Non-Vanishing of Modular L-Functions on a Disc

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Abstract

This paper studies non-vanishing of L-functions associated to holomorphic cusp forms of weight k and level N at various points inside the critical strip. We will establish lower bounds in terms of level for the number of holomorphic cusp forms whose twisted L-functions with a fixed Dirichlet character do not vanish on a certain disc inside the critical strip.

1 Introduction

Non-vanishing of L-functions on a disc has been studied in various context in the recent years. In the context of Dirichlet L-functions P. Elliott [6] proved that there are infinitely many Dirichlet L-functions $L(s, \chi_p)$ (χ_p is a Dirichlet character mod p (prime)) which are uniformly bounded below by $c(\log p)^{\frac{1}{2}}$ in the disc $|s - \frac{1}{2}| \leq (\log p)^{-(1+\epsilon)}$, and so do not vanish there. This result has been improved by R. Balasubramanian in [2]. He proved that the number of Dirichlet L-functions $L(s, \chi_p)$ that do not vanish in the disc $|s - \frac{1}{2}| \leq (\log p)^{-(1+\epsilon)}$ is bounded below by $cp(\log p)^{-2}$. Also, in [3] R. Balasubramanian and K. Murty studied non-vanishing of Dirichlet L-functions in the disc $|s - \sigma_j| \leq 2(\log p)^{-1}$, where $\sigma_j = \frac{1}{2} + \frac{j}{\log p}$ and $2 \leq j \leq \frac{\log p}{2} - 2$. They proved that for a positive proportion of the characters χ_p (mod p), $L(s, \chi_p)$ does not have a real zero in the region $\frac{1}{2} + \frac{c}{\log p} \leq Re(s) < 1$. Here, $c > 0$ is an absolute constant and p is a sufficiently large prime.

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In this paper we prove an analogue of the above results in the context of modular L-functions. We are interested in the zeros of $L_f(s,\chi)$ in the critical strip $\frac{k}{2} < Re(s) < \frac{k+1}{2}$, where $L_f(s,\chi)$ is the twisted L-function associated to newform f and Dirichlet character χ. Generalized Riemann Hypothesis predicts that $L_f(s,\chi)$ is non-zero in this strip. One of the known results in the subject is given by K. Murty and T. Stefanicki [7]. They proved that at least $Y^{\frac{2}{3}-\epsilon}$ quadratic twists $L_f(s,\chi_d)$ ($|d| \leq Y,\ d \equiv 1\ (mod\ 4)$) attached to holomorphic newforms and $Y^{\frac{2}{3}-\epsilon}$ attached to Maass newforms do not vanish inside the disc $|s-s_0| < (\log Y)^{-(1+\epsilon)}$ for any $\epsilon > 0$ and any point s_0 inside the critical strip (the exponent $\frac{2}{3}$ can in fact be improved now to 1 using improved character sum estimates of Heath-Brown as in the work of Perelli and Pomykala [8]).

Here, we prove the following theorem.

Theorem 1 *Let $s_0 = \sigma_0 + it_0$ be a point in the strip $\frac{k}{2} < Re(s) < \frac{k+1}{2}$ and let C_N be the disc with center s_0 and radius $r_N = \underline{o}(1)$ (i.e. $r_N \to 0$ when $N \to \infty$). Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q,N) = 1$. Then there are positive constants $C_{\sigma_0,k}$ (depending only on k and σ_0) and C_{s_0,q,k,r_N} (depending on q, k, s_0 and r_N) such that for prime $N > C_{s_0,q,k,r_N}$ there exist at least $C_{\sigma_0,k}N(\log N)^{-1}$ newforms f of weight k and level N for which $L_f(s,\chi) \neq 0$ for all $s \in C_N$.*

The methodology of the proof is based on a comparison of mean values. In sections 3 and 4, we derive asymptotic formulae for $L_f(s_f,\chi)$ and $|L_f(s_f,\chi)|^2$ on average, where s_f is an arbitrary point in the disc C_N. To do this first we derive the asymptotic formulae for a fixed point s_0 in the critical strip (Lemmas 5 and 7). These are analogous of results given by W. Duke [4] for the center of critical strip. Then an application of Cauchy's integral formula gives us the asymptotic formulae on a disc (Propositions 1 and 2). This technique has already been applied by P. Elliott, B. Balasubramanian and B. Balasubramanian-K. Murty for Dirichlet L-functions. Finally we have to deal with the contribution of oldforms, we apply the technique developed by the author in [1] to overcome this difficulty. In section 5 we finish off the proof of Theorem 1 by an application of the Cauchy-Schwarz inequality.

Finally, with a slight modification of our previous results, we establish asymptotic formulae for $L_f(s_f,\chi)$ and $|L_f(s_f,\chi)|^2$ on average, where s_f is an arbitrary point in the disc C_N

with center on the critical line $s = \frac{k}{2} + it$, and as a result we prove the following non-vanishing theorem.

Theorem 2 *Let $s_0 = \frac{k}{2} + it_0$ and let C_N be the disc with center s_0 and radius $r_N = \frac{1}{(\log N)^{4+\epsilon}}$ ($\epsilon > 0$). Suppose that χ is a fixed primitive Dirichlet character mod q such that $(q, N) = 1$. Then there are positive constants C_k (depending only on k) and $C_{t_0,q,k,\epsilon}$ (depending on q, k, t_0 and ϵ) such that for prime $N > C_{t_0,q,k,\epsilon}$ there exists at least $C_k N (\log N)^{-2}$ newform f of weight k and level N for which $L_f(s, \chi) \neq 0$ for all $s \in C_N$.*

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2 Preliminaries

In this section we review some basic facts concerning modular forms and set up our notation.

Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ with trivial character. The space $S_k(N)$ has an inner product (Petersson inner product)

$$< f, g >= \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $\mathcal{H}$ denotes the upper half plane. For any $f \in S_k(N)$ let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz), \; e(z) = e^{2\pi i z}$$

be the Fourier expansion of f at $i\infty$.

Let χ be a primitive Dirichlet character mod q with $(q, N) = 1$, then the twisted L-function associated to f and χ is defined by

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) a_f(n)}{n^s}.$$

The twisted L-function is given by an absolutely convergent series on the half-plane $Re(s) > \frac{k+1}{2}$ and it has an analytic continuation to the whole plane. Moreover, if f is a newform (in Atkin-Lehner sense), then $L_f(s,\chi)$ has an Euler product valid on $Re(s) > \frac{k+1}{2}$ and it satisfies the following functional equation

$$\left(\frac{q\sqrt{N}}{2\pi}\right)^s \Gamma(s)L_f(s,\chi) = \epsilon_\chi \left(\frac{q\sqrt{N}}{2\pi}\right)^{k-s} \Gamma(k-s)L_f(k-s,\bar{\chi}) \tag{1}$$

where $\epsilon_\chi = \epsilon_f\chi(N)\tau(\chi)^2q^{-1}$with $\epsilon_f = \pm 1$ (the root number of f) which depends only on f and $\tau(\chi)$ is the Gauss sum.

Let $\{T_p\ (p \not| N),\ U_q\ (q|N)\}$ be the collection of the classical Hecke operators and let $W_q\ (q|N)$ be the "W operator" of Atkin and Lehner. In 1983 A. Pizer introduced the operators C_q on $S_k(N)$ for $q|N$, such that the action of C_q on the new part of $S_k(N)$ is the same as the action of the classical U_q operators. More precisely he defined C_q as

$$C_q = U_q + W_qU_qW_q + q^{\frac{k}{2}-1}W_q \quad\ if\ \ q\|N$$

$$C_q = U_q + W_qU_qW_q \quad\ if\ \ q^2|N.$$

Then he showed that $T_p\ (p \not| N),\ C_q\ (q|N)$ form a commuting family of Hermitian operators. Using this, he proved ([9], Theorem 3.10) the following result.

Theorem *There exists a basis $f_i(z)$ $(1 \leq i \leq dim\ S_k(N))$ of $S_k(N)$ such that each $f_i(z)$ is an eigenform for all the T_p and C_q operators with $p \not| N$ and $q|N$. Let $f(z) = \sum_{n=1}^\infty a_f(n)e(z)$ be an element of this basis. Then $a_f(1) \neq 0$ and assuming $f(z)$ is normalized so that $a_f(1) = 1$, we have $f|T_p = a_f(p)f$ for all $p \not| N$, $f|C_q = a_f(q)f$ for all $q|N$, and $a_f(nm) = a_f(n)a_f(m)$ whenever $(n,m) = 1$. Furthermore $f(z)$ is an eigenform for all W_q operators, $q|N$. Finally, if $g(z) \in S_k(N)$ is an eigenform for all the T_p and C_q operators with $p \not| N$ and $q|N$, then $g(z) = cf_i(z)$ for some $c \in \mathbb{C}^*$ and some unique i, $1 \leq i \leq dim\ S_k(N)$.*

Now let $\mathcal{F}_N$ be the set of all normalized ($a_f(1) = 1$) newforms in $S_k(N)$ and let $\mathcal{P}_N$ be the basis of $S_k(N)$ given by the above theorem. The elements of $\mathcal{P}_N$ form an orthogonal basis (with respect to Petersson inner product) for $S_k(N)$, any $f \in \mathcal{P}_N$ has real Fourier coefficient and $L_f(s,\chi)$ satisfies the functional equations (1). Moreover, we can show that the action of C_q on $S_k(N)^{new}$ is the same as the action of U_q (see [9], Remark 2.9). This shows that $\mathcal{F}_N \subset \mathcal{P}_N$.

For the Fourier coefficients of a newform f we have the Deligne bound

$$|a_f(n)| \leq \mathbf{d}(n)n^{\frac{k-1}{2}}$$

where $\mathbf{d}(n)$ is the divisor function. In the case that N is a prime, we have the following estimation of the Fourier coefficients of $f \in \mathcal{P}_N$.

Lemma 1 *Suppose N is prime and $f \in \mathcal{P}_N$. Then*

$$|a_f(n)| \leq c_0 n^{\frac{k}{2}}$$

where c_0 is an absolute constant independent of f.

Proof: Propositions 3.6 and 3.4 of [9] imply that if $f \in \mathcal{P}_N - \mathcal{F}_N$, then

$$f(z) = h(z) \pm N^{\frac{k}{2}} h(Nz)$$

where h is the normalized newform of weight k and level 1 associated to f. Now the result follows from the Deligne bound for the newforms (see [1], Lemma 2.2 for the details). □

Finally, since $\mathcal{P}_N$ forms an orthogonal basis of $S_k(N)$, the Fourier coefficients of its elements are semi-orthogonal in the following sense:

Lemma 2 *Let $\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}<f,f>}$ and let $\delta_{m,n}$ be the Kronecker delta. For m and n positive integers we have the inequality*

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}} \frac{a_f(n)}{\sqrt{n^{k-1}}} - \delta_{m,n} \right| \leq M \mathbf{d}(N) N^{\frac{1}{2}-k} (m,n)^{\frac{1}{2}} \sqrt{(mn)^{k-1}}$$

where M is a constant depending only on k and $\mathbf{d}(N)$ is the number of the divisors of N.

Proof: See [4] Lemma 1. □

3 Mean estimation

In this section we will find an asymptotic formula for

$$\sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi)$$

where s_f is a variable point in the disc with center $s_0 = \sigma_0 + it_0$ ($\frac{k}{2} < \sigma_0 < \frac{k+1}{2}$) and radius $r_N = \underline{o}(1)$.

Lemma 3 *For any $x > 0$ and $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ where $\frac{k-1}{2} \le \sigma_0 \le \frac{k+1}{2}$, let*

$$W(s_0, x) = \frac{1}{2\pi i} \int_{(\frac{5}{4})} \Gamma(s + s_0) x^{-s} \frac{ds}{s}$$

and

$$\mathcal{A}_{f,\chi}(x, s_0) = \sum_{n \ge 1} \chi(n) a_f(n) n^{-s_0} W(s_0, \frac{2\pi n}{x})$$

where χ is a fixed primitive Dirichlet character mod q with $(q, N) = 1$. Then we have

$$\Gamma(s_0) L_f(s_0, \chi) = \mathcal{A}_{f,\chi}(x, s_0) + \epsilon_\chi \left(\frac{q\sqrt{N}}{2\pi} \right)^{k-2s_0} \mathcal{A}_{f,\bar{\chi}}(\frac{q^2 N}{x}, k - s_0)$$

where ϵ_χ is the root number of $L_f(s, \chi)$.

Proof: From the definition of $W(s_0, x)$ it is clear that

$$\mathcal{A}_{f,\chi}(x, s_0) = \frac{1}{2\pi i} \int_{(\frac{5}{4})} L_f(s + s_0, \chi) (\frac{x}{2\pi})^s \Gamma(s + s_0) \frac{ds}{s}.$$

Changing the line of integration from $\frac{5}{4}$ to $-\frac{5}{4}$ and using the functional equation (1) yields

$$\mathcal{A}_{f,\chi}(x, s_0) = \Gamma(s_0) L_f(s_0, \chi) + \epsilon_\chi \left(\frac{q\sqrt{N}}{2\pi} \right)^{k-2s_0} \frac{1}{2\pi i} \int_{(-\frac{5}{4})} L_f(k - s - s_0, \bar{\chi}) (\frac{2\pi x}{q^2 N})^s \Gamma(k - s - s_0) \frac{ds}{s}.$$

Now changing variables $s \to -s$ implies the result. □

Lemma 4 *Under the assumptions of Lemma 3*

$$W(s_0, x) \ll x^{\sigma_0 - 1} e^{-x}$$

when $x \to \infty$ and

$$W(s_0, x) \ll_k 1$$

when $x \to 0$.

Proof: We have

$$W(s_0, x) = \frac{1}{2\pi i} \int_{(\frac{5}{4})} \left(\int_0^\infty e^{-t} t^{s+s_0-1} dt \right) x^{-s} \frac{ds}{s} = \int_x^\infty t^{s_0-1} e^{-t} dt.$$

Therefore

$$|W(s_0, x)| = \left| \int_x^\infty t^{s_0-1} e^{-t} dt \right| \le \int_x^\infty t^{\sigma_0-1} e^{-t} dt.$$

Now the first result follows from the estimation of the last integral using integration by parts.

The second result is clear since $|W(s_0, x)| \le \Gamma(\sigma_0)$ when $x \to 0$. □

Lemma 5 *Let χ be a fixed primitive Dirichlet character mod q with $(q,N) = 1$ and let $s_0 = \sigma_0 + it_0$ be a point in the strip $\frac{k-1}{2} < Re(s) \leq \frac{k+1}{2}$. Then we have*

$$\sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi) = 1 + O\left(\frac{1}{|\Gamma(s_0)|} N^{\frac{1}{2}-\sigma_0} (\log N)^{k-\sigma_0}\right) + O\left(\frac{1}{|\Gamma(s_0)|} N^{\frac{k-12}{2}-\sigma_0} (\log N)^{k-\sigma_0-1}\right)$$

for N prime. The implied constant depends only on q and k.

Proof: Choosing $x = q^2 N \log N$ in Lemma 3 gives

$$\mathcal{A}_{f,\bar{\chi}}(\frac{Nq^2}{x}, k - s_0) = \sum_{n\geq 1} \bar{\chi}(n) a_f(n) n^{s_0-k} W(k - s_0, 2\pi n \log N).$$

Using Lemma 4 and Lemma 1 we have

$$\left|\mathcal{A}_{f,\bar{\chi}}(\frac{1}{\log N}, k - s_0)\right| \leq \sum_{n\geq 1} |a_f(n)| n^{\sigma_0-k} |W(k - s_0, 2\pi n \log N)|$$

$$\leq \sum_{n\geq 1} c_0 n^{\frac{k}{2}} n^{\sigma_0-k} (2\pi n \log N)^{k-\sigma_0-1} e^{-2\pi n \log N} = c_0 (2\pi \log N)^{k-\sigma_0-1} \sum_{n\geq 1} \frac{n^{\frac{k}{2}-1}}{(N^{2\pi})^n}.$$

Therefore from Lemma 3 we get

$$\Gamma(s_0) \sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi) = \sum_{f\in\mathcal{P}_N} \omega_f \mathcal{A}_{f,\chi}(x, s_0) + (\sum_{f\in\mathcal{P}_N} \omega_f) O_{q,k}(N^{-6+\frac{k}{2}-\sigma_0} (\log N)^{k-\sigma_0-1}).$$

From this, we have

$$\Gamma(s_0) \sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi) - \Gamma(s_0) = \sum_{n\geq 1} \chi(n) (\sum_{f\in\mathcal{P}_N} \omega_f \frac{a_f(n)}{n^{\frac{k-1}{2}}} - \delta_{1,n}) W(s_0, \frac{2\pi n}{q^2 N \log N}) n^{\frac{k-1}{2}-s_0}$$

$$+ W(s_0, \frac{2\pi}{q^2 N \log N}) - \Gamma(s_0) + (\sum_{f\in\mathcal{P}_N} \omega_f) O_{q,k}(N^{-6+\frac{k}{2}-\sigma_0} (\log N)^{k-\sigma_0-1}).$$

Note that

$$W(s_0, \frac{2\pi}{q^2 N \log N}) - \Gamma(s_0) = \int_0^{\frac{2\pi}{q^2 N \log N}} t^{s_0-1} e^{-t} dt = O_{q,k}((N \log N)^{-\sigma_0}).$$

Also from, Lemma 2 for $m = n = 1$ follows that

$$\sum_{f\in\mathcal{P}_N} \omega_f = 1 + O(N^{\frac{1}{2}-k}).$$

By applying $m = 1$ in Lemma 2 and using the above identities, we have

$$\left|\Gamma(s_0) \left(\sum_{f\in\mathcal{P}_N} \omega_f L_f(s_0,\chi) - 1\right)\right| \leq M_1 \frac{N^{\frac{1}{2}-k}}{(N \log N)^{\sigma_0-1}} \sum_{n\geq 1} n^{k-2} e^{-\frac{2\pi n}{q^2 N \log N}} + M_2 (N \log N)^{-\sigma_0}$$

$$+M_3N^{-6+\frac{k}{2}-\sigma_0}(\log N)^{k-\sigma_0-1}$$

where M_1, M_2 and M_3 are constants depending on q and k. This proves the desired result. □

Proposition 1 *Let $s_0=\sigma_0+it_0$ be a point in the strip $\frac{k}{2}<Re(s)<\frac{k+1}{2}$ and let Γ and C_N be the circles with center (σ_0,t_0) and radius $R=\frac{1}{2}min\{\frac{k+1}{2}-\sigma_0,\sigma_0-\frac{k}{2}\}$ and $r_N=o(1)$ respectively. Then for N prime*

$$\sum_{f\in\mathcal{P}_N}\omega_fL_f(s_f,\chi)=1+O_{q,k}\left(\frac{1}{|\Gamma(s_0)|}N^{-\frac{1}{2}}\right)+O_{q,k,s_0}\left(\frac{r_N}{R-r_N}N^{-\frac{1}{2}}\right)$$

where s_f is an arbitrary point in C_N.

Proof: By Cauchy's integral formula for any $s_f\in C_N$, we have

$$L_f(s_f,\chi)-L_f(s_0,\chi)=\frac{1}{2\pi i}\int_\Gamma L_f(w,\chi)(\frac{1}{w-s_f}-\frac{1}{w-s_0})dw$$

where Γ traversed in the counter clockwise direction. Therefore

$$\sum_{f\in\mathcal{P}_N}\omega_fL_f(s_f,\chi)=\sum_{f\in\mathcal{P}_N}\omega_fL_f(s_0,\chi)+\frac{1}{2\pi i}\int_\Gamma\left(\sum_{f\in\mathcal{P}_N}\omega_fL_f(w,\chi)\right)\frac{s_f-s_0}{(w-s_f)(w-s_0)}dw. \tag{2}$$

Now using Lemma 5 yields

$$\left|\frac{1}{2\pi i}\int_\Gamma\left(\sum_{f\in\mathcal{P}_N}\omega_fL_f(w,\chi)\right)\frac{s_f-s_0}{(w-s_f)(w-s_0)}dw\right|\le\frac{r_N}{R-r_N}O_{q,k,s_0}\left(N^{-\frac{1}{2}}\right). \tag{3}$$

Note that here we used the fact that $\frac{1}{2\pi i}\int_\Gamma\frac{s_f-s_0}{(w-s_f)(w-s_0)}dw=0$. Applying (3) and Lemma 5 in (2) completes the proof. □

4 Mean square estimation

In this section we are going to find an asymptotic formula for the average values of $|L_f(s_f,\chi)|^2$ where s_f is a variable point in a disc with centre $s_0=\sigma_0+it_0$ ($\frac{k}{2}<\sigma_0<\frac{k+1}{2}$) and radius $r_N=\underline{o}(1)$. We start with writing $|L_f(s_0,\chi)|^2$ as a sum of two convergent series.

Let $|L_f(s_0,\chi)|^2=\sum_{l\geq1}b_f(l)l^{-\sigma_0}$ so that

$$b_f(l)=\sum_{mn=l}\chi(n)\bar{\chi}(m)a_f(n)a_f(m)(\frac{m}{n})^{it_0}. \tag{4}$$

For $x > 0$ and $s_0 = \sigma_0 + it_0 \in \mathbb{C}$ where $\frac{k-1}{2} \leq \sigma_0 \leq \frac{k+1}{2}$, define

$$\mathcal{B}_f(x, s_0) = \sum_{l \geq 1} \frac{b_f(l)}{l^{\sigma_0}} Z(s_0, \frac{l}{x}) \tag{5}$$

where

$$Z(s_0, x) = \frac{1}{2\pi i} \int_{(\frac{5}{4})} (2\pi)^{-2s} \Gamma(s + s_0) \Gamma(s + \bar{s_0}) x^{-s} \frac{ds}{s}. \tag{6}$$

Using Deligne's bound in (4) and standard estimates for $Z(s_0, x)$ shows that (5) is absolutely convergent.

Lemma 6 *Let $f \in \mathcal{P}_N$ and suppose that χ is a primitive Dirichlet character mod q with $(q, N) = 1$. For any $x > 0$ we have*

$$|\Gamma(s_0) L_f(s_0, \chi)|^2 = \mathcal{B}_f(x, s_0) + \left(\frac{q^2 N}{4\pi^2}\right)^{k - 2\sigma_0} \mathcal{B}_f(\frac{(q^2 N)^2}{x}, k - \bar{s_0}).$$

Proof: From (6) we have

$$\mathcal{B}_f(x, s_0) = \frac{1}{2\pi i} \int_{(\frac{5}{4})} (2\pi)^{-2s} \Gamma(s + s_0) \Gamma(s + \bar{s_0}) L_f(s + s_0, \chi) L_f(s + \bar{s_0}, \bar{\chi}) x^s \frac{ds}{s}.$$

By changing the line of integration from $\frac{5}{4}$ to $-\frac{5}{4}$ and using the functional equation (1) we get

$$\mathcal{B}_f(x, s_0) = |\Gamma(s_0) L_f(s_0, \chi)|^2 + \left(\frac{q^2 N}{4\pi^2}\right)^{k - 2\sigma_0}$$

$$\int_{(-\frac{5}{4})} (2\pi)^{2s} \Gamma(k - s - s_0) \Gamma(k - s - \bar{s_0}) L_f(k - s - s_0, \bar{\chi}) L_f(k - s - \bar{s_0}, \chi) (\frac{x}{(q^2 N)^2})^s \frac{ds}{s}.$$

Now changing variables $s \to -s$ yields the result. $\square$

We estimate $\mathcal{B}_f(x, s_0)$ on average. From (4) and (5) it follows that

$$\sum_{f \in \mathcal{P}_N} \omega_f \mathcal{B}_f(x, s_0) = \sum_{f \in \mathcal{P}_N} \omega_f \sum_{l \geq 1} b_f(l) l^{-\sigma_0} Z(s_0, \frac{l}{x})$$

$$= \sum_{m,n \geq 1} \chi(n) \bar{\chi}(m) Z(s_0, \frac{mn}{x}) \frac{(\frac{m}{n})^{it_0}}{(mn)^{\sigma_0 - \frac{k-1}{2}}} \sum_{f \in \mathcal{P}_N} \omega_f \frac{a_f(m)}{\sqrt{m^{k-1}}} \frac{a_f(n)}{\sqrt{n^{k-1}}}$$

$$= \sum_{n \geq 1} |\chi(n)|^2 Z(s_0, \frac{n^2}{x}) \frac{1}{n^{2\sigma_0 - k + 1}} + R \tag{7}$$

where

$$R \ll N^{\frac{1}{2}-k} \sum_{m,n\geq 1} Z(\sigma_0, \frac{mn}{x})(m,n)^{\frac{1}{2}}(mn)^{-\sigma_0+k-1}. \tag{8}$$

Note that here we are using the inequality $|Z(s_0,x)| \leq Z(\sigma_0,x)$. This is true since by writing Γ functions in terms of integrals in (6) and interchanging the order of integration, we have

$$Z(s_0,x) = \int_0^\infty {t_1}^{s_0-1} e^{-t_1} \left(\int_{\frac{4\pi^2 x}{t_1}}^\infty e^{-t_2} {t_2}^{\bar{s_0}-1} dt_2 \right) dt_1.$$

Applying the triangle inequality in the above identity implies the desired inequality.

Using the definition of $Z(s_0,x)$, the first term in (7) is equal to

$$\frac{1}{2\pi i}\int_{(\frac{5}{4})} L(2s+2\sigma_0-k+1,\chi_0)(2\pi)^{-2s}\Gamma(s+s_0)\Gamma(s+\bar{s_0})x^s\frac{ds}{s}$$

where χ_0 is the principal character mod q and $L(s,\chi_0)=\zeta(s)\prod_{p|q}(1-\frac{1}{p^s})$. Now we assume that $\sigma_0 \neq \frac{k}{2}$, since the integrand has simple poles at $s=0$ and $s=\frac{k}{2}-\sigma_0$, by moving the line of integration from $\frac{5}{4}$ to $-\frac{1}{2}$, the integral is equal to

$$|\Gamma(s_0)|^2\prod_{p|q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1)+\frac{\prod_{p|q}\left(1-\frac{1}{p}\right)(2\pi)^{2\sigma_0-k}}{k-2\sigma_0}\Gamma(\frac{k}{2}+it_0)\Gamma(\frac{k}{2}-it_0)x^{\frac{k}{2}-\sigma_0}$$
$$+O_{\sigma_0,q,k}(x^{-\frac{1}{2}}). \tag{9}$$

Now in (7) we estimate the reminder term R. We calculate

$$\sum_{m,n\geq 1} Z(\sigma_0,\frac{mn}{x})(m,n)^{\frac{1}{2}}(mn)^{-\sigma_0+k-1}.$$

It is

$$\frac{1}{2\pi i}\int_{(\frac{k+1}{2})}(2\pi)^{-2s}(\Gamma(s+\sigma_0))^2x^s\left(\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s+\sigma_0-k+1)}\right)\frac{ds}{s}.$$

Note that since the integrand does not have any pole in the strip $\frac{5}{4} < Re(s) < \frac{k+1}{2}$, we can move the line of integration from $\frac{5}{4}$ to $\frac{k+1}{2}$. From [4] Lemma 4, we know that

$$\sum_{m,n\geq 1}(m,n)^{\frac{1}{2}}(mn)^{-(s+\sigma_0-k+1)} = \frac{\zeta(2s+2\sigma_0-2k+\frac{3}{2})\zeta(s+\sigma_0-k+1)^2}{\zeta(2s+2s_0-2k+2)}.$$

Applying this identity to the above integral and moving the line of integration from $\frac{k+1}{2}$ to $k-\sigma_0-\epsilon$ ($\epsilon>0$) yields

$$\sum_{m,n\geq 1} Z(\sigma_0,\frac{mn}{x})(m,n)^{\frac{1}{2}}(mn)^{-(s+\sigma_0-k+1)} \sim C_{\sigma_0,k}\ x^{k-\sigma_0}\log x \tag{10}$$

and by (8), $R \ll N^{\frac{1}{2}-k} x^{k-\sigma_0} \log x$. Therefore we have

$$\sum_{f\in\mathcal{P}_N} \omega_f \mathcal{B}_f(x,s_0) = |\Gamma(s_0)|^2 \prod_{p|q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1) + \frac{\prod_{p|q}\left(1-\frac{1}{p}\right)(2\pi)^{2\sigma_0-k}}{k-2\sigma_0}$$

$$\Gamma(\frac{k}{2}+it_0)\Gamma(\frac{k}{2}-it_0)x^{\frac{k}{2}-\sigma_0} + O_{\sigma_0,q,k}(x^{-\frac{1}{2}}) + O_{\sigma_0,k}(N^{\frac{1}{2}-k}x^{k-\sigma_0}\log x). \qquad (11)$$

Lemma 7 *Let χ be a fixed primitive Dirichlet character mod q with $(q,N)=1$ and let $s_0=\sigma_0+it_0$ where $\frac{k}{2}<\sigma_0\le\frac{k+1}{2}$. Then we have*

$$\sum_{f\in\mathcal{P}_N} \omega_f |L_f(s_0,\chi)|^2 = \prod_{p|q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1) + c_1 N^{\frac{k}{2}-\sigma_0} + O_{s_0,q,k}(N^{-\frac{1}{2}})$$

for N prime. Here, c_1 depends on s_0, q and k.

Proof: Choosing $x=q^2N$ in Lemma 6 and applying (11) in it, proves the Lemma. □

Proposition 2 *Under the assumptions of Proposition 1*

$$\sum_{f\in\mathcal{P}_N} \omega_f |L_f(s_f,\chi)|^2 = \prod_{p|q}\left(1-\frac{1}{p^{2\sigma_0-k+1}}\right)\zeta(2\sigma_0-k+1) + c_1 N^{\frac{k}{2}-\sigma_0} + +O_{s_0,q,k}(N^{-\frac{1}{2}})$$

$$+O_{\sigma_0,k}\left(\frac{r_N}{R-r_N}\right) + O_{s_0,q,k}\left(\frac{r_N}{R-r_N}N^{\frac{k}{2}-\sigma_0+R}\right).$$

Here, c_1 depends on s_0, q and k.

Proof: We have

$$\left|\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2 - \sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_0,\chi)|^2\right| \le \sum_{f\in\mathcal{P}_N}\omega_f\left||L_f(s_f,\chi)|^2-|L_f(s_0,\chi)|^2\right|$$

$$\le \sum_{f\in\mathcal{P}_N}\omega_f\left|{L_f}^2(s_f,\chi)-{L_f}^2(s_0,\chi)\right|.$$

By applying Cauchy's integral formula and Lemma 7, the last expression equals to

$$\sum_{f\in\mathcal{P}_N}\omega_f\left|\frac{1}{2\pi i}\int_\Gamma {L_f}^2(w,\chi)\frac{s_f-s_0}{(w-s_f)(w-s_0)}dw\right| \le \frac{r_N}{R-r_N}\left(O_{\sigma_0,k}(1)+O_{s_0,q,k}(N^{\frac{k}{2}-\sigma_0+R})\right).$$

This shows that

$$\sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_f,\chi)|^2 = \sum_{f\in\mathcal{P}_N}\omega_f|L_f(s_0,\chi)|^2 + \frac{r_N}{R-r_N}\left(O_{\sigma_0,k}(1)+O_{s_0,q,k}(N^{\frac{k}{2}-\sigma_0+R})\right). \quad (12)$$

Now applying Lemma 7 in (12) completes the proof. □

5 Proof of Theorem 1

We need the following estimation of ω_f.

Proposition 3 *For N prime we have*

$$\omega_f \ll_k \begin{cases} \log N/N & ; \ f \in \mathcal{F}_N \\ 1/N & ; \ f \in \mathcal{P}_N - \mathcal{F}_N \end{cases} . \tag{13}$$

Proof: See [4], Proposition 4 for the case $f \in \mathcal{F}_N$. If $f \in \mathcal{P}_N - \mathcal{F}_N$ then

$$f(z) = h(z) \pm N^{\frac{k}{2}} h(Nz)$$

as we mentioned in the proof of Lemma 1. Now the result follows from the fact that

$$< f, f > = < h(z) \pm N^{\frac{k}{2}} h(Nz), h(z) \pm N^{\frac{k}{2}} h(Nz) >$$

is bounded below by a constant multiple of N (See [1], Proposition 5.3 for the details). □

Now we can prove our theorem. Set

$$\mathcal{E}_N = \{ f \in \mathcal{P}_N : \ L_f(s, \chi) \neq 0, \ for \ all \ s \ in \ C_N \} .$$

Proposition 1 shows that $\mathcal{E}_N \neq \emptyset$ for large N. Now if $f \in \mathcal{P}_N - \mathcal{E}_N$ we choose s_f such that $L_f(s_f, \chi) = 0$. With this choice of s_f for elements of $\mathcal{P}_N - \mathcal{E}_N$ and arbitrary choice of s_f in C_N for elements of $\mathcal{E}_N$ and applying the Cauchy-Schwarz inequality and (13), we get

$$\left| \sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi) \right|^2 = \left| \sum_{f \in \mathcal{E}_N} \omega_f L_f(s_f, \chi) \right|^2 \leq \left(\sum_{f \in \mathcal{E}_N \cap \mathcal{F}_N} \omega_f + \sum_{f \in \mathcal{E}_N - \mathcal{F}_N} \omega_f \right) \left(\sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 \right)$$

$$\ll \left(\sharp \{ f \in \mathcal{F}_N ; \ L_f(s, \chi) \neq 0 \ for \ all \ s \in C_N \} \frac{\log N}{N} + 2 dim S_k(1) \frac{1}{N} \right) \sum_{f \in \mathcal{P}_N} \omega_f |L_f(s_f, \chi)|^2 . \tag{14}$$

Theorem 1 follows by applying propositions 1 and 2 in (14). □

6 Proof of Theorem 2

We first establish the analogues of Proposition 1, Lemma 7 and Proposition 2 for a point s_0 on the critical line $\sigma = \frac{k}{2}$.

Proposition $1'$ *Let N be prime, and let Γ and C_N be the circles with center $(\frac{k}{2}, t_0)$ and radius R_N and r_N respectively. Suppose that $0 < r_N < R_N < \frac{1}{2}$, and $\frac{r_N}{R_N} N^{R_N} (\log N)^{R_N} = \underline{o}\left(\frac{N^{\frac{1}{2}}}{\log N}\right)$. Then*

$$\sum_{f \in \mathcal{P}_N} \omega_f L_f(s_f, \chi) = 1 + O_{q,k}\left(\frac{1}{\Gamma(\frac{k}{2}+t_0)} N^{-\frac{1}{2}} \log N\right) + O_{q,k,t_0}\left(\frac{r_N}{R_N - r_N} N^{R_N - \frac{1}{2}} (\log N)^{R_N+1}\right)$$

where s_f is an arbitrary point in C_N.

Proof: It is similar to the proof of Proposition 1. □

Lemma $7'$ *Let χ be a fixed primitive Dirichlet character mod q with $(q, N) = 1$ and let $s_0 = \frac{k}{2} + it_0$. Then*

$$\sum_{f \in \mathcal{P}_N} \omega_f \left|L_f(\frac{k}{2} + it_0, \chi)\right|^2 = \prod_{p|q} (1 - \frac{1}{p}) \log N + c_1 + O_{t_0,q,k}(N^{-\frac{1}{2}} \log N)$$

for N prime. Here, c_1 depends on t_0, q and k.

Proof: The proof is exactly similar to the proof of Lemma 7. The result follows by observing that

$$\frac{1}{2\pi i} \int_{(\frac{5}{4})} L(2s+1, \chi_0)(2\pi)^{-2s} \Gamma(s + \frac{k}{2} + it_0) \Gamma(s + \frac{k}{2} - it_0) x^s \frac{ds}{s}$$

has a double pole at $s = \frac{k}{2}$ which contributes $\log N$ to the main term (see [1], Proposition 4.2 for the details). □

Lemma 8 *Let Γ be a circle with center $(\frac{k}{2}, t_0)$ and radius $0 < R_N < \frac{1}{2}$, and let w be a point on (or inside) Γ. Then if $\sigma = Re(w) \geq \frac{k}{2}$,*

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} (\log N)^4$$

and if $\sigma = Re(w) \leq \frac{k}{2}$,

$$\sum_{f \in \mathcal{P}_N} \omega_f |L_f(w, \chi)|^2 \ll_{k,q,t_0} N^{k-2\sigma} (\log N)^4.$$

Proof: First we assume that $\sigma = Re(w) \geq \frac{k}{2}$. Choosing $x = q^2 N \log N$ in Lemma 3 gives

$$\Gamma(w) L_f(w, \chi) = \sum_{n \geq 1} \frac{\chi(n) a_f(n)}{n^w} W(w, \frac{2\pi n}{q^2 N \log N}) + O_{q,k}(N^{-6+\frac{k}{2}-\sigma} (\log N)^{k-\sigma+1}).$$

Now by applying the upper bound of Lemma 1 for $a_f(n)$ and the upper bound of Lemma 4 for $W(w,.)$, we deduce that

$$\sum_{n>q^2N(\log N)^2} \frac{\chi(n)a_f(n)}{n^w} W(w, \frac{2\pi n}{q^2N\log N}) = O_{q,k}\left(N^{-5+\frac{k}{2}-\sigma}(\log N)^{k-\sigma}\right).$$

Therefore

$$\Gamma(w)L_f(w,\chi) = \sum_{n\le q^2N(\log N)^2} \frac{\chi(n)}{n^w} W(w, \frac{2\pi n}{q^2N\log N})a_f(n) + O_{q,k}\left(N^{-5}(\log N)^{\frac{k}{2}}\right). \quad (15)$$

We know that for complex numbers c_n

$$\sum_{f\in\mathcal{P}_N} \omega_f \Big|\sum_{n\le X} c_n a_f(n)\Big|^2 = \Big(1+O(N^{-1}X\log X)\Big)\sum_{n\le X} n^{k-1}|c_n|^2$$

with an absolute implied constant (see [5], Theorem 1). Applying this identity for $X = Nq^2(\log N)^2$, $c_n = \frac{\chi(n)}{n^w}W(w, \frac{2\pi n}{q^2N\log N})$, and using Lemma 4 imply that

$$\sum_{f\in\mathcal{P}_N} \omega_f \left|\sum_{n\le q^2N(\log N)^2} c_n a_f(n)\right|^2 = O_{q,k}\left((\log N)^3 \sum_{n\le q^2N(\log N)^2} \frac{1}{n^{2\sigma-k+1}}\right) = O_{q,k}\left((\log N)^4\right).$$

This together with (15) proves the lemma.

If $\sigma = Re(w) < \frac{k}{2}$ the results from the functional equation of $|L_f(w,\chi)|^2$. □

Proposition $2'$ *Let N be prime, and let Γ and C_N be the circles with center $(\frac{k}{2}, t_0)$ and radius R_N and r_N respectively. Suppose that $0 < r_N < R_N < \frac{1}{2}$ and $\frac{r_N N^{2R_N}}{R_N} = \underline{o}\left(\frac{1}{(\log N)^3}\right)$. Then*

$$\sum_{f\in\mathcal{P}_N} \omega_f |L_f(s_f,\chi)|^2 = \prod_{p|q}(1-\frac{1}{p})\log N + c_1 + + O_{t_0,q,k}(N^{-\frac{1}{2}}\log N) + O_{t_0,q,k}\left(\frac{r_N N^{2R_N}(\log N)^4}{R_N - r_N}\right)$$

where s_f is an arbitrary point in C_N and c_1 depends on t_0, q and k.

Proof: From the proof of Proposition 2, we know that

$$\sum_{f\in\mathcal{P}_N} \omega_f |L_f(s_f,\chi)|^2 = \sum_{f\in\mathcal{P}_N} \omega \Big|L_f(\frac{k}{2}+it,\chi)\Big|^2 + \sum_{f\in\mathcal{P}_N} \omega_f \left|\frac{1}{2\pi i}\int_\Gamma L_f{}^2(w,\chi)\frac{s_f - s_0}{(w-s_f)(w-s_0)}d\omega\right|.$$

The result follows by applying Lemma $7'$ in the above identity and the fact that by Lemma 8

$$\sum_{f\in\mathcal{P}_N} \omega_f \left|\frac{1}{2\pi i}\int_\Gamma L_f{}^2(w,\chi)\frac{s_f - s_0}{(w-s_f)(w-s_0)}d\omega\right| \le \frac{r_N}{R_N - r_N} O_{t_0,q,k}\left(N^{2R_N}(\log N)^4\right). \ □$$

Now in Proposition $1'$ and $2'$, let $R_N = \frac{1}{\log N}$ and $r_N = \frac{1}{(\log N)^{4+\epsilon}}$. We then proceed in a way similar to the proof of Theorem 1 and finally Theorem 2 follows by applying Proposition $1'$ and Proposition $2'$ in (14). □

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