BOUNDS FOR ORDERS OF ZEROS OF A CLASS OF EISENSTEIN SERIES AND THEIR APPLICATIONS ON DUAL PAIRS OF ETA QUOTIENTS

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ABSTRACT. Let k be an even positive integer, p be a prime and m be a nonnegative integer. We find an upper bound for orders of zeros (at cusps) of a linear combination of classical Eisenstein series of weight k and level p^m . As an immediate consequence we find the set of all eta quotients that are linear combinations of these Eisenstein series and hence the set of all eta quotients of level p^m whose derivatives are also eta quotients.

1. INTRODUCTION

For an element z in the upper half-plane of the complex numbers, let $q := e^{2\pi i z}$. The classical Eisenstein series are defined by

$$E_k(z) := \frac{-B_k}{2k} + \sum_{n \ge 1} \sigma_{k-1}(n) q^n, \tag{1.1}$$

where $k \ge 2$ is an even integer, B_k is the *k*-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{0 \le d|n} d^{k-1}$. (Here we use the normalization given in [10, p. 88] for the Eisenstein series $E_k(z)$.) Let

$$\mathcal{E}_k(N) := \begin{cases} \langle E_2(z) - dE_2(dz); \ 1 < d \mid N \rangle_{\mathbb{Q}} & \text{if } k = 2, \\ \langle E_k(dz); \ 1 \le d \mid N \rangle_{\mathbb{Q}} & \text{otherwise.} \end{cases}$$

Let $M_k(\Gamma_0(N))$ be the space of modular forms of weight k for $\Gamma_0(N)$. Then it is known that for all even $k \ge 2$ the space $\mathcal{E}_k(N)$ is a subset of $M_k(\Gamma_0(N))$.

Some infinite products that can be expressed explicitly as infinite sums are elements of $\mathcal{E}_k(N)$. For example we have

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{20}}{(1-q^n)^8(1-q^{4n})^8} = 8E_2(z) - 32E_2(4z),$$

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2 (1-q^{4n})^4 (1-q^{6n})^6}{(1-q^n)^2 (1-q^{3n})^2 (1-q^{12n})^4} = 2E_2(z) - 3E_2(2z) + 4E_2(4z) + 9E_2(6z) - 36E_2(12z),$$

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and

$$\left(\prod_{n=1}^{\infty} \frac{(1-q^{2n})^5}{(1-q^n)^2(1-q^{4n})^2}\right)^{4k} = \frac{-2k}{B_{2k}} \left(\frac{(-1)^k}{2^{2k}-1} E_{2k}(z) - \frac{((-1)^k+1)}{2^{2k}-1} E_{2k}(2z) + \frac{2^{2k}}{2^{2k}-1} E_{2k}(4z)\right) + \sum_{n=1}^{\infty} O(n^k)q^n,$$

where the first identity can be deduced from Jacobi's formula for the representation by four squares [5], the second identity is from Williams's paper [11, Table 1, No. 24], and the last one is given by Ramanujan [8, Sec. 25] and proven by Mordell [7].

By using [1, Corollary 2.1] the first two indentities above induce the differential identities

$$D\left(q\prod_{n=1}^{\infty}\frac{(1-q^{4n})^8}{(1-q^n)^8}\right) = q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{20}}{(1-q^n)^{16}}$$

and

$$\begin{split} D\left(q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3 (1-q^{12n})^6}{(1-q^n)^4 (1-q^{4n})^2 (1-q^{6n})^3}\right) \\ &= 2q^2 \prod_{n=1}^{\infty} \frac{(1-q^{2n})^5 (1-q^{4n})^2 (1-q^{6n})^3 (1-q^{12n})^2}{(1-q^n)^6 (1-q^{3n})^2}. \end{split}$$

Here, $D := q \frac{d}{dq}$ denotes the Ramanujan differential operator. Additionally, the differential identity

$$E_2(z)^2 = \frac{5}{12}E_4(z) - \frac{1}{2}D(E_2(z))$$

appears in the works of Besge, Glaisher and Ramanujan independently (see [4] for the references). In all these examples, one can replace z by tz ($t \in \mathbb{N}$) and obtain another product-to-sum formula. To avoid this triviality; and to avoid counting the same example more than once we define the set

$$\mathcal{P}_k(N) := \mathcal{E}_k(N) \backslash \mathcal{O}_k(N),$$

where

$$O_k(N) := \bigcup_{1 < d \mid N} \left(\mathcal{E}_k(N/d) \cup \{ f(dz); f(z) \in \mathcal{E}_k(N/d) \} \right).$$

Let R(N) denote a complete set of inequivalent cusps of $\Gamma_0(N)$ and for $r \in R(N)$, let $v_r(f)$ denote the order of vanishing of f at the cusp r. Letting $f(z) \in \mathcal{P}_k(p^m)$, in this paper we find the following upper bound for the sum of orders of vanishings of f(z) at cusps in $R(p^m)$.

Theorem 1.1. Let p be a prime and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $k \ge 2$ be even. If $f(z) \in \mathcal{P}_k(p^m)$, then we have

$$\sum_{r \in R(p^m)} v_r(f) < \begin{cases} 1 & \text{if } p^m = 1, \\ 4 & \text{if } p^m = 4, \\ p^{[(m-1)/2]}(p^{(m-1)-2[(m-1)/2]} + 1) & \text{if } p^m \neq 1 \text{ or } 4 \end{cases}$$

Remarks 1.2. (1) For $p^m \neq 4$, the proof of Theorem 1.1 more generally establishes that if $f(z) \in \mathcal{P}_k(p^m)$, then $v_r(f) \leq 1$ at any cusp $r \in R(p^m)$. Note that

$$\#R(p^m) = p^{[(m-1)/2]}(p^{(m-1)-2[(m-1)/2]} + 1),$$

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see [3, Corollary 6.3.24.(b)].

- (2) The bounds given by Theorem 1.1 do not depend on the weight k. Therefore, by valence formula, as the weight increases, the proportion of the zeros at the cusps compared to all the zeros decreases.
- (3) If we let the field of coefficients in the definition of $\mathcal{E}_k(N)$ to be \mathbb{C} , then Theorem 1.1 holds for

$$f(z) = \sum_{t|p^m} r_t E_k(tz) \in \mathcal{P}_k(p^m),$$

unless m is even and $\omega \frac{r_{p^m}}{r_1} \neq -p^{mk/2}$ where ω is a certain $p^{m/2}$ th root of unity defined in the proof of Lemma 3.1. In this case the bound on $\sum_{r \in R(p^m)} v_r(f)$ can be bigger because one of the arguments in the proof of Lemma 3.1 may fail in certain cases.

We next describe an application of Theorem 1.1. The Dedekind eta function is defined by the infinite product

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

An eta quotient of level N is defined to be of the form

$$f(z) = \prod_{t|N} \eta(tz)^{r_t}, \tag{1.2}$$

where the exponent r_t are integers. The weight attached to this eta quotient is $k = \sum_{0 < t | N|} r_t/2$. Notice that we do not require level to be the lowest common multiple of t for which $r_t \neq 0$, therefore the level of an eta quotient in our approach is not unique. This gives us a certain freedom in our discussions and does not affect the completeness of our results.

We say an eta quotient f is primitive if there is no eta quotient g such that f(z) = g(dz) for some integer d > 1. A pair (f,g) of eta quotients is called a *dual pair* if f is of weight -k and g is of weight k + 2, for some non-negative integers k, and the (k + 1)-th derivative of f is a nonzero constant multiple of g. In [2] the problem of finding all dual pairs of eta quotients (f,g) on $\Gamma_0(N)$ for which f is a primitive eta quotients is studied. Theorem 1.1 has an immediate application on finding dual pairs of eta quotients (f,g) of weight (0,2). These are the eta quotients whose derivatives are also eta quotients (or constant multiple of eta quotients). In [2] the set of all such primitive eta quotients on $\Gamma_0(N)$ with squarefree levels N is given; in [1] a set of 203 dual pairs of weight (0, 2) was given and conjectured to be the complete set for all N. Additionally in [1] it is established that every such pair is induced by the eta quotients in $\mathcal{E}_2(N)$. Since eta quotients have all their zeros (or poles) at cusps, as a direct consequence of our Theorem 1.1 we establish the complete set of eta quotients of prime power levels whose derivatives are also eta quotients. This extends the results of [2] on dual pairs of weight (0, 2) for squarefree levels to prime power levels.

As noted all zeros (or poles) of eta quotients are at the cusps. Hence, by the valence formula, for an eta quotient f(z) of level p^m we have $\sum_{r \in R(p^m)} v_r(f) = \frac{k}{12}(p^m + p^{m-1})$ (see [2, Lemma 2.1]). Therefore a comparison with the upper bound given by Theorem 1.1 and investigations among possible pairs of (k, p^m) give us the following statement.

Theorem 1.3. Let p be prime, $m \in \mathbb{N}_0$ and $k \in \mathbb{N}$ be even. Then there is no eta quotient in $\mathcal{P}_k(p^m)$ unless $(k, p^m) = (2, 4), (2, 8), (2, 9), (2, 16), (4, 2)$ or (4, 4).

In Corollary 1.4 below the *trivial extensions* mean the following:

If $f(z) \in \mathcal{E}_k(N)$, then $cf(t_1z) \in \mathcal{E}_k(t_2N)$ for all $t_2 \in \mathbb{N}$, $t_1 \mid t_2$ and $c \in \mathbb{Q}$.

Corollary 1.4. *Let* p *be prime,* $m \in \mathbb{N}_0$ *.*

(1) The set of eta quotients

$$\begin{cases} \frac{\eta^{8}(z)}{\eta^{4}(z)}, \frac{\eta^{8}(4z)}{\eta^{4}(2z)}, \frac{\eta^{20}(2z)}{\eta^{8}(z)\eta^{8}(4z)}, \frac{\eta^{4}(z)\eta^{10}(4z)}{\eta^{6}(2z)\eta^{4}(8z)}, \frac{\eta^{10}(2z)\eta^{4}(8z)}{\eta^{4}(z)\eta^{6}(4z)}, \\ \frac{\eta^{6}(2z)\eta^{6}(4z)}{\eta^{4}(z)\eta^{4}(8z)}, \frac{\eta^{4}(z)\eta^{4}(8z)}{\eta^{2}(2z)\eta^{2}(4z)}, \frac{\eta^{10}(3z)}{\eta^{3}(z)\eta^{3}(9z)}, \frac{\eta^{2}(z)\eta^{8}(4z)\eta(8z)}{\eta^{5}(2z)\eta^{2}(16z)}, \\ \frac{\eta(2z)\eta^{8}(4z)\eta^{2}(16z)}{\eta^{2}(z)\eta^{5}(8z)}, \frac{\eta(2z)\eta^{6}(4z)\eta(8z)}{\eta^{2}(z)\eta^{2}(16z)}, \frac{\eta^{2}(z)\eta^{10}(4z)\eta^{2}(16z)}{\eta^{5}(2z)\eta^{5}(8z)} \end{cases}$$

is the complete set of eta quotients in $\mathcal{E}_2(p^m)$ (up to trivial extensions). (2) The set of eta quotients

$$\left\{\frac{\eta^{16}(2z)}{\eta^8(z)}, \frac{\eta^{40}(2z)}{\eta^{16}(z)\eta^{16}(4z)}, \frac{\eta^8(z)\eta^8(4z)}{\eta^8(2z)}, \frac{\eta^{16}(z)}{\eta^8(2z)}\right\}$$

is the complete set of eta quotients in $\mathcal{E}_4(p^m)$ (up to trivial extensions). (3) If k > 4, then there are no eta quotients in $\mathcal{E}_k(p^m)$.

It is known that the eta quotients in $\mathcal{E}_2(N)$ are intimately related to the eta quotients of level N whose derivatives are eta quotients. More precisely, Lemma 2.1 of [1] establishes a one-to-one correspondence between the eta quotients of the form $\sum_{1 < \delta | N} r_{\delta} (E_2(z) - \delta E_2(\delta z)) \in \mathcal{E}_2(N)$ and the eta quotients $\prod_{0 < \delta | N} \eta^{r_{\delta}}(\delta z)$ with $r_1 = -\sum_{1 < \delta | N} r_{\delta}$ whose derivatives are also eta quotients.

Corollary 1.5. Let p be prime, $m \in \mathbb{N}_0$. The set of eta quotients

$$\begin{cases} \frac{\eta^{8}(z)\eta^{16}(4z)}{\eta^{24}(2z)}, \frac{\eta^{3}(2z)}{\eta^{2}(z)\eta(4z)}, \frac{\eta^{8}(4z)}{\eta^{8}(z)}, \frac{\eta^{4}(z)\eta^{2}(4z)\eta^{4}(8z)}{\eta^{10}(2z)}, \frac{\eta^{5}(4z)}{\eta^{2}(z)\eta(2z)\eta^{2}(8z)}, \\ \frac{\eta^{2}(2z)\eta^{4}(8z)}{\eta^{4}(z)\eta^{2}(4z)}, \frac{\eta^{7}(2z)\eta^{2}(8z)}{\eta^{2}(z)\eta^{7}(4z)}, \frac{\eta^{3}(9z)}{\eta^{3}(z)}, \frac{\eta^{2}(z)\eta^{2}(4z)\eta^{2}(16z)}{\eta^{5}(2z)\eta(8z)}, \\ \frac{\eta(2z)\eta^{5}(8z)}{\eta^{2}(z)\eta^{2}(4z)\eta^{2}(16z)}, \frac{\eta^{5}(2z)\eta^{2}(16z)}{\eta^{2}(z)\eta^{5}(8z)}, \\ \end{cases}$$

is the complete set (up to trivial extensions) of eta quotients of level p^m whose derivatives are also eta quotients.

In Corollary 1.5 the trivial extensions mean the following:

If f(z) is an eta quotient whose derivative is also an eta quotient (or a constant multiple of an eta quotient), then $f^{\ell}(tz)$ is an eta quotient whose derivative is also an eta quotient (or a constant multiple of an eta quotient) for all $t, \ell \in \mathbb{N}$.

There is a previously known example of an eta quotient whose second derivative is also an eta quotient

$$D^{2}\left(\frac{\eta^{2}(2z)}{\eta^{4}(z)}\right) = 4\frac{\eta^{18}(2z)}{\eta^{12}(z)}.$$

From Corollary 1.4(2) we see that

$$\left(\frac{\eta^{18}(2z)}{\eta^{12}(z)}\right) \left| \left(\frac{\eta^2(2z)}{\eta^4(z)}\right) = \frac{\eta^{16}(2z)}{\eta^8(z)} \in \mathcal{E}_4(4).$$

This is not a coincidence. In fact we will show that any integer solution (r_1, r_2, r_4) of the system

$$\begin{cases} r_1 + r_2 + r_4 = -2, \\ \frac{5}{12}r_1^2 + \frac{1}{3}r_1r_2 = s_1, \\ \frac{5}{3}r_2^2 + \frac{4}{3}r_1r_2 + \frac{1}{2}r_1r_4 + \frac{4}{3}r_2r_4 = s_2, \\ \frac{20}{3}r_4^2 + \frac{8}{3}r_1r_4 + \frac{16}{3}r_2r_4 = s_4, \end{cases}$$

where $\sum_{0 < \delta|4} s_{\delta} E_4(\delta z) \in \mathcal{E}_4(4)$, gives rise to an eta quotient of weight -1 and level 4 for which its second derivative is an eta quotient. As a consequence of this we classify all the level 4 eta quotients whose second derivatives are also eta quotients.

Theorem 1.6. Up to trivial extensions the only level 4 eta quotient whose second derivative is also an eta quotient is $\frac{\eta^2(2z)}{\eta^4(z)}$.

The *trivial extensions* in Theorem 1.6 mean the following:

If f(z) is an eta quotient whose second derivative is also an eta quotient (or a constant multiple of an eta quotient), then f(tz) is an eta quotient whose second derivative is also an eta quotient (or a constant multiple of an eta quotient) for all $t \in \mathbb{N}$.

The arguments we use to prove Theorem 1.6 will not hold for eta quotients in most of the levels other than 4 because formulas analogous to (4.6)–(4.8) in other levels almost always involve cusp forms of weight 4. So the second derivative of an eta quotient in general may not be in $\mathcal{E}_4(N)$.

The organization of the rest of the paper is as follows. In the next section we derive the Fourier series expansions of $E_k(tz)$ at each cusp using the expansion at $i\infty$ of $E_k(z)$ and some matrix relations. We employ these expansions and use the description of the Eisenstein series in $\mathcal{P}_k(p^m)$ to prove Theorem 1.3. In Section 3 we prove Theorems 1.1 and 1.3, and Corollaries 1.4 and 1.5 by using the results of Section 2. Finally in Section 4 we work on the second derivatives of eta quotients and prove Theorem 1.6.

2. The Fourier series expansions of $E_k(tz)$ at cusps a/c

Let $t \in \mathbb{N}$. To prove Theorem 1.1 we need to derive the Fourier series expansion of $E_k(tz)$ at a/c where $a, c \in \mathbb{Z}$ satisfy gcd(a, c) = 1. We follow an approach similar to the proof of [6, Proposition 2.1] where constant terms of Dedekind eta function is calculated. For notational convenience let us denote

$$E_{k,t}(z) := E_k(tz) = \frac{-B_k}{2k} + \sum_{n \ge 1} \sigma_{k-1}(n)q^{tn}.$$

Define

$$\iota_k := \begin{cases} 1 & \text{if } k = 2, \\ 0 & \text{otherwise} \end{cases}$$

Set

$$q_{c,N} := e^{2\pi i \operatorname{gcd}(c^2,N)z/N}$$

and

$$\omega_{M,t} := \begin{cases} 1 & \text{if } c \equiv 0 \pmod{t}, \\ e^{(\frac{-2\pi i \gcd(t,c)df}{t})} & \text{if } c \not\equiv 0 \pmod{t}, \end{cases}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$

Lemma 2.1. Let a/c be a rational cusp of $\Gamma_0(N)$, where (a, c) = 1. Let b and d be two integers such that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If $f(z) \in \mathcal{E}_k(N)$, then the Fourier series expansion of $f(z) = \sum_{t|N} r_t E_k(tz)$ at the cusp a/c is given by

$$(cz+d)^{-k}f(Mz) = (cz+d)^{-k}\sum_{t|N} r_t E_{k,t}(Mz)$$
$$= \sum_{t|N} \sum_{n>0} a_n(c,t) r_t \omega_{M,t}^n q_{c,N}^{n \operatorname{gcd}(t,c)^2 N/t \operatorname{gcd}(c^2,N)},$$

where $a_n(c, t) \neq 0$ are rational numbers given explicitly in the proof below.

Proof. It is known that

$$E_k(Mz) = E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k E_k(z) + \iota_k \frac{\mathrm{i}c}{4\pi}(cz+d)$$
(2.1)

for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (see [3, Corollary 5.2.17.(b)]). We have

$$E_{k,t}(Mz) = E_k(tMz) = E_k\left(\frac{atz+bt}{cz+d}\right).$$
(2.2)

Now let $e := \frac{at}{\gcd(t,c)}$ and $g := \frac{c}{\gcd(t,c)}$. It is clear that $e, g \in \mathbb{Z}$ and since $\gcd(e, g) = 1$, there exists $f, h \in \mathbb{Z}$ such that

$$eh - fg = 1. \tag{2.3}$$

Therefore $\begin{pmatrix} e & f \\ g & h \end{pmatrix} \in SL_2(\mathbb{Z})$. Hence using (2.3) we obtain

$$\begin{pmatrix} at & bt \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} aht - cf & bht - df \\ 0 & -bgt + de \end{pmatrix}.$$
 (2.4)

Putting (2.2) and (2.4) together we obtain

$$E_{k,t}(Mz) = E_k \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} aht - cf & bht - df \\ 0 & -bgt + de \end{pmatrix} z \right).$$

Let

$$z_M := \frac{(aht - cf)z + (bht - df)}{-bgt + de}$$

Then using aht - cf = gcd(t, c) we get

$$gz_M + h = \frac{\gcd(t,c)}{t}(cz+d).$$
 (2.5)

Observe that

$$-bgt + de = t/\gcd(t,c).$$
(2.6)

Now, by employing (2.5), (2.6), and (2.1) we have

$$E_{k,t}(Mz) = \left(\frac{\gcd(t,c)}{t}\right)^k (cz+d)^k E_k\left(\frac{\gcd(t,c)^2 z - \gcd(t,c)df}{t}\right) + \iota_k \frac{\mathrm{i}c}{4\pi t}(cz+d),$$

where in the calculations we use (2.3) and

$$E_2(z+1) = E_2(z).$$

Additionally, if $c \equiv 0 \pmod{t}$, then $gcd(t, c)df \equiv 0 \pmod{t}$. That is, we have

$$E_{k,t}(Mz) = \begin{cases} \left(\frac{\gcd(t,c)}{t}\right)^{k} (cz+d)^{k} E_{k}\left(\frac{\gcd(t,c)^{2}z}{t}\right) \\ +\iota_{k}\frac{ic}{4\pi t}(cz+d), & \text{if } c \equiv 0 \pmod{t}, \\ \left(\frac{\gcd(t,c)}{t}\right)^{k} (cz+d)^{k} E_{k}\left(\frac{\gcd(t,c)^{2}z - \gcd(t,c)df}{t}\right) \\ +\iota_{k}\frac{ic}{4\pi t}(cz+d), & \text{if } c \not\equiv 0 \pmod{t}. \end{cases}$$
(2.7)

Applying (1.1) in (2.7) we obtain the following Fourier series expansion of $E_k(tz)$ at a/c:

$$E_{k,t}(Mz) = \begin{cases} \left(\frac{\gcd(t,c)}{t}\right)^{k} (cz+d)^{k} \left(\frac{-B_{k}}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)e^{2\pi i n \gcd(t,c)^{2}z/t}\right) \\ +\iota_{k}\frac{ic}{4\pi t}(cz+d), & \text{if } c \equiv 0 \pmod{t}, \\ \left(\frac{\gcd(t,c)}{t}\right)^{k} (cz+d)^{k} \left(\frac{-B_{k}}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n)e^{2\pi i n(\frac{\gcd(t,c)^{2}z-\gcd(t,c)df}{t})}\right) \\ +\iota_{k}\frac{ic}{4\pi t}(cz+d), & \text{if } c \not\equiv 0 \pmod{t}. \end{cases}$$
(2.8)

Note that the width of the cusp a/c in $\Gamma_0(N)$ is given by $\frac{N}{\gcd(c^2,N)}$ ([3, Corollary 6.3.24.(a)]) and recall that we define

$$q_{c,N} = e^{2\pi i \gcd(c^2, N) z/N}.$$

Employing this in (2.8), we obtain

$$(cz + d)^{-k} E_{k,t}(Mz) = \begin{cases} \sum_{n \ge 0} a_n q_{c,N}^{n \gcd(t,c)^2 N/t \gcd(c^2,N)} + \iota_k \frac{ic}{4\pi t(cz+d)} & \text{if } c \equiv 0 \pmod{t}, \\ \sum_{n \ge 0} a_n e^{(\frac{-2\pi i n \gcd(t,c)df}{t})} q_{c,N}^{n \gcd(t,c)^2 N/t \gcd(c^2,N)} + \iota_k \frac{ic}{4\pi t(cz+d)} & \text{if } c \neq 0 \pmod{t}, \end{cases}$$
(2.9)

where, for a fixed k, a_n depends on c and t and given by

$$a_{n} := a_{n}(c, t) = \begin{cases} \left(\frac{\gcd(t, c)}{t}\right)^{k} \cdot \frac{-B_{k}}{2k} & \text{if } n = 0, \\ \left(\frac{\gcd(t, c)}{t}\right)^{k} \sigma_{k-1}(n) & \text{if } n > 0, \end{cases}$$
(2.10)

and hence $a_n(c, t) \in \mathbb{Q}$ and $a_n(c, t) \neq 0$ for all $n \in \mathbb{N}_0$. Now recall that

$$\omega_{M,t} = \begin{cases} 1 & \text{if } c \equiv 0 \pmod{t}, \\ e^{(\frac{-2\pi i \gcd(t,c)df}{t})} & \text{if } c \not\equiv 0 \pmod{t}. \end{cases}$$

Using this and noting that $a_n(c, t)$ is a function of c, t and k, we write (2.9) as

$$(cz+d)^{-k}E_{k,t}(Mz) = \sum_{n\geq 0} a_n(c,t)\omega_{M,t}^n q_{c,N}^{n\gcd(t,c)^2N/t\gcd(c^2,N)} + \iota_k \frac{ic}{4\pi t(cz+d)}.$$
(2.11)

Now, let $f(z) \in \mathcal{E}_k(N)$, then we have

$$f(z) = \sum_{t|N} r_t E_k(tz),$$
 (2.12)

where, when k = 2, r_t satisfy

$$\sum_{t|N} \frac{r_t}{t} = 0,$$
 (2.13)

and $\iota_k = 0$ for all even $k \ge 4$, that is, for all even $k \ge 2$ we have

$$\iota_k \sum_{t|N} r_t \frac{\mathrm{i}c}{4\pi t(cz+d)} = \iota_k \frac{\mathrm{i}c}{4\pi (cz+d)} \sum_{t|N} \frac{r_t}{t} = 0.$$

Therefore from (2.11) and (2.12) we have the desired result.

3. PROOF OF THEOREM 1.1

In this section we let $N = p^m$ for p a prime and m a positive integer. Then for any $f(z) \in \mathcal{E}_k(p^m)$ we have

$$f(z) = \sum_{t \mid p^m} r_t E_{k,t}(z),$$

where $r_t \in \mathbb{Q}$ and should satisfy (2.13) when k = 2. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then for all $c \mid p^m$, by Lemma 2.1, we have

$$(cz+d)^{-k}f(Mz) = \sum_{t|p^m} \sum_{n\geq 0} a_n(c,t)r_t \omega_{M,t}^n q_{c,p^m}^{n\gcd(t,c)^2 p^m/t \gcd(c^2,p^m)}$$

Since all divisors of p^m are of the form p^i , for $0 \le i \le m$ we have

$$(p^{i}z+d)^{-k}f(Mz) = \sum_{j=0}^{m} \sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{p^{j}}\omega_{M,p^{j}}^{n}q_{p^{i},p^{m}}^{n\operatorname{gcd}(p^{j},p^{i})^{2}p^{m}/p^{j}\operatorname{gcd}(p^{2i},p^{m})}.$$
(3.1)

On the other hand we have

$$\omega_{M,p^{j}} = \begin{cases} 1 & \text{if } p^{i} \equiv 0 \pmod{p^{j}} \\ e^{-2\pi i \operatorname{gcd}(p^{j},p^{i})df/p^{j}} & \text{if } p^{i} \not\equiv 0 \pmod{p^{j}} \end{cases} = \begin{cases} 1 & \text{if } i \geq j, \\ e^{-2\pi i \operatorname{gcd}(p^{j},p^{i})df/p^{j-i}} & \text{if } i < j, \end{cases}$$

and

$$\gcd(p^{j}, p^{i})^{2} p^{m} / p^{j} \gcd(p^{2i}, p^{m}) = \begin{cases} p^{j} & \text{if } i \geq j \text{ and } i \geq m/2, \\ p^{2i-j} & \text{if } i < j \text{ and } i \geq m/2, \\ p^{m+j-2i} & \text{if } i \geq j \text{ and } i < m/2, \\ p^{m-j} & \text{if } i < j \text{ and } i < m/2. \end{cases}$$

We put these into (3.1) to obtain

$$(p^{i}z+d)^{-k}f(Mz) = \begin{cases} \sum_{i\geq j} \sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{pj}q_{p^{i},p^{m}}^{np^{j}} \\ + \sum_{i< j} \sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{pj}e^{-2\pi i n df/p^{j-i}}q_{p^{i},p^{m}}^{np^{2i-j}} & \text{if } i\geq m/2, \end{cases} \\ \sum_{i\geq j} \sum_{n\geq 0} \sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{pj}q_{p^{i},p^{m}}^{np^{m+j-2i}} \\ + \sum_{i< j} \sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{pj}e^{-2\pi i n df/p^{j-i}}q_{p^{i},p^{m}}^{np^{m-j}} & \text{if } i< m/2. \end{cases}$$
(3.2)

Lemma 3.1. Let $f(z) \in \mathcal{E}_k(p^m)$ where $p^m \neq 4$. If $v_{a/p^i}(f(z)) > 1$ for some $0 \le i \le m$, then either $r_1 = 0$ or $r_{p^m} = 0$.

Proof. Let $f(z) \in \mathcal{E}_k(p^m)$. Let $0 \le i < m/2$. By (3.2) the Fourier series expansion of f(z) at a/p^i is given by

$$(p^{i}z+d)^{-k}f(Mz) = \sum_{i\geq j}\sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{p^{j}}q_{p^{i},p^{m}}^{np^{m+j-2i}} + \sum_{i< j}\sum_{n\geq 0} a_{n}(p^{i}, p^{j})r_{p^{j}}e^{-2\pi i ndf/p^{j-i}}q_{p^{i},p^{m}}^{np^{m-j}}$$

If $v_{a/p^i}(f(z)) > 1$, then the coefficient of q_{p^i,p^m}^1 in the Fourier series expansion of f(z) at a/p^i is 0, i.e., $a_1(p^i, p^m)r_{p^m}e^{-2\pi i n df/p^{m-i}} = 0$. Since $a_1(p^i, p^m)e^{-2\pi i n df/p^{m-i}} \neq 0$, we must have $r_{p^m} = 0$. Arguing similarly we obtain

$$\begin{aligned} r_{p^m} &= 0 & \text{if } i < m/2, \\ r_1 &= 0 & \text{if } i > m/2, \\ a_1(p^{m/2}, 1)r_1 + a_1(p^{m/2}, p^m)e^{-2\pi \mathrm{i} df/p^{m/2}}r_{p^m} &= 0 & \text{if } i = m/2. \end{aligned}$$

Since, in notation of Lemma 2.1, $c = p^{m/2}$, $t = p^m$, ad - bc = 1, and eh - fg = 1, we have gcd(df, p) = 1. Therefore when $p \neq 2$ and $m \neq 2$, $e^{-2\pi i df/p^{m/2}}$ is not a real number. Now since $a_1(p^{m/2}, 1), a_1(p^{m/2}, p^m) \in \mathbb{Q}$ we have $r_1 = 0$ and $r_{p^m} = 0$. Therefore the statement follows.

Proof of Theorem 1.1. Assume $p^m \neq 4$. Let $f(z) \in \mathcal{P}_k(p^m)$. Assume, for sake of contradiction, that $v_{a/p^i}(f(z)) > 1$ for some cusp a/p^i where $0 \le i \le m$ and $gcd(a, p^i) = 1$. Then by Lemma 3.1 we have either $r_1 = 0$ or $r_{p^m} = 0$. If $r_1 = 0$, then there exists a $g(z) \in \mathcal{E}_k(p^{m-1})$ such that f(z) = g(pz), that is, $f(z) \in \mathcal{O}_k(p^m)$. This contradicts to f(z) being in $\mathcal{P}_k(p^m)$. If $r_{p^m} = 0$, then $f(z) \in \mathcal{E}_k(p^{m-1})$ again a contradiction with f(z) being in $\mathcal{P}_k(p^m)$. Therefore we must have $v_{a/p^i}(f(z)) \le 1$ for all cusps a/p^i where $0 \le i \le m$ and $gcd(a, p^i) = 1$. Since f is not a cusp form, for at least one $r \in R(p^m)$ we have $v_r(f(z)) = 0$. Then the theorem follows from observing

$$#R(p^m) = p^{[(m-1)/2]}(p^{(m-1)-2[(m-1)/2]} + 1),$$

see [3, Corollary 6.3.24.(b)], together with the fact that all $r \in R(p^m)$ can be chosen to be in the form a/p^i .

Now we prove the case p = 2 and m = 2. In this case let us fix

$$R(4) = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{4}\right\}.$$

By arguments similar to the proof of Lemma 3.1 if $v_{1/1}(f(z)) > 1$ or $v_{1/4}(f(z)) > 1$, then $r_1 = 0$ or $r_4 = 0$, that is, f(z) cannot be in $\mathcal{P}_k(4)$. Therefore, we must have $v_{1/1}(f(z)) \le 1$ and $v_{1/4}(f(z)) \le 1$. On the other hand if $v_{1/2}(f(z)) > 2$, then from the expansion at 1/2, formula (3.2) for p = 2, m = 2, i = 1, we have

$$(2z+d)^{-k}f(Mz) = \sum_{n\geq 0} a_n(2,1)r_1q_{2,4}^n + \sum_{n\geq 0} a_n(2,2)r_2q_{2,4}^{2n} + \sum_{n\geq 0} a_n(2,4)r_4e^{-2\pi i ndf/2}q_{2,4}^n.$$

Observing that $df \equiv 1 \pmod{2}$ yields

$$a_0(2, 1)r_1 + a_0(2, 2)r_2 + a_0(2, 4)r_4 = 0,$$

$$a_1(2, 1)r_1 - a_1(2, 4)r_4 = 0,$$

$$a_2(2, 1)r_1 + a_1(2, 2)r_2 + a_2(2, 4)r_4 = 0.$$

Putting the values of a_n 's from (2.10) in the the above equations and simplifying them we have

$$r_1 + r_2 + r_4/2^k = 0,$$

$$r_1 - r_4/2^k = 0,$$

$$(1 + 2^{k-1})r_1 + r_2 + (1 + 2^{k-1})r_4/2^k = 0.$$

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Thus $r_1, r_2, r_4 = 0$. Therefore, if $v_{1/2}(f(z)) > 2$, then f(z) = 0.

Hence, if $f(z) \in \mathcal{P}_k(4)$, then $v_{1/1}(f(z)) \le 1$, $v_{1/4}(f(z)) \le 1$ and $v_{1/2}(f(z)) \le 2$. Also since f(z) is not a cusp form $v_r(f(z))$ has to be 0 for at least one $r \in R(4)$. Thus, we have $\sum_{r \in R(4)} v_r(f(z)) < 4$. \Box

Proof of Theorem 1.3. Let $f \in \mathcal{P}_k(p^m)$ be an eta quotient. Then we have

$$\sum_{r \in R(p^m)} v_r(f) = \begin{cases} \frac{k}{12} & \text{if } m = 0, \\ \frac{k}{12}(p^m + p^{m-1}) & \text{if } m \neq 0, \end{cases}$$

see [2, Lemma 2.1].

We observe that $\sum_{r \in R(p^m)} v_r(f)$ is greater than or equal to the upper bound given by Theorem 1.1 unless $(k, p^m) = (2, 1), (4, 1), (6, 1), (8, 1), (10, 1), (2, 2), (2, 4), (2, 8), (2, 16), (4, 2), (4, 4), (6, 2), (6, 4), (2, 3), (2, 9), (4, 3), (2, 5), (2, 25), (2, 7). This means unless <math>(k, p^m)$ is one of the pairs above, $f \in \mathcal{P}_k(p^m)$ will have at least one zero in the upper half-plane, that is, f cannot be an eta quotient. We run the algorithms given in [1] and see that there are no eta quotients in $\mathcal{P}_k(p^m)$ when $(k, p^m) = (2, 1), (4, 1), (6, 1), (8, 1), (10, 1), (2, 2), (6, 2), (6, 4), (2, 3), (4, 3), (2, 5), (2, 25), (2, 7).$

Proof of Corollary 1.4. Notice that for any $f \in \mathcal{E}_k(p^m)$ there is a $g \in \mathcal{P}_k(p^{m'})$ such that $0 \le m' \le m$ and f is a trivial extension of g. By Theorem 1.3, we have

$$(k, p^{m'}) \in \{(2, 4), (2, 8), (2, 9), (2, 16), (4, 2), (4, 4)\}.$$

Now we can derive the finite set of all the eta quotients of weight k and level $p^{m'}$ in $\mathcal{P}_k(p^{m'})$ by employing Algorithm 4.2 in [1].

Proof of Corollary 1.5. By Lemma 2.1 of [1] the desired set can be obtained by finding the antiderivatives of the eta quotients given in part (1) of Corollary 1.4. Now Algorithm 4.3 of [1] derives the required antiderivatives.

4. SECOND ORDER DERIVATIVES

In this section we prove Theorem 1.6. We start by determining the weight of an eta quotient whose second derivative is also an eta quotient.

Proposition 4.1. If f is an eta quotient of weight k for which $D^2(f)/f$ is an eta quotient of weight ℓ , then k = -1 and $\ell = 4$.

The following lemma is needed for the proof of this proposition.

Lemma 4.2. Let f be a nonzero holomorphic function defined on the upper half-plane. Suppose that for an $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have

$$f(Mz) = \psi_M (cz + d)^k f(z),$$
 (4.1)

where k is an integer and ψ_M is a root of unity depending on M. Then

$$(cz+d)^{-4}\frac{D^2(f(Mz))}{f(Mz)} = \frac{D^2(f(z))}{f(z)} + \frac{k+1}{\pi i}\frac{D(f(z))}{f(z)}\left(\frac{c}{cz+d}\right) - \frac{k(k+1)}{4\pi^2}\left(\frac{c}{cz+d}\right)^2.$$
 (4.2)

Proof. This is a consequence of differentiation of (4.1) twice and employing the transformation property (4.1). (For a general formula for the *m*-th derivative of a modular form see [9, Proposition 3.1].)

Proof of Proposition 4.1. Since *f* is an eta quotient of weight *k*, then (4.1) holds for any $M \in \Gamma_0(R)$ for a suitable non-negative integer *R* and for corresponding 24-th roots of unity ψ_M 's. Thus (4.2) holds for such *M*'s. Now assume that $D^2(f)/f$ is also an eta quotient of weight ℓ . Suppose that this eta quotient has level *N* (a multiple of *R*) and thus satisfies

$$\frac{D^2(f(Mz))}{f(Mz)} = \chi_M(cz+d)^\ell \frac{D^2(f(z))}{f(z)}$$
(4.3)

for some 24-th root of unity χ_M , depending on M, and for all matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Now, for a fixed z in the upper half-plane, applying (4.3) in (4.2) and rearranging the terms yields

$$\left(\chi_M(cz+d)^\ell - (cz+d)^4\right) \frac{D^2(f(z))}{f(z)} = (k+1)\left(kc^2\alpha(cz+d)^2 + c\beta(cz+d)^3\right)$$
(4.4)

for some fixed constants $\alpha, \beta \in \mathbb{C}$. (Note that since *z* is fixed, the value of D(f(z))/f(z) is absorbed in β .) Next assume that $\ell \ge 0$ and $\ell \ne 4$. Observe that the set *S* of matrices $M = \begin{pmatrix} * & * \\ N & d \end{pmatrix} \in \Gamma_0(N)$ for which $\chi_0(Nz + d)^\ell - (Nz + d)^4 \ne 0$, for one of the 24-th roots of unity χ_0 , is infinite. Assuming that $k \ne -1$, the equation (4.4), for c = N and $M \in S$, can be re-written as

$$\frac{1}{k+1} \frac{D^2(f(z))}{f(z)} = \frac{kN^2 \alpha (Nz+d)^2 + N\beta (Nz+d)^3}{\chi_0 (Nz+d)^\ell - (Nz+d)^4}.$$
(4.5)

Now for a fixed z in the upper half-plane, the right-hand side of (4.5) is equal to a fixed non-zero complex number γ for infinitely many values of d. This is a contradiction as the non-trivial polynomial equation

$$(kN^2\alpha)X^2 + (N\beta)X^3 - (\gamma\chi_0)X^\ell + \gamma X^4 = 0$$

has finitely many solutions. The other cases can be analyzed in a similar fashion to conclude that if both *f* and $D^2(f)/f$ are eta quotients, then k = -1, $\ell = 4$, and $\chi_M = 1$.

We continue with listing some identities that will be useful in the proof:

$$E_2(z)^2 = \frac{5}{12}E_4(z) - \frac{1}{2}D(E_2(z)), \qquad (4.6)$$

$$E_2(z)E_2(2z) = \frac{1}{12}E_4(z) + \frac{1}{3}E_4(2z) - \frac{1}{8}D(E_2(z)) - \frac{1}{4}D(E_2(2z)),$$
(4.7)

and

$$E_2(z)E_2(4z) = \frac{1}{48}E_4(z) + \frac{1}{16}E_4(2z) + \frac{1}{3}E_4(4z) - \frac{1}{16}D(E_2(z)) - \frac{1}{4}D(E_2(4z)),$$
(4.8)

first of which is due to Besge, Glaisher and Ramanujan independently and for the latter two see [4, Theorems 2 and 4]. Additionally we note that using (4.6) and replacing z by 2z we obtain

$$E_2(2z)^2 = \frac{5}{12}E_4(2z) - \frac{1}{4}D(E_2(2z)), \qquad (4.9)$$

using (4.6) and replacing z by 4z we obtain

$$E_2(4z)^2 = \frac{5}{12}E_4(4z) - \frac{1}{8}D(E_2(4z)), \qquad (4.10)$$

and using (4.7) and replacing z by 2z we obtain

$$E_2(2z)E_2(4z) = \frac{1}{12}E_4(2z) + \frac{1}{3}E_4(4z) - \frac{1}{16}D(E_2(2z)) - \frac{1}{8}D(E_2(4z)).$$
(4.11)

Now we prove Theorem 1.6.

Proof of Theorem 1.6. If $f(z) = \eta^{r_1}(z)\eta^{r_2}(2z)\eta^{r_4}(4z)$, then by taking logarithmic derivative two times and employing the identity

$$D(\log(\eta(z))) = -E_2(z)$$

we obtain

$$\frac{D^2(f(z))}{f(z)} = D(-r_1E_2(z) - 2r_2E_2(2z) - 4r_4E_2(4z))$$

$$+ (-r_1E_2(z) - 2r_2E_2(2z) - 4r_4E_2(4z))^2.$$

Next we use (4.6)–(4.11) and obtain

$$\begin{aligned} \frac{D^2(f(z))}{f(z)} &= E_4(z) \left(\frac{5}{12} r_1^2 + \frac{1}{3} r_1 r_2 \right) + E_4(2z) \left(\frac{5}{3} r_2^2 + \frac{4}{3} r_1 r_2 + \frac{1}{2} r_1 r_4 + \frac{4}{3} r_2 r_4 \right) \\ &+ E_4(4z) \left(\frac{20}{3} r_4^2 + \frac{8}{3} r_1 r_4 + \frac{16}{3} r_2 r_4 \right) \\ &+ D(E_2(z)) \left(-r_1 - \frac{1}{2} r_1(r_1 + r_2 + r_4) \right) \\ &+ D(E_2(2z)) \left(-2r_2 - r_2(r_1 + r_2 + r_4) \right) \\ &+ D(E_2(4z)) \left(-4r_4 - 2r_4(r_1 + r_2 + r_4) \right). \end{aligned}$$

Noting that, by Proposition 4.1, $r_1 + r_2 + r_4 = -2$ must hold, we have

$$\begin{aligned} \frac{D^2(f(z))}{f(z)} &= E_4(z) \left(\frac{5}{12} r_1^2 + \frac{1}{3} r_1 r_2 \right) \\ &+ E_4(2z) \left(\frac{5}{3} r_2^2 + \frac{4}{3} r_1 r_2 + \frac{1}{2} r_1 r_4 + \frac{4}{3} r_2 r_4 \right) \\ &+ E_4(4z) \left(\frac{20}{3} r_4^2 + \frac{8}{3} r_1 r_4 + \frac{16}{3} r_2 r_4 \right). \end{aligned}$$

Therefore $\frac{D^2(f(z))}{f(z)} \in \mathcal{E}_4(4)$. The result follows by investigating all the eta quotients in $\mathcal{E}_4(4)$ that are found by using the algorithms given in [1].

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