^{(w_a)}$ to $D'^{(w_{b+1})}$. Since $D^{(v_a)}$ also meets $D'(w_a)$ in at least one vertex, all arcs from $D^{(v_a)}$ to $D^{(v_{b+1})}$ must also have color k. This completes the proof that all arcs from $D^{(v_a)}$ to $D^{(v_b)}$ have color k whenever $b > a$, if $v_b \neq v_a$. Reversing the direction of the arcs and replacing k by k' throughout the two inductive arguments that we have just concluded, will prove that all arcs from $D^{(v_b)}$ to $D^{(v_a)}$ have color k' whenever $b > a$, if $v_b \neq v_a$.

If all of the vertices v_i (where i is an integer) were distinct, then either $k' = k$ (completing the proof), or the vertices in this chain would induce a subgraph isomorphic to F, a contradiction. We now show that even if not all of the vertices in the chain are distinct, we must have $k' = k$, completing the proof. Suppose that there is some v_i such that $v_i = v_j$ for some $j \neq i$. Without loss of generality, assume $j < i$. Then since $v_j \neq v_{j+1}$ and $v_i = v_j$, we must have $i \geq j + 2 > j + 1$. Now, by the conclusion of the previous paragraph, since $j + 1 > j$ and $v_j \neq v_{j+1}$, all arcs from

v_j to v_{j+1} must have color k. However, since $i > j+1$ and $v_i = v_j \neq v_{j+1}$, all arcs from v_i to v_{j+1} must have color k'. Since $v_i = v_j$, we must conclude that $k' = k$. $\square$

Lemma 3. *Let C, C', D and D' be color digraphs. Suppose that $X = C \wr D = C' \wr D'$. Suppose further that there is some vertex v of C for which $D^{(v)}$ is neither a union of copies of D', nor contained within a copy of D'.*

Whenever there is some color k, $0 \leq k \leq r$, for which the k-complement of D is disconnected, then C' has a neighbour-equivalent k-pair of vertices, and the k-complement of D' is disconnected.

Proof. Let v be a vertex of C for which $D^{(v)}$ is neither a union of copies of D', nor contained within a copy of D'. Let w be a vertex of C' such that $D'^{(w)} \cap D^{(v)} \neq \emptyset$, $D'^{(w)} \setminus D^{(v)} \neq \emptyset$, and $D^{(v)} \setminus D'^{(w)} \neq \emptyset$. Let v' be a vertex of C such that $D^{(v')} \cap D'^{(w)} \neq \emptyset$. Since we have replaced non-arcs by arcs of color 0, and since $X = C \wr D$, all arcs from $D^{(v)}$ to $D^{(v')}$ have the same color, k, say.

By Lemma 1, all arcs from vertices of $D^{(v)} \setminus D'^{(w)}$ to vertices of $D^{(v)} \cap D'^{(w)}$ have color k also. Notice that this means that the k'-complement of D is connected for every $k' \neq k$, even if $k = 0$.

Under the assumption that there is some k', $0 \leq k' \leq r$, for which the k'-complement of D is disconnected, we must have $k' = k$, and if the color of the arcs from $D^{(v)} \cap D'^{(w)}$ to $D^{(v)} \setminus D'^{(w)}$is not k, then the k-complement of D is also connected, a contradiction. So there are arcs of color k in both directions between every vertex of $D^{(v)} \setminus D'^{(w)}$ and every vertex of $D^{(v)} \cap D'^{(w)}$. In particular, for any vertex $w' \neq w$ of C' for which $D^{(v)} \cap D'^{(w')} \neq \emptyset$, there are arcs of color k in both directions between every vertex of $D'^{(w)}$ and every vertex of $D'^{(w')}$.

This is enough to allow us to use Lemma 1, with $D'^{(w)}$ and $D^{(v)}$ taking on each others' roles, to conclude that all arcs in either direction between vertices of $D'^{(w)} \cap D^{(v)}$ and vertices of $D'^{(w)} \setminus D^{(v)}$, have color k. Since both of these sets are nonempty, the k-complement of D' is disconnected.

Finally, we establish that w and w' are a neighbour-equivalent k-pair. We have already shown that the arcs between w and w' in C' have color k. If w'' is any color k' out-neighbour of w, let v'' be any vertex of C for which $D^{(v'')} \cap D'^{(w'')} \neq \emptyset$. Then all arcs from $D'^{(w)}$ to $D'^{(w'')}$ have color k', and the various nonempty intersections establish that this is equivalent to all arcs from $D^{(v)}$ to $D^{(v'')}$ having color k', which in turn is equivalent to all arcs from $D'^{(w')}$ to $D'^{(w'')}$ having color k'; that is, w'' is a color k' out-neighbour of w'. An analogous argument can be used to show that w and w' have the same in-neighbours of any color. Hence w and w' are a neighbour-equivalent k-pair of vertices, as claimed. $\square$

We can now characterise when $\mathrm{Aut}(X \wr Y) \cong \mathrm{Aut}(X) \wr \mathrm{Aut}(Y)$.

Theorem 4. *Let $X \cong C' \wr D'$ be a color digraph with r colors, colors $1, \ldots, r$, where D' is not isomorphic to a proper induced subdigraph of itself. Suppose that X contains no induced subdigraph isomorphic to F (where the arcs of F may have any two distinct* *colors). Replace non-arcs by arcs of a new color, color 0. Then*

$$\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$$
$$\Leftrightarrow$$
$$\forall 0 \le k \le r, \ \textit{if } C' \textit{ has a neighbour-equivalent } k\textit{-pair of vertices,}$$
$$\textit{then the } k\textit{-complement of } D' \textit{ is connected.}$$

Proof. We first prove that if $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$, then for every $0 \le k \le r$, C' having a neighbour-equivalent k-pair of vertices implies that the k-complement of D' is connected. We use the contrapositive. Suppose that there is some k such that there exists a neighbour-equivalent k-pair of vertices of C', which we will call v_0 and v_1, and that the k-complement of D' is disconnected. Let U be a component of the k-complement of D'.

Denote every vertex of X in the natural way as a pair (v, w), where $v \in V(C')$, $w \in V(D')$. Define $\psi : V(X) \to V(X)$ by

$$\psi((v,w)) = \begin{cases} (v,w), \text{ if } w \notin V(U) \text{ or if } v \neq v_0, v_1 \\ (v_{1-i}, w), \text{ if } w \in V(U), \text{ and either } v = v_0 \text{ or } v = v_1. \end{cases}$$

Then it is straightforward to verify that $\psi \in \mathrm{Aut}(X)$, but $\psi \notin \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$, so $\mathrm{Aut}(X) \neq \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$.

Now we prove the converse, also by contrapositive. Suppose that $\mathrm{Aut}(X) \neq \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. As mentioned earlier in this paper, it is clear that $\mathrm{Aut}(C') \wr \mathrm{Aut}(D') \le \mathrm{Aut}(X)$, so we must have some automorphism $\psi \in \mathrm{Aut}(X) \setminus \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. We now use Lemma 2, with C' and D' taking their own roles, while the graphs taking the roles of C and D are obtained by looking at the structure of the image of $C' \wr D'$ under ψ. That is, copies of D' map to copies of D under ψ. Since $\psi \notin \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ and D' is not isomorphic to a proper induced subdigraph of itself, there must be some vertices $v, v', v'' \in V(C')$, with $v \neq v'$ for which $\psi(D'^{(v'')})$ meets both $D'^{(v)}$ and $D'^{(v')}$ nontrivially. So the conditions of the lemma are satisfied, and we conclude that there is some k for which the k-complement of D is disconnected.

This gives us the conditions of Lemma 3, so we can use that lemma to conclude that C' has a neighbour-equivalent k-pair of vertices, and the k-complement of D' is disconnected. This completes the proof. $\square$

Remark. It is necessary to forbid the subdigraph F in this result. Suppose that the colors of the arcs of F are 0 and 1. If we define the color digraph D on two vertices x and y to have an arc of color 0 from x to y and an arc of color 1 from y to x, then $F \wr D \cong F$. Label the vertices of this digraph with the integers as in the definition of F. Now, the map that takes i to $i+1$ for every integer i, is an automorphism of $F \wr D$, but each copy of D from $F \wr D$ has one vertex mapped within this copy of D, and the other vertex mapped to a different copy of D. Since copies of D are not preserved, this automorphism is not an element of $\mathrm{Aut}(F) \wr \mathrm{Aut}(D)$, and hence $\mathrm{Aut}(F \wr D) \neq \mathrm{Aut}(F) \wr \mathrm{Aut}(D)$.

Now that we have a characterisation of when $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, but on the surface of it, this gives us little information about what $\mathrm{Aut}(C \wr D)$ might look like, if it is not $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$. This is the question we consider in the next section.

2. What else could $\mathrm{Aut}(C \wr D)$ be?

Quite often, it is the case that even if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, there are nontrivial color digraphs C' and D' for which $C' \wr D' \cong C \wr D$ and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. Later in this section, we will determine precisely which color digraphs have this property. Before doing so, however, we will provide a result that gives the form of $\mathrm{Aut}(C \wr D)$ in some generality. We will assume that D is finite.

Notation 2. *In the next few results, it will prove convenient to have a special notation for the colour digraph on n vertices that has an arc of color k from every vertex to every other vertex (that is, the complete digraph on n vertices, all of whose arcs have color k). We denote this by K_n^k.*

Let Γ be a permutation group acting on the set Ω. Then for any partition $\mathcal{P}$ of the elements of Ω, we let $\mathrm{fix}_\Gamma(\mathcal{P})$ denote the subgroup of Γ that fixes every set $P \in \mathcal{P}$ setwise.

By Theorem 4, if we have $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, we must have some color k for which C has a neighbour-equivalent k-pair of vertices, and the k-complement of D is disconnected. Suppose that the k-complement of D is disconnected, and let U be a connected component of the k-complement of D. Then we must have arcs of color k in both directions between every vertex of U and every vertex of D that is not in U, and therefore the k'-complement of D is connected for every $k' \neq k$, even if $k = 0$. This has shown that if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, then the color k for which C has a neighbour-equivalent k-pair of vertices and the k-complement of D is disconnected, is unique. Henceforth, k will be used exclusively to denote this color.

With this fixed k, we consider the neighbour-equivalent k-classes of C. These form a partition of the vertices of C. We denote this partition by $\mathcal{P}$.

Let $\mathcal{B}$ be the set of connected components of the k-complement of D; we partition $\mathcal{B}$ into subsets $\mathcal{B}_1, \ldots, \mathcal{B}_m$ where all of the components in $\mathcal{B}_i$ are isomorphic for every $1 \leq i \leq m$, and m is the number of nonisomorphic components of the k-complement of D. For each $1 \leq i \leq m$, let $B_i \in \mathcal{B}_i$ be any one copy of the component in this set of isomorphic components. Then it is straightforward to see that

$$\mathrm{Aut}(D) = \bigtimes_{1 \leq i \leq m} \left(\mathcal{S}_{\mathcal{B}_i} \wr \mathrm{Aut}(B_i)\right),$$

a direct product of wreath products. (Here, and throughout, $\mathcal{S}_\Omega$ denotes the symmetric group acting on the elements of the set Ω.)

We are now ready to give the form of $\mathrm{Aut}(C \wr D)$.

Theorem 5. *For any color digraphs C and D, where D is finite and $C \wr D$ contains no induced subdigraph isomorphic to F (where the arcs of F may have any two distinct colors),*

$$\mathrm{Aut}(C \wr D) = \left(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\right) \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

where $1_{\mathrm{Aut}(D)}$ denotes the identity element of $\mathrm{Aut}(D)$.

Before proving this theorem, some comments are appropriate.

There is some redundancy in the group $\left(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\right) \left[\bigtimes_{1 \le i \le m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right]$. To see this, we clarify how the group $\left(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\right) \left[\bigtimes_{1 \le i \le m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right]$ acts on the vertices of $C \wr D$. For each of the m nonisomorphic connected components B_i of the k-complement of D, let B_i' denote the induced subgraph of D with the same vertices as B_i, so B_i' is isomorphic to the k-complement of B_i. Then $\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i)$ takes all of the vertices of C in each of the neighbour-equivalent k-classes of C in turn, and permutes all components isomorphic to B_i' in each copy of D that corresponds to these vertices of C. Since this is done to each of the m nonisomorphic connected components independently, this produces all of the direct products of wreath products. We then have $\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}$ acting as usual on the vertices of $C \wr D$. The redundancy occurs because each of the m nonisomorphic components of the k-complement of D has been permuted independently within each neighbour-equivalent k-class of C, and then each copy of D is permuted as a set by the action of $\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}$.

It is possible to remove this redundancy. Notice that every neighbour-equivalent k-class of C consists of a K_i^k, and if we delete the edges of this K_i^k, each of the vertices in this equivalence class has exactly the same in-neighbours and out-neighbours of every color, as every other vertex in the equivalence class. Therefore we have $\mathrm{fix}_{\mathrm{Aut}(C)}(\mathcal{P}) = \bigtimes_{P \in \mathcal{P}} \mathcal{S}_P$.

We could therefore write $\mathrm{Aut}(C \wr D)$ as

$$(\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}) \left[\bigtimes_{1 \le i \le m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

where $\mathrm{Aut}_0(C)$ is a permutation group for which $\mathrm{Aut}(C) = \mathrm{Aut}_0(C) \ltimes \mathrm{fix}_{\mathrm{Aut}(C)}(\mathcal{P})$. This notation has the advantage that it can be written as a semi-direct product: this group is in fact

$$(\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}) \ltimes \left[\bigtimes_{1 \le i \le m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

since $\left[\bigtimes_{1 \le i \le m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right] \lhd \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$ (this will be shown in our proof) and the redundancy has been eliminated. However, difficulties in choosing the precise action of $\mathrm{Aut}_0(C)$ make this method of eliminating redundancy seem somewhat artificial, so we have left in the redundancy.

With these comments in mind, we proceed with the proof of the theorem.

Proof. Let $\mathcal{Q}$ be a partition of the vertices of D into sets of vertices, each of which induces a connected component of the k-complement of D. Then we let $\mathcal{Q}'$ be a partition of the vertices of $C \wr D$, where for each $Q \in \mathcal{Q}$, and for each $v \in V(C)$, there is a set $Q'_v \in \mathcal{Q}'$, namely $Q'_v = \{(v,w) : w \in Q\}$. We claim that the partition $\mathcal{Q}'$ is preserved by every element of $\mathrm{Aut}(C \wr D)$, by which we mean that if $g \in \mathrm{Aut}(C \wr D)$, $(v,w) \in Q'_v$, and $g((v,w)) \in Q'' \in \mathcal{Q}'$, then $g((v,w')) \in Q''$ for every $(v,w') \in Q'_v$.

Towards a contradiction, suppose that there were some $g \in \mathrm{Aut}(C \wr D)$ that did not preserve the partition $\mathcal{Q}'$. Then there must be some $Q'_v \in \mathcal{Q}'$ for which there exist $Q', Q'' \in \mathcal{Q}'$ with $Q' \neq Q''$, such that $g(Q'_v) \cap Q' \neq \emptyset$, and $g(Q'_v) \cap Q'' \neq \emptyset$. Recall that each element of $\mathcal{Q}'$ is a set of vertices of $C \wr D$ in some copy of D that corresponds to the vertices of a connected component of the k-complement of D. Therefore, there exists some vertex v' of C for which $Q' \subset V(D^{(v')})$. If $g(Q'_v) \subset V(D^{(v')})$, then since the vertices of Q' form a connected component of the k-complement of D, $C \wr D$ must have every possible arc of color k in both directions between $g(Q'_v) \cap Q'$ and $g(Q'_v) \setminus Q'$. Since both of these sets are nonempty, this leads to the contradiction that $g(Q'_v)$ induces a disconnected subgraph of the k-complement of D.

If, on the other hand, $g(Q'_v) \not\subset V(D^{(v')})$, we may assume $Q'' \subset V(D^{(v'')})$ for some $v'' \neq v'$. We let $D^{(v')}$, $D^{(v'')}$, and $g(D^{(v)})$ take the roles of $D^{(v)}$, $D^{(v')}$, and $D'^{(w)}$, respectively, in Lemma 2. Then since k is the only color for which the k-complement of D is disconnected, the lemma tells us that every arc in both directions between $D^{(v')}$ and $D^{(v'')}$ has color k. In particular, $C \wr D$ must have every possible arc of color k in both directions between $g(Q'_v) \cap D^{(v')}$ and $g(Q'_v) \setminus D^{(v')}$. Since both of these sets are nonempty, this again leads to the contradiction that $g(Q'_v)$ induces a disconnected subgraph of the k-complement of D. We conclude that the partition $\mathcal{Q}'$ is indeed preserved by every element of $\mathrm{Aut}(C \wr D)$.

With this fact in hand, it is straightforward to verify that

$$\left[\bigtimes_{1\leq i\leq m}\left(\bigtimes_{P\in\mathcal{P}} \mathcal{S}_{\mathcal{B}_i\times P} \wr \mathrm{Aut}(B_i)\right)\right] = \mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P}).$$

Since $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P})$ is the kernel of the projection of $\mathrm{Aut}(C \wr D)$ onto the partition $\mathcal{P}$, this group is in fact normal in $\mathrm{Aut}(C \wr D)$, as we claimed in the observations that preceded this proof.

Since $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P}) \triangleleft \mathrm{Aut}(C\wr D)$, every automorphism in $\mathrm{Aut}(C\wr D)$ can be formed by combining an automorphism in $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P})$ with an automorphism that permutes sets of the partition $\mathcal{P}$ according to some automorphism of C; as mentioned in our observations, $\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}$ would provide a semi-direct product since redundancy would be eliminated, but we certainly have

$$\mathrm{Aut}(C \wr D) \leq (\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}) \left[\bigtimes_{1\leq i\leq m}\left(\bigtimes_{P\in\mathcal{P}} \mathcal{S}_{\mathcal{B}_i\times P} \wr \mathrm{Aut}(B_i)\right)\right].$$

Since both of the groups that make up the product on the right have been shown to be subgroups of $\mathrm{Aut}(C \wr D)$, we have the desired equality. $\square$

Although it is of interest to have determined this exact form of the automorphism group of any wreath product color digraph, the expression at which we have arrived is not always as enlightening as it could be. For many wreath products of color digraphs $C \wr D$, it turns out that if $\mathrm{Aut}(C \wr D) \neq \mathcal{S}_n$ for some n then it is possible to find nontrivial color digraphs C' and D' for which $C' \wr D' \cong C \wr D$, and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C' \wr D') = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. That is to say, that in these cases, if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, it merely means that we have made the wrong choices for C and D, the factors of our wreath product.

The next result characterises precisely which finite color digraphs C and D have the property that, if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, either $C \wr D \cong K_n^k$ for some color k and some n, or there are nontrivial color digraphs C' and D' for which $C' \wr D' \cong C \wr D$, and $\mathrm{Aut}(C' \wr D') = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$.

Proposition 6. *Let C and D be finite color digraphs, C having n_1 vertices, and D having n_2 vertices, with $n_1 n_2 = n$, and $X = C \wr D$. The conditions on C and D that follow are both necessary and sufficient to ensure that*

$$\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D) \Rightarrow$$
$$(\mathrm{Aut}(X) = \mathcal{S}_n, \textit{ or } \exists \textit{ nontrivial } C', D' \textit{ such that } C' \wr D' \cong X \textit{ and } \mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D'))$$

The conditions are: For any color k for which C has a neighbour-equivalent k-pair of vertices and the k-complement of D is not connected, at least one of the following must hold:

(1) *$D \cong K_{n_2}^k$ and $C \cong K_{n_1}^k$; or*
(2) *$D \cong D'' \wr D'$ for some nontrivial D'' and D', $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, and: if there is some k' for which C has a neighbour-equivalent k'-pair of vertices and there is some vertex v of D'' that forms a singleton component of the k'-complement of D'', then the k'-complement of D' is connected; or*
(3) *$C \cong C' \wr C''$ for some nontrivial C' and C'', and $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$.*

Proof. We begin by showing that the conditions are sufficient.

Suppose that the conditions hold, and that $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$. Then by Theorem 4, there is some k for which C has a neighbour-equivalent k-pair of vertices and the k-complement of D is not connected. We break the proof down into cases, according to which of the three conditions holds.

Case 1. $D \cong K_{n_2}^k$ and $C \cong K_{n_1}^k$. Then $X = C \wr D \cong K_n^k$, so $\mathrm{Aut}(X) = \mathcal{S}_n$, completing the proof in this case.

Case 2. $D \cong D'' \wr D'$ for some nontrivial D'' and D', $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, and: if there is some k' for which C has a neighbour-equivalent k'-pair of vertices and there is some vertex v of D'' that forms a singleton component of the k'-complement of D'', then the k'-complement of D' is connected.

We claim that $\mathrm{Aut}(X) = \mathrm{Aut}(C \wr D'') \wr \mathrm{Aut}(D')$. Since D'' and D' are nontrivial, so is C' where $C' = C \wr D''$, and (since wreath products are associative) clearly $X \cong C' \wr D'$, so establishing our claim will be sufficient to complete the proof in this case. Again, we will use the conditions in Theorem 4 to establish our claim.

Suppose that for some color k', C' has a neighbour-equivalent k'-pair of vertices, which we will call v_0 and v_1. Recall that $C' = C \wr D''$. If v_0 and v_1 are in the same copy of D'' within C', then choosing corresponding vertices v_0' and v_1' in D'', we must have v_0' and v_1' being a neighbour-equivalent k'-pair of vertices, so D'' has a neighbour-equivalent k'-pair of vertices. Now, since $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, Theorem 4 forces the k'-complement of D' to be connected.

If, on the other hand, v_0 and v_1 are in different copies of D'' within C', then the vertices v_0' and v_1' of C corresponding to these copies of D'' must have the property that v_0' and v_1' are a neighbour-equivalent k'-pair, so C has a neighbour-equivalent k'-pair of vertices. Furthermore, since there are arcs of color k' in both directions between v_0 and the copy of D'' in C' that contains v_1, there must be arcs in both directions between v_1 and every other vertex in this copy of D''. So the vertex in D'' corresponding to v_1 will be the special vertex v described in this case, and we may therefore assume that the k'-complement of D' is connected.

We have shown that C' having a neighbour-equivalent k'-pair of vertices forces the k'-complement of D' to be connected, so by Theorem 4, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ and we are done.

Case 3. $C \cong C' \wr C''$ for some nontrivial C' and C'', and $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$.

We claim that $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'' \wr D)$. Since C' and C'' are nontrivial, so is D' where $D' = C'' \wr D$, and (since wreath products are associative) clearly $X \cong C' \wr D'$, so establishing our claim will be sufficient to complete the proof in this case. Again, we will use the conditions in Theorem 4 to establish our claim.

Since $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$, we have that for any color k', C' having a neighbour-equivalent k'-pair of vertices implies that the k'-complement of C'' is connected. But if the k'-complement of C'' is connected, then the k'-complement of $C'' \wr D$, which is the same as the k'-complement of D', will also certainly be connected, so Theorem 4 again tells us that $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ and we are done.

Now we show that the conditions are necessary. We consider all of the ways in which the assumption

$$\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D) \Rightarrow$$
$$(\mathrm{Aut}(X) = \mathcal{S}_n, \text{ or } \exists \text{ nontrivial } C', D' \text{ such that } C' \wr D' \cong X \text{ and } \mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D'))$$

can be satisfied, and show that for each, the conditions must hold.

First, if $\mathrm{Aut}(X) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, then by Theorem 4, for every color k, C having a neighbour-equivalent k-pair of vertices implies that the k-complement of D is connected, so the premise of the conditions never occurs, and therefore the conditions are vacuously satisfied.

If $\mathrm{Aut}(X) = \mathcal{S}_n$, then there must be some color k for which $X \cong K_n^k$. Hence we must have $C \cong K_{n_1}^k$ and $D \cong K_{n_2}^k$. Notice that C has no neighbour-equivalent k'-pair of vertices for any $k' \neq k$, so the premise of our conditions can only be satisfied by the color k. We have shown that in this case, condition (1) is satisfied whenever the premise holds.

Finally, if there exist nontrivial C' and D' such that $X \cong C \wr D = C' \wr D'$ and $\mathrm{Aut}(X) \cong \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$, then since $\mathrm{Aut}(X) \ncong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, Theorem 4 tells us that there must be some color k for which the k-complement of D is disconnected. So by Lemma 3, if there were a copy of D that were neither a union of copies of D', nor contained within a copy of D', then C' has a neighbour-equivalent k-pair of vertices and the k-complement of D' is disconnected, but by Theorem 4, this is a contradiction. Hence any copy of D must either be a union of copies of D', or contained within a copy of D'.

Suppose first that every copy of D is a union of copies of D' (size constraints make it impossible to have some copies of D being unions of copies of D', while others are contained in a copy of D'). Since $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D') \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, the union must be nontrivial. Then since $C \wr D \cong C' \wr D'$, we must in fact have $D \cong D'' \wr D'$ for some nontrivial D'' (we already have D' nontrivial, by assumption). So $C' \cong C \wr D''$.

Now, using Theorem 4, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ is equivalent to for any k', C' having a neighbour-equivalent k'-pair of vertices implies that the k'-complement of D' is connected. Notice that D'' having a neighbour-equivalent k'-pair of vertices forces C' to have a neighbour-equivalent k'-pair of vertices, and therefore the k'-complement of D' is connected. But this is the same (by Theorem 4) as saying that $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, the first part of condition (2).

Suppose that there is some k' for which C has a neighbour-equivalent k'-pair of vertices and some vertex v of D'' that has arcs of color k' to and from every other vertex of D''. Then in C', take the copies of v in two copies of D corresponding to vertices in C that are a neighbour-equivalent k'-pair; these two vertices will be a neighbour-equivalent k'-pair in C'. So C' has a neighbour-equivalent k'-pair of vertices, and again the k'-complement of D' is connected. This is precisely what remained to be shown of condition (2).

Finally, we suppose that every copy of D is contained within a copy of D', so every copy of D' is a union of copies of D, and again since $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D') \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, the union must be nontrivial. Then since $C \wr D \cong C' \wr D'$, we must in fact have $D' \cong C'' \wr D$ for some nontrivial C'', and $C \cong C' \wr C''$ (we already have C' nontrivial, by assumption).

Now, using Theorem 4, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ is equivalent to for any k', C' having a neighbour-equivalent k'-pair of vertices implies that the k'-complement of D' is connected. Notice that the k'-complement of D' being connected forces the k'-complement of C'' to be connected, since $D' \cong C'' \wr D'$. But this has shown (using Theorem 4) that $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$, and so condition (3) holds. □

It may not be easy to see precisely which color digraphs satisfy the condition given in Proposition 6. In fact, although it is possible to show that vertex-transitive color digraphs satisfy this condition, a direct proof of a stronger result turns out to be shorter.

Theorem 7. *For any finite vertex-transitive color digraph $X \cong C \wr D$, if $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$ then there are some natural numbers $r > 1$ and $s > 1$, and some color k, for which $C \cong C' \wr K_r^k$, $D \cong K_s^k \wr D'$, and $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr (\mathcal{S}_{rs} \wr \mathrm{Aut}(D'))$.*

Proof. By Theorem 4, since $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, there is some color k for which C has a neighbour-equivalent k-pair of vertices and the k-complement of D is disconnected.

Since X (and therefore C) is vertex-transitive and finite, every neighbour-equivalent k-class of C has the same size, r, and since C has a neighbour-equivalent k-pair of vertices, we have $r > 1$. Therefore, every neighbour-equivalent k-class of C induces a sub-color-digraph of C that is isomorphic to K_r^k. Since each vertex in an equivalence class has exactly the same neighbours as any other vertex in that equivalence class, we have $C \cong C' \wr K_r^k$ for some vertex-transitive color digraph C'.

Since D is also vertex-transitive, every connected component of the k-complement of D is isomorphic. If we give the name D' to the induced sub-color-digraph of D that corresponds to the vertices in a connected component of the k-complement of D, we have $D \cong K_s^k \wr D'$, where s is the number of connected components of the k-complement of D (greater than 1, since the k-complement of D is disconnected).

Hence $X \cong C' \wr K_r^k \wr K_s^k \wr D' \cong C' \wr K_{rs}^k \wr D'$.

Notice that k is the only color for which the k-complement of $K_{rs}^k \wr D'$ is disconnected, and since each K_r^k was a neighbour-equivalent k-class of C, we have C' having no neighbour-equivalent k-pairs of vertices. Hence by Theorem 4, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(K_{rs}^k \wr D')$.

Now, the only color k' for which K_{rs}^k has a neighbour-equivalent k'-pair of vertices, is $k' = k$, and since each D' corresponded to the vertices of a connected component of the k-complement of D, we must have the k-complement of D' connected. Hence by Theorem 4, $\mathrm{Aut}(K_{rs}^k \wr D') = \mathrm{Aut}(K_{rs}^k) \wr \mathrm{Aut}(D') = \mathcal{S}_{rs} \wr \mathrm{Aut}(D')$.

Combining the conclusions of the last two paragraphs, we have $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr (\mathcal{S}_{rs} \wr \mathrm{Aut}(D'))$, as desired. □

3. Isomorphisms of Wreath Products of Cayley Digraphs of Abelian Groups

In recent years, a great deal of work has been directed towards solving the *Cayley isomorphism* problem. That is, given any two isomorphic Cayley (di)graphs Γ and Γ' of a group G, is it true that there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(\Gamma) = \Gamma'$? If the answer to the preceding question is yes for every Γ' isomorphic to Γ, then we say that Γ is a *CI-(di)graph of G*. If any two isomorphic Cayley (di)graphs of G are isomorphic by a group automorphism of G, we say that G *is a CI-group with respect to (di)graphs*. This problem was first proposed in 1967 by Ádám [1] in a less general form when he conjectured that $\mathbb{Z}_n$ was a CI-group with respect to graphs. The reader is referred to [16] for a recent survey of this problem. Here, we will be concerned with the isomorphism problem for Cayley digraphs that can be written as a wreath product. Intuitively, if Γ_1 is a CI-(di)graph of G_1, and Γ_2 is a CI-(di)graph of G_2, then surely $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph $G_1 \times G_2$. This, however, is not true as the following example shows. Before turning to this example, we will need Babai's well-known characterization of the CI property [3] (we remark that Alspach and Parsons

[2] also obtained this criterion, although in a less general form). If G is a group, then G_L is the left regular representation of G. If $H \leq G$, we let $\bar{H}_L = \{g_L \in G_L : g \in H\}$.

Lemma 8. *For a Cayley (di)graph Γ of G the following are equivalent:*

(1) Γ *is a CI-(di)graph,*
(2) *given a permutation $\varphi \in S_G$ such that $\varphi^{-1}G_L\varphi \leq \mathrm{Aut}(\Gamma)$, G_L and $\varphi^{-1}G_L\varphi$ are conjugate in* $\mathrm{Aut}(\Gamma)$.

Example 9. *Let p be a prime. Then there exists a Cayley (di)graph Γ of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ such that $\Gamma = \Gamma_1 \wr \Gamma_1$, where Γ_1 is a CI-(di)graph of $\mathbb{Z}_p$ and Γ_2 is a CI-(di)graph of $\mathbb{Z}_{p^2}$, but Γ is not a CI-(di)graph of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$.*

Proof. We first claim that $\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$ contains regular subgroups R_1 and R_2 isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ that are not conjugate in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$.

Define $\tau_1, \tau_2, \rho_1, \rho_2 : \mathbb{Z}_p^3 \to \mathbb{Z}_p^3$ by
$\tau_1(i,j,k) = (i+1, j+b_i, k)$,
$\tau_2(i,j,k) = (i+1, j, k)$,
$\rho_1(i,j,k) = (i,j,k+1)$, and
$\rho_2(i,j,k) = (i, j+1, k+c_j)$,
where $b_i = 0$ if $i \neq p-1$ and $b_{p-1} = 1$, and $c_j = 0$ if $j \neq p-1$ and $c_{p-1} = 1$. It is straightforward to verify that $|\tau_1| = |\rho_2| = p^2$, $|\tau_2| = |\rho_1| = p$, and $R_1 = \langle \rho_1, \tau_1 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2} \cong \langle \rho_2, \tau_2 \rangle = R_2$. Note that $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$ admits a unique complete block system $\mathcal{B}$ consisting of p blocks of size p^2 formed by the orbits of $1_{S_p} \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. Furthermore, $\mathrm{fix}_{R_1}(\mathcal{B}) = \langle \tau_1^p, \rho_1 \rangle$ and $\mathrm{fix}_{R_2}(\mathcal{B}) = \langle \rho_2 \rangle$. Let $\delta \in \mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. Then $\delta(\mathcal{B}) = \mathcal{B}$ so that $\mathrm{fix}_{\delta^{-1}R_2\delta}(\mathcal{B})$ is cyclic while $\mathrm{fix}_{R_1}(\mathcal{B})$ is not cyclic. Hence R_1 and R_2 are not conjugate in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$ as claimed.

It thus only remains to show that there exists Cayley (di)graphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ whose automorphism groups contain $\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$ and are contained in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. This though, is easy to accomplish using the literature. First, Alspach and Parsons [2] have determined necessary and sufficient conditions for a Cayley (di)graph of $\mathbb{Z}_{p^2}$ to be a CI-digraph of $\mathbb{Z}_{p^2}$ (including when the full automorphism groups contains $\mathbb{Z}_p \wr \mathbb{Z}_p$), and Gu and Li [10] have determined for which values of m all Cayley graphs of $\mathbb{Z}_{p^2}$ that are regular of degree m are CI-graphs. □

Lemma 10. *Let G and H be groups and $J \leq S_G$, $K \leq S_H$ be 2-closed permutation groups that contain G_L and H_L respectively. Suppose that any two regular subgroups of J isomorphic to G are conjugate in J and any two regular subgroups of K isomorphic to H are conjugate in K. Let $\varphi \in S_G \wr S_H$ such that $\varphi^{-1}(G \times H)_L\varphi \leq J \wr K$. Then $\varphi^{-1}(G \times H)_L\varphi$ and $(G \times H)_L$ are conjugate in $J \wr K$. Furthermore, $\varphi^{-1}\bar{G}_L\phi$ and $\bar{G}_L$ are also conjugate in $J \wr K$.*

Proof. It is straightforward to verify that $(G \times H)_L = \bar{G}_L \times \bar{H}_L$ so that $(G \times H)_L \leq J \wr K$. As $\varphi \in S_G \wr S_H$, $\varphi(g,h) = (\sigma(g), \omega_g(h))$, $\sigma \in S_G$, $\omega_g \in S_H$. Let $\mathcal{B}$ be the complete block system of $J \wr K$ formed by the orbits of $1_{S_G} \wr K$. As any two regular subgroups of J isomorphic to G are conjugate in J, there exists $\delta \in J \wr K$ such that

$\delta^{-1}\varphi^{-1}(G\times H)_L\varphi\delta/\mathcal{B} = (G\times H)_L/\mathcal{B} = \bar{G}_L$. Replacing $\varphi\delta$ by φ, we assume without loss of generality that $\varphi^{-1}(G\times H)_L\varphi/\mathcal{B} = G_L$. Hence $\varphi(g,h) = (g'_L\alpha(g),\omega_g(h))$ for some $g'\in G$ and $\alpha\in\mathrm{Aut}(G)$. Define $\psi : G\times H\to G\times H$ by $\psi(g,h) = (\alpha^{-1}(g),h)$. Then $\psi\in\mathrm{Aut}(G\times H)\cap(S_G\wr S_H)$. Furthermore, $\varphi^{-1}(G\times H)_L\varphi$ is conjugate to $(G\times H)_L$ in $J\wr K$ if and only if $\psi^{-1}\varphi^{-1}(G\times H)_L\varphi\psi$ is conjugate to $(G\times H)_L$ in $\psi^{-1}J\wr K\psi = \alpha J\alpha^{-1}\wr K$. Replacing $\alpha J\alpha^{-1}$ by J and $\varphi\psi$ by φ, we assume without loss of generality that $\alpha = 1$. As $\bar{G}_L\le J\wr K$, we may also assume that $g' = 1$. Hence $\varphi(g,h) = (g,\omega_g(h))$.

As any two regular subgroups of K isomorphic to H are conjugate in H, by standard arguments there exists $\delta\in 1_{S_G}\wr K$ such that $\delta^{-1}\varphi^{-1}\bar{H}_L\varphi\delta|_B\le\bar{H}_L|_B$ for every $B\in\mathcal{B}$. By replacing φ with $\varphi\delta$, we assume that $\varphi^{-1}\bar{H}_L\varphi|_B\le\bar{H}_L|_B$ for every $B\in\mathcal{B}$, so that $\varphi(g,h) = (g,(h'_g)_L\alpha_g(h))$, $h'_g\in H$, $\alpha_g\in\mathrm{Aut}(H)$. As $H_L|_B\in J\wr K$ for every $B\in\mathcal{B}$, we assume that $h'_g = 1_H$ for every $g\in G$, so that $\varphi(g,h) = (g,\alpha_g(h))$. Define $\iota : G\times H\to G\times H$ by $\iota(g,h) = (g,\alpha^{-1}_{1_G}(h))$. Then $\iota\in\mathrm{Aut}(G\times H)\cap(S_G\wr S_H)$. By considering $\iota^{-1}J\wr K\iota$ instead of $J\wr K$, we assume that $\alpha_{1_G} = 1$. Let Γ be an orbital digraph of $J\wr K$. Note that as $\alpha_{1_G} = 1$, we have that $\varphi|_{B_1}(\Gamma[B_1]) = \Gamma[B_1]$, where $B_1\in\mathcal{B}$ with $1_G\in B_1$. As both Γ and $\varphi(\Gamma)$ are Cayley digraphs of $G\times H$, we conclude that $\varphi|_B(\Gamma[B]) = \Gamma[B]$ for every $B\in\mathcal{B}$. Whence $\varphi|_B\in\mathrm{Aut}(\Gamma[B])$ for every $B\in\mathcal{B}$. As every orbital digraph of K can be written in the form $\Gamma'[B]$, where Γ' is an orbital digraph of $J\wr K$ and $B\in\mathcal{B}$, we conclude that $\varphi|_B\in K^{(2)} = \bar{K}|_B\cong K$. Thus $\varphi\in J\wr K$, and $\varphi\varphi^{-1}(G\times H)_L\varphi\varphi^{-1} = (G\times H)_L$ as required. That $\bar{G}_L$ and $\varphi^{-1}\bar{G}_L\varphi$ are conjugate in $J\wr K$ follows immediately from the fact that as $\varphi\in S_G\wr S_H$ and $J\wr K\le S_G\wr S_H$, there exists $\delta\in J\wr K$ such that $\varphi\delta\in N_{S_{G\times H}}((G\times H)_L)\cap(S_G\wr S_H) = N_{S_G}(G_L)\times N_{S_H}(H_L)$. $\square$

Theorem 11. *Let Γ_1 be a CI-digraph of H and Γ_2 be a CI-digraph of K, where H and K are abelian groups such that $\gcd(|H|,|K|) = r$. If whenever $p|r$ is prime, then every Sylow p-subgroup of H and K is elementary abelian, respectively, then $\Gamma_1\wr\Gamma_2$ is a CI-digraph of $G = H\times K$.*

Proof. It is straightforward to verify that $(H\times K)_L = \bar{H}_L\times\bar{K}_L$ so that $\Gamma_1\wr\Gamma_2$ is a Cayley digraph of $H\times K$. Let $\varphi\in S_G$ be such that $\varphi^{-1}G_L\varphi\le\mathrm{Aut}(\Gamma_1\wr\Gamma_2)$. We first show that there exists $\delta\in\mathrm{Aut}(\Gamma_1\wr\Gamma_2)$ such that $\langle G_L,\delta^{-1}\varphi^{-1}G_L\varphi\delta\rangle\le\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$.

If $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$, we may take $\delta = 1$. We thus assume that $\mathrm{Aut}(\Gamma)\ne\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$. By Theorem 7 we have that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma'_1)\wr(S_{rs}\wr\mathrm{Aut}(\Gamma'_2))$ for some $r,s>1$, where Γ_1 and Γ_2 are appropriate wreath products. It is not then difficult to see that there exists $\delta\in\mathrm{Aut}(\Gamma)$ such that $\langle G_L,\delta^{-1}\varphi^{-1}G_L\varphi\delta\rangle\le(\mathrm{Aut}(\Gamma'_1)\wr S_r)\wr(S_s\wr\mathrm{Aut}(\Gamma'_2)$. We thus assume without loss of generality (replacing $\delta\varphi$ by φ) that $\varphi^{-1}G_L\varphi\le\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$. Note that $\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$ admits a unique complete block system $\mathcal{B}$ of $|G|$ blocks of size $|H|$ formed by the orbits of $\bar{K}_L$.

Let $r = p_1^{a_1}\cdots p_m^{a_m}$ be the prime power decomposition of r. Let P_i be a Sylow p_i-subgroup of H and Q_i be a Sylow p_i subgroup of K, $1\le i\le m$. Then $H = H'\times\Pi_{i=1}^m P_i$, and $K = K'\times\Pi_{i=1}^m Q_i$, where $\gcd(|H'|,r) = 1$, $\gcd(|K'|,r) = 1$, and $\gcd(|H'|,|K'|) = 1$. Note that every Sylow subgroup of $G/(H'\times K')$ is elementary abelian by hypothesis. Now, as $\varphi^{-1}G_L\varphi\le\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$ and G is abelian, there exists $\hat{K} \leq G$ such that $\mathcal{B}$ is formed by the orbits of $\varphi^{-1}\hat{K}\varphi$. Note that $K' \leq \hat{K}$. Let $\hat{H} \leq G$ such that $\hat{H} \times \hat{K} = G$. Similarly, observe that $H' \leq \hat{H}$. Then $\varphi^{-1}\hat{H}_L\varphi/\mathcal{B} \leq \mathrm{Aut}(\Gamma_1)$ and $\varphi^{-1}\hat{H}_L^{-1}\varphi/\mathcal{B} \cong \varphi^{-1}\hat{H}_L\varphi$. As every Sylow subgroup of $G/(H' \times K')$ is elementary abelian and $K' \leq \hat{K}$, $H' \leq \hat{K}$, we have that $\hat{K} \cong K$ and $\hat{H} \cong H$. It is then easy to see that there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^{-1}\hat{K}\alpha = K$ and $\alpha^{-1}\hat{H}\alpha = H$. Note that Γ and $\varphi(\Gamma)$ are isomorphic by a group automorphism of G if and only if $\alpha^{-1}(\Gamma)$ and $\varphi(\Gamma)$ are isomorphic by a group automorphism of G. We may then, by replacing Γ with $\alpha^{-1}(\Gamma)$, assume that $\hat{K} = K$ and $\hat{H} = H$. But then $\varphi \in S_H \wr S_K$ so that by Lemma 10 G_L and $\varphi^{-1}G_L\varphi$ are conjugate in $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$. The result follows by Lemma 8. □

The following result is now immediate.

Corollary 12. *Let H and K be abelian groups such that every Sylow subgroup of H and K is elementary abelian. If Γ_1 is a CI-(di)graph of H and Γ_2 is a CI-(di)graph of K, then $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph of $H \times K$.*

Corollary 13. *Let H and K be abelian groups such that* $\gcd(|H|, |K|) = r$. *Then the following are equivalent:*

1. *whenever Γ_1 is a CI-digraph of H and Γ_2 is a CI-digraph of K, then $\Gamma_1 \wr \Gamma_2$ is a CI-digraph of $H \times K$,*
2. *if p divides r is prime, then every Sylow p-subgroup of H and K is elementary abelian.*

Proof. That (2) implies (1) follows directly from Theorem 11. To show that (1) implies (2), suppose that a Sylow p-subgroup of H or K is not elementary abelian for some prime $p|r$. Then G must contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. By Example 9, there a Cayley (di)graph Γ of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ which can be written as a wreath product of a Cayley (di)graph of $\mathbb{Z}_p$ and a Cayley (di)graph of $\mathbb{Z}_{p^2}$ and can also be written as a wreath product of a Cayley (di)graph of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. It is then not difficult to see that $|H| \cdot |K|/p^3$ disjoint copies of Γ is a Cayley (di)graph of $H \times K$ that is not a CI-(di)graph of $H \times K$ (as Γ is not a CI-digraph of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$), a contradiction. □

It is possible that a stronger result is true. We would like to propose the following conjecture.

Conjecture 14. *Let H and K be abelian groups, Γ_1 a Cayley (di)graph of H, and Γ_2 a Cayley (di)graph of K. If Γ_1 is not a Cayley (di)graph of an abelian group with more elementary divisors than H and Γ_2 is not a Cayley (di)graph of an abelian group with more elementary divisors than K, then $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph of $H \times K$.*

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