AMBIGUOUS SOLUTIONS OF A PELL EQUATION

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ABSTRACT. It is known that if the negative Pell equation $X^2 - DY^2 = -1$ is solvable (in integers), and if (x, y) is its solution with the smallest positive x and y, then all of its solutions (x_n, y_n) are given by the formula

$$x_n + y_n \sqrt{D} = \pm(x + y\sqrt{D})^{2n+1}$$

for $n \in \mathbb{Z}$. Furthermore, a theorem of Walker from 1967 states that if the equation $aX^2 - bY^2 = \pm 1$ is solvable, and if (x, y) is its solution with the smallest positive x and y, then all of its solutions (x_n, y_n) are given by

$$x_n \sqrt{a} + y_n \sqrt{b} = \pm(x\sqrt{a} + y\sqrt{b})^{2n+1}$$

for $n \in \mathbb{Z}$. We prove a unifying theorem that includes both of these results as special cases. The key observation is a structural theorem for the non-trivial ambiguous classes of the solutions of the (generalized) Pell equations $X^2 - DY^2 = \pm N$. We also provide a criterion for determination of the non-trivial ambiguous classes of the solutions of Pell's equations.

1. INTRODUCTION

Consider the (generalized) *Pell equation*

$$X^2 - DY^2 = \pm N \tag{1}$$

for square-free $D > 1$ and $N > 0$, both integers, where the sign in the right-hand side of the equation is either positive or negative. A pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ that satisfies (1) is called a solution of (1). Let

$$S(D; N) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z};\ x^2 - Dy^2 = N\}$$

be the set of solutions of $X^2 - DY^2 = N$. This set has fascinated mathematicians for centuries, in particular in the basic case $N = 1$. This case was studied by Indian mathematicians and a method for solving it, the so called *chakravala*, was developed starting by Brahmagupta in the 7th century C.E.. About a thousand years later William Brouncker, an English mathematician, with a method similar to *chakravala* solved (1) for $N = 1$ and positive sign. Euler mistakenly attributed this method to John Pell, another English mathematician, and thus, the name "Pell" stuck to this equation. For information regarding the history of the Pell equation see [12, Chapter II, Section XIII] and [2, Chapter XII].

We continue with a review of some known results related to solutions of (1) when $N = 1$ and the sign is either positive or negative.

2. RELEVANT PRELIMINARIES AND ASSERTIONS

It is known that the equation

$$X^2 - DY^2 = 1 \tag{2}$$

has infinitely many solutions. Moreover, the set

$$P(D; 1) := \{x + y\sqrt{D};\ (x, y) \in S(D; 1)\},$$

The second author was supported by an NSERC Undergraduate Summer Research Award (USRA). Research of both authors is partially supported by NSERC.

with multiplication, forms a group generated by -1 and $x_1 + y_1\sqrt{D}$, where $x_1 + y_1\sqrt{D}$ is of infinite order. That is, $P(D; 1) \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$. For a proof of these facts see [4, Theorems 8.5 and 8.6].

We can determine the solution (x_1, y_1) by choosing the smallest positive x_1 and y_1 such that $(x_1, y_1) \in S(D; 1)$. We name this unique solution in $S(D; 1)$, the *fundamental solution* of (2). (See [3] for an informal introduction to the algorithms for finding the fundamental solution of (2), described by the celebrated cattle problem of Archimedes as a motivating example.)

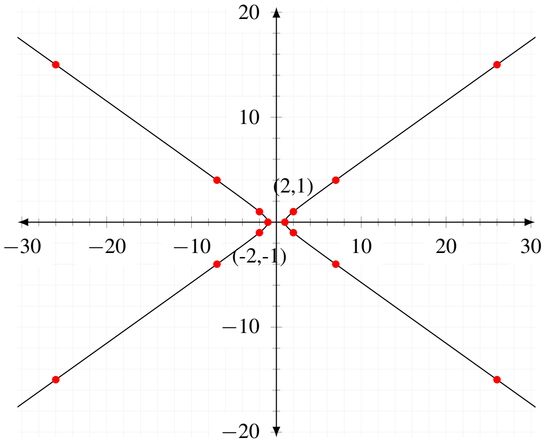

FIGURE 1. The Pell equation $X^2 - 3Y^2 = 1$, which has the fundamental solution $(2, 1)$.

As an example, Figure 1 illustrates the equation $X^2 - 3Y^2 = 1$. Each red point is a solution of $X^2 - 3Y^2 = 1$, which has the fundamental solution $(2, 1)$. The solutions $(1, 0)$ and $(-1, 0)$, corresponding to $Y = 0$, are called the *trivial* solutions of this equation.

The solutions of the *negative* Pell equation

$$X^2 - DY^2 = -1 \tag{3}$$

provides another example of a structural theorem. Let $S(D, -N)$ and $P(D, -N)$ be defined similar to the above. In this case $S(D; -1)$ can be empty, however for the values of D for which $S(D; -1)$ is non-empty the solutions of (3) are intimately related to the solutions of (2). The following is proved in [4, Theorem 8.7, p. 202].

Proposition 1. *If* (3) *is solvable, and if* (x, y) *is its solution with the smallest positive* x *and* y*, then*

$$u_1 + v_1\sqrt{D} = (x + y\sqrt{D})^2,$$

where (u_1, v_1) *is the fundamental solution of the Pell equation*

$$U^2 - DV^2 = 1.$$

Moreover, all solutions of (3) *are given by the formula*

$$x_n + y_n\sqrt{D} = \pm(x + y\sqrt{D})^{2n+1}$$

for $n \in \mathbb{Z}$.

As a consequence of the group structure of $P(D; 1)$ and the above theorem, we observe that if (3) is solvable, then $P(D; 1) \cup P(D; -1)$, generated by -1 and $x + y\sqrt{D}$, is a multiplicative group which contains $P(D; 1)$ as an index 2 subgroup and $P(D; -1)$ as a coset of $P(D; 1)$. Note that as an abstract group $P(D; 1) \cup P(D; -1) \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$.

Next, consider the Pell-type equation

$$aX^2 - bY^2 = \pm 1, \tag{4}$$

where $a, b > 1$ are integers and ab is square-free . The following is proved in [11, Theorems 6 and 9].

Proposition 2 (Walker, 1967). *If* (4) *is solvable, and if* (x, y) *is its solution with the smallest positive x and y, then*

$$u_1 + v_1\sqrt{ab} = (x\sqrt{a} + y\sqrt{b})^2,$$

where (u_1, v_1) *is the fundamental solution of the Pell equation*

$$U^2 - abV^2 = 1.$$

Moreover, all of its solutions (x_n, y_n) *are given by*

$$x_n\sqrt{a} + y_n\sqrt{b} = \pm(x\sqrt{a} + y\sqrt{b})^{2n+1}$$

for $n \in \mathbb{Z}$.

In [11, p. 507], it is mentioned that the first assertion of the above proposition may have been obtained by K. Petr [8] by use of continued fractions.

In this note we provide a general result that gives Propositions 1 and 2 as special cases. More precisely, we describe conditions under which the solutions of a Pell or a Pell-type equation, if exist, are in the form described in Propositions 1 and 2. We call the collection of such solutions a *non-trivial ambiguous class* of solutions of a Pell or a Pell-type equation. (We will see that a Pell equation may have solutions that are not in a non-trivial ambiguous class.) In the rest of this article we describe a structural theorem for such solutions, moreover we show how to determine the existence of such classes of solutions for any Pell equation.

We now briefly describe the common approach in study of solutions of the generalized Pell equation (1) (for more details see [7, Section 58] and [9]). Two solutions $(x, y), (x', y')$ of (1) are said to be *associated* if there exists a solution (u, v) of

$$U^2 - DV^2 = 1 \tag{5}$$

for which

$$x + y\sqrt{D} = (u + v\sqrt{D})(x' + y'\sqrt{D}).$$

It can be shown that association is an equivalence relation on the solutions of (1), and that each non-empty equivalence class, henceforth called an *association class*, has infinitely many elements. A useful criterion for association is the following.

Lemma 3 ([7, page 205]). *Two solutions* $(x, y), (x', y')$ *of* (1) *are associated if and only if* $N \mid xy' - x'y$ *and* $N \mid xx' - Dyy'$.

For each non-empty association class, we can find a unique representative by first choosing the solution (x, y) in the class with the least non-negative value of y and then, if this choice is not yet unique, choosing $x > 0$. Such a representative is called the *fundamental solution of the association class*. Notice that the fundamental solution of a class may be in the form $(x, 0)$ or $(0, y)$. We call such solutions *trivial*. Any association class which contains a trivial solution will be called a *trivial class*. In particular, observe that the equation (2) has the fundamental solution (x_1, y_1) with $x_1 > 0$ and $y_1 > 0$ (in the classical sense), but it has $(1, 0)$ as the fundamental solution of its association class of solutions. That is, the class of solutions of (2) forms a trivial class.

Bounds on the size of of the fundamental solution of an association class are well-known.

Proposition 4 (Tchebichef, 1851). *Let* (u_1, v_1) *be the fundamental solution of the equation*

$$U^2 - DV^2 = 1$$

and let (x, y) *be the fundamental solution of an association class of equation* (1) *with the positive sign. Then*

$$0 < |x| \leqslant x_0 = \sqrt{\frac{(u_1 + 1)N}{2}} \text{ and } 0 \leqslant y \leqslant y_0 = \sqrt{\frac{(u_1 - 1)N}{2D}}. \tag{6}$$

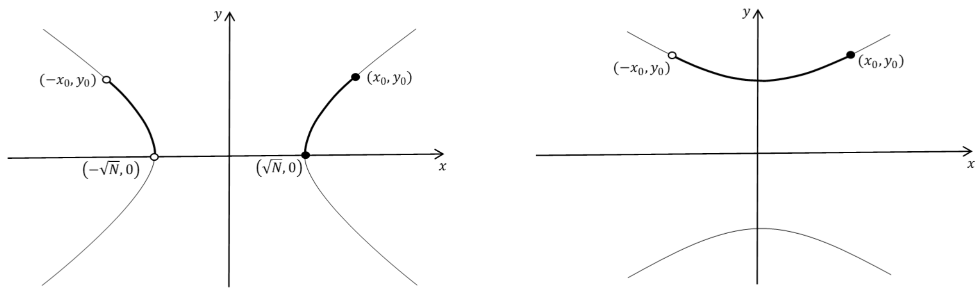

FIGURE 2. The lattice points on the highlighted segments of the hyperbolas $X^2 - DY^2 = \pm N$ are the fundamental solutions of the association classes.The values of x_0 and y_0 are given in Proposition 4.

If, instead, the sign in (1) *is negative, then*

$$0 \leqslant |x| \leqslant x_0 = \sqrt{\frac{(u_1 - 1)N}{2}} \text{ and } 0 < y \leqslant y_0 = \sqrt{\frac{(u_1 + 1)N}{2D}}. \tag{7}$$

The above proposition was first proved by Tchebichef [10] and later rediscovered by Nagell ([7, Theorems 108 and 108a]). Note that as a direct corollary of this proposition we conclude that the number of association classes is finite.

Recently Matthews, Robertson, and Srinivasan [6] proved that the conditions given in Proposition 4 is also sufficient. More precisely, Theorem 4.1 of [6] states the following assertions.

Proposition 5 (Matthews-Robertson-Srinivasan, 2015)**.** *(i) The lattice points on the highlighted segment of the horizontal hyperbola $X^2 - DY^2 = N$ in Figure 2 are the fundamental solutions of the association classes of the solutions of $X^2 - DY^2 = N$.*

(ii) The lattice points on the highlighted segment of the vertical hyperbola $X^2 - DY^2 = -N$ in Figure 2 are the fundamental solutions of the association classes of the solutions of $X^2 - DY^2 = -N$.

Note that the above proposition provides an effective algorithm for finding the fundamental solutions (and thus all solutions) of (1). In addition it states that, the number of association classes are equal to the number of lattice points on the highlighted segments in Figure 2.

We next observe that any solution (x, y) with $x > 0$, $y > 0$ of (1) determines four solutions of (1): (x, y) itself, its two *conjugates* $(-x, y)$ and $(x, -y)$, and its *negation*, $(-x, -y)$. Since $(-1, 0)$ is a solution of (5), a solution (x, y) of (1) is always associated with $(-x, -y)$. On the other hand, if (x, y) is in the same class as $(-x, y)$[1], then we say that this class is *ambiguous*. Certainly, every trivial class is ambiguous.

[1] Equivalently, in the same class as $(x, -y)$.

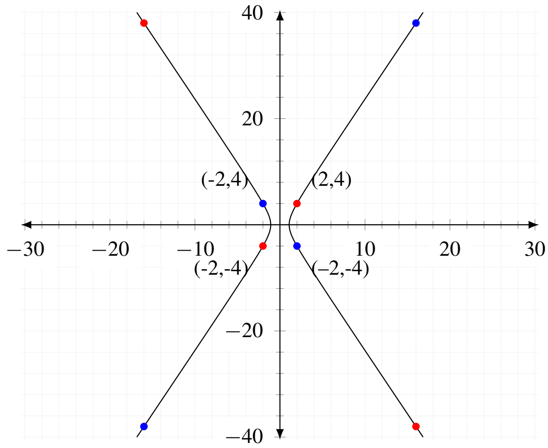

FIGURE 3. The equation $17X^2 - 3Y^2 = 20$ has two association classes. The two fundamental solutions are $(2,4)$ and $(-2,4)$.

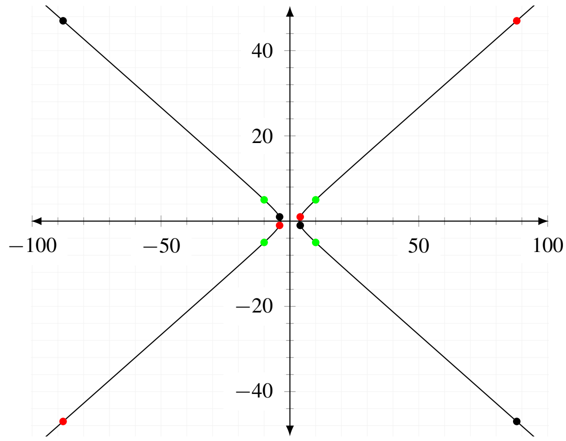

FIGURE 4. The equation $2X^2 - 7Y^2 = 25$ has three association classes. Solutions in the class with the fundamental solution $(10,5)$ are associated with their conjugates, while those in the classes with the fundamental solutions $(4,1)$ or $(-4,1)$ are not.

Figure 3 describes an equation with two non-ambiguous association classes. An example of an equation with an ambiguous class is given in Figure 4, in which one of the three association classes is an ambiguous class. With the help of Propositions 4 and 5, one may tabulate the number of association classes among the solutions of (1) for fixed N and D, and classify them according to if they are ambiguous or not. Table 5 provides such a tabulation, where, for pairs of N and D we record in the pair (C,A), the total number of association classes C, the number of ambiguous classes A, and mark the occurrence of the trivial classes with an asterisk. The values are computed using Maple™ [2] [5].

FIGURE 5. For N and square-free D fixed, the pair (C,A) lists the number of classes C and ambiguous classes A of (1). If one of the ambiguous classes is trivial, A is marked with an asterisk.

-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	$-N:D:N$	1	2	3	4	5	6	7	8	9	10
(0,0)	(1,1)	(1,1*)	(2,0)	(0,0)	(0,0)	(1,1)	(0,0)	(1,1*)	(1,1)	2	(1,1*)	(1,1)	(0,0)	(1,1*)	(0,0)	(0,0)	(2,0)	(1,1)	(1,1*)	(0,0)
(0,0)	(0,0)	(1,1)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1*)	(1,1)	(0,0)	3	(1,1*)	(0,0)	(0,0)	(1,1*)	(0,0)	(1,1)	(0,0)	(0,0)	(1,1*)	(0,0)
(0,0)	(1,1)	(0,0)	(0,0)	(0,0)	(1,1*)	(3,1)	(0,0)	(0,0)	(1,1)	5	(1,1*)	(0,0)	(0,0)	(3,1*)	(1,1)	(0,0)	(0,0)	(0,0)	(1,1*)	(0,0)
(0,0)	(0,0)	(1,1)	(0,0)	(1,1*)	(2,0)	(0,0)	(0,0)	(1,1)	(0,0)	6	(1,1*)	(0,0)	(1,1)	(1,1*)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1*)	(2,0)
(0,0)	(0,0)	(0,0)	(1,1*)	(2,0)	(0,0)	(0,0)	(2,0)	(0,0)	(0,0)	7	(1,1*)	(1,1)	(0,0)	(1,1*)	(0,0)	(0,0)	(0,0)	(1,1)	(3,1*)	(0,0)
(1,1*)	(3,1)	(0,0)	(0,0)	(2,0)	(0,0)	(1,1)	(0,0)	(0,0)	(1,1)	10	(1,1*)	(0,0)	(0,0)	(1,1*)	(0,0)	(2,0)	(0,0)	(0,0)	(3,1*)	(1,1)
(2,0)	(0,0)	(1,1)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1)	(0,0)	11	(1,1*)	(0,0)	(0,0)	(1,1*)	(2,0)	(0,0)	(0,0)	(0,0)	(1,1*)	(0,0)
(0,0)	(3,1)	(0,0)	(0,0)	(0,0)	(0,0)	(3,1)	(2,0)	(0,0)	(1,1)	13	(1,1*)	(0,0)	(2,0)	(3,1*)	(0,0)	(0,0)	(0,0)	(0,0)	(3,1*)	(0,0)
(2,0)	(0,0)	(0,0)	(1,1)	(0,0)	(2,0)	(0,0)	(0,0)	(0,0)	(0,0)	14	(1,1*)	(1,1)	(0,0)	(1,1*)	(0,0)	(0,0)	(0,0)	(1,1)	(1,1*)	(0,0)
(0,0)	(0,0)	(0,0)	(0,0)	(1,1)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	15	(1,1*)	(0,0)	(0,0)	(1,1*)	(0,0)	(0,0)	(0,0)	(0,0)	(1,1*)	(1,1)
(0,0)	(1,1)	(2,0)	(0,0)	(0,0)	(0,0)	(1,1)	(0,0)	(0,0)	(1,1)	17	(1,1*)	(0,0)	(0,0)	(1,1*)	(0,0)	(0,0)	(0,0)	(2,0)	(1,1*)	(0,0)
(2,0)	(0,0)	(1,1)	(0,0)	(0,0)	(0,0)	(0,0)	(2,0)	(1,1)	(0,0)	19	(1,1*)	(0,0)	(0,0)	(1,1*)	(2,0)	(2,0)	(0,0)	(0,0)	(3,1*)	(0,0)

3. THE MAIN RESULT

The upper bounds in (6) and (7) of Proposition 4 are crucial in the identification of non-trivial ambiguous classes. The following is our main result.

Theorem 6. *For equation*

$$X^2 - DY^2 = \pm N$$

the following hold:

[2] Maple is a trademark of Waterloo Maple, inc.

(i) *A non-trivial ambiguous class has the fundamental solution* (x, y) *if and only if* (x, y) *is a solution with* $x > 0$ *that attains the upper bounds of Proposition 4.*

(ii) *All solutions* (x_n, y_n) *in a non-trivial ambiguous class with the fundamental solution* (x, y) *are given by the formula*

$$x_n + y_n\sqrt{D} = \frac{\pm(x + y\sqrt{D})^{2n+1}}{N^n},$$

where n is any integer.

Proof. We give the proof for the case that the sign in (1) is positive, the arguments can be easily adjusted for the case that the sign in (1) is negative.

(i) Let (x, y) be a solution given by

$$x = \sqrt{\frac{(u_1 + 1)N}{2}} \text{ and } y = \sqrt{\frac{(u_1 - 1)N}{2D}}.$$

Then $x^2 + Dy^2 = Nu_1$ and $2xy = N|v_1|$. Since the right-hand sides of both of these equations are divisible by N, Lemma 3 establishes that (x, y) and $(-x, y)$ are associated (that is, (x, y) is a solution in an ambiguous class). Now, suppose (x', y') is the fundamental solution of this class and $(x', y') \neq (x, y)$. (Note that since this class is ambiguous, $x' > 0$ and $y' \geqslant 0$.) Then we have

$$x + y\sqrt{D} = (x' + y'\sqrt{D})(u_1 + v_1\sqrt{D})^i \tag{8}$$

for some positive integer $i \geqslant 1$. This, in turn, implies that there must exist a solution (w, z) for which

$$w + z\sqrt{D} = (x' + y'\sqrt{D})(u_1 + v_1\sqrt{D})^{i-1}. \tag{9}$$

Clearly, (w, z) is in the same class as (x, y) and $w > 0$, $z \geqslant 0$. Furthermore, from (8) and (9), and $u_1^2 - Dv_1^2 = 1$, we have

$$(x + y\sqrt{D})(u_1 - v_1\sqrt{D}) = w + z\sqrt{D},$$

and so $z = u_1 y - v_1 x$. However, noting that $v_1 = \sqrt{(u_1^2 - 1)/D}$,

$$z = u_1 y - v_1 x = \sqrt{\frac{(u_1 - 1)N}{2D}}\left(u_1 - \sqrt{(u_1 + 1)^2}\right) = -y < 0.$$

This is a contradiction since z was known to be non-negative. Therefore, (x, y) is the fundamental solution of its (ambiguous) class. Since $u_1 \neq \pm 1$, then this class is also non-trivial.

For the converse, suppose that (x, y) is the fundamental solution of a non-trivial ambiguous class. Note that this implies that $x > 0$ and $y > 0$. Then there must be some solution (u, v) to $U^2 - DV^2 = 1$ for which

$$x + y\sqrt{D} = (u + v\sqrt{D})(x - y\sqrt{D}).$$

Thus,

$$(x + y\sqrt{D})^2 = Nu + Nv\sqrt{D}. \tag{10}$$

Comparing the rational and irrational parts on both sides of the previous equation, we determine that

$$x^2 + Dy^2 = Nu \text{ and } 2xy = Nv. \tag{11}$$

Since $x^2 - N = Dy^2$, the first equation in (11) implies that $x = \sqrt{\frac{(u+1)N}{2}}$. By Proposition 4 we have $x \leqslant x_0 = \sqrt{\frac{(u_1+1)N}{2}}$ and thus $u \leqslant u_1$. Hence, u must either be u_1 or 1, since (u_1, v_1) is the fundamental solution of $U^2 - DV^2 = 1$. However, if $u = 1$, then $v = 0$, so (11) implies that

$y = 0$, which contradicts the assumption that (x, y) was the fundamental solution of a non-trivial ambiguous class. Therefore, $u = u_1$ and, consequently, $v = v_1$. (Note that $v > 0$, since $2xy = Nv$ and $xy > 0$.) Solving

$$\left(\sqrt{\frac{(u+1)N}{2}}\right)^2 - Dy^2 = N$$

for y shows that $y = \sqrt{\frac{(u_1-1)N}{2D}}$, and therefore $(x, y) = \left(\sqrt{\frac{(u_1+1)N}{2}}, \sqrt{\frac{(u_1-1)N}{2D}}\right)$. This completes the proof of part (i) if the sign in (1) is positive.

(ii) Let (x, y) be the fundamental solution of a non-trivial ambiguous class. So, by the steps similar to the above argument leading to (10) and the arguments after that, we have

$$\frac{(x + y\sqrt{D})^2}{N} = u_1 + v_1\sqrt{D}, \tag{12}$$

where (u_1, v_1) is the fundamental solution of the equation $U^2 - DV^2 = 1$. Any solution (x', y') in the same class as (x, y) must satisfy, for some integer n,

$$x' + y'\sqrt{D} = \pm(u_1 + v_1\sqrt{D})^n(x + y\sqrt{D}).$$

Therefore, by employing (12), we have

$$x' + y'\sqrt{D} = \frac{\pm(x + y\sqrt{D})^{2n+1}}{N^n}. \tag{13}$$

□

4. Corollaries of Theorem 6

We now describe some consequences of Theorem 6.

Corollary 7. *If a Pell equation has only one association class which is also non-trivial, then the class is a non-trivial ambiguous class and thus the solutions of this equation are given by the formula stated in part (ii) of Theorem 6. In particular, this is true for the equation $X^2 - DY^2 = p$, where p is a prime that divides $2D$ and for $X^2 - DY^2 = -p$, where $p \neq D$ is a prime dividing $2D$, provided these equations have solutions.*

Proof. If a Pell equation has only one association class, then all of its solutions are associated. Therefore every solution is associated with its conjugate, i.e., the class is ambiguous. If this class is also non-trivial, then Theorem 6 establishes that these solutions have the specified form. The second claim is a consequence of the fact that, by Theorem 110 of [7], each of the stated equations have only one association class which is also non-trivial. □

The following provides a generalization of the group structure assertion stated after Proposition 1.

Corollary 8. *If $X^2 - DY^2 = N^2$ has exactly one association class and $X^2 - DY^2 = -N^2$ has exactly one association class, then*

$$(1/N)P(D; N^2) \cup (1/N)P(D; -N^2) \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}.$$

Proof. Note that since D is square-free then any association class of $X^2 - DY^2 = -N^2$ is non-trivial. Now if $X^2 - DY^2 = -N^2$ has only one association class, then by Corollary 7 this class is a non-trivial ambiguous class and thus, by Theorem 6, all of its solutions are given in integer n by

$$\frac{\pm(x + y\sqrt{D})^{2n+1}}{N^{2n}}, \tag{14}$$

where (x, y) is the fundamental solution of the association class of $X^2 - DY^2 = -N^2$. From (12), we recall $(x+y\sqrt{D})/N^2 = u_1+v_1\sqrt{D}$, where (u_1, v_1) is the fundamental solution of $U^2-DV^2 = 1$. On the other hand, any equation of the form $X^2 - DY^2 = N^2$ must have a trivial association class with the fundamental solution $(N, 0)$. If this is the only association class, then all solutions of $X^2 - DY^2 = N^2$ have the form

$$\pm (N + 0\sqrt{D})(u_1 + v_1\sqrt{D})^n = \pm N\frac{(x + y\sqrt{D})^{2n}}{N^{2n}} \tag{15}$$

for integer n. Hence, we have, from (14) and (15), that

$$\frac{1}{N}P(D; N^2) \cup \frac{1}{N}P(D; -N^2) = \left\{ \pm\frac{(x + y\sqrt{D})^i}{N^i} \; ; i \in \mathbb{Z} \right\}$$

is a multiplicative group generated by -1 and $(x + y\sqrt{D})/N$. □

Note that Corollary 8 for $N = 1$ gives the structural result derived from Proposition 1 together with the group structure of $P(D; 1)$.

Next we consider the equation

$$X^2 - abY^2 = \pm ac, \tag{16}$$

for integers $a, b > 1$, $c > 0$, and square-free ab, a special case of (1). It can be checked that the solutions (x, y) of (16) are in one-to-one correspondence with the solutions $(x/a, y)$ of

$$aX^2 - bY^2 = \pm c. \tag{17}$$

Additionally, we can define the concept of the association class for the solutions of (17). We call two solutions (x, y) and (x', y') of (17) *associated*, if (ax, y) and (ax', y') are two associated solutions of (16). This leads to association classes of the solutions of (17). An association class of (17) is called an *ambiguous* association class and with the fundamental solution (x, y) if the solutions of (16) associated to the solution (ax, y) is an ambiguous association class of solutions of (16) with the fundamental solution (ax, y). The *trivial* and *non-trivial* association classes of (17) can be defined in a similar manner. Therefore, we may extend the results of Theorem 6 to equations of the form (17). In particular, we have the following structural result for a non-trivial ambiguous class of (17).

Corollary 9. *If (x, y) is the fundamental solution of a non-trivial ambiguous class of* (17)*, then all of the solutions (x_n, y_n) in it are given by*

$$x_n\sqrt{a} + y_n\sqrt{b} = \frac{\pm(x\sqrt{a} + y\sqrt{b})^{2n+1}}{c^n}$$

where n is an integer.

Proof. Let (x, y) be the fundamental solution of a non-trivial association class of (17). Then, (ax, y) is the fundamental solution of a non-trivial association class of (16). Thus, by part (ii) of Theorem 6, we have

$$ax_n + y_n\sqrt{ab} = \frac{\pm(ax + y\sqrt{ab})^{2n+1}}{(ac)^n}$$

for any integer n. Simplifying the above formula yields the result. □

If $c = 1$, the conditions in Lemma 3 hold for any two solutions of (16) (Note that for any solution (x, y) of (16), we have $a \mid x$). Therefore, (17) must have at most one association class, which is ambiguous and non-trivial. Thus, for $c = 1$, Corollary 9 reduces to the second assertion of Proposition 2. Therefore, Corollary 9 provides a generalization of [11, Theorem 9].

The explicit expression of the solutions of the Pell-type equation (4) has been applied in several number theoretical problems over the years. For example in [1] the expression is used in studying a conjecture of Erdős on the density of the consecutive square-full numbers. As a natural direction for further investigations, one may explore the applications of the explicit expression given in Corollary 9 in certain problems related to square-full numbers.

Acknowledgements. This research was done in the summer of 2015 as part of an NSERC Undergraduate Summer Research Award held by the second author at the University of Lethbridge. The authors would like to thank Kirsty Chalker for producing Figure 2. The authors would also like to thank the referee for the valuable comments and suggestions. After this research was done, we found an unpublished manuscript of Matthews and Robertson in "https://citeseerx.ist.psu.edu/" titled "On the converse of a theorem of Nagell and Tchebicheff", in which the assertion (i) of Theorem 6 was stated and proved as Theorem 2 in the manuscript.

REFERENCES

[1] Kevser Aktaş and M. Ram Murty, *Fundamental units and consecutive squarefull numbers*, Int. J. Number Theory **13** (2017), no. 1, 243–252, DOI 10.1142/S1793042117500142. MR3573422

[2] Leonard Eugene Dickson, *History of the theory of numbers. Vol. II: Diophantine analysis*, Chelsea Publishing Co., New York, 1966. MR0245500

[3] H. W. Lenstra, *Solving the Pell equation*, Notices Amer. Math. Soc. **49** (2002), no. 2, 182–192. MR1875156

[4] William J. LeVeque, *Fundamentals of number theory*, Dover Publications, Inc., Mineola, NY, 1996. Reprint of the 1977 original. MR1382656

[5] *Maple 2015.0*, Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.

[6] Keith R. Matthews, John P. Robertson, and Anitha Srinivasan, *On fundamental solutions of binary quadratic form equations*, Acta Arith. **169** (2015), no. 3, 291–299, DOI 10.4064/aa169-3-4. MR3361224

[7] Trygve Nagell, *Introduction to number theory*, Second edition, Chelsea Publishing Co., New York, 1964. MR0174513

[8] K. Petr, Časopis Pěst. Mat. **56** (1926/27), 58–66.

[9] Bengt Stolt, *On the Diophantine equation* $u^2 - Dv^2 = \pm 4N$, Ark. Mat. **2** (1952), 1–23, DOI 10.1007/BF02591379. MR49918

[10] P. Tchebichef, *Sur les formes quadratiques*, J. Math. Pures Appl. **16** (1851), 257–282.

[11] D. T. Walker, *On the diophantine equation* $mX^2 - nY^2 = \pm 1$, Amer. Math. Monthly **74** (1967), 504–513, DOI 10.2307/2314877. MR211954

[12] André Weil, *Number theory*, Birkhäuser Boston, Inc., Boston, MA, 1984. An approach through history; From Hammurapi to Legendre. MR734177

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