On Non-Vanishing of Convolution of Dirichlet Series

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Abstract

We study the non-vanishing on the line $Re(s) = 1$ of the convolution series associated to two Dirichlet series in a certain class of Dirichlet series. The non-vanishing of various L-functions on the line $Re(s) = 1$ will be simple corollaries of our general theorems.

Let $f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} \hat{a}_g(n)e^{2\pi inz}$ be cusp forms of weight k and level N with trivial character. Let $L_f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$ and $L_g(s) = \sum_{n=1}^{\infty} a_g(n)n^{-s}$ be the L-functions associated to f and g, respectively, where $a_f(n) = \hat{a}_f(n)/n^{\frac{k-1}{2}}$ and $a_g(n) = \hat{a}_g(n)/n^{\frac{k-1}{2}}$. Let

$$L(f \otimes g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$. In [11] Rankin established the analytic continuation of $L(f \otimes g, s)$ (see Theorem 1.5). Rankin's Theorem has numerous number theoretic applications. In [10], Rankin used this theorem to prove the non-vanishing of the modular L-function associated to the discriminant function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

on the line $Re(s) = 1$. In fact, Rankin's argument establishes the non-vanishing of L-functions associated to eigenforms for the points on the line $Re(s) = 1$, except the point $s = 1$. In [9], Ogg proved that the same result is true for $s = 1$. Moreover, he showed the following.

Theorem 0.1 (Ogg) *If f and g are eigenforms with respect to the family of the Hecke operators for $\Gamma_0(N)$ and $\langle f, g\rangle = 0$, then $L(f \otimes g, 1) \neq 0$. Here $\langle f, g\rangle$ denote the Petersson inner product of f and g.*

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In this paper we prove similar non-vanishing results (Theorem 2.3, Theorem 3.5 and Theorem 4.2) for the convolution of two Dirichlet series belonging to a certain family of Dirichlet series $\mathcal{S}^*$ (see Definitions 1.1 and 1.2). Our theorems are quite general and clearly demonstrate the close connection between the analytic continuation of a Dirichlet series and its various convolutions to the left of its half plane of convergence and its non-vanishing on the line $Re(s) = 1$. More precisely, for two Dirichlet series F and $G \in \mathcal{S}^*$ with Euler products

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right) \quad \text{and} \quad G(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_G(p^k)}{p^{ks}}\right)$$

valid on $Re(s) > 1$, we define the *Euler product convolution* of F and G as

$$(F \otimes G)(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right).$$

We say $F \in \mathcal{S}^*$ is $\otimes$*-simple* in $Re(s) \geq \sigma_0$, if $F \otimes F$ has an analytic continuation to $Re(s) \geq \sigma_0$, except for a possible simple pole at $s = 1$. One of our main results is the following.

Theorem 2.3 *Let* $F,\ G \in \mathcal{S}^*$ *be* $\otimes$*-simple in* $Re(s) \geq 1$ *and* $t \neq 0$*. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) If $F \otimes G$ *has an analytic continuation to the line* $Re(s) = 1$ *and* $(F \otimes G)(s) = 0$ *if and only if* $(F \otimes G)(\bar{s}) = 0$ *for any* s *on the line* $Re(s) = 1$*, then* $(F \otimes G)(1 + it) \neq 0$.

Note that this result does not say anything about the value of $(F \otimes G)(s)$ at $s = 1$. to deal with this case, in Section 3 we prove a non-vanishing theorem, valid on the line $Re(s) = 1$, for Euler product convolution of two Dirichlet series in $\mathcal{S}^*$ with completely multiplicative coefficients (Theorem 3.5). Finally in Section 4 for Dirichlet series with general coefficients we prove the following.

Theorem 4.2 *Let* $\sigma_0 < 1$*, and assume the following:*

(i) F *and* G *(as elements of* $\mathcal{S}^*$*) are* $\otimes$*-simple in* $Re(s) > \sigma_0$*;*

(ii) $F \otimes G$ *has an analytic continuation to the half-plane* $Re(s) > \sigma_0$*;*

(iii) At least one of $F \otimes F$*,* $G \otimes G$*, or* $F \otimes G$ *has zeros in the strip* $\sigma_0 < Re(s) < 1$.

Then $(F \otimes G)(1 + it) \neq 0$ *for all real* t.

Our general theorems have several applications. The non-vanishing of various classical L-functions will be simple corollaries of our general theorems (see Corollaries 2.4, 2.6 and 4.4). Moreover, as a consequence of our theorems, we will be able to extend Ogg's theorem to the line $Re(s) = 1$ (Corollary 2.6, *(iv)*). Another application will result in an extension of Ogg's non-vanishing result to the line $Re(s) = 1$ and for eigenforms with characters (Corollary 4.4, *(iv)*). Corollary 4.4, *(ii)* gives a generalization of the non-vanishing result of Rankin to eigenforms with characters. Finally non-vanishing of twisted symmetric square L-functions on the line $Re(s) = 1$

(Corollary 4.4, *(v)*) is a simple consequence of our theorems. Our general theorems could also be applied to the L-functions associated to number fields, however, in applications of this paper we restrict ourselves to Dirichlet and modular L-functions. For results of these types in the context of automorphic forms and representations see [4], [12] and [13].

Our approach in this paper is motivated by [8] and section 8.4 of [6].

Notation In this paper we use the following notations:

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$: the Riemann zeta-function,

$\zeta_q(s) = \prod_{p|q}(1 - 1/p)\zeta(s)$: the Riemann zeta-function with the Euler p-factors corresponding to $p \mid N$ removed,

$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)/n^s$: the Dirichlet L-function associated to a Dirichlet character χ,

$S_k(N)$: the space of cusp forms of weight k and level N with the trivial character,

$S_k(N, \psi)$: the space of cusp forms of weight k and level N with character ψ where $\psi(-1) = (-1)^k$,

$\langle f, g\rangle = \int_{D_0(N)} f(z)\overline{g(z)}y^{k-2}dxdy$: the Petersson inner product of f, $g \in S_k(N, \psi)$. Here, $D_0(N)$ is a fundamental domain for the congruence subgroup $\Gamma_0(N)$,

$L_f(s) = \sum_{n=1}^{\infty} a_f(n)/n^s$: the L-function associated to a cusp form $f \in S_k(N, \psi)$,

$L_{f,\chi}(s) = \sum_{n=1}^{\infty} a_f(n)\chi(n)/n^s$: the twisted L-function associated to a cusp form $f \in S_k(N, \psi)$ and a Dirichlet character χ,

$L(f \otimes g, s) = L_{\psi_1\bar{\psi}_2}(2s) \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}/n^s$: the Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$, where $f \in S_k(N, \psi_1)$ and $g \in S_k(N, \psi_2)$,

$L(sym^2 f, s) = L(f \otimes f, s)/\zeta_N(s)$: the symmetric square L-function associated to a normalized eigenform f in $S_k(N)$,

$L_\chi(f \otimes g, s) = L_{\psi_1\bar{\psi}_2\chi^2}(2s) \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}\chi(n)/n^s$: the twisted Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$, where $f \in S_k(N, \psi_1)$, $g \in S_k(N, \psi_2)$ and χ is a Dirichlet character,

$L_\chi(sym^2 f, s) = L_\chi(f \otimes \bar{f}, s)/L_{\psi\chi}(s)$: the twisted symmetric square L-function associated to a normalized eigenform f with character ψ and a Dirichlet character χ.

Note that in the above definitions, we assume that $Re(s) > 1$ and for a normalized eigenform f we have $a_f(1) = 1$.

1 A Class of Dirichlet Series

We consider the following class of Dirichlet Series.

Definition 1.1 *The class $\mathcal{S}^{*}$[1] is the family of Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ ($Re(s) > 1$) satisfying the following properties:*

[1] We use this notation to emphasis the relation of this class to the Selberg class $\mathcal{S}$. Note that $\mathcal{S} \subset \mathcal{S}^*$. For the definition of the Selberg class $\mathcal{S}$ see [6], Chapter 8.

(a) (Euler Product): For $Re(s) > 1$, we have

$$F(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\right);$$

(b) (Ramanujan's Hypothesis): For any fixed $\epsilon > 0$,

$$a_F(n) = O(n^{\epsilon})$$

where the implied constant may depend upon ϵ.

(c) (Analytic Continuation): $F(s)$ has an analytic continuation to the line $Re(s) = 1$, except for a possible pole at point $s = 1$.

For $F \in \mathcal{S}^*$, we define

$$\bar{F}(s) = \overline{F(\bar{s})} = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{\overline{b_F(p^k)}}{p^{ks}}\right).$$

We continue by defining a convolution operation on $\mathcal{S}^*$.

Definition 1.2 *For $F,\ G \in \mathcal{S}^*$, the Euler product convolution of F and G is defined as*

$$(F \otimes G)(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right).$$

The following lemma shows that this operation is well-defined on the half plane $Re(s) > 1$.

Lemma 1.3 *For $F,\ G$ in $\mathcal{S}^*$, $(F \otimes G)(s)$ is convergent for $Re(s) > 1$.*

Proof Let $\epsilon > 0$. One can show that $|a_F(n)| \leq c(\epsilon)n^{\epsilon}$ implies

$$|b_F(p^k)| \leq \frac{c(\epsilon)(2^k - 1)p^{k\epsilon}}{k} \tag{1}$$

(see [6] Exercise 8.2.9). Now suppose that $\sigma = Re(s) \geq 1 + 3\epsilon$. By applying (1) and the expansion

$$-\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \tag{2}$$

valid for $|z| < 1$, we have

$$\exp\left(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\right) \ll \exp\left(\sum_{k=1}^{\infty} \frac{(2^k - 1)^2 p^{2k\epsilon}/k}{p^{k\sigma}}\right) \ll \left(1 - \frac{4}{p^{\sigma - 2\epsilon}}\right)^{-1}.$$

Since $\sigma - 2\epsilon \geq 1 + \epsilon > 1$, the product $\prod_p \left(1 - \frac{4}{p^{\sigma-2\epsilon}}\right)$ is convergent. □

The next lemma will enable us to express several classical L-functions of number theory as Euler product convolution of two simpler L-functions. This lemma plays an important role in the applications of our general theorems.

Lemma 1.4 *(i)* $\zeta(s)$ *is in* $\mathcal{S}^*$*, and for any* F *in* $\mathcal{S}^*$*, we have*

$$(F \otimes \zeta)(s) = F(s).$$

(ii) For F *in* $\mathcal{S}^*$*, we have*

$$(\zeta \otimes F)(s) = \bar{F}(s).$$

(iii) If χ *is a Dirichlet character (mod q), then* $L_\chi(s)$ *is in* $\mathcal{S}^*$*, and*

$$(L_\chi \otimes L_\chi)(s) = \zeta_q(s).$$

(iv) Let f *be a normalized eigenform in* $S_k(N, \psi)$*. Then* $L_f(s)$ *is in* $\mathcal{S}^*$*, and*

$$(L_f \otimes L_\chi)(s) = L_{f,\overline{\chi}}(s).$$

(v) For any two normalized eigenforms $f \in S_k(N, \psi_1)$ *and* $g \in S_k(N, \psi_2)$ *and Dirichlet characters* χ_1 *and* χ_2*,* $L_{f,\chi_1}(s)$ *and* $L_{g,\chi_2}(s)$ *are in* $\mathcal{S}^*$*, and*

$$(L_{f,\chi_1} \otimes L_{g,\chi_2})(s) = L_{\chi_1\bar{\chi}_2}(f \otimes g, s).$$

Proof We only prove the identity in part *(v)*. Using (2) we have

$$L_{f,\chi_1}(s) = \prod_p (1 - a_f(p)\chi_1(p)p^{-s} + \psi_1(p)\chi_1(p)^2 p^{-2s})^{-1}$$

$$= \prod_p (1 - \alpha_f(p)\chi_1(p)p^{-s})^{-1}(1 - \beta_f(p)\chi_1(p)p^{-s})^{-1} = \prod_p \exp\left(\sum_{l=1}^{\infty} \frac{(\alpha_f(p)^l + \beta_f(p)^l)\chi_1(p)^l/l}{p^{ls}}\right)$$

where $\alpha_f(p)+\beta_f(p) = a_f(p)$, $\alpha_f(p)\beta_f(p) = \psi_1(p)$. We have also a similar product representation for $L_{g,\chi_2}(s)$. So

$$(L_{f,\chi_1} \otimes L_{g,\chi_2})(s) = \prod_p \exp\left(\sum_{l=1}^{\infty} \frac{(\alpha_f(p)^l + \beta_f(p)^l)(\overline{\alpha_g(p)}^l + \overline{\beta_g(p)}^l)\chi_1(p)^l\overline{\chi_2(p)}^l/l}{p^{ls}}\right)$$

$$= \prod_p (1 - \alpha_f(p)\overline{\alpha_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}(1 - \alpha_f(p)\overline{\beta_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}$$

$$\times \ (1 - \beta_f(p)\overline{\alpha_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}(1 - \beta_f(p)\overline{\beta_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}.$$

Applying the identity $a_f(p^l) = a_f(p)a_f(p^{l-1}) - \psi_1(p)a_f(p^{l-2})$ (and a similar one for the coefficients of g) repeatedly yields

$$a_f(p^l)\overline{a_g(p^l)} - a_f(p)a_f(p^{l-1})\overline{a_g(p)a_g(p^{l-1})} + (\overline{\psi_2(p)}a_f(p)^2 + \psi_1(p)\overline{a_g(p)}^2 - 2\psi_1(p)\overline{\psi_2(p)})a_f(p^{l-2})\overline{a_g(p^{l-2})}$$
$$-\psi_1(p)\overline{\psi_2(p)}a_f(p)a_f(p^{l-3})\overline{a_g(p)a_g(p^{l-3})} + \psi_1(p)^2\overline{\psi_2(p)}^2 a_f(p^{l-4})\overline{a_g(p^{l-4})} = 0.$$

Using this we arrive at

$$(1 - \psi_1(p)\overline{\psi_2(p)}\chi_1(p)^2\overline{\chi_2(p)}^2 p^{-2s})^{-1} \sum_{l=0}^{\infty} \frac{a_f(p^l)\overline{a_g(p^l)}\chi_1(p^l)\overline{\chi_2(p^l)}}{p^{ls}}$$
$$= \prod_p (1 - \alpha_f(p)\overline{\alpha_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}(1 - \alpha_f(p)\overline{\beta_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}$$
$$\times\ (1 - \beta_f(p)\overline{\alpha_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}(1 - \beta_f(p)\overline{\beta_g(p)}\chi_1(p)\overline{\chi_2(p)}p^{-s})^{-1}.$$

Therefore

$$L_{\chi_1\bar{\chi}_2}(f \otimes g, s) = L_{\psi_1\bar{\psi}_2\chi_1{}^2\bar{\chi}_2^2}(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}\chi_1(n)\overline{\chi_2(n)}}{n^s}$$
$$= (1 - \psi_1(p)\overline{\psi_2(p)}\chi_1(p)^2\overline{\chi_2(p)}^2 p^{-2s})^{-1} \sum_{l=0}^{\infty} \frac{a_f(p^l)\overline{a_g(p^l)}\chi_1(p^l)\overline{\chi_2(p^l)}}{p^{ls}} = (L_{f,\chi_1} \otimes L_{g,\chi_2})(s).$$

This completes the proof. □

In our applications, we also need the following theorem of Rankin [11].

Theorem 1.5 (Rankin) *Let* $f \in S_k(N, \psi_1)$ *and* $g \in S_k(N, \psi_2)$. *Let*

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s + k - 1)L(f \otimes g, s).$$

Then both $L(f \otimes g, s)$ *and* $\Phi(s)$ *are entire if* $\psi_1 \neq \psi_2$ *or* $\langle f, g\rangle = 0$. *Otherwise, for* $N = 1$ *they are analytic everywhere except that* $L(f \otimes g, s)$ *has a simple pole at* $s = 1$ *and* $\Phi(s)$ *has simple poles at points* $s = 0$ *and* 1, *and for* $N > 1$ *both* $L(f \otimes g, s)$ *and* $\Phi(s)$ *are analytic except a simple pole at* $s = 1$.

2 Mertens's Method

In 1898 Mertens gave a proof for the non-vanishing of $\zeta(s)$ on the line $Re(s) = 1$. Mertens's proof depends upon the choice of a suitable trigonometric inequality. This line of proof is adaptable for establishing the non-vanishing of various L-functions. For example in [10], Rankin used this method to prove the non-vanishing of $L_f(s)$ on the line $Re(s) = 1$, $s \neq 1$, where f is an eigenform for $\Gamma_0(N)$. Another example is the proof of the following lemma, due to K. Murty [7], which, similar to Mertens's proof, depends on a certain trigonometric inequality.

Lemma 2.1 *Let $f(s)$ be a complex function satisfying the following:*

(i) $f(s)$ is analytic in $Re(s) > 1$ and non-zero there;

(ii) $\log f(s)$ can be written as a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with $b_n \geq 0$ for $Re(s) > 1$;

(iii) On the line $Re(s) = 1$, $f(s)$ is analytic except for a pole of order $e \geq 0$ at $s = 1$. Then, if $f(s)$ has a zero on the line $Re(s) = 1$, the order of that zero is bounded by $\frac{e}{2}$.

Proof See [7], Lemma 3.2. □

Here by employing the above lemma we prove a conditional theorem regarding the non-vanishing of $(F \otimes G)(s)$ on the punctured line $Re(s) = 1$ $(s \neq 1)$. The following definition describes one of the main conditions of our theorem.

Definition 2.2 *For $F \in \mathcal{S}^*$ and $\sigma_0 \leq 1$, we say F is $\otimes$-simple in $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), if $F \otimes F$ has an analytic continuation to $Re(s) > \sigma_0$ (resp. $Re(s) \geq \sigma_0$), except for a possible simple pole at $s = 1$.*

The following theorem is the main result of this section.

Theorem 2.3 *Let $F,\ G \in \mathcal{S}^*$ be $\otimes$-simple in $Re(s) \geq 1$ and $t \neq 0$. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) If $F \otimes G$ has an analytic continuation to the line $Re(s) = 1$ and $(F \otimes G)(s) = 0$ if and only if $(F \otimes G)(\bar{s}) = 0$ for any s on the line $Re(s) = 1$, then $(F \otimes G)(1 + it) \neq 0$.

Proof *(i)* Let $f(s) = (F \otimes F)(s)$. We have

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k)|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 2.1 with $e = 1$. Therefore, the order of the vanishing of $f(s)$ at point $1 + it$ is $\leq \frac{1}{2}$. This means that $(F \otimes F)(1 + it) \neq 0$.

(ii) Let

$$f(s) = (F \otimes F)(s)\ (F \otimes G)(s)\ (G \otimes F)(s)\ (G \otimes G)(s).$$

Since for $t \neq 0$, all the factors of $f(s)$ have finite values at point $1 + it$, in order to prove that $(F \otimes G)(1 + it) \neq 0$, it suffices to show that $f(1 + it) \neq 0$. Note that

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k) + b_G(p^k)|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 2.1 with $e \leq 2$, and therefore, the order of the vanishing of $f(s)$ at point $1+it$ is ≤ 1. Now suppose that $f(1+it) = 0$. Thus,

$$(F \otimes F)(1+it)\ (F \otimes G)(1+it)\ \overline{(F \otimes G)(1-it)}\ (G \otimes G)(1+it) = 0.$$

Since by part *(i)*, $(F \otimes F)(1+it) \neq 0$ and $(G \otimes G)(1+it) \neq 0$, it follows that $(F \otimes G)(1+it) = 0$. This is a contradiction, otherwise, the order of the vanishing of $f(s)$ at point $1+it$ should be 2. □

Note In Theorem 2.3 in fact we can have $(F \otimes G)(1) = 0$. To see this, Let $F(s) = \sum_{n=1}^{\infty} \frac{(-1)^{\Omega(n)}}{n^s}$ and $G(s) = \zeta(s)$, where $\Omega(n)$ is the total number of prime factors of n. Then we have $(F \otimes G)(s) = \dfrac{\zeta(2s)}{\zeta(s)}$ and so $(F \otimes G)(1) = 0$.

Corollary 2.4 *Let $f \in S_k(N, \psi)$ be a normalized eigenform for $\Gamma_0(N)$ and let $t \neq 0$. Then*
(i) $\zeta(1+it) \neq 0$.
(ii) $L(f \otimes f, 1+it) \neq 0$.
(iii) For trivial ψ we have $L(sym^2 f, 1+it) \neq 0$. Here t is any real number including zero.

Proof *(i)* This is a consequence of part *(i)* of Theorem 2.3 with $F(s) = \zeta(s)$.

(ii) From part *(v)* of Lemma 1.4 we have $(L_f \otimes L_f)(s) = L(f \otimes f, s)$. By Theorem 1.5 we know that $L(f \otimes f, s)$ is entire except a simple pole at $s = 1$. Thus $L_f(s)$ is $\otimes$-simple in the whole plane. So $L_f(s)$ satisfies all the conditions of part *(i)* of Theorem 2.3 and we have

$$L(f \otimes f, 1+it) = (L_f \otimes L_f)(1+it) \neq 0.$$

(iii) Note that $L(sym^2 f, s) = L(f \otimes f, s)/\zeta_N(s)$. So the result follows from part *(i)* and *(iii)* for $t \neq 0$. For $t = 0$, $L(sym^2 f, 1)$ in a non-zero multiple of $\langle f, f \rangle$ (see [11], Theorem 3 (iii)) and therefore it is non-vanishing. □

Corollary 2.5 *If $F = \bar{F} \in \mathcal{S}^*$ is analytic and $\otimes$-simple in $Re(s) \geq 1$, then $F(1+it) \neq 0$ for $t \neq 0$.*

Proof This is a simple consequence of part *(ii)* of the previous theorem with $G(s) = \zeta(s)$. □

Corollary 2.6 *Let $f \in S_k(N)$ be an eigenform for $\Gamma_0(N)$, let χ be a real non-trivial Dirichlet character (mod q), and let $t \neq 0$. Then*
(i) $L_\chi(1+it) \neq 0$.
(ii) $L_f(1+it) \neq 0$.
(iii) $L_{f,\chi}(1+it) \neq 0$.
(iv) Suppose $g \in S_k(N)$ is also an eigenform for $\Gamma_0(N)$. If $\langle f, g \rangle = 0$, then $L(f \otimes g, 1+it) \neq 0$.

Proof Note that without loss of generality, we can assume that f is normalized.

(i) By part *(iii)* of Lemma 1.4, $L_\chi(s)$ is $\otimes$-simple in the whole plane. Since $L_\chi(s)$ is analytic on the line $Re(s) = 1$, by Corollary 2.5 we have the desired results.

(ii) Part *(v)* of Lemma 1.4 and Theorem 1.5 imply that $L_f(s)$ is $\otimes$-simple. Since $L_f(s)$ is analytic on the line $Re(s) = 1$, by Corollary 2.5 $L_f(1 + it) \neq 0$.

(iii) As we showed in part (i) and (ii) $L_\chi(s)$ and $L_f(s)$ are $\otimes$-simple. Now since f is an eigenform and χ is real, the coefficients of $L_{f,\chi}(s)$ are real. Also $L_{f,\chi}(s)$ is the L-function associated to a cusp form of weight k and level q^2N (see [5], p. 127, Proposition 17 (b)). So, $L_{f,\chi}(s)$ is analytic on the line $Re(s) = 1$. Therefore, by part *(iv)* of Lemma 1.4 and part *(ii)* of Theorem 2.3,

$$L_{f,\chi}(1 + it) = (L_f \otimes L_\chi)(1 + it) \neq 0.$$

(iv) Note that the coefficients of eigenforms are real and $L_f(s)$ and $L_g(s)$ are $\otimes$-simple in the whole plane. If $\langle f, g\rangle = 0$, by Theorem 1.5 $L(f \otimes g, s)$ is actually an entire function. Therefore, by part *(v)* of Lemma 1.4 and part *(ii)* of Theorem 2.3, we have

$$L(f \otimes g, 1 + it) = (L_f \otimes L_g)(1 + it) \neq 0.$$

This completes the proof. □

3 Ingham's Method

One of the main facts regarding Dirichlet series with positive coefficients is the following result of Landau.

Lemma 3.1 (Landau) *A Dirichlet series with non-negative coefficients has a singularity at its abscissa of convergence.*

Proof See [6], Exercise 2.5.14. □

In this section, we will show that for two Dirichlet series in $\mathcal{S}^*$ with completely multiplicative coefficients[2], one can apply this lemma of Landau to prove a non-vanishing result, valid on the line $Re(s) = 1$, for the convolution series. Our result is a generalization of Ingham's proof of the non-vanishing of the Riemann zeta-function on the line $Re(s) = 1$ [3]. To do this, we start with recalling some results regarding Dirichlet series with completely multiplicative coefficients and completely multiplicative arithmetic functions.

[2]This means $a_F(mn) = a_F(m)a_F(n)$ for every m and n.

Lemma 3.2 *For $F, G \in \mathcal{S}^*$ with completely multiplicative coefficients,*

$$(F \otimes G)(s) = \sum_{n=1}^{\infty} \frac{a_F(n)\overline{a_G(n)}}{n^s}.$$

Proof We have

$$(F \otimes G)(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{a_F(p)^k \left(\overline{a_G(p)}\right)^k /k}{p^{ks}} \right) = \prod_p \left(1 - a_F(p)\overline{a_G(p)}p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_F(n)\overline{a_G(n)}}{n^s}.$$

□

Definition 3.3 *If $f(n)$ is an arithmetic function, the formal L-series attached to $f(n)$ is defined by*

$$L(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

If $g(n)$ is also an arithmetic function, the Dirichlet convolution of $f(n)$ and $g(n)$ is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}).$$

The following identity of formal L-series, due to J. Borwein and Choi [1], will be fundamental in the proof of the main result of this section.

Lemma 3.4 *Let f_1, f_2, g_1, g_2 be completely multiplicative arithmetic functions. Then we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n)(f_2 * g_2)(n)}{n^s} = \frac{L(f_1 f_2, s)L(g_1 g_2, s)L(f_1 g_2, s)L(f_2 g_1, s)}{L(f_1 f_2 g_1 g_2, 2s)}.$$

Proof See [1], Theorem 2.1. □

We are ready to state and prove the main result of this section.

Theorem 3.5 *Let F, $G \in \mathcal{S}^*$ be two Dirichlet series with completely multiplicative coefficients. Also assume the following:*

(i) F and G are $\otimes$-simple in $Re(s) > \frac{1}{2}$;

(ii) $F \otimes G$ has an analytic continuation to $Re(s) > \frac{1}{2}$;

(iii) $(F \otimes G) \otimes (F \otimes G)$ is analytic for $Re(s) > 1$ and has a pole at $s = 1$.

(iv) $(F \otimes F)(s)$, $(G \otimes G)(s)$ and $(F \otimes G)(s)$ have finite limits as $s \to \frac{1}{2}^+$ [3].

Then, $(F \otimes G)(1 + it) \neq 0$ for all t.

[3] This means $s = \sigma + it \to \frac{1}{2} + it$ for any t as $\sigma \to \frac{1}{2}^+$.

Proof Let

$$F(s)=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s},\ G(s)=\sum_{n=1}^{\infty}\frac{a_G(n)}{n^s}$$

and suppose that $(F\otimes G)(1+it_0)=0$ for a real t_0. Let

$$f_1(n)=a_F(n)n^{-it_0},\ f_2(n)=\overline{a_F(n)}n^{it_0},\ g_1(n)=a_G(n),\ g_2(n)=\overline{a_G(n)},$$

and for $Re(s)>1$, consider the following Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}\frac{|(f_1*g_1)(n)|^2}{n^s}=\sum_{n=1}^{\infty}\frac{(f_1*g_1)(n)(f_2*g_2)(n)}{n^s}.$$

Since f_1 and f_2 are completely multiplicative, by Lemma 3.2 we have

$$L(f_1f_2,s)=\sum_{n=1}^{\infty}\frac{|a_F(n)|^2}{n^s}=(F\otimes F)(s).$$

Similarly, we can derive the following

$$L(g_1g_2,s)=(G\otimes G)(s),\quad L(f_1g_2,s)=(F\otimes G)(s+it_0),\quad L(f_2g_1,s)=(G\otimes F)(s-it_0),$$

and

$$L(f_1f_2g_1g_2,2s)=\left[(F\otimes G)\otimes(F\otimes G)\right](2s).$$

So, by Lemma 3.4 and for $Re(s)>1$, we have

$$f(s)=\frac{(F\otimes F)(s)\ (F\otimes G)(s+it_0)\ (G\otimes F)(s-it_0)\ (G\otimes G)(s)}{\left[(F\otimes G)\otimes(F\otimes G)\right](2s)}.$$

Now by assumption of $(F\otimes G)(1+it_0)=0$ we have in fact the analyticity of $f(s)$ for $Re(s)>\frac{1}{2}$, and since the coefficients in the series are non-negative, by Lemma 3.1 the Dirichlet series representing $f(s)$ is convergent for $Re(s)>\frac{1}{2}$. So, for $\eta>0$, we have

$$f\left(\frac{1}{2}+\eta\right)=\sum_{n=1}^{\infty}\frac{|(f_1*g_1)(n)|^2}{n^{\frac{1}{2}+\eta}}\geq 1.$$

However, since $(F\otimes G)\otimes(F\otimes G)$ has a pole at $s=1$,

$$\left[(F\otimes G)\otimes(F\otimes G)\right]\left(2\left(\frac{1}{2}+\eta\right)\right)=\left[(F\otimes G)\otimes(F\otimes G)\right](1+2\eta)\to\infty$$

as $\eta\to 0^+$. This shows that

$$\lim_{\eta\to 0^+} f\left(\frac{1}{2}+\eta\right)=0,$$

which is a contradiction. □

By choosing $G(s) = \zeta(s)$ in the previous theorem, we have

Corollary 3.6 *Let $F \in \mathcal{S}^*$ be analytic and $\otimes$-simple in $Re(s) > \frac{1}{2}$. If the coefficients of F are completely multiplicative and $F(s)$ together with $(F \otimes F)(s)$ have finite limits as $s \to \frac{1}{2}^+$, then $F(1+it) \neq 0$, for all $t \in \mathbb{R}$.*

The following non-vanishing results are simple consequences of the previous corollary.

Corollary 3.7 *Let χ be a non-trivial Dirichlet character and let $f(n)$ be a completely additive[4] arithmetic function and let $t \in \mathbb{R}$. Then*

(i) $L_\chi(1+it) \neq 0$.

(ii) If $\sum_{n \le x} (-1)^{f(n)} \chi(n) = O(x^\delta)$ for $\delta < \frac{1}{2}$, then $L(s) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{f(n)} \chi(n)}{n^s}$ is analytic in $Re(s) > \delta$ and $L(1+it) \neq 0$.

4 Ogg's Method

In section 2, we proved a general non-vanishing result on the line $Re(s) = 1$, however our result were applicable mostly for Dirichlet series with real coefficients and also it did not cover the point $s = 1$. In the previous section we overcome these difficulties for the case of Dirichlet series with completely multiplicative coefficients. In this section, we consider the extension of the results of Section 3 to Dirichlet series with general coefficients. Our approach in this section is motivated by a paper of Ogg [9]. The following lemma describes the basic ingredient of this approach.

Lemma 4.1 *Let $f(s)$ be a complex function that satisfies the following:*

(i) $f(s)$ is analytic on the half-plane $Re(s) > \sigma_0$;

(ii) $\log f(s)$ has a representation in terms of a Dirichlet series with non-negative coefficients on the half-plane $Re(s) > \sigma_1$ ($\sigma_1 > \sigma_0$).

Then $f(s) \neq 0$ for $Re(s) > \sigma_0$.

Proof Let σ_2 be the largest real zero of f ($\sigma_0 < \sigma_2 \le \sigma_1$). Since $\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ for $Re(s) > \sigma_1$ ($c(n) \ge 0$), and since $\log f(s)$ is analytic in a neighborhood of the segment $\sigma_2 < \sigma \le \sigma_1$, then by Lemma 3.1, we have $\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ for $Re(s) > \sigma_2$. Thus,

$$\log |f(\sigma)| = Re(\log f(\sigma)) = \log f(\sigma) = \sum_{n=1}^{\infty} \frac{c(n)}{n^\sigma} \ge 0$$

[4]This means $f(mn) = f(m)f(n)$ holds for all m and n.

for $\sigma > \sigma_2$. Therefore, $|f(\sigma)| \geq 1$ for $\sigma > \sigma_2$. This contradicts the assumption $f(\sigma_2) = 0$, and therefore f has no real zero $\sigma > \sigma_0$. So $\log f(s)$ is analytic on the interval $(\sigma_0,\ \sigma_1]$, and Lemma 3.1 in fact shows that $\log f(s)$ exists and is analytic for $Re(s) > \sigma_0$. This means that $f(s)$ is non-zero for $Re(s) > \sigma_0$. □

Here, we prove the main result of this section.

Theorem 4.2 *Let $\sigma_0 < 1$, and assume the following:*

(i) F and G (as elements of $\mathcal{S}^*$) are $\otimes$-simple in $Re(s) > \sigma_0$;

(ii) $F \otimes G$ has an analytic continuation to the half-plane $Re(s) > \sigma_0$;

(iii) At least one of $F \otimes F$, $G \otimes G$, or $F \otimes G$ has zeros in the strip $\sigma_0 < Re(s) < 1$.

Then $(F \otimes G)(1 + it) \neq 0$ for all real t.

Proof Suppose that $(F \otimes G)(1 + it_0) = 0$, and let

$$f(s) = (F \otimes F)(s)\ (F \otimes G)(s + it_0)\ (G \otimes F)(s - it_0)\ (G \otimes G)(s).$$

First of all note that $G \otimes F$ is analytic for $Re(s) > \sigma_0$. Since $(F \otimes G)(1 + it_0) = 0$, then $(G \otimes F)(1 - it_0) = 0$, and since $s = 1$ is a pole of order ≤ 1 for both $F \otimes F$ and $G \otimes G$, we conclude that $f(s)$ is analytic at point $s = 1$, and therefore, analytic for $Re(s) > \sigma_0$. Now note that for $Re(s) > 1$,

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k) + b_G(p^k)p^{ikt_0}|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

where $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of the Corollary 4.1 with $\sigma_1 = 1$, and therefore, $f(s) \neq 0$ for $Re(s) > \sigma_0$. This contradicts our assumption in *(iii)*. □

The following corollary gives an extension of Corollary 3.6 to the Dirichlet series with general coefficients.

Corollary 4.3 *Let $F \in \mathcal{S}^*$ be analytic and $\otimes$-simple in $Re(s) \geq \frac{1}{2}$, then $F(1 + it) \neq 0$.*

Proof Let $G(s) = \zeta(s)$. Note that $F \otimes G = F$ and note that $\zeta(s)$ has zeros in the half-plane $Re(s) \geq 1/2$ (see [2], p. 97). Thus, $F(1 + it) = (F \otimes G)(1 + it) \neq 0$. □

Corollary 4.4 *Let $f \in S_k(N, \psi_1)$ and $g \in S_k(N, \psi_2)$ be eigenforms for $\Gamma_0(N)$, let χ be a non-trivial Dirichlet character (mod q) and let t be any real number. Then*

(i) $L_\chi(1 + it) \neq 0$.

(ii) $L_f(1 + it) \neq 0$.

(iii) $L_{f,\chi}(1+it) \neq 0$.

(iv) If $\psi_1 \neq \psi_2$ *or* $\langle f, g\rangle = 0$, *then* $L(f \otimes g, 1+it) \neq 0$.

(v) Let $\bar{f}_{\bar{\chi}}(z) = \sum_{n=1}^{\infty} \overline{a_f(n)\chi(n)} e^{2\pi i n z}$. *Then if* $\psi\chi$ *is not a real character of order 2 or* $\int_{D_0(Nq^2)} f(z)\overline{\bar{f}_{\bar{\chi}}(z)} y^{k-2} dxdy = 0$, *we have* $L_\chi(f \otimes \bar{f}, 1+it) \neq 0$ *and* $L_\chi(sym^2 f, 1+it) \neq 0$. *Here* $D_0(Nq^2)$ *is a fundamental domain for* $\Gamma_0(Nq^2)$.

Proof *(i), (ii)* Both are simple consequences of Corollary 4.3. Note that $L_\chi(s)$ and $L_f(s)$ are entire and $\otimes$-simple in the whole plane.

(iii) Note that $L_{\bar{\chi}}(s)$ and $L_f(s)$ are $\otimes$-simple. Also by part *(iv)* of Lemma 1.4, we have $L_{f,\chi}(s) = (L_f \otimes L_{\bar{\chi}})(s)$. We know that $L_{f,\chi}(s)$ is a cusp form of weight k, level Nq^2 and character $\psi_1\chi^2$ (see [5], p. 127), so $(L_f \otimes L_\chi)(s)$ is entire. Also note that $(L_{\bar{\chi}} \otimes L_{\bar{\chi}})(s) = \zeta_q(s)$ has in fact infinitely many zeros (see [2], p. 97). So, all the conditions of Theorem 4.2 are met and therefore, $L_{f,\chi}(1+it) = (L_f \otimes L_{\bar{\chi}})(1+it) \neq 0$.

(iv) By Theorem 1.5 we can show that conditions *(i)* and *(ii)* of Theorem 4.2 are satisfied. The result will be obtained if we only show that $L(f \otimes g, s)$ has a zero in the half-plane $Re(s) < 1$. Again by Theorem 1.5, if $\psi_1 \neq \psi_2$ or $\langle f, g\rangle = 0$, then

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s)$$

is analytic at $s = 0$. Since $\Gamma(s)$ has a pole at $s = 0$, then $L(f \otimes g, 0) = 0$.

(v) First of all note that $\bar{f}_{\bar{\chi}} \in S_k(Nq^2, \psi\chi^2)$ (see [5], p. 127) and $L_\chi(f \otimes \bar{f}, s) = L(f \otimes \bar{f}_{\bar{\chi}}, s)$. So under the given conditions, by part *(iv)* we have $L_\chi(f \otimes \bar{f}, 1+it) \neq 0$. This together with part *(i)* imply that $L_\chi(sym^2 f, 1+it) = L_\chi(f \otimes \bar{f}, 1+it)/L_{\psi\chi}(1+it) \neq 0$. □

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