Straight-Line Drawings on Restricted Integer Grids in Two and Three Dimensions

by

S. Felsner, G. Liotta and S.K. Wismath

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Stefan Felsner, *Giuseppe Liotta*, and *Stephen Wismath*

[1] Freie Universität Berlin, Fachbereich Mathematik und Informatik, Takustr. 9, 14195 Berlin, Germany. `felsner@inf.fu-berlin.de`.
[2] Dipartimento di Ingegneria Elettronica e dell'Informazione, Università degli Studi di Perugia, Via Duranti, 06100 Perugia, Italy. `liotta@diei.ing.unipg.it`.
[3] Dept. of Mathematics and Computer Science, U. of Lethbridge, Alberta, T1K-3M4 Canada. `wismath@cs.uleth.ca`.

Abstract. This paper investigates the following question: Given an integer grid ϕ, where ϕ is a proper subset of the integer plane or a proper subset of the integer 3d space, which graphs admit straight-line crossing-free drawings with vertices located at the grid points of ϕ? We characterize the trees that can be drawn on a two dimensional $c \cdot n \times k$ grid, where k and c are given integer constants, and on a two dimensional grid consisting of k parallel horizontal lines of infinite length. Motivated by the results on the plane we investigate restrictions of the integer grid in 3 dimensions and show that every outerplanar graph with n vertices can be drawn crossing-free with straight lines in linear volume on a grid called a prism. This prism consists of $3n$ integer grid points and is universal – it supports all outerplanar graphs of n vertices. This is the first algorithm that computes crossing-free straight line 3d drawings in linear volume for a non-trivial family of planar graphs. We also show that there exist planar graphs that cannot be drawn on the prism and that extension to a $n \times 2 \times 2$ integer grid, called a box, does not admit the entire class of planar graphs.

1 Introduction

This paper deals with crossing-free straight-line drawings of planar graphs in 2 and 3 dimensions. Given a graph G , we constrain the vertices in a drawing of G to be located at integer grid points and aim at computing drawings whose area/volume is small. The interest in these two requirements is motivated in part by the fact that the screen of a computer is an integer grid of limited size and by the fact that a drawing algorithm should not be affected by round-off errors when representing the output coordinates. Also, the increasing demand of visualization algorithms to draw and browse very large networks makes it natural to study what families of graphs can be entirely visualized on a two dimensional screen and to investigate how much can one benefit from the third dimension to represent the overall structure of a huge graph in a small portion of a virtual 3d environment.

A rich body of literature has been published on computing straight-line drawings of graphs, such that the vertices are the intersection points of an integer 2d grid and the overall area of the drawing is kept small. Typically, papers that deal with this subject focus on lower bounds on the area required by straight line drawings of specific classes of graphs and on the design of algorithms that possibly match these lower bounds. A very limited list of mile-stone papers in this field includes the works by de Fraysseix, Pach, and Pollack [7, 8] and by Schnyder [19] who independently showed that every n-vertex triangulated planar graph has a crossing-free straight-line drawing such that the vertices are at grid points, the size of the grid is $O(n) \times O(n)$, and that this is worst case optimal; the work by Kant [15, 16], Chrobak and Kant [3], Schnyder and Trotter [20], Felsner [11] and Chrobak, Goodrich, and Tamassia [4] who studied convex grid drawings of triconnected planar graphs in an integer grid of quadratic area; and the many papers proving that linear or almost-linear area bounds can be achieved for classes of trees, including the result by Garg, Goodrich and Tamassia [12] and the result by Chan [2]. Summarizing tables and more references can be found in the book by Di Battista, Eades, Tamassia, and Tollis [9].

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While the problem of computing small-sized crossing-free straight-line drawings in the plane has a long tradition, its 3d counterpart has become the subject of much attention only in recent years. Chrobak, Goodrich, and Tamassia [4] gave an algorithm for constructing 3d convex drawings of triconnected planar graphs with $O(n)$ volume and non-integer coordinates. Cohen, Eades, Lin and Ruskey [6] showed that every graph admits a straight-line crossing-free 3d drawing on an integer grid of $O(n^3)$ volume, and proved that this is asymptotically optimum. Calamoneri and Sterbini [1] showed that all 2-, 3-, and 4-colourable graphs can be drawn in a 3d grid of $O(n^2)$ volume with $O(n)$ aspect ratio and proved a lower bound of $\Omega(n^{1.5})$ on the volume of such graphs. For r-colourable graphs, Pach, Thiele and Tóth [17] showed a bound of $\theta(n^2)$ on the volume. Garg, Tamassia, and Vocca [13] showed that all 4-colorable graphs (and hence all planar graphs) can be drawn in $O(n^{1.5})$ volume and with $O(1)$ aspect ratio but using a grid model where the coordinates of the vertices may not be integer.

In this paper we study the problem of computing drawings of graphs on integer 2d or 3d grids that have small area/volume. The area/volume of a drawing Γ is measured as the number of grid points contained in or on a *bounding box* of Γ, *i.e.* the smallest axis-aligned box enclosing Γ. Note that along each side of the bounding box the number of grid points is one more than the actual length of the side.

We approach the drawing problem with the following point of view: Instead of "squeezing" a drawing onto a small portion of a grid of unbounded dimensions, we assume that a grid of *specified* dimensions is given and we consider what the graphs are whose drawings fit that restricted grid. For example, it is well-known that there are families of graphs that require $\Omega(n^2)$ area to be drawn in the plane, the canonical example being a sequence of $n/3$ nested triangles (see [8, 5, 19]). Such graphs can be drawn on the surface of a 3 dimensional triangular prism of linear volume and using integer coordinates. Thus a natural question is whether there exist specific restrictions of the 3d integer grid of linear volume that can support straight-line crossing-free drawings of meaningful families of graphs. For planar graphs the best known results for 3 dimensional crossing-free straight-line drawings on an integer grid are by Calamoneri and Sterbini [1] who show $O(n^2)$ volume for general planar graphs and by Eades, Lin and Ruskey [6] who show $O(n \log n)$ volume for trees. By following the above described approach we have been able to design the first algorithms that draw significant families of planar graphs in an integer 3d grid requiring only $O(n)$ volume.

The main contributions of the present paper are combinatorial characterizations and negative results on the drawability of graphs on 2d and 3d restricted integer grids and new drawing algorithms for some classes of graphs. An overview of the results is as follows.

- We characterize those trees that can be drawn on an integer restricted 2d grid consisting of k consecutive infinite horizontal grid lines (for a given positive integer k) and where edges can connect either collinear vertices or vertices that are one unit apart in their y-coordinates (notice that this type of drawing is different form the so-called k-level drawings [14] where vertices on the same level are not permitted to be connected by an edge). We also present a linear time recognition and drawing algorithm for this class of trees.
- We study those trees that can be drawn on an integer restricted 2d grid of dimensions $c \cdot n \times k$, where c and k are two given integer positive constants and n is the number of vertices of the tree. In this case we relax one of our drawing constraints to allow adjacent vertices to be more than one unit apart in their y-coordinates, and show that in spite of this relaxation the family of drawable trees coincides with those studied within the drawing convention of the previous item. A consequence of our characterization is that for any given k and c there always exist some trees that are not drawable on the $c \cdot n \times k$ grid.
- Motivated by the results on restricted integer 2d grids we explore the capability of restricted integer 3d grids for supporting linear volume drawings of graphs. In particular, we focus on two types of 3d integer grids to be defined subsequently, both having linear volume, called the *prism* and the *box*. We show that all outerplanar graphs can be drawn in linear volume on a prism. Note that this is the first result on 3d straight-line drawings of a significant class of planar graphs that achieves linear volume with integer coordinates.
- We further explore the class of graphs that can be drawn on a prism by asking whether the prism is a *universal* integer 3d grid for all planar graphs. We answer this question in the negative by exhibiting examples of planar graphs that cannot be drawn on a prism. We also investigate the relationship between prism-drawable and Hamiltonian graphs.

- We extend our study to box-drawability and present a characterization of the box-drawable graphs. While the box would appear to be a much more powerful grid than the prism, we prove that not all planar graphs are box-drawable.

The remainder of the paper is organized as follows. Preliminaries and basic definitions are in Section 2. The characterization of drawable trees for restricted integer 2d grids can be found in Section 3. Section 4 presents the linear volume algorithm for outerplanar graphs. Combinatorial properties of the graphs that can be drawn on the surface of a prism and on the surface of a box are studied in Sections 5 and 6. Final remarks, directions for further research and open problems can be found in Section 7.

2 Preliminaries

We assume familiarity with basic graph drawing, and computational geometry terminology; see for example [18, 9]. Since in the remainder of the paper we shall study crossing-free straight-line drawings of planar graphs, from now on we shall simply talk about "graphs" to mean "planar graphs" and about "drawings" to mean "crossing-free straight line drawings". We use the terms "vertex" and "edge" for both the graph and its drawing.

We will draw graphs such that vertices are located at integer grid points. The dimensions of a grid are specified as the number of different grid points along each side of a bounding box of the grid. In 2 dimensions, a $p \times q$ grid consists of p grid points along the x-axis and of q grid points along the y-axis. In 3 dimensions, a $p \times q \times r$ grid consists of p grid points along the x-axis, q grid points along the y-axis, and r grid points along the z-axis; p, q and r are referred to as the x-, y-, and *z-dimension* of the grid, respectively.

We shall deal with the following grids and drawings.

- A 2d 1-*track* (or simply a *track*) is a $\infty \times 1$ grid; a 1-*track* drawing of a graph G is a drawing of G where the vertices are at distinct grid points of the track.
- A 2d *strip* is a $\infty \times 2$ grid; note that a strip consists of two tracks. A *strip drawing* of a graph G is a drawing of G with the vertices located at distinct grid points of the strip and the edges either connect vertices on the same track or connect vertices on different tracks.
- Let k be a given positive integer value. A 2d *k-track* grid is a $\infty \times k$ grid consisting of k consecutive parallel tracks. A *k-track drawing* of a graph G is a drawing of G where the vertices are at distinct grid points of the k-track and edges are only permitted between vertices that are either on the same track or that are one unit apart in their y-coordinates.
- Let k and c be two given positive integer values. In a *$c \cdot n \times k$-grid drawing* of a graph G the vertices are located at distinct grid points and the edges can connect any pair of vertices on that grid.
- We will also study two different types of $n \times 2 \times 2$ grids. A *box* is a $n \times 2 \times 2$ grid where each side of the bounding box is also a grid line. Therefore, a box has four tracks which lie on two parallel planes and are one grid unit apart from each other. A *prism* is a $n \times 2 \times 2$ grid obtained by removing a track from a box. Figure 1 shows an example of a box of size $6 \times 2 \times 2$ and an example of a prism.

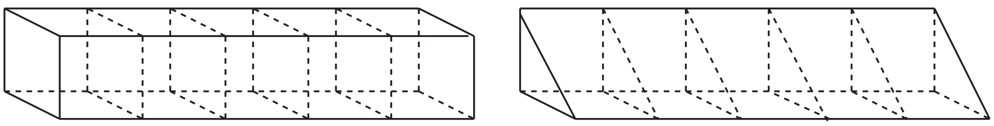

Fig. 1. A Box and a Prism

Note that k-track drawings differ from the so-called k-level drawings (see, e.g. [14]) as in a k-track drawing (consecutive) vertices on the same track are permitted to be joined by an edge and the given graph is undirected.

Let ϕ be one of the grids defined above. We say that a graph G is *ϕ drawable* if G admits a ϕ drawing Γ where each vertex is mapped to a distinct grid point of ϕ.

Property 1. A graph is 1-track drawable if and only if it is a simple path

While in a k-track drawing no edge can connect vertices that are on non-consecutive tracks, in a $c \cdot n \times k$-grid this is allowed. As the following property shows, this difference has immediate consequences on the families of k-track drawable and $c \cdot n \times k$-grid drawable graphs.

Property 2. Let c, k be two positive integers. There exist graphs with n vertices that are $c \cdot n \times k$-grid drawable but are not k-track drawable.

Proof. The graph K_4 has a $1 \cdot n \times 3$-grid drawing but it does not have a 3-track drawing.

3 Two-Dimensional Restricted Grids

In this section we characterize the family of k-track drawable trees and the family of $c \cdot n \times k$-grid drawable trees. In contrast to Property 2, we show that these two families of trees are actually the same. We also give linear time recognition and drawing algorithms for these trees. The approach is as follows. We first study strip-drawable trees, then we extend the result to the k-track grid, and finally we show that the result also holds on a $c \cdot n \times k$ grid.

3.1 Strip-Drawable Trees

By Property 1 we have that all paths are strip-drawable, since they are in fact 1-track drawable. A tree is defined as *2-strict* if it contains a vertex of degree greater than or equal to three. An immediate consequence of Property 1 is the following.

Property 3. A 2-strict tree is not 1-track drawable.

An edge is defined as a *core edge* if its removal results in two 2-strict components. For an edge $e = (u, v)$, we refer to the two subtrees resulting from its removal as T_u and T_v.

Lemma 1. *Core edges are connected.*

Proof. (sketch) Let $e_1 = (u, v)$ and $e_2 = (w, x)$ be 2 core edges and consider any edge on the tree between them. Each such edge receives one 2-strict component from T_u and one from T_x and thus must be core.

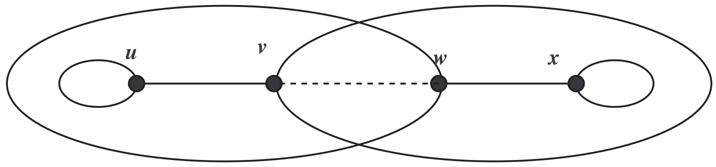

Fig. 2. Core edges are connected

Lemma 2. *A tree is strip drawable if and only if its core edges form a path.*

Proof. (sketch) ($\Rightarrow$) (by contradiction) By the previous lemma, if the core edges do not form a path, then there is a vertex v with at least 3 incident core edges $(v, a), (v, b), (v, c)$ – see Figure 3. If the subtrees T_a, T_b, T_c are drawable then by Property 3 their associated drawings $\Gamma_a, \Gamma_b, \Gamma_c$ each require 2 tracks. There is no location for v that permits a crossing-free connection to all 3 subdrawings.

($\Leftarrow$) If the core edges form a path of at least 2 vertices, then draw them consecutively on track t_1. Consider an arbitrary non-core edge $e = (u, v)$ with u on track t_1. Since e is non-core, T_v must *not* be 2-strict and is thus 1-track drawable. Therefore v can be placed on track t_2 with the drawing of T_v also on the same track as in Figure 4.

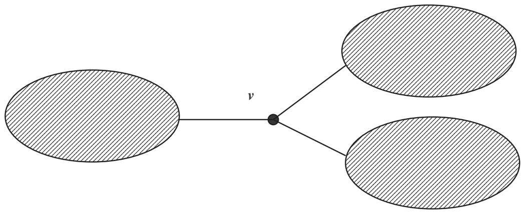

Fig. 3. T is not strip-drawable if the core edges are not a path

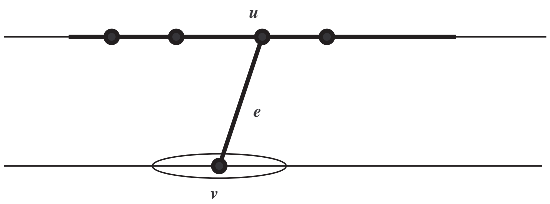

Fig. 4. Drawing a tree on a strip

There is one degenerate case to consider. If there are no core edges (*i.e.* a path of length 0), then either the tree has no vertex of degree 3 and is in fact drawable on a single track, or there exists at most one vertex v with neighbours $w_1, w_2, \ldots w_k$ and each T_{w_i} is *not* 2-strict. Each of the subtrees can thus be drawn on track t_2 and v on track t_1 as in Figure 5.

Lemma 3. *Let T be a tree with n vertices. There exists an $O(n)$-time algorithm that recognizes whether T is strip-drawable and, if so, computes a strip drawing of T.*

Proof. (sketch) Note that a tree is 2-strict iff it has more than 2 leaves; thus counting leaves is the crucial operation. First the core edges must be established and then the path condition on the core edges checked.

With each edge $e = (u, v)$ we associate 2 counters: l_u will be the number of leaves in T_u, and l_v will be the number of leaves in T_v. Let l be the number of leaves in the entire tree T. Then clearly $l_u + l_v = l$. By the previous observation, e is a core edge iff both l_u and $l_v > 2$.

Choose an arbitrary non-leaf vertex r as a root. Each vertex v reports the number of leaves in the subtree below it to its parent u – thus establishing l_u for the edge (u, v) and hence l_v. If v has no children then it is a leaf and reports 1. A simple recursive function can be used to implement this counting step in linear time.

Finally, checking that the core edges form a path is also easily accomplished in linear time as is the production of a strip drawing.

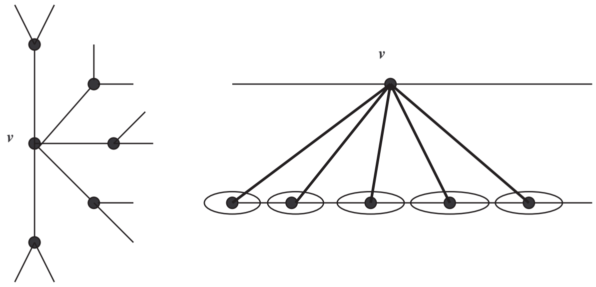

Fig. 5. A degenerate core path

We can summarize Lemmas 2 and 3 as follows.

Theorem 1. *A tree T with n vertices is strip drawable if and only if its core edges form a path. Furthermore, there exists an $O(n)$-time algorithm that determines whether T is strip drawable and, if so computes a strip drawing of T.*

3.2 k-track and $c \cdot n \times k$-grid Drawable Trees

The results of Theorem 1 can be extended to k tracks by generalizing some of the concepts of the previous section. A tree is *k-strict* if it contains a vertex adjacent to at least three subtrees that are $(k-1)$-strict. An edge is a *k-core edge* if its removal results in two k-strict components. The proofs of some of the following lemmas are similar to the case when $k = 2$ and are omitted in this extended abstract.

Property 4. A k-strict tree is not $(k-1)$-track drawable.

Lemma 4. *k-core edges are connected.*

Lemma 5. *A tree is k-track drawable if and only if the k-core edges form a path.*

The following lemma shows that the families of k-track drawable trees and $c \cdot n \times k$-grid drawable trees coincide.

Lemma 6. *Let c, k be two positive integer. A tree T is $c \cdot n \times k$-grid drawable if and only if it is k-track drawable.*

Proof. (sketch) Since a k-track drawing is a restricted form of a $c \cdot n \times k$ grid drawing where c is any given positive integer constant, it suffices to show that if T has a $c \cdot n \times k$ grid drawing, then the k-core edges form a path which can be shown by contradiction – the proof follows the form of Lemma 2. If the core edges do not form a path, then there is some vertex v with three incident core edges whose subtrees each critically require k grid lines and there is no location for v to attach to all three.

Theorem 2. *Let T be a tree with n vertices and let c and k be two positive integer constants. The following three statements are equivalent:*
1. *T is $c \cdot n \times k$-grid drawable.*
2. *T is k-track drawable.*
3. *the k-core edges of T form a path.*

Furthermore, there exists an $O(n)$-time algorithm that determines whether T satisfies the above conditions and, if so, computes a k-track and a $c \cdot n \times k$-grid drawing of T.

One consequence of the previous theorem is the existence of non-drawable trees – ternary trees for example provide the critical strict components.

Corollary 1. *The complete ternary tree of height k is not drawable on a $c \cdot n \times (k-1)$ grid, for any positive integer c.*

Proof. (sketch) Such a tree is k-strict and hence not $(k-1)$-track drawable by Property 4. Therefore, by Theorem 2 the tree is not $c \cdot n \times (k-1)$-grid drawable, for any given integer c such that $c \geq 1$.

We conclude the section with two remarks concerning the natural problem of extending Theorem 2 to graphs other than trees.

Remarks:
- It is surprising that the class of trees that are drawable on k tracks is exactly the class of trees drawable on an $c \cdot n \times k$ grid. However this result does not immediately extend more generally to planar graphs as for example, the complete graph K_4 is drawable on an $n \times 3$ grid but not on 3 tracks.
- There are certainly strip drawable graphs that are not trees. It is clear that all strip drawable graphs are outerplanar as they have an embedding with all vertices on 2 tracks and thus on the exterior face. The *maximal* outerplanar graphs that are strip drawable can also be established in linear time.

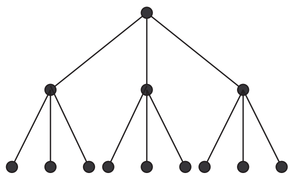

Fig. 6. Complete ternary trees of height k are neither $(k-1)$-track drawable nor in an $c \cdot n \times (k-1)$-grid drawable.

Lemma 7. *A maximal outerplanar graph with $n > 3$ vertices is strip-drawable iff it has exactly 2 vertices of degree 2, and this can be determined in $O(n)$ time.*

Proof. In any strip drawing of a maximal outerplanar graph the faces between the 2 tracks form a consecutive sequence with exactly 2 faces that are leftmost and rightmost, each of which contains exactly one vertex of degree 2 and two vertices of higher degree. All other vertices must appear on the interior of the spine and hence have degree strictly greater than 2.

4 Three-Dimensional Drawings of Outerplanar Graphs

In the previous section Theorem 2 and Corollary 1 showed that, for any given positive integers c, k, there is no $c \cdot n \times k$ grid that supports all trees of n vertices. This motivates us to investigate whether there are 3 dimensional restricted grids that support all trees. As it turns out, the situation in 3 dimensions is distinctly different. In this section we show that all outerplanar graphs are prism-drawable by providing a linear time algorithm that computes this drawing. This is the first known three-dimensional straight-line drawing algorithm for the class of outerplanar graphs that achieves $O(n)$ volume on an integer grid.

A high level description of our drawing algorithm, called `Algorithm Prism Draw`, is as follows. Let G be an outerplanar graph with a specified *outerplanar embedding*, i.e. a circular ordering of the edges incident around each vertex such that all vertices of G belong to the external face. (Such an embedding can be computed in linear time). `Algorithm Prism Draw` computes a prism drawing of G by executing two main steps. Firstly a 2d drawing of G is computed on a grid that consists of $O(n)$ horizontal tracks and such that adjacent vertices are at grid points whose y-coordinates differ by at most one; this is done by visiting G in a breadth-first fashion. Secondly, the drawing is "wrapped" on the faces of a prism by folding it along the tracks.

`Algorithm Prism Draw`

input: An outerplanar graph G with a given outerplanar embedding.
output: A prism drawing of G.

`Step 1. The 2d Drawing Phase:` A 2d grid drawing Γ of G where vertices are assigned to distinct tracks is computed as follows.

- Add a dummy vertex d on the external face and an edge connecting d to an arbitrary vertex v.
- mark d
- i:=0
- currx :=0
- draw v on track t_0 by setting X(v) := currx; Y(v) :=i
- currx := currx +1
- mark v
- while there are unmarked vertices of G do
  - visit the vertices on track t_i from left to right and for each encountered vertex u do
    - let w be a marked neighbour of u in G
    - visit the neighbours of u in counterclockwise order starting from w, and for each encountered vertex r such that r is unmarked do
      - draw r on track t_{i+1} by setting X(r) := currx; Y(r) := i+1

· currx := currx +1
· mark r
- i := i+1

`Step 2: The 3d Wrapping Phase:` A prism drawing Γ' is obtained by folding Γ along its tracks as follows.
- for each vertex v of Γ define its coordinates $X'(v)$, $Y'(v)$ and $Z'(v)$ in Γ' by setting:
  - $X'(v) := X(v)$
  - if $Y(v) = 0, 1 \, mod \, 3$ then $Y'(v) := 0$, else $Y'(v) := 1$
  - if $Y(v) = 0, 2 \, mod \, 3$ then $Z'(v) := 0$, else $Z'(v) := 1$

`End of Algorithm Prism Draw`

Figure 7 shows an example of the output of Step 1 of the algorithm.

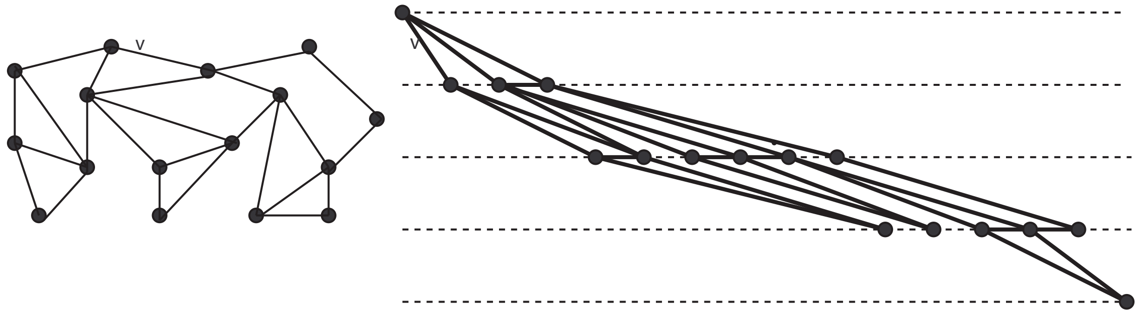

Fig. 7. An outerplanar graph drawn by `Step 1` of `Algorithm Prism Draw`.

Lemma 8. *Let G be an outerplanar graph with a given outerplanar embedding.* `Algorithm Prism Draw` *computes a prism drawing of G.*

Proof. (sketch) To prove the correctness of `Algorithm Prism Draw` we observe the following about `Step 1`.
- No two vertices of Γ are assigned the same X-coordinate.
- Every vertex assigned to track t_{i+1} has a neighbour on track t_i, for $i \geq 0$: The only vertices assigned to track t_{i+1} are those that are not yet marked/assigned and with a neighbour on track t_i.

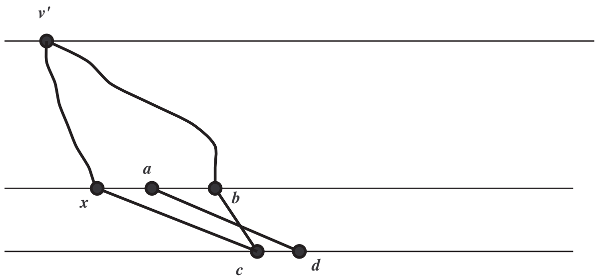

Fig. 8. Illustration for the proof of Lemma 8.

– No pair of edges between vertices on adjacent tracks intersect: By contradiction. Suppose i is the smallest subscript such that there is an intersection between vertices on track t_i and t_{i+1}. Let a and b be the 2 leftmost vertices on track t_i that intersect and let d and c be the respective endpoints on track t_{i+1}. Refer to Figure 8. By the previous 2 arguments, there exists a path from a to v and from b to v. If these 2 paths intersect, it must be at a vertex v' on some lower indexed track. Now since all unmarked neighbours of a were assigned to track t_{i+1} before those of b, it must be that vertex c is also adjacent to some other vertex x on track t_i to the left of b. There are 3 possible locations for x: to the left of a, at a (i.e. $a = x$), or between a and b. Each of these locations of x on this track leads to a violation of the outerplanar specification of the input graph.

Concerning `Step 2 (3d Wrapping Phase)`, observe that the coordinate assignment is such that the vertices of Γ' are grid points of a $n \times 2 \times 2$ grid; also they all belong to just three of the four possible tracks of the $n \times 2 \times 2$ grid. Therefore, the vertices of Γ' are drawn as grid points of a prism. Also, no two edges of Γ' intersect because, as shown above, each subdrawing of Γ' induced by vertices on two different tracks is a strip drawing and because the vertices on each track have distinct X-coordinates.

Based on Lemma 8 and on the observation that `Algorithm Prism Draw` runs in linear time since it is essentially a breadth-first traversal of the graph, we can conclude the following.

Theorem 3. *Every outerplanar graph G with n vertices admits a crossing-free straight line grid drawing in 3 dimensions in optimal $O(n)$ volume. Furthermore, there exists an algorithm that computes such a drawing of G in $O(n)$ time and with the vertices of G drawn on the grid points of a prism.*

5 Prism-Drawable Graphs

Since by Theorem 3 all outerplanar graphs can be drawn on a prism, it is natural to investigate the class of graphs that are prism-drawable. For example, note that the family of planar graphs consisting of a sequence of nested triangles and that are known to require $\Omega(n^2)$ area in the plane, can be drawn on the prism (and thus have $O(n)$ volume). It is also clear that since any drawing on the prism can be augmented by edges to form a convex polytope, by the theorem of Steinitz only planar graphs are prism-drawable. In this section, we give a characterization for the prism-drawable graphs and show that not all planar graphs are in this class. Figure 9 shows a graph, and its prism drawing.

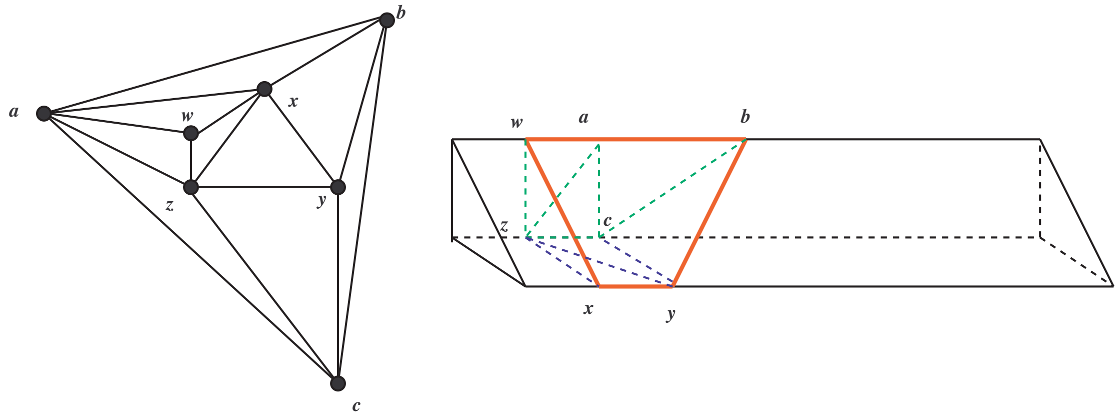

Fig. 9. A prism-drawable graph G and its drawing

5.1 Characterization of Prism-Drawable Graphs

An essential prerequisite of our characterization of prism-drawable graphs, is the study of the strip-drawable graphs since a prism effectively consists of three strips.

We define a *spine* in a graph as a sequence of adjacent vertices with no chord. The characterization of strip-drawable graphs proposed in this section notes that in a strip drawing, there must exist 2 potential spines and that edges incident to vertices on both spines must not cross. This formulation will be generalized in subsequent sections to larger grid sets.

Theorem 4. *A graph G is strip-drawable iff it is possible to augment G with edges to produce a graph G' which contains two pairs of adjacent vertices r, b and r', b' and there exists a spine from r to r', a spine from b to b' with all vertices of G on the two spines and if there exists an edge (r_i, b_j) then there are no edges of the form (r_k, b_l) with ($k < i$ and $l > j$) or ($k > i$ and $l < j$).*

Proof. (sketch) Refer to Figure 10. Given a strip drawing, it is clear that edges can be added along the 2 tracks to form the 2 spines, and to add an edge between the leftmost pair on the two tracks and between the rightmost pair on the 2 tracks. Since no pair of edges intersect, the non-crossing conditions are maintained.

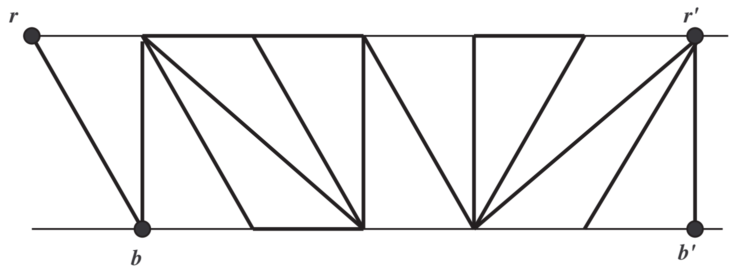

Fig. 10. Strip characterization

Given an augmented graph with the required properties, a valid strip drawing is obtained by drawing each spine on a separate track, and the edge conditions ensure there are no crossings. Vertices on each spine can be placed at consecutive integral X-coordinates thus ensuring that the strip is at most of length n.

The characterization of prism drawable graphs generalizes Theorem 4 to 3 dimensions, namely it must be possible to augment a given graph to obtain 3 spines with 2 lids (3 cycles) and between each pair of spines the non-crossing condition on edges must hold.

Theorem 5. *A graph G is prism-drawable iff it is possible to augment G with edges to produce a graph G' which contains two three cycles r, b, g and r', b', g' and there exists a spine from r to r', a spine from b to b', a spine from g to g' with all vertices of G on the three spines and for each pair of spines $x \to x'$ and $y \to y'$ ($x, y = r, b, g$, $x \neq y$), if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with ($k < i$ and $l > j$) or ($k > i$ and $l < j$).*

Proof. (sketch) A prism consists of 3 strips. Any drawing on a prism must observe the strip conditions between each pair of spines as established in theorem 4.

In the rest of the paper it will be convenient to imagine the three spines as coloured red, blue and green.

5.2 Prism-Drawabity and Planarity

In this section we show there exist 1-connected, 2-connected and 3-connected planar graphs that are not prism-drawable.

Define a graph G as *strictly prism-drawable* if it is prism-drawable and all prism drawings of G have at least one edge on each facet of the prism. Note that K_4 is strictly prism-drawable. Our goal is to show the existence of series-parallel graphs that are not prism-drawable, and the graph P of Figure 11 is a more useful strictly prism-drawable graph for this purpose.

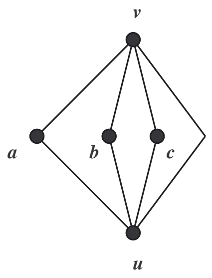

Fig. 11. A strictly prism-drawable graph P

Lemma 9. *The graph P in Figure 11 is strictly prism-drawable.*

Proof. (sketch) It is easy to verify that P has three spines that satisfy the characterization of prism-drawability.

We need to show that P is *strictly* prism-drawable. Note that P cannot be drawn so that all edges belong to the same facet of the prism as P is not outerplanar. Therefore, P must have a vertex on each of the red, green and blue spines.

Secondly, note that the vertices u and v cannot be both on the same spine, which we prove by contradiction: Assume that u and v are both green. Note that neither a nor b nor c can be green, as otherwise the green path would contain a chord. Thus two of these vertices are on a non-green spine. Assume that a and b are red. This forces a crossing. The cases where b and c are red or a and c are red follow the same argument. Therefore, u and v must have different colours.

Because P needs all three colors, at least one of a, b, c is red. Suppose a is red. Now the 3-cycle (a, u, v) has its three vertices on different spines, which implies that a prism drawing of P will have (a, v) (a, u) and (v, u) on different facets.

Lemma 10. *Let G be a 1-connected graph that has a cut vertex whose removal separates the graph into h strictly prism-drawable components ($h \geq 3$); then G is not prism-drawable.*

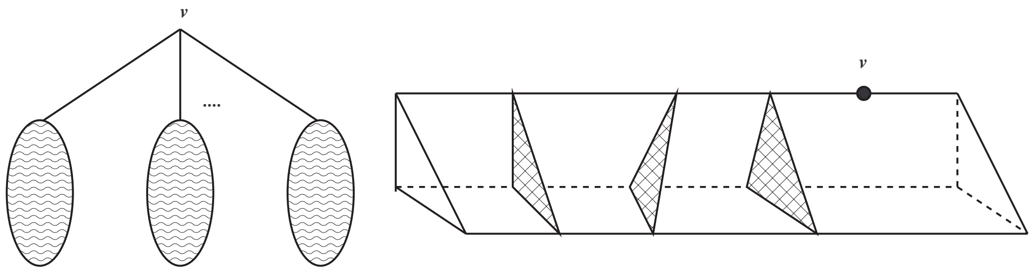

Fig. 12. A non-prism-drawable graph

Proof. (sketch) Figure 12 outlines the necessary construction. Consider a prism drawing Γ_i of G_i. Γ_i has a 3-cycle which defines a plane that intersects all three facets of the prism, because G_i is strictly prism-drawable. Thus, each Γ_i slices the prism into $h + 1$ slices. Now there is no location for v that permits it to be connected crossing-free to all Γ_i without crossing at least one 3-cycle.

It is not critical that v be a cut vertex in the previous proof – any vertex connected to 3 or more strictly prism-drawable subgraphs will suffice and thus, the previous lemma generalizes as follows – its proof follows an argument similar to that of Lemma 10 and is omitted in this extended abstract.

Lemma 11. *Let G be a graph with subgraphs $G_1, G_2, \ldots G_h$ ($h \geq 3$) such that: $G_i \subset G$ ($1 \leq i \leq h$), $G_i \cap G_j = \emptyset$ ($1 \leq i, j \leq h$), and G_i is strictly prism-drawable. If there exists a vertex $v \epsilon G - \{G_1, G_2, \ldots, G_h\}$, such that each G_i has a vertex adjacent to v, then G is not prism-drawable.*

A consequence of the previous lemmas is the following.

Theorem 6. *There exist series-parallel graphs that are not prism-drawable.*

Proof. (sketch) The graph in Figure 13 has the required properties.

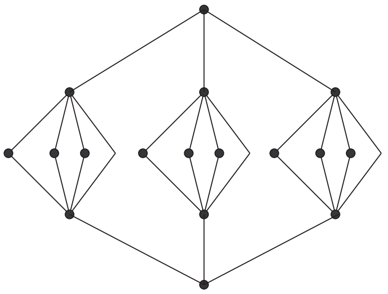

Fig. 13. A series-parallel graph that is not prism-drawable

As noted in the introduction, there are maximal planar graphs that are prism-drawable. The following theorem investigates prism drawability of maximal planar graphs.

Theorem 7. *Not all maximal planar graphs are prism-drawable. Also, the family of maximal planar prism drawable graphs is a proper subset of the family of Hamiltonian planar graphs.*

Proof. (sketch) It can easily be seen that all prism drawings of maximal planar graphs are Hamiltonian (an example of the cycle is outlined in Figure 14). Since there exist maximal planar graphs which are not

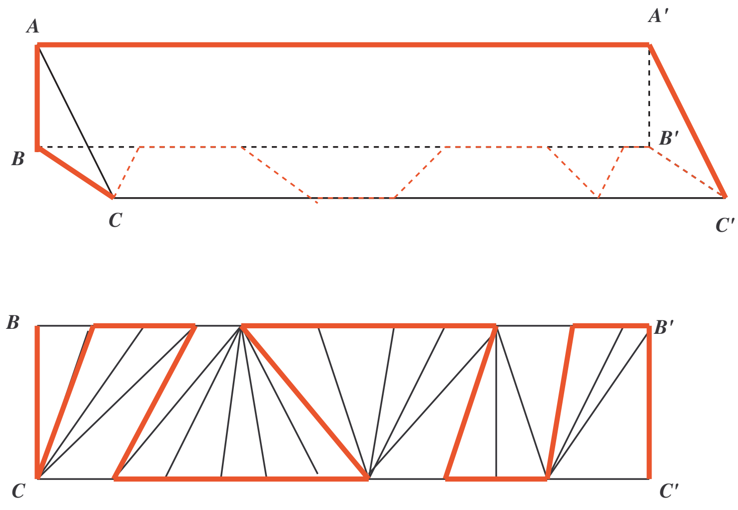

Fig. 14. Hamiltonian Cycle on a Prism. Above: General form. Below: bottom strip

Hamiltonian it immediately follows that not all maximal planar graphs are prism-drawable.

It is also possible to use the spine characterization more directly to produce planar graphs that are not prism-drawable – even Hamiltonian. Consider the Hamiltonian graph G of Figure 15. Consider the vertices a, b, c. Each must be either a start or end vertex of some spine (otherwise there exists a chord). Without loss of generality, assume 2 are starts. However no 2 share the same face and hence there is no way to form the initial 3 cycle for the spines.

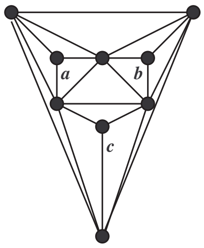

Fig. 15. A Hamiltonian planar graph G that is not prism-drawable.

6 Box-Drawable Graphs

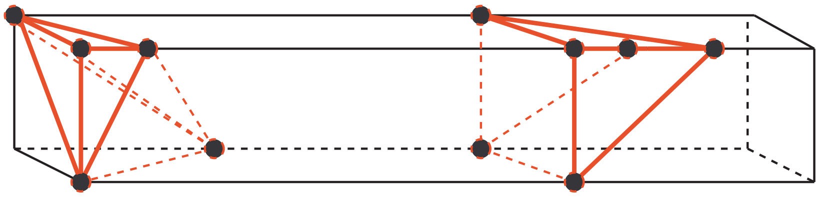

Fig. 16. K_5 and $K_{3,3}$ drawn on a box

The family of box drawable graphs is clearly a superset of the class of prism drawable graphs. Furthermore, there exist non-planar graphs that are box drawable; for example K_5 and $K_{3,3}$ are box-drawable as shown in figure 16. Note however that K_6 is not box-drawable and we will show that there exist planar graphs (and even series-parallel graphs) that are not box-drawable. We refer to the 6 possible strips on which edges can be drawn as the *facets* of the box although two such facets appear inside the bounding box.

Theorem 8. *A graph G is box-drawable iff it is possible to augment G with edges to produce a graph G' which contains two four cycles r, b, g, o and r', b', g', o' and there exists a spine from r to r', a spine from b to b', a spine from g to g' , and a spine from o to o' with all vertices of G on the four spines and for each pair of spines $x \to x'$ and $y \to y'$, if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with $(k < i$ and $l > j)$ or $(k > i$ and $l < j)$.*

Proof. (sketch) As in the previous characterizations for the strip and prism, a box drawing can be augmented to complete the spines and the 6 facets must form valid strips. In this case there appears to be an extra condition necessary to ensure that the 2 pairs of strips on the 2 diagonally opposite spines do not intersect. As shown in Figure 17, these 2 pairs of spines can be separated to ensure no diagonal crossings appear in the interior of the box.

By using Theorem 8 examples of graphs that are not box drawable can be constructed.

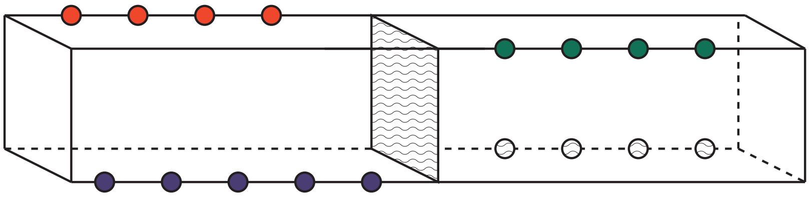

Fig. 17. The 2 pairs of diagonally opposite spines can be separated

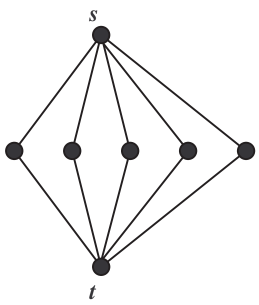

Fig. 18. Graph F for lemma 12

Lemma 12. *In the graph F in Figure 18, the vertices s and t cannot appear on the same spine of any box drawing. Furthermore, at least 2 vertices $v_1, \ldots v_5$ cannot share the same spine as s or t.*

Proof. *(sketch)* If s and t are on the same spine then 2 of the 5 vertices $v_1, \ldots v_5$ must lie on the same spine which would force a crossing.

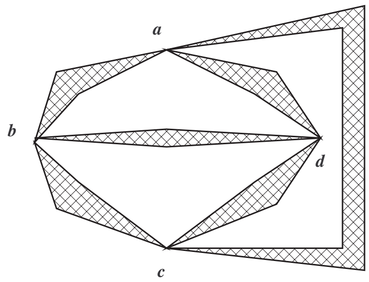

Fig. 19. Graph H for lemma 13

Lemma 13. *The graph H in figure 19 which consists of 6 copies of the graph F, has the property that in any box drawing, vertices a, b, c, d lie on 4 different spines and all 6 facets of the box contain an edge.*

Proof. *(sketch)* By the previous lemma, each pair of vertices a, b, c, d are forced to be on different spines.

Theorem 9. *Not all planar graphs are box-drawable.*

Proof. *(sketch)* The planar graph E in Figure 20 which consists of 3 copies of graph H connected to a vertex x, is not box-drawable since the 3 components slice the box and prevent the vertex labelled x from attaching to all 3 components simultaneously.

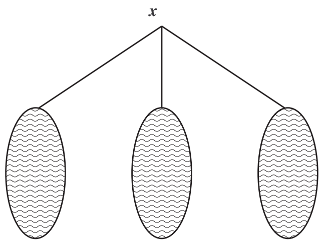

Fig. 20. A planar graph E that is not box-drawable

7 Conclusions, Extensions and Open Problems

In this paper we showed that all outerplanar graphs can be drawn in linear volume on a prism – a restriction of the 3 dimensional integer grid. We gave efficient characterizations of the trees that are drawable in 2 dimensions on an $n \times k$ grid. Proofs that certain classes of planar graphs are not prism-drawable nor box-drawable were also provided.

Although the problem of finding a universal integer 3d grid of linear volume that supports crossing-free straight line drawings of all planar graphs is still far from being solved, the drawing techniques and characterization results of this paper may provide a critical starting point for attacking such an ambitious research programme. We believe that the results on the restricted 2d grid are not only useful preliminary results for the study in the three dimensional space, but they may also shed some new light on the problem of drawing trees in linear area on the plane.

There remain several interesting problems and directions for further research. We conclude the paper by mentioning some of those that in our opinion are among the most interesting.

1. Can all outerplanar graphs be drawn in linear area on a 2d integer grid? Does there exist a 2d universal grid set of linear area that supports all outerplanar graphs?
2. Characterize the graphs drawable on an $n \times k$ grid.
3. Can all planar graphs be drawn in linear volume on a 3 dimensional integer grid? Does there exist a 3d universal grid set of linear volume that supports all planar graphs?
4. **Aspect Ratio**: Our results about linear volume come at the expense of aspect ratio. Is it possible to achieve both linear volume and $o(n)$ aspect ratio for outerplanar graphs? We conjecture that it is in fact not possible in 2 dimensions to simultaneously attain linear area and $O(1)$ aspect ratio for some classes of planar graphs.

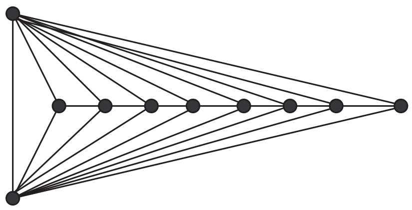

Fig. 21. A graph S_n with poor aspect ratio

Conjecture 1. There is no fixed constant k for which the family of graphs S_n (in Figure 21) can be drawn in a 2-dimensional integer grid of size $k\sqrt{n} \times \sqrt{n}$.

Note that the graph S_n can be drawn on a $n \times 3$ grid (and hence in linear area but with *linear* aspect ratio).

5. **Grid Drawings and Pathwidth**: The notion of the *pathwidth* of a graph has been well-studied in the graph theory literature – for definitions, see for example the book by Diestel [10]. It is not difficult to show that for trees there is a strong connection between pathwidth and grid drawings as summarized in the following propositions.

   Proposition 1. *For a tree T, pathwidth(T)*$=\min_k$ *(T is drawable on a $n \times k$ grid).*

   Proposition 2. *If G is a planar graph then pathwidth$(G) \leq \min_k$ (G is drawable on a k-strip).*

   We have not been able to determine whether there are planar graphs with *pathwidth*$(G) < \min_k$ (G is drawable on a k-strip). If not, a linear time algorithm for recognizing graphs of pathwidth $\leq k$ would provide us with a linear time algorithm to recognize k-strip drawable graphs. We believe that 2-strip drawable graphs can be recognized in polynomial time, however the complexity of the recognition of k-strip drawable graphs for $k \geq 3$ remains open.

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References

1. T. Calamoneri and A. Sterbini. Drawing 2-, 3-, and 4-colorable graphs in $o(n^2)$ volume. In S. North, editor, *Graph Drawing (Proc. GD '96)*, volume 1190 of *Lecture Notes Comput. Sci.*, pages 53–62. Springer-Verlag, 1997.
2. T.M. Chan. A near-linear area bound for drawing binary trees. In *Proc. 10th Annu. ACM-SIAM Sympos. on Discrete Algorithms.*, pages 161–168, 1999.
3. M. Chrobak and G. Kant. Convex grid drawings of 3-connected planar graphs. *Internat. J. Comput. Geom. Appl.*, 7(3):211–223, 1997.
4. Marek Chrobak, Michael T. Goodrich, and Roberto Tamassia. Convex drawings of graphs in two and three dimensions. In *Proc. 12th Annu. ACM Sympos. Comput. Geom.*, pages 319–328, 1996.
5. Marek Chrobak and Shin ichi Nakano. Minimum-width grid drawings of plane graphs. *Comput. Geom. Theory Appl.*, 11:29–54, 1998.
6. R. F. Cohen, P. Eades, T. Lin, and F. Ruskey. Three-dimensional graph drawing. *Algorithmica*, 17:199–208, 1997.
7. H. de Fraysseix, J. Pach, and R. Pollack. Small sets supporting Fary embeddings of planar graphs. In *Proc. 20th Annu. ACM Sympos. Theory Comput.*, pages 426–433, 1988.
8. H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.
9. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing.* Prentice Hall, Upper Saddle River, NJ, 1999.
10. Reinhard Diestel. *Graph theory.* Graduate Texts in Mathematics. 173. Springer, 2000. Transl. from the German. 2nd ed.
11. Stefan Felsner. Convex drawings of planar graphs and the order dimension of 3-polytopes. *Order – accepted to appear.*
12. A. Garg, M. T. Goodrich, and R. Tamassia. Planar upward tree drawings with optimal area. *Internat. J. Comput. Geom. Appl.*, 6:333–356, 1996.
13. A. Garg, R. Tamassia, and P. Vocca. Drawing with colors. In *Proc. 4th Annu. European Sympos. Algorithms*, volume 1136 of *Lecture Notes Comput. Sci.*, pages 12–26. Springer-Verlag, 1996.
14. Michael Juenger and Sebastian Leipert. Level planar embedding in linear time. In J. Kratochvil, editor, *Graph Drawing (Proc. GD '99)*, volume 1731 of *Lecture Notes Comput. Sci.*, pages 72–81. Springer-Verlag, 1999.
15. G. Kant. A new method for planar graph drawings on a grid. In *Proc. 33rd Annu. IEEE Sympos. Found. Comput. Sci.*, pages 101–110, 1992.
16. G. Kant. Drawing planar graphs using the canonical ordering. *Algorithmica*, 16:4–32, 1996. (special issue on Graph Drawing, edited by G. Di Battista and R. Tamassia).
17. János Pach, Torsten Thiele, and Géza Tóth. Three-dimensional grid drawings of graphs. In G. Di Battista, editor, *Graph Drawing (Proc. GD '97)*, volume 1353 of *Lecture Notes Comput. Sci.*, pages 47–51. Springer-Verlag, 1997.
18. F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, 3rd edition, October 1990.
19. W. Schnyder. Embedding planar graphs on the grid. In *Proc. 1st ACM-SIAM Sympos. Discrete Algorithms*, pages 138–148, 1990.
20. W. Schnyder and W. T. Trotter. Convex embeddings of 3-connected plane graphs. *Abstracts of the AMS*, 13(5):502, 1992.