Uniform Distribution of Zeros of Dirichlet Series

Amir Akbary and M. Ram Murty*

Abstract

We consider a class of Dirichlet series which is more general than the Selberg class. Dirichlet series in this class, have meromorphic continuation to the whole plane and satisfy a certain functional equation. We prove, under the assumption of a certain hypothesis concerning the density of zeros on average, that the sequence formed by the imaginary parts of the zeros of a Dirichlet series in this class is uniformly distributed mod 1. We also give estimations for the discrepancy of this sequence.

1 Introduction

Let $\{x\}$ be the fractional part of x. The sequence (γ_n) of real numbers is said to be uniformly distributed mod 1 if for any pair a and b of real numbers with $0 \leq a < b \leq 1$ we have

$$\lim_{N\to\infty} \frac{\#\{n \leq N;\ a \leq \{\gamma_n\} < b\}}{N} = b - a.$$

To determine whether a sequence of real numbers is uniformly distributed we have the following widely applicable criterion.

Weyl's Criterion (Weyl, 1914) *The sequence (γ_n), $n = 1, 2, \cdots$, is uniformly distributed mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m \gamma_n} = 0, \quad \text{for all integers } m \neq 0.$$

*Research of both authors is partially supported by NSERC.

For a proof see [8], Theorem 2.1.

Here we are interested in studying the uniform distribution mod 1 of the sequence formed by the imaginary parts of the zeros of an arithmetic or geometric L-series. In the case of the Riemann zeta function, Rademacher observed in [16] that under the assumption of the Riemann Hypothesis the sequence $(\alpha\gamma_n)$ is uniformly distributed in the interval $[0, 1)$, where α is a fixed non-zero real number and γ_n runs over the imaginary parts of zeros of $\zeta(s)$. Later Hlawka [6] proved this assertion unconditionally. Moreover, for $\alpha = \frac{\log x}{2\pi}$, where x is an integer, Hlawka proved that the discrepancy of the set $\{\{\alpha\gamma_n\} : \ 0 < \gamma_n \le T\}$ is $O(1/\log T)$, under the assumption of the Riemann Hypothesis, and that it is $O(1/\log\log T)$ unconditionally. We emphasize that Hlawka's result does not cover the case corresponding to $\alpha = 1$, since in this case $x = e^{2\pi}$ is a transcendental number by a classical theorem of Gelfond. Finally in [3], Fujii proved that the discrepancy is $O(\log\log T/\log T)$ unconditionally for any non-zero α.

In this paper, we consider generalization of these results to a large class of Dirichlet series which includes the Selberg class. In Section 2 we define and study some elementary properties of the elements of this class. Dirichlet series in this class, have multiplicative coefficients, have meromorphic continuation to the whole plane and satisfy a certain functional equation. Let $\tilde{\mathcal{S}}$ denote this class, and let $\beta + i\gamma$ denote a zero of an element F of this class. Let $N_F(T)$ be the number of zeros of F with $0 \le \beta \le 1$ and $0 \le \gamma \le T$. We introduce the following.

Average Density Hypothesis *We say that $F \in \tilde{\mathcal{S}}$ satisfies the Average Density Hypothesis if*

$$\sum_{\substack{0\le\gamma\le T\\ \beta>\frac{1}{2}}} (\beta - \frac{1}{2}) = o(N_F(T)),$$

as $T \to \infty$.

Let $N_F(\sigma, T)$ be the number of zeros of F with $0 \le \gamma \le T$ and $\beta \ge \sigma$. The Riemann Hypothesis for F states that $N_F(\sigma, T) = 0$, for $\sigma > \frac{1}{2}$. A Density Hypothesis for F, which is weaker than the Riemann Hypothesis, usually refers to a desired upper bound for $N_F(\sigma, T)$ which holds uniformly for $\frac{1}{2} \le \sigma \le 1$. Several formulations of such hypothesis have been given in the literature. One can easily show that

$$\sum_{\substack{0\le\gamma\le T\\ \beta>\frac{1}{2}}} (\beta-\frac{1}{2}) = \int_{\frac{1}{2}}^{1} N_F(\sigma,T)\ d\sigma$$

(see Section 4). This explain why we call the above statement an Average Density Hypothesis.

Let (γ_n), $\gamma_n \geq 0$, be the sequence formed by imaginary parts of the zeros of F (ordered increasingly). In Section 3, by employing Weyl's criterion for uniform distribution, we prove the following.

Theorem 6 *Let $F \in \tilde{\mathcal{S}}$. Suppose that F satisfies the Average Density Hypothesis, then for $\alpha \neq 0$, $(\alpha\gamma_n)$ is uniformly distributed mod 1.*

Consequently under the analogue of the Riemann Hypothesis for this class, the imaginary parts of zeros of elements of this class are uniformly distributed mod 1. However, proving these facts unconditionally is a new and a difficult problem in analytic number theory.

To establish some unconditional results, in Section 4 we prove that if there is a real $k > 0$ such that the k-th moment of F satisfies a certain bound then the Average Density Hypothesis is true for F. Such moment bound is known (unconditionally) for several important group of Dirichlet series. As a consequence of this observation, in Section 5 we prove that Theorem 6 is true (unconditionally) for the classical Dirichlet L-series, L-series attached to modular forms and L-series attached to Maass wave forms. To extend these results further seems to be difficult. For example, can one show such a result for zeros of Dedekind zeta function attached to an arbitrary number field K? In the case that K is abelian over $\mathbb{Q}$, we are able to do this. If the field is not abelian over $\mathbb{Q}$, the results can be extended in extremely special cases (for example in the case that K is a dihedral extension of $\mathbb{Q}$). The problem is intimately related to proving the Average Density Hypothesis for these Dirichlet series.

Finally we derive estimations for the discrepancy of the sequence in Theorem 6. Our main result here (Theorem 17) can be considered as an extension of Hlawka's result [6] for the Riemann zeta function to the elements of $\tilde{\mathcal{S}}$. Unlike Hlawka's, our result covers the case $\alpha = 1$ too. The main ingredients of the proof are a uniform version of an explicit formula of Landau (Proposition 15), and the Erdös-Turán inequality.

2 A Class of Dirichlet Series

Let $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_1 = 1$, be a Dirichlet series with multiplicative coefficients which is absolutely convergent for $\Re(s) > 1$. Then $F(s)$ has an absolutely convergent Euler product on $\Re(s) > 1$. More precisely,

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left(\sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} \right) = \prod_p F_p(s), \quad \text{for } \Re(s) > 1. \tag{1}$$

We also assume that

$$F_p(s) \neq 0 \ \text{ on } \ \Re(s) > 1, \quad \text{for any } p. \tag{2}$$

Since $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is absolutely convergent for $\Re(s) > 1$, then for any $\epsilon > 0$, we have

$$\sum_{n \le x} |a_n| \le \sum_{n \le x} |a_n| (\frac{x}{n})^{1+\epsilon} \ll_\epsilon x^{1+\epsilon},$$

and so

$$a_n \ll_\epsilon n^{1+\epsilon}. \tag{3}$$

This implies that $\log F_p(s)$ has a Dirichlet series representation in the form

$$\log F_p(s) = \sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}, \quad \text{for } \Re(s) > c_p, \tag{4}$$

where c_p is a positive number which depends on p.

Lemma 1 *b_{p^k} is given by the recursion*

$$b_{p^k} = a_{p^k} - \frac{1}{k} \sum_{j=1}^{k-1} j b_{p^j} a_{p^{k-j}},$$

where $b_p = a_p$.

Proof Note that by differentiating (4), we have

$$\sum_{k=0}^{\infty} \frac{a_{p^k} \log p^k}{p^{ks}} = \left(\sum_{k=0}^{\infty} \frac{a_{p^k}}{p^{ks}} \right) \left(\sum_{k=1}^{\infty} \frac{b_{p^k} \log p^k}{p^{ks}} \right),$$

for $\Re(s) > c_p + 1$. Now the result follows by equating the p^k-th coefficient of two sides of the above identity. □

Lemma 2 *Let $\epsilon > 0$. Then $b_{p^k} \ll_\epsilon (p^k)^{2+\epsilon}$.*

Proof It is enough to prove that

$$b_{p^k} \ll_\epsilon \frac{2^k - 1}{k}(p^k)^{1+\epsilon},$$

which easily can be derived by employing the recursion of Lemma 1 for b_{p^k}, bound (3), and induction on k. $\square$

Notation Let

$$b_n = \begin{cases} b_{p^k} & \text{if} \ \ n = p^k \\ 0 & \text{otherwise} \end{cases}.$$

In light of (3) and Lemma 2, we can assume that $a_n \ll_\eta n^\eta$, for some $\eta < \frac{3}{2}$, and $b_n \ll_\vartheta n^\vartheta$, for some $\vartheta < \frac{5}{2}$. Also it is clear that in (2), we can assume $c_p = \vartheta$ for any p.

From now on we fix a $0 < \theta < \frac{5}{2}$ such that $b_n \ll n^{\theta - \epsilon}$ for some $\epsilon > 0$. So

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

is absolutely convergent for $\Re(s) \geq \theta + 1$. By differentiating this identity, we get

$$-\frac{F'}{F}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{n^s},$$

for $\Re(s) \geq 1 + \theta$, where

$$\Lambda_F(n) = \begin{cases} b_n \log n & \text{if} \ n = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Definition Let $\tilde{\mathcal{S}}$ be the class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad a_1 = 1,$$

which satisfy (1) and (2), and moreover, they satisfy the following.

(Analytic continuation) For some integer $m \geq 0$, $(s-1)^m F(s)$ extends to an entire function of finite order.

(Functional equation) There are numbers $Q > 0$, $\alpha_j > 0$, $r_j \in \mathbb{C}$ such that

$$\Phi(s) = Q^s \prod_{j=1}^{d} \Gamma(\alpha_j s + r_j) F(s)$$

satisfies the functional equation

$$\Phi(s) = \epsilon \bar{\Phi}(1-s)$$

where ϵ is a complex number with $|\epsilon| = 1$ and $\bar{\Phi}(s) = \overline{\Phi(\bar{s})}$.

This class is larger than the Selberg class $\mathcal{S}$ (see [17], and [9] for more information regarding the Selberg class). There are two main differences between $\tilde{\mathcal{S}}$ and $\mathcal{S}$. First of all in $\mathcal{S}$ we assume that the Ramanujan Hypothesis holds. More precisely, for an element in $\mathcal{S}$, we have $a_n \ll n^\eta$ where $\eta > 0$ is any fixed positive number, and $\Re(r_j) \geq 0$. Secondly, for an element of Selberg class, we have $b_n \ll n^\vartheta$, for some $\vartheta < \frac{1}{2}$. For an element of $\tilde{\mathcal{S}}$, we do not have these restrictions on a_n, b_n and r_j. Note that since $F(s) \neq 0$ on $\Re(s) > 1$, we have $\Re(r_j) \geq -\alpha_j$.

From now on we assume that $F \in \tilde{\mathcal{S}}$. We recall some facts regarding the zeros of F. We call a zero of F, a trivial zero, if it is located at the poles of the gamma factor of the functional equation of F. They can be denoted by $\rho = -\frac{k+r_j}{\alpha_j}$ with $k = 0,\ 1, \cdots$ and $j = 1, \cdots, r$. The zeros of $F(s)$ in the strip $0 \leq \sigma \leq 1$ are called non-trivial. By employing the functional equation we see that if ρ is a non-trivial zero of F then $1 - \bar{\rho}$ is also a non-trivial zero of F. In other words the non-trivial zeros of F are symmetric with respect to the line $\sigma = 1/2$. The Riemann Hypothesis for F is the assertion that all the non-trivial zeros of F are located on the line $\sigma = \frac{1}{2}$. From now on $\rho = \beta + i\gamma$, denotes a non-trivial zero of F.

Let

$$N_F(T) = \#\{\rho = \beta + i\gamma :\ F(\rho) = 0,\ 0 \leq \beta \leq 1,\ 0 \leq \gamma \leq T\}.$$

It is known that

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_F T + O_F(\log T)$$

with suitable constants d_F and c_F (see [9], formula (2.4)). We also recall a generalization of an explicit formula of Landau, due to M. R. Murty and V. K. Murty, which states that for $x > 1$ and $T \to \infty$,

$$\sum_{0 \leq \gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda_F(x) + O_{F,x}(\log T), \tag{5}$$

where

$$\Lambda_F(x) = \begin{cases} b_x \log x & \text{if } x = p^k \\ 0 & \text{otherwise} \end{cases}.$$

Landau proved the above formula for the Riemann zeta function. For a proof in the case of functions in the Selberg class (and similarly for the functions in $\tilde{\mathcal{S}}$) see [12]. Also with a simple observation regarding the symmetry of the non-trivial zeros of F respect to $\sigma = \frac{1}{2}$, for $0 < x < 1$, we have

$$\begin{aligned} \sum_{0 \le \gamma \le T} x^{\rho} &= \sum_{0 \le \gamma \le T} x^{1-\bar{\rho}} = x \overline{\sum_{0 \le \gamma \le T} \left(\frac{1}{x}\right)^{\rho}} \\ &= -\frac{T}{2\pi} x \Lambda_F(\frac{1}{x}) + O_{F,x}(\log T). \end{aligned} \tag{6}$$

We also need the following three lemmas.

Lemma 3 *Let $F \in \tilde{\mathcal{S}}$. We have*

$$N_F(T+1) - N_F(T) = O(\log T).$$

The implied constant depends only on F.

Proof This is the analogue of Lemma 4 of [14]. □

Lemma 4 *Let $F \in \tilde{\mathcal{S}}$. Let $s = \sigma + iT$ denote a point in the complex plane and $\rho = \beta + i\gamma$ denote a non-trivial zero of F. Then there is $T_0 > 0$, such that for $-\frac{5}{2} \le \sigma \le \frac{7}{2}$ and $T \ge T_0$, where T does not coincide with the ordinate of a zero of F, we have*

$$\frac{F'}{F}(s) = \sum_{|\gamma - T| < 1} \frac{1}{s - \rho} + O(\log T).$$

The implied constant depends only on F.

Proof This is the analogue of Lemma 5 of [14]. □

Lemma 5 *Let $F \in \tilde{\mathcal{S}}$. Let $\sigma_0 < 0$ be fixed. For $\sigma > 1$ set $\bar{F}(s) = \sum_{n=1}^{\infty} \frac{\bar{a}_n}{n^s}$. Then there is $T_0 > 0$ such that for $t \geq T_0$, we have*

$$\frac{F'}{F}(\sigma_0 + it) = -\frac{\bar{F}'}{\bar{F}}(1 - \sigma_0 - it) - 2\log Q + \sum_j c_j \log(d_j + ie_j t) + O\left(\frac{1}{t}\right).$$

Here c_j and e_j are real constants which depend only on F and d_j is a complex constant. d_j and the implied constant depend on F and σ_0.

Proof This is a consequence of logarithmically differentiating the functional equation of F and applying the asymptotic

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O\left(\frac{1}{|s|}\right),$$

which holds as $|s| \to \infty$ in the sector $-\pi + \eta < \arg s < \pi - \eta$ for any fixed $\eta > 0$ (see [11], Exercise 6.3.17). $\square$

3 Uniform Distribution

We are ready to state and prove our main result.

Theorem 6 *Let $F \in \tilde{\mathcal{S}}$. Suppose that F satisfies the Average Density Hypothesis, then for $\alpha \neq 0$, $(\alpha\gamma_n)$ is uniformly distributed mod 1.*

Proof By the Weyl criterion, to prove the uniform distribution of $(\alpha\gamma_n)$, for nonzero integer m, we should consider the exponential sum

$$\sum_{0 \leq \gamma \leq T} e^{2\pi i m\alpha\gamma} = \sum_{0 \leq \gamma \leq T} x^{i\gamma},$$

where $x = e^{2\pi m\alpha}$. We have the identity

$$\frac{1}{N_F(T)} \sum_{0 \leq \gamma \leq T} x^{i\gamma} = \frac{x^{-\frac{1}{2}}}{N_F(T)} \left(\sum_{0 \leq \gamma \leq T} x^{\beta + i\gamma} + \sum_{0 \leq \gamma \leq T} \left(x^{\frac{1}{2} + i\gamma} - x^{\beta + i\gamma} \right) \right). \tag{7}$$

We assume that $x > 1$. So by the mean value theorem, and the fact that the non-trivial zeros of F are symmetric respect to $\sigma = \frac{1}{2}$, we have

$$\sum_{0 \leq \gamma \leq T} \left(x^{\frac{1}{2}+i\gamma} - x^{\beta+i\gamma} \right) \ll \sum_{0 \leq \gamma \leq T} \left| x^{\frac{1}{2}} - x^{\beta} \right| \ll x \log x \sum_{\substack{0 \leq \gamma \leq T \\ \beta > \frac{1}{2}}} (\beta - \frac{1}{2}). \tag{8}$$

Now by applying (5) and (8) in (7), we have

$$\frac{1}{N_F(T)} \sum_{0 \leq \gamma \leq T} x^{i\gamma} \quad \ll_{x,F} \quad \frac{1}{N_F(T)} \left(T + \sum_{\substack{0 \leq \gamma \leq T \\ \beta > \frac{1}{2}}} (\beta - \frac{1}{2}) \right).$$

From here since $N_F(T) \sim c_0 T \log T$, for some fixed constant c_0, and F satisfies the Average Density Hypothesis, we have

$$\sum_{0 \leq \gamma \leq T} x^{i\gamma} = o(N_F(T)).$$

The same result is also true if $x < 1$, we basically repeat the same argument and apply (6) instead of (5).

So, by Weyl's criterion, $(\alpha\gamma_n)$ is uniformly distributed mod 1. $\square$

Corollary 7 *Under the analogue of the Riemann Hypothesis for F, $(\alpha\gamma_n)$, $\alpha \neq 0$, is uniformly distributed mod 1 where $\alpha \neq 0$.*

4 Moment Hypothesis $\rightarrow$ Average Density Hypothesis

We introduce the following hypothesis.

Moment Hypothesis *We say that $F \in \tilde{\mathcal{S}}$ satisfies the Moment Hypothesis if there exists a real $k > 0$ such that*

$$M_F(k,T) = \frac{1}{T} \int_0^T |F(\frac{1}{2} + it)|^{2k} \, dt = O_{k,F}\left(\exp(\psi(T))\right)$$

for some $\psi(T)$, where $\psi(T)$ is a positive real function such that $\psi(T) = o(\log T)$.

Our goal in this section is to prove that this hypothesis implies the Average Density Hypothesis. Using this in the next section we give several examples of Dirichlet series that satisfy the Moment Hypothesis and so by Theorem 6, the imaginary parts of their zeros are uniformly distributed mod 1.

Let

$$N_F(\sigma, T) = \#\{\rho = \beta + i\gamma : \ F(\rho) = 0, \ \beta \geq \sigma, \ 0 \leq \gamma \leq T\}.$$

We note that since the non-trivial zeros of F are symmetric respect to $\sigma = \frac{1}{2}$, we have

$$\begin{aligned} \sum_{\substack{0 \leq \gamma \leq T \\ \beta > \frac{1}{2}}} (\beta - \frac{1}{2}) &= \sum_{\substack{0 \leq \gamma \leq T \\ \beta > \frac{1}{2}}} \int_{\frac{1}{2}}^{\beta} d\sigma \\ &= \int_{\frac{1}{2}}^{1} N_F(\sigma, T) \ d\sigma. \end{aligned}$$

Next let R be the rectangle bounded by the lines $t = 0$, $t = T$, $\sigma = \sigma$, and $\sigma = 1 + \theta$ ($\frac{1}{2} \leq \sigma \leq 1 + \theta$, and θ is defined in Section 2). Then by an application of the residue theorem (and the usual halving convention regarding the number of zeros or poles on the boundary), we have

$$N_F(\sigma, T) - \frac{m_F}{2} = \frac{1}{2\pi i} \int_R \frac{F'(s)}{F(s)} ds, \tag{9}$$

where m_F is the order of pole of F at $s = 1$. Now let R_1 be the part of R traversed in the positive direction from σ to $\sigma + iT$ and let R_2 be the part of R traversed in the positive direction from $\sigma + iT$ to σ. Then by integrating (9) from $\frac{1}{2}$ to $1 + \theta$ with respect to σ and splitting the integral over R we have

$$2\pi i \int_{\frac{1}{2}}^{1+\theta} \Big(N_F(\sigma, T) - \frac{m_F}{2} \Big) \, d\sigma = \int_{\frac{1}{2}}^{1+\theta} d\sigma \int_{R_1} \frac{F'(s)}{F(s)} ds + \int_{\frac{1}{2}}^{1+\theta} d\sigma \int_{R_2} \frac{F'(s)}{F(s)} ds. \tag{10}$$

We choose $T_0 < 1$ and $T - 1 < T_1 < T + 1$ such that T_0 and T_1 are not equal to an ordinate of a zero of F. Let R_1' be the part of the rectangle bounded by $t = T_0$, $t = T_1$, $\sigma = \sigma$ and $\sigma = 1 + \theta$, traversed from $\sigma + iT_0$ to $\sigma + iT_1$. Then since, by Lemma 3, the number of zeros of F with ordinate between $T - 1$ and $T + 1$ is $\ll \log T$, we have

$$\begin{aligned} \int_{R_1} \frac{F'(s)}{F(s)} ds &= \int_{R_1'} \frac{F'(s)}{F(s)} ds + O(\log T) \\ &= \log F(\sigma + iT_1) - \log F(\sigma + iT_0) + O(\log T). \end{aligned} \tag{11}$$

We also have

$$\int_{\frac{1}{2}}^{1+\theta} d\sigma \int_{R_2} \frac{F'(s)}{F(s)} ds = -\int_0^T idt \int_{\frac{1}{2}}^{1+\theta} \frac{F'(\sigma+it)}{F(\sigma+it)} d\sigma$$
$$= \int_0^T \left(\log F(\frac{1}{2}+it) - \log F(1+\theta+it) \right) idt. \qquad (12)$$

Applying (11) and (12) in (10) and considering only the imaginary part of the resulting identity yields

$$2\pi \int_{\frac{1}{2}}^{1+\theta} \left(N_F(\sigma,T) - \frac{m_F}{2} \right) \, d\sigma = \int_{\frac{1}{2}}^{1+\theta} \arg \, F(\sigma+iT_1) \, d\sigma - \int_{\frac{1}{2}}^{1+\theta} \arg \, F(\sigma+iT_0) \, d\sigma$$
$$+ \int_0^T \log |F(\frac{1}{2}+it)| \, dt - \int_0^T \log |F(1+\theta+it)| \, dt$$
$$+ O(\log T). \qquad (13)$$

We note that

$$\int_0^T \log |F(1+\theta+it)| \, dt = \Re \left(\sum_{n=1}^{\infty} \frac{b_n}{n^{1+\theta}} \frac{n^{-iT}-1}{-i\log n} \right) = O(1).$$

Also we know that

$$\arg \, F(\sigma+iT_1) = O(\log T), \quad \text{and} \quad \arg \, F(\sigma+iT_0) = O(1),$$

(see [9], formula (2.4)). By applying these estimations in (13) we arrive at the following lemma.

Lemma 8 *As $T \to \infty$, we have*

$$\int_{\frac{1}{2}}^{1} N_F(\sigma,T) \, d\sigma = \frac{1}{2\pi} \int_0^T \log |F(\frac{1}{2}+it)| \, dt + O(\log T).$$

The implied constant depends on F.

In the sequel we need a special case of Jensen's inequality, which states that for any non-negative continuous function $f(t)$

$$\frac{1}{b-a} \int_a^b \log f(t) \, dt \le \log \left\{ \frac{1}{b-a} \int_a^b f(t) dt \right\}.$$

Proposition 9 *Let $F \in \tilde{\mathcal{S}}$. If F satisfies the Moment Hypothesis, then F satisfies the Average Density Hypothesis.*

Proof From Lemma 8, Jensen's inequality and the Moment Hypothesis, we have

$$\begin{aligned}
\int_{\frac{1}{2}}^{1} N_F(\sigma, T)\, d\sigma &= \frac{1}{2\pi} \int_0^T \log |F(\frac{1}{2} + it)|\, dt + O(\log T) \\
&= \frac{1}{4\pi k} \int_0^T \log |F(\frac{1}{2} + it)|^{2k}\, dt + O(\log T) \\
&\leq \frac{T}{4\pi k} \log \left\{ \frac{1}{T} \int_0^T |F(\frac{1}{2} + it)|^{2k}\, dt \right\} + O(\log T) \\
&\ll_{k,F} T\psi(T),
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{\substack{0 \leq \gamma \leq T \\ \beta > \frac{1}{2}}} (\beta - \frac{1}{2}) &= \int_{\frac{1}{2}}^{1} N_F(\sigma, T)\, d\sigma \\
&\ll_{k,F} T\psi(T).
\end{aligned}$$

$\square$

Note: From the proof of the previous proposition it is clear that the desired bound on $\int_{\frac{1}{2}}^{1} N_F(\sigma, T)\, d\sigma$ can be achieved under the assumption of a non-trivial upper bound for the mean value of $\log |F|$. This weaker assumption can be deduced from a non-trivial upper bound on any positive moment of F.

5 Examples

Let $\zeta(s)$ be the Riemann zeta function. For a primitive Dirichlet character mod q, we denote its associated Dirichlet L-series by $L(s, \chi)$. Let $L(s, f)$ be the L-series associated to a holomorphic cusp newform of weight k and level N with nebentypus ϕ, and $L(s, g)$ be the L-series associated to an even Maass cusp newform of weight zero and level N with nenentypus ϕ.

Proposition 10 *The moment Hypothesis is true for $\zeta(s)$, $L(s, \chi)$, $L(s, f)$, and $L(s, g)$.*

Proof For $k = 1$, it is known that

$$M(1, T) \ll T \log T$$

for these L-series. In fact, in all cases more precise asymptotic formulae are known. See [7] for $\zeta(s)$, [10] for $L(s, \chi)$, [18] for $L(s, f)$, and [19] for $L(s, g)$. □

Corollary 11 *The sequences $(\alpha\gamma_n)$, $\alpha \neq 0$, for $\zeta(s)$, $L(s, \chi)$, $L(s, f)$, and $L(s, g)$ are uniformly distributed mod 1.*

Proof One can show that these L-series are in $\tilde{\mathcal{S}}$. Now the result follows from Proposition 10, Proposition 9, and Theorem 6. □

Remark If χ_1 is an imprimitive character mod ℓ, the assertion of the previous corollary remains true for $L(s, \chi_1)$. To see this, note that

$$L(s, \chi_1) = \prod_{p \mid \frac{\ell}{q}} \left(1 - \frac{\chi(p)}{p^s}\right) L(s, \chi) = P(s, \chi) L(s, \chi),$$

where χ is a primitive Dirichlet character mod q $(q \mid \ell)$. Since the zeros of $1 - \chi(p)/p^s$ are in the form $i\gamma'_m = i(t_0 + 2m\pi/\log p)$ for fixed t_0 and $m \in \mathbb{N}$, then the total number of zeros of $P(s, \chi)$ up to height T is $\ll T$. Therefore $\sum_{0 \le \gamma' \le T} e^{2\pi i m \alpha \gamma'} \ll T$. Now the uniform distribution assertion follows by Weyl's criterion.

The following simple observation will be useful in constructing examples of Dedekind zeta functions whose zeros are uniformly distributed.

Proposition 12 *(i) Let (a_n) and (b_n) be two increasing (resp. decreasing) sequences of real numbers, and let (c_n) be the union of these two sequences ordered increasingly (resp. decreasingly). If (a_n) and (b_n) are uniformly distributed mod 1, then (c_n) is also uniformly distributed mod 1.*

(ii) For F, $G \in \tilde{\mathcal{S}}$, if the sequences $(\alpha\gamma_{F,n})$ and $(\alpha\gamma_{G,n})$, $\alpha \neq 0$, formed from imaginary parts of zeros of F and G, are uniformly distributed, then the same is true for the sequence $(\alpha\gamma_{FG,n})$ formed from imaginary parts of zeros of FG.

Proof (i) Without loss of generality, we assume that (a_n) and (b_n) are increasing. For $t > 0$, let $n_c(t)$ be the number of elements of (c_n) not exceeding t. So $n_c(t) = n_a(t) + n_b(t)$. We denote a general term of (c_n) by c. We have

$$\begin{aligned}\frac{|\sum_{c\le t} e^{2\pi imc}|}{n_c(t)} &= \frac{|\sum_{a\le t} e^{2\pi ima} + \sum_{b\le t} e^{2\pi imb}|}{n_a(t)+n_b(t)} \\ &\le \frac{|\sum_{a\le t} e^{2\pi ima}|}{n_a(t)} + \frac{|\sum_{b\le t} e^{2\pi imb}|}{n_b(t)}.\end{aligned}$$

Now the result follows from Weyl's criterion.

(ii) This is clear from (i), since the set of zeros of FG is a union of the zeros of F and the zeros of G. □

Corollary 13 *Let K be an abelian number field. Then $(\alpha\gamma_n)$, $\alpha \neq 0$, for the Dedekind zeta function $\zeta_K(s)$ is uniformly distributed mod 1.*

Proof Since K is abelian, $\zeta_K(s)$ can be written as a product of Dirichlet L-series associated to some primitive Dirichlet characters ([15], Theorem 8.6). So the result follows from Corollary 11 and Proposition 12. □

Corollary 14 *Let K be a finite abelian extension of a quadratic number field k. Then $(\alpha\gamma_n)$, $\alpha \neq 0$, for the Dedekind zeta function $\zeta_K(s)$ is uniformly distributed mod 1.*

Proof Since $Gal(K/k)$ is abelian, we can write

$$\zeta_K(s) = \prod_{\psi} L(s,\psi),$$

where by Artin's reciprocity, ψ's denote the Hecke characters associated to certain grössencharacters of k ([5], Theorems 9-2-2 and 12-3-1). We know that corresponding to ψ there is a cuspidal automorphic representation π of $GL_1(\mathbb{A}_k)$ such that $L(s,\psi) = L(s,\pi)$ (see [4], Section 6.A). On the other hand since $k/\mathbb{Q}$ is quadratic, the automorphic induction map exists [1]. In other words, there is a cuspidal automorphic representation $I(\pi)$ of $GL_2(\mathbb{A}_\mathbb{Q})$ such that $L(s,\pi) = L(s,I(\pi))$. So

$$\zeta_K(s) = \prod_{I(\pi)} L(s,I(\pi)).$$

However it is known that the cuspidal automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ correspond to holomorphic or Maass forms (see [4], Section 5.C), so the result follows from Corollary 11 and Proposition 12. □

6 Discrepancy

We define the discrepancy of the sequence $(\alpha\gamma_n)$ by

$$D^*_{F,\alpha}(T) = \sup_{0\le\beta\le 1}\left|\frac{\#\{0\le\gamma\le T;\ 0\le\{\alpha\gamma\}<\beta\}}{N_F(T)}-\beta\right|.$$

In this section we employ the Erdös-Turán inequality to establish an upper bound in terms of α and T for $D^*_{F,\alpha}(T)$. The main tool needed is a uniform (in terms of x) version of (5). The following proposition gives a uniform version of Landau's formula.

Proposition 15 *Let $F\in\tilde{\mathcal{S}}$. Let $x\ge 2$ and n_x be the closest integer to x. (If x is a half-integer, we set $n_x=[x]+1$.) Then we have*

$$\sum_{0\le\gamma\le T} x^{\rho} = \delta_{x,T} + O(x^{1+\theta}\log T),$$

where

$$\delta_{x,T} = -\frac{T}{2\pi}\Lambda_F(x) \quad \text{if} \quad x\in\mathbb{N},$$

and

$$\delta_{x,T} \ll \Lambda_F(n_x)\min\{T, \frac{1}{|\log\frac{x}{n_x}|}\} \quad \text{if} \quad x\notin\mathbb{N}.$$

The implied constants depend only on F. Recall that $0<\theta<\frac{5}{2}$ is such that $b_n\ll n^{\theta-\epsilon}$ for some $\epsilon>0$.

Proof We follow Proposition 1 of [13] closely. We choose T and T_0 such that

$$T > T_0 > \max_{1\le j\le d}\left|\frac{\Im(r_j)}{\alpha_j}\right|,$$

moreover we assume that T_0 is large enough such that the assertions of Lemmas 4 and 5 are satisfied. Also we assume that T_0 and T are not the ordinate of a zero of F.

Next we consider the rectangle $R=R_1\cup(-R_2)\cup(-R_3)\cup R_4$ (oriented counter-clockwise), where

$R_1:\ (1+\theta)+it,\quad T_0\le t\le T,$
$R_2:\ \sigma+iT,\quad -\theta\le\sigma\le 1+\theta,$
$R_3:\ -\theta+it,\quad T_0\le t\le T,$

$R_4: \ \sigma + iT_0, \quad -\theta \le \sigma \le 1+\theta.$

By the residue theorem, we have

$$\frac{1}{2\pi i}\int_R \frac{F'}{F}(s)x^s ds = \sum_{T_0 \le \gamma \le T} x^\rho. \tag{14}$$

Here ρ runs over the zeros of $F(s)$ inside the rectangle R (considered with multiplicities). Let $I_i = \frac{1}{2\pi i}\int_{R_i}$. We have

$$\begin{aligned} I_1 &= -\frac{1}{2\pi}\int_{T_0}^{T}\sum_{m=1}^{\infty}\Lambda_F(m)\left(\frac{x}{m}\right)^{1+\theta+it} dt \\ &= -\frac{1}{2\pi}\Lambda_F(n_x)\left(\frac{x}{n_x}\right)^{1+\theta}\int_{T_0}^{T}\left(\frac{x}{n_x}\right)^{it} dt + O\left(x^{1+\theta}\sum_{m\neq n_x}\frac{|\Lambda_F(m)|}{m^{1+\theta}}\left|\int_{T_0}^{T}\left(\frac{x}{m}\right)^{it} dt\right|\right) \\ &= I_{x,T} + O\left(x^{1+\theta}\sum_{m\neq n_x}\frac{|\Lambda_F(m)|}{m^{1+\theta}}\frac{1}{\left|\log\frac{x}{m}\right|}\right), \end{aligned}$$

where

$$I_{x,T} = -\frac{1}{2\pi}\Lambda_F(n_x)\left(\frac{x}{n_x}\right)^{1+\theta}\int_{T_0}^{T}\left(\frac{x}{n_x}\right)^{it} dt.$$

Next we note that

$$x^{1+\theta}\sum_{m\neq n_x}\frac{|\Lambda_F(m)|}{m^{1+\theta}}\frac{1}{\left|\log\frac{x}{m}\right|} \ll x^{1+\theta}\sum_{m\neq n_x}\frac{|\Lambda_F(m)|}{m^{1+\theta}}\frac{1}{\left|\log\frac{n_x}{m}\right|}.$$

We split this series into ranges $m \le \frac{n_x}{2}$, $\frac{n_x}{2} < m < 2n_x$, and $m \ge 2n_x$, denoting them as $\sum_1$, $\sum_2$ and $\sum_3$. Now it is easy to see that

$$\sum_1 + \sum_3 \ll x^{1+\theta}.$$

For the second sum, we note that if $|z| < 1$, then

$$|-\log(1-z)| \ge \frac{1}{2}|z|.$$

By employing this inequality in $\sum_2$, we have

$$\sum_2 \ll \sum_{\substack{\frac{n_x}{2} < m < 2n_x \\ m \neq n_x}} |\Lambda_F(m)|\left|\frac{n_x}{n_x - m}\right| \ll x^{1+\theta}.$$

Next from Lemma 4, we have

$$-I_2 = \frac{1}{2\pi i}\int_{1+\theta+iT}^{-\theta+iT} \frac{F'}{F}(s)x^s ds = \sum_{|\gamma - T|<1} \int_{1+\theta+iT}^{-\theta+iT} \frac{x^s}{s-\rho} ds + O(x^{1+\theta}\log T). \tag{15}$$

Let C_T be the circle with center $\frac{1}{2}+iT$ and radius $\frac{1}{2}+\theta$. We denote the upper (respectively lower) semi-circle of C_T by C_T^+ (respectively C_T^-). We assume that $\gamma < T$, then

$$\int_{(1+\theta)+iT}^{-\theta+iT} \frac{x^s}{s-\rho} ds = \int_{C_T^+} \frac{x^s}{s-\rho} ds,$$

where we consider C_T^+ in clockwise direction. Now since $|s-\rho| > \theta$, the integral over C_T^+ is $O(x^{1+\theta})$. A similar result is true for $\gamma > T$, in this case we consider the lower semi-circle C_T^-. Finally we note that by Lemma 3, the number of terms in $\sum_{|\gamma-T|<1}$ is $O(\log T)$. So applying these estimations in (15) yields

$$I_2 \ll x^{1+\theta}\log T.$$

Next we estimate I_3. From Lemma 5 we have

$$\begin{aligned} I_3 &= \frac{1}{2\pi}\int_{T_0}^{T} \frac{F'}{F}(1+\theta-it)x^{-\theta+it}dt + \frac{\log Q}{\pi}\int_{T_0}^{T} x^{-\theta+it}dt \\ &+ \frac{1}{2\pi}\sum_j c_j \int_{T_0}^{T} x^{-\theta+it}\log(d_j+ie_jt)dt + O(x^{-\theta}\log T) \\ &= I_{31}+I_{32}+I_{33}+O(x^{-\theta}\log T). \end{aligned} \tag{16}$$

We have

$$I_{31} = -x^{-\theta}\sum_{n=1}^{\infty}\frac{\Lambda_F(n)}{n^{1+\theta}}\left(\frac{1}{2\pi}\int_{T_0}^{T}(xn)^{it}dt\right) \ll x^{-\theta}\sum_{n=1}^{\infty}\frac{|\Lambda_F(n)|}{n^{1+\theta}\log{(xn)}} \ll x^{-\theta},$$

and

$$I_{32} = \frac{\log Q}{\pi}x^{-\theta}\int_{T_0}^{T} x^{it}dt \ll \frac{x^{-\theta}}{\log x}.$$

Also an application of integration by parts results in

$$I_{33} = \frac{1}{2\pi}\sum_j c_j \int_{T_0}^{T} x^{-\theta+it}\log(d_j+ie_jt)dt \ll \frac{x^{-\theta}\log T}{\log x}.$$

Applying these bounds in (16) yields

$$I_3 \ll \log T.$$

Finally since $\frac{F'}{F}(s)$ is bounded on R_4, we have

$$I_4 \ll \int_{-\theta}^{1+\theta} x^{\sigma} d\sigma \ll x^{1+\theta}.$$

Now applying the estimations for I_1, I_2, I_3, and I_4 in (14) yield

$$\sum_{T_0 \le \gamma \le T} x^{\rho} = I_{x,T} + O(x^{1+\theta} \log T).$$

The result follows from this, together with the facts that the number of zeros of F with $0 \le \gamma \le T_0$ is finite and

$$\int_{T_0}^{T} \left(\frac{x}{n_x}\right)^{it} dt$$

is $T - T_0$ if $x = n_x$ and it is

$$\ll \min\{T, \frac{1}{|\log \frac{x}{n_x}|}\}.$$

□

Next we consider the following hypothesis. We say that $F \in \tilde{S}$ satisfies the Refined Average Density Hypothesis if

$$\sum_{\substack{0 \le \gamma \le T \\ \beta > \frac{1}{2}}} (\beta - \frac{1}{2}) = o(T\psi(T)),$$

for some $\psi(T)$, where $\psi(T)$ is a positive real function such that $\psi(T) = o(\log T)$.

Corollary 16 *In Proposition* 15 *under the assumption of the Refined Average Density Hypothesis for F, we have*

$$\sum_{0 \le \gamma \le T} x^{i\gamma} = \frac{\delta_{x,T}}{x^{\frac{1}{2}}} + O(x^{\frac{1}{2}+\theta} \max\{\log T, T\psi(T)\}).$$

Moreover, under the assumption of the Riemann Hypothesis for F, we have

$$\sum_{0 \le \gamma \le T} x^{i\gamma} = \frac{\delta_{x,T}}{x^{\frac{1}{2}}} + O(x^{\frac{1}{2}+\theta} \log T).$$

Proof This is evident from (7), (8), and Proposition 15. $\square$

We are ready to state and prove the main result of this section.

Theorem 17 *Let $F \in \tilde{\mathcal{S}}$. Assume that $\alpha \geq \frac{\log 2}{2\pi}$, and let $x = e^{2\pi\alpha}$.*

(i) If F satisfies the Refined Average Density Hypothesis with $\psi(T) \gg 1$ then

$$D^*_{F,\alpha}(T) \ll_F \frac{\alpha}{\log\left(\frac{\log T}{\psi(T)}\right)}.$$

(ii) Assume that $0 < \theta < \frac{1}{2}$, then under the assumption of the Riemann Hypothesis for F,

$$D^*_{F,\alpha}(T) \ll_F \frac{\alpha}{\log T}.$$

(iii) Let x be an algebraic number that is not a k-th root of a natural number for any k, then under the assumption of the Riemann Hypothesis for F,

$$D^*_{F,\alpha}(T) \ll_{F,\alpha} \frac{1}{\log T}.$$

θ can be $\geq \frac{1}{2}$ in (iii).

Proof (i) From the Erdös-Turán inequality, for any integer K, we have

$$D^*_{F,\alpha}(T) \leq \frac{1}{K+1} + \frac{3}{N_F(T)} \sum_{k=1}^{K} \frac{1}{k} \left| \sum_{0\leq\gamma\leq T} e^{2\pi i k\alpha\gamma} \right|.$$

So by applying Corollary 16, we have

$$\begin{aligned}
D^*_{F,\alpha}(T) &\ll \frac{1}{K} + \frac{1}{N_F(T)} \sum_{k=1}^{K} \frac{1}{k} \left| \sum_{0\leq\gamma\leq T} x^{ik\gamma} \right| \\
&\ll \frac{1}{K} + \frac{1}{N_F(T)} \left(T \sum_{k=1}^{K} \frac{\Lambda_F(n_{x^k})}{k x^{\frac{k}{2}}} + T\psi(T) \sum_{k=1}^{K} \frac{x^{k(\theta+\frac{1}{2})}}{k} \right) \\
&\ll \frac{1}{K} + \frac{1}{N_F(T)} \left(T \sum_{k=1}^{K} \frac{x^{k(\theta-\frac{1}{2})} \log x^k}{k} + T\psi(T) \sum_{k=1}^{K} \frac{x^{k(\theta+\frac{1}{2})}}{k} \right) \\
&\ll \frac{1}{K} + \frac{\psi(T)}{\log T} x^{K(\theta+\frac{1}{2})} \log K.
\end{aligned}$$

Now the result follows by choosing

$$K = \frac{\log\left(\frac{\log T}{\psi(T)}\right)}{(2\theta+1)\log x}.$$

(ii) By the Erdös-Turán inequality and Corollary 16, we have

$$\begin{aligned}
D^*_{F,\alpha}(T) &\ll \frac{1}{K} + \frac{1}{N_F(T)}\sum_{k=1}^{K}\frac{1}{k}\left|\sum_{0\le\gamma\le T} x^{ik\gamma}\right| \\
&\ll \frac{1}{K} + \frac{1}{N_F(T)}\left(T\sum_{k=1}^{K}\frac{\log x}{x^{k(\frac{1}{2}-\theta)}} + \log T\sum_{k=1}^{K}\frac{x^{k(\theta+\frac{1}{2})}}{k}\right) \\
&\ll \frac{1}{K} + \frac{\log x}{x^{\frac{1}{2}-\theta}\log T} + \frac{1}{T}x^{K(\theta+\frac{1}{2})}\log K.
\end{aligned}$$

Now the result follows by choosing

$$K = \frac{\log T}{(\theta+1)\log x}.$$

(iii) We have

$$\begin{aligned}
D^*_{F,\alpha}(T) &\ll \frac{1}{K} + \frac{1}{N_F(T)}\left(\sum_{k=1}^{K}\frac{1}{k}\left|\sum_{0\le\gamma\le T} x^{ik\gamma}\right|\right) \\
&\ll \frac{1}{K} + \frac{1}{N_F(T)}\left(\sum_{k=1}^{K}\frac{\Lambda_F(n_{x^k})}{kx^{\frac{k}{2}}|k\log x - \log n_{x^k}|} + \log T\sum_{k=1}^{K}\frac{x^{k(\theta+\frac{1}{2})}}{k}\right) \\
&\ll_x \frac{1}{K} + \frac{Ke^{cK}}{T}.
\end{aligned}$$

We choose

$$K = \frac{\log T}{c+1}$$

to get the result. Here, we use Baker's theorem to get a lower bound for a linear form in logarithms of algebraic numbers. More precisely by Baker's theorem [2], we have

$$|k\log x - \log n_{x^k}| > e^{-ak},$$

where a is a constant which depends on x. □

Acknowledgment: We thank the referee for the careful reading of the paper.

References

[1] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies 120, Princeton University Press, 1990.

[2] A. Baker, A central theorem in transcendence theory, *Diophantine Approximation and Its Applications*, 1–23, Academic Press, 1973.

[3] A. Fujii, Some problems of Diophantine approximation in the theory of the Riemann zeta function (III), *Comment. Math. Univ. St. Pauli*, **42** (1993), 161–187.

[4] S. S. Gelbart, *Automorphic forms on adele groups*, Annals of Math. Studies 83, Princeton University Press, 1975.

[5] L. J. Goldstein, *Analytic number theory*, Prentice-Hall Inc., 1971.

[6] E. Hlawka, Über die Gleichverteilung gewisser Folgen, welche mit den Nullstellen der Zetafuncktionen zusammenhängen, *Österr. Akad. Wiss., Math.-Naturw. Kl. Abt. II* **184** (1975), 459–471.

[7] G. H. Hardy and J. E. Littlewood, Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes, *Acta Mathematica*, **41** (1918), 119–96.

[8] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Dover Publication Inc., Mineola, New York, 2006.

[9] J. Kaczorowski and A. Perelli, The Selberg class: a survey, *Number theory in progress, Vol.* 2, 953–992, de Gruyter, Berlin, 1999.

[10] Y. Motohashi, A note on the mean value of the zeta and L-functions. II, *Proc. Japan Acad. Ser. A Math. Sci.*, **61** (1985), 313–316.

[11] M. R. Murty, *Problems in analytic number theory*, Springer-Verlag, New York, 2001.

[12] M. R. Murty and V. K. Murty, Strong multiplicity one for Selberg's class, *C. R. Acad. Sci. Paris Sér. I Math.*, **319** (1994), 315–320.

[13] M. R. Murty and A. Perelli, The pair correlation of zeros of functions in the Selberg class, *Internat. Math. Res. Notices*, (1999) No. 10, 531–545.

[14] V. K. Murty, Explicit formulae and the Lang-Trotter conjecture, *Rocky Mountain Journal of Mathematics*, **15** (1985) 535–551.

[15] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers, third edition*, Springer-Verlag, Berlin Heidelberg, 2004.

[16] H. Rademacher, Fourier analysis in number theory, *Symposium on harmonic analysis and related integral transforms: Final technical report*, Cornell Univ., Ithica, N.Y. (1956), 25 pages; also in: *H. Rademacher, Collected Works*, pp. 434–458.

[17] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, *Collected Works, Vol.* 2, 47–63, Springer-Verlag, 1991.

[18] Q. Zhang, Integral mean values of modular L-functions, *J. Number Theory*, **115** (2005), 100–122.

[19] Q. Zhang, Integral mean values of Maass L-functions, *Internat. Math. Res. Notices*, (2006), Article ID 41417, 1–19.

Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive West, Lethbridge, Alberta, T1K 3M4, CANADA
E–mail address: amir.akbary@uleth.ca

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, K7L 3N6, CANADA
E–mail address: murty@mast.queensu.ca