ON $L^{(r+1)}(\pi, 1/2)$

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Abstract. Let $r$ be the order of vanishing of the automorphic $L$-function $L(\pi, s)$ at $s = 1/2$. We study the non-vanishing of the derivative of order $r + 1$ of $L(\pi, s)$ at $s = 1/2$.

Résumé. Soit $r$ l’ordre d’annulation de la fonction $L$ automorphe $L(\pi, s)$ à $s = 1/2$. Nous étudions la non-annulation de la dérivée d’ordre $r + 1$ de $L(\pi, s)$ à $s = 1/2$.

1. Introduction

Let $F$ be a number field of degree $d$ and $\pi = \otimes \pi_\nu$ be an irreducible cuspidal automorphic representation of $GL_m$ over $F$ with unitary central character and contragradient representation $\tilde{\pi}$. Let $L(\pi, s)$ and $L(\tilde{\pi}, s)$ be the associated $L$-functions to $\pi$ and $\tilde{\pi}$. We have

$$L(\tilde{\pi}, s) = L(\pi, s).$$

It is known that $L(\pi, s)$ and $L(\tilde{\pi}, s)$ satisfy the functional equation

$$q^{s/2} L(\pi_{\infty}, s)L(\pi_v, s) = \omega q^{(1-s)/2} L(\pi_{\infty}, 1-s)L(\tilde{\pi}, 1-s),$$

where the positive integer $q$ is the conductor of $\pi$, $\omega$ is the root number (a complex number of modulus 1) and

$$L(\pi_{\infty}, s) = \prod_{j=1}^{md} \pi^{-s/2}\Gamma\left(\frac{s + \mu_j}{2}\right), \quad L(\tilde{\pi}_{\infty}, s) = \prod_{j=1}^{md} \pi^{-s/2}\Gamma\left(\frac{s + \bar{\mu}_j}{2}\right).$$

Note that each side of the equation (1) represents a meromorphic function in the whole complex plane with at most two simple poles. Moreover, by a theorem of Luo, Rudnick and Sarnak [LRS], we have

$$\Re \mu_j \geq \frac{1}{m^2 + 1} - \frac{1}{2}, \quad j = 1, \ldots, md,$$

which implies that $L(\pi_\infty, s)$ and $L(\tilde{\pi}_\infty, s)$ are analytic and non-zero on the half plane $\Re s > \frac{1}{2} - \frac{1}{md+1}$. Let

$$r = \text{ord}_{s=1/2} L(\pi, s).$$

So we have $L^{(i)}(\pi, 1/2) = 0$ for $0 \leq i \leq (r - 1)$ and $L^{(r)}(\pi, 1/2) \neq 0$. In this note we will exploit the functional equation (1) to investigate the possible values of $L^{(r+1)}(\pi, 1/2)$. In fact, we show that in several cases this value is non-zero.

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2. The Main Lemma

The following lemma is a generalization of Exercise 5.5.22 of [M].

**Lemma 2.1.** With the above notation we have
\[
\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} + \frac{L^{(r+1)}(\tilde{\pi}, 1/2)}{L^{(r)}(\tilde{\pi}, 1/2)} = -(r+1) \left( \log q + \frac{L'(\pi, 1/2)}{L(\pi, 1/2)} + \frac{L'((\pi), 1/2)}{L((\pi), 1/2)} \right).
\]

**Proof.** Let \( A(s) = q^{1/2}L(\pi, s, \sigma) \) and \( B(s) = q^{1/2}L(\tilde{\pi}, s) \). So from (1) we have
\[
A(1/2)L^{(r)}(\pi, 1/2) = \omega(-1)^r B(1/2)L^{(r)}(\tilde{\pi}, 1/2).
\]

Similarly from (1) we have
\[
A(1/2)L^{(r+1)}(\pi, 1/2) + (r+1)A'(1/2)L^{(r)}(\pi, 1/2)
\]
\[
= \omega(-1)^{r+1} \left( B(1/2)L^{(r+1)}(\tilde{\pi}, 1/2) + (r+1)B'(1/2)L^{(r)}(\tilde{\pi}, 1/2) \right).
\]

Dividing the latter equation to the former one yields
\[
\frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} + \frac{L^{(r+1)}(\tilde{\pi}, 1/2)}{L^{(r)}(\tilde{\pi}, 1/2)} = -(r+1) \left( \frac{A'(1/2)}{A(1/2)} + \frac{B'(1/2)}{B(1/2)} \right).
\]

Now the result follows by calculating the logarithmic derivative of \( A(s)B(s) = q^{1/2}L(\pi, s)\) at \( s = 1/2 \). \( \square \)

The following corollary is a direct consequence of the previous lemma.

**Corollary 2.2.**
\[
q \neq \exp \left( - \left( \frac{L'(\pi, 1/2)}{L(\pi, 1/2)} + \frac{L'(\pi, 1/2)}{L(\pi, 1/2)} \right) \right) \Rightarrow L^{(r+1)}(\pi, 1/2) \neq 0.
\]

From now on let \( \psi(s) = \Gamma'/(\pi, s) \) and \( \mu_j = \sigma_j + it_j \). An automorphic representation \( \pi \) is called **tempered** if \( \sigma_j = 0 \) for \( j = 1, \cdots, md \). We have the following

**Corollary 2.3.** If \( \pi \) is tempered we have
\[
\Re \left( \frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} \right) = 0 \iff q = (2\pi)^{md} \exp \left( \frac{\pi}{2} \sum_{j=1}^{md} \text{sech}(\pi t_j) \right) \exp \left( - \sum_{j=1}^{md} \Re \left( \frac{1}{2} + it_j \right) \right).
\]

**Proof.** From Lemma 2.1, we have
\[
\Re \left( \frac{L^{(r+1)}(\pi, 1/2)}{L^{(r)}(\pi, 1/2)} \right) = 0 \iff q = \exp \left( - \left( \frac{L'(\pi, 1/2)}{L(\pi, 1/2)} + \frac{L'((\pi), 1/2)}{L((\pi), 1/2)} \right) \right).
\]

\[
\iff q = \pi^{md} \exp \left( - \frac{1}{2} \sum_{j=1}^{md} \left( \psi(1/4 + it_j/2) + \psi(1/4 - it_j/2) \right) \right)
\]

(2) \[
\iff q = (2\pi)^{md} \exp \left( \frac{\pi}{2} \sum_{j=1}^{md} \tan \left( \frac{\pi}{4} + it_j/2 \right) \right) \exp \left( - \sum_{j=1}^{md} \psi(1/2 + it_j) \right).
\]
The last equivalence is a consequence of calculating the logarithmic derivative of the identity
\[(3) \quad \left(\cos\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right)\Gamma(s) = \sqrt{\pi} 2^{s-1} \Gamma\left(\frac{s}{2}\right)\]
([D], p. 73) at \(s = 1/2 + it_j\). Next note that
\[(4) \quad \Re\left(\tan\left(\frac{\pi}{4} + i\frac{\pi t_j}{2}\right)\right) = \text{sech}(\pi t_j).\]
Now the results follows from (2), (4) and the fact that
\[\frac{L'\left(\pi, 1/2\right)}{L\left(\pi, 1/2\right)} + \frac{L'\left(\hat{\pi}, 1/2\right)}{L\left(\hat{\pi}, 1/2\right)}\]
is real. \(\square\)

In the rest of this paper, we apply the above corollaries in some special cases and as a result we prove that for Dirichlet \(L\)-functions and Modular \(L\)-functions \(L^{(r+1)}(\pi, 1/2) \neq 0\). We also investigate the situation for the \(L\)-functions associated to Maass forms.

3. \(\text{GL}(1)\)

**Proposition 3.1.** If \(\chi\) is a primitive Dirichlet character mod \(q\), then \(L^{(r+1)}(\chi, 1/2) \neq 0\).

**Proof.** We have
\[L(\pi, s) = L\left(\hat{\pi}, s\right) = \begin{cases} \frac{\pi^{-s/2}}{2} \Gamma\left(\frac{s}{2}\right) & \text{if } \chi(-1) = 1 \\ \frac{\pi^{-s/2}}{2} \Gamma\left(\frac{s}{2} + 1\right) & \text{if } \chi(-1) = -1 \end{cases} .\]

Then
\[\exp\left(-\left(\frac{L'\left(\pi, 1/2\right)}{L\left(\pi, 1/2\right)} + \frac{L'\left(\hat{\pi}, 1/2\right)}{L\left(\hat{\pi}, 1/2\right)}\right)\right) = \begin{cases} \pi e^{-\psi(1/2)} & \text{if } \chi(-1) = 1 \\ \pi e^{-\psi(3/4)} & \text{if } \chi(-1) = -1 \end{cases} = \begin{cases} 8\pi e^{\gamma - \frac{3}{2}} & \text{if } \chi(-1) = 1 \\ 8\pi e^{\gamma - 3} & \text{if } \chi(-1) = -1 \end{cases} .\]

Here \(\gamma\) is the Euler constant and we used the identities \(\psi\left(\frac{1}{2}\right) = -\gamma - \frac{3}{2} - 3\log 2\) and \(\psi\left(\frac{3}{4}\right) = -\gamma + \frac{3}{2} - 3\log 2\) (see [W]). Now Corollary 2.2 implies the result. \(\square\)

4. \(\text{GL}(2)\), **Holomorphic Case**

**Proposition 4.1.** If \(f\) is a holomorphic cuspidal newform of weight \(k\) and level \(q\) and nebentypus \(\chi\), then \(L^{(r+1)}(f, 1/2) \neq 0\).

**Proof.** By employing the Legendre duplication formula, we have
\[L(\pi, s) = L\left(\hat{\pi}, s\right) = \pi^{-s/2} \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) = \frac{\sqrt{\pi}}{2^{k-2}} (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) .\]

So
\[\exp\left(-\left(\frac{L'\left(\pi, 1/2\right)}{L\left(\pi, 1/2\right)} + \frac{L'\left(\hat{\pi}, 1/2\right)}{L\left(\hat{\pi}, 1/2\right)}\right)\right) = (2\pi)^3 \exp\left(-2\psi(k/2)\right) .\]
We know that
\[ \psi(z) = \log z + O(1/|z|) \]
for \(|z| \to \infty\) in the sector \(-\pi + \delta < \arg z < \pi - \delta\) for any fixed \(\delta > 0\) (see [M], Exercise 6.3.17). So
\[ \lim_{k \to \infty} (2\pi)^2 \exp(-2\psi(k/2)) = 0. \]

More precisely by evaluating \(f(k) = (2\pi)^2 \exp(-2\psi(k/2))\) for integer \(k\), using Maple, we can see that for integer \(1 \leq k \leq 13\), \(f(k)\) is not an integer and for integer \(k \geq 14\) we have \(0 < f(k) < 1\). So \(f(k)\) never is an integer and therefore \(q \neq f(k)\). Thus Corollary 2.2 implies the result. \(\square\)

5. \(\text{GL}(2)\), \textbf{Real Analytic Case}

For \(t \geq 0\), let \(g(t) = (2\pi)^2 \exp\left(\pi \text{sech}(\pi t)\right) \exp(-2\Re\left(\psi\left(\frac{1}{2} + it\right)\right))\). From (5) and \(\lim_{t \to \infty} \text{sech}(t) = 0\) we have
\[ \lim_{t \to \infty} g(t) = 0. \]

By employing Maple one can show that \(g(0) = 46,368.09\cdots\) and \(0 < g(t) < 1\) for \(t \geq 6.29\). The following is the graph of \(g(t)\) for \(0 \leq t \leq 0.7\).

![Graph of g(t) for 0 ≤ t ≤ 0.7](graph.png)

From here it is clear that for any integer \(1 \leq q \leq 46,368\) there exists a unique \(0 < t_q < 6.29\) such that \(g(t_q) = q\).

**Proposition 5.1.** If \(f\) is an even Maass cuspidal newform of weight zero and level \(q\), nebentypus \(\chi\) and eigenvalue \(\lambda\), then under the assumption of the Selberg Eigenvalue Conjecture we have the following.

(i) If \(q \leq 46,368\), then
\[ \Re\left(\frac{L^{(r+1)}(f,1/2)}{L^{(r)}(f,1/2)}\right) = 0 \iff \lambda = \frac{1}{4} + t_q^2. \]

(ii) If \(q \geq 46,369\) or \(\lambda \geq 6.54\), then
\[ L^{(r+1)}(f,1/2) \neq 0. \]

**Proof.** Let \(\lambda\) be the eigenvalue corresponding to \(f\), then \(\lambda = \frac{1}{4} + t^2\). In this case for the \(\infty\) factors in the functional equation (1) we have \(\mu_1 = it\) and \(\mu_2 = -it\). More precisely, we have
\[ L(\pi \infty, s) = L(\pi \infty, s) = \pi^{-s} \Gamma\left(\frac{s + it}{2}\right) \Gamma\left(\frac{s - it}{2}\right). \]
From the Selberg Eigenvalue Conjecture we know that $t$ is real, so by Corollary 2.3 and the definition of the function $g(t)$ we have

$$\Re \left( \frac{L^{(r+1)}(f, 1/2)}{L^{(r)}(f, 1/2)} \right) = 0 \iff q = (2\pi)^{r} \exp(\pi \text{sech}(\pi t)) \exp \left( -2\Re \left( \psi \left( \frac{1}{2} + it \right) \right) \right) \iff q = g(t).$$

But $q = g(t)$ if and only if $t = t_q$. \hfill \Box

6. Conclusion

Our observations here indicate that in several cases $L^{(r+1)}(\pi, 1/2) \neq 0$. We end this note by raising the following question.

**Question** Is there an automorphic representation $\pi$ such that $L^{(r+1)}(\pi, 1/2) = 0$?

References


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