ON $L^{(r+1)}(\pi,1/2)$

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ABSTRACT. Let r be the order of vanishing of the automorphic L-function $L(\pi, s)$ at $s = 1/2$. We study the non-vanishing of the derivative of order $r+1$ of $L(\pi, s)$ at $s = 1/2$.

RÉSUMÉ. Soit r l'ordre d'annulation de la fonction L automorphe $L(\pi, s)$ à $s = 1/2$. Nous étudions la non-annulation de la dérivée d'ordre $r+1$ de $L(\pi, s)$ à $s = 1/2$.

1. INTRODUCTION

Let F be a number field of degree d and $\pi = \otimes_\nu \pi_\nu$ be an irreducible cuspidal automorphic representation of GL_m over F with unitary central character and contragradient representation $\tilde{\pi}$. Let $L(\pi, s)$ and $L(\tilde{\pi}, s)$ be the associated L-functions to π and $\tilde{\pi}$. We have

$$L(\tilde{\pi}, s) = \overline{L(\pi, \bar{s})}.$$

It is known that $L(\pi, s)$ and $L(\tilde{\pi}, s)$ satisfy the functional equation

$$q^{s/2} L(\pi_\infty, s) L(\pi, s) = \omega q^{(1-s)/2} L(\tilde{\pi}_\infty, 1-s) L(\tilde{\pi}, 1-s), \tag{1}$$

where the positive integer q is the conductor of π, ω is the root number (a complex number of modulus 1) and

$$L(\pi_\infty, s) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s+\mu_j}{2}\right), \; L(\tilde{\pi}_\infty, s) = \prod_{j=1}^{md} \pi^{-s/2} \Gamma\left(\frac{s+\bar{\mu}_j}{2}\right).$$

Note that each side of the equation (1) represents a meromorphic function in the whole complex plane with at most two simple poles. Moreover, by a theorem of Luo, Rudnick and Sarnak [LRS], we have

$$\Re\mu_j \geq \frac{1}{m^2+1} - \frac{1}{2}, \; j = 1, \cdots, md,$$

which implies that $L(\pi_\infty, s)$ and $L(\tilde{\pi}_\infty, s)$ are analytic and non-zero on the half plane $\Re s > \frac{1}{2} - \frac{1}{m^2+1}$. Let

$$r = \mathrm{ord}_{s=1/2} L(\pi, s).$$

So we have $L^{(i)}(\pi, 1/2) = 0$ for $0 \leq i \leq (r-1)$ and $L^{(r)}(\pi, 1/2) \neq 0$. In this note we will exploit the functional equation (1) to investigate the possible values of $L^{(r+1)}(\pi, 1/2)$. In fact, we show that in several cases this value is non-zero.

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2. The Main Lemma

The following lemma is a generalization of Exercise 5.5.22 of [M].

Lemma 2.1. *With the above notation we have*

$$\frac{L^{(r+1)}(\pi,1/2)}{L^{(r)}(\pi,1/2)}+\frac{L^{(r+1)}(\tilde{\pi},1/2)}{L^{(r)}(\tilde{\pi},1/2)}=-(r+1)\left(\log q+\frac{L'(\pi_\infty,1/2)}{L(\pi_\infty,1/2)}+\frac{L'(\tilde{\pi}_\infty,1/2)}{L(\tilde{\pi}_\infty,1/2)}\right).$$

Proof. Let $A(s)=q^{s/2}L(\pi_\infty,s)$ and $B(s)=q^{s/2}L(\tilde{\pi}_\infty,s)$. So from (1) we have

$$A(1/2)L^{(r)}(\pi,1/2)=\omega(-1)^r B(1/2)L^{(r)}(\tilde{\pi},1/2).$$

Similarly from (1) we have

$$A(1/2)L^{(r+1)}(\pi,1/2)+(r+1)A'(1/2)L^{(r)}(\pi,1/2)$$
$$=\omega(-1)^{r+1}\left(B(1/2)L^{(r+1)}(\tilde{\pi},1/2)+(r+1)B'(1/2)L^{(r)}(\tilde{\pi},1/2)\right).$$

Dividing the latter equation to the former one yields

$$\frac{L^{(r+1)}(\pi,1/2)}{L^{(r)}(\pi,1/2)}+\frac{L^{(r+1)}(\tilde{\pi},1/2)}{L^{(r)}(\tilde{\pi},1/2)}=-(r+1)\left(\frac{A'(1/2)}{A(1/2)}+\frac{B'(1/2)}{B(1/2)}\right).$$

Now the result follows by calculating the logarithmic derivative of $A(s)B(s)=q^sL(\pi_\infty,s)L(\tilde{\pi}_\infty,s)$ at $s=1/2$. □

The following corollary is a direct consequence of the previous lemma.

Corollary 2.2.

$$q\neq\exp\left(-\left(\frac{L'(\pi_\infty,1/2)}{L(\pi_\infty,1/2)}+\frac{L'(\tilde{\pi}_\infty,1/2)}{L(\tilde{\pi}_\infty,1/2)}\right)\right)\Rightarrow L^{(r+1)}(\pi,1/2)\neq 0.$$

From now on let $\psi(s)=\Gamma'(s)/\Gamma(s)$ and $\mu_j=\sigma_j+it_j$. An automorphic representation π is called *tempered* if $\sigma_j=0$ for $j=1,\cdots,md$. We have the following.

Corollary 2.3. *If π is tempered we have*

$$\Re\left(\frac{L^{(r+1)}(\pi,1/2)}{L^{(r)}(\pi,1/2)}\right)=0\iff$$
$$q=(2\pi)^{md}\exp\left(\frac{\pi}{2}\sum_{j=1}^{md}\operatorname{sech}(\pi t_j)\right)\exp\left(-\sum_{j=1}^{md}\Re\left(\psi(\frac{1}{2}+it_j)\right)\right).$$

Proof. From Lemma 2.1, we have

$$\Re\left(\frac{L^{(r+1)}(\pi,1/2)}{L^{(r)}(\pi,1/2)}\right)=0\iff q=\exp\left(-\left(\frac{L'(\pi_\infty,1/2)}{L(\pi_\infty,1/2)}+\frac{L'(\tilde{\pi}_\infty,1/2)}{L(\tilde{\pi}_\infty,1/2)}\right)\right).$$

$$\iff q=\pi^{md}\exp\left(-\frac{1}{2}\sum_{j=1}^{md}\left(\psi(\frac{1}{4}+i\frac{t_j}{2})+\psi(\frac{1}{4}-i\frac{t_j}{2})\right)\right)$$

$$\iff q=(2\pi)^{md}\exp\left(\frac{\pi}{2}\sum_{j=1}^{md}\tan\left(\frac{\pi}{4}+i\frac{\pi t_j}{2}\right)\right)\exp\left(-\sum_{j=1}^{md}\psi(\frac{1}{2}+it_j)\right). \tag{2}$$

The last equivalence is a consequence of calculating the logarithmic derivative of the identity

$$\left(\cos \frac{\pi s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \Gamma(s) = \sqrt{\pi}\ 2^{s-1}\ \Gamma\left(\frac{s}{2}\right) \tag{3}$$

([D], p. 73) at $s = 1/2 + it_j$. Next note that

$$\Re\left(\tan\left(\frac{\pi}{4} + i\frac{\pi t_j}{2}\right)\right) = \text{sech } (\pi t_j). \tag{4}$$

Now the results follows from (2), (4) and the fact that

$$\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}$$

is real. □

In the rest of this paper, we apply the above corollaries in some special cases and as a result we prove that for Dirichlet L-functions and Modular L-functions $L^{(r+1)}(\pi, 1/2) \neq 0$. We also investigate the situation for the L-functions associated to Maass forms.

3. GL(1)

Proposition 3.1. *If χ is a primitive Dirichlet character mod q, then $L^{(r+1)}(\chi, 1/2) \neq 0$.*

Proof. We have

$$L(\pi_\infty, s) = L(\tilde{\pi}_\infty, s) = \begin{cases} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) & \text{if} \quad \chi(-1) = 1 \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right) & \text{if} \quad \chi(-1) = -1 \end{cases}.$$

Then

$$\begin{aligned} \exp\left(-\left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}\right)\right) &= \begin{cases} \pi e^{-\psi(1/4)} & \text{if} \quad \chi(-1) = 1 \\ \pi e^{-\psi(3/4)} & \text{if} \quad \chi(-1) = -1 \end{cases} \\ &= \begin{cases} 8\pi e^{\gamma + \frac{\pi}{2}} & \text{if} \quad \chi(-1) = 1 \\ 8\pi e^{\gamma - \frac{\pi}{2}} & \text{if} \quad \chi(-1) = -1 \end{cases}. \end{aligned}$$

Here γ is the Euler constant and we used the identities $\psi(\frac{1}{4}) = -\gamma - \frac{\pi}{2} - 3\log 2$ and $\psi(\frac{3}{4}) = -\gamma + \frac{\pi}{2} - 3\log 2$ (see [W]). Now Corollary 2.2 implies the result. □

4. GL(2), Holomorphic Case

Proposition 4.1. *If f is a holomorphic cuspidal newform of weight k and level q and nebentypus χ, then $L^{(r+1)}(f, 1/2) \neq 0$.*

Proof. By employing the Legendre duplication formula, we have

$$\begin{aligned} L(\pi_\infty, s) = L(\tilde{\pi}_\infty, s) &= \pi^{-s}\Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right)\Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) \\ &= \frac{\sqrt{\pi}}{2^{\frac{k-3}{2}}}(2\pi)^{-s}\Gamma\left(s + \frac{k-1}{2}\right). \end{aligned}$$

So

$$\exp\left(-\left(\frac{L'(\pi_\infty, 1/2)}{L(\pi_\infty, 1/2)} + \frac{L'(\tilde{\pi}_\infty, 1/2)}{L(\tilde{\pi}_\infty, 1/2)}\right)\right) = (2\pi)^2 \exp\left(-2\psi(k/2)\right).$$

We know that

$$\psi(z) = \log z + O(1/|z|) \tag{5}$$

for $|z| \to \infty$ in the sector $-\pi + \delta < \arg z < \pi - \delta$ for any fixed $\delta > 0$ (see [M], Exercise 6.3.17). So

$$\lim_{k\to\infty} (2\pi)^2 \exp\left(-2\psi(k/2)\right) = 0.$$

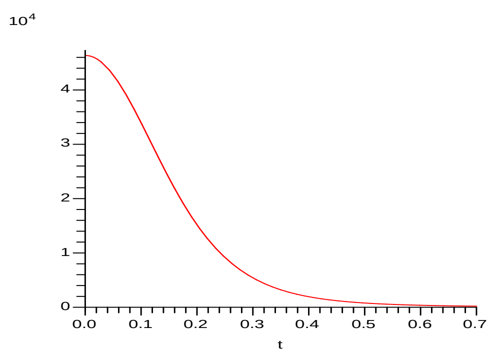
More precisely by evaluating $f(k) = (2\pi)^2 \exp\left(-2\psi(k/2)\right)$ for integer k, using Maple, we can see that for integer $1 \le k \le 13$, $f(k)$ is not an integer and for integer $k > 13$ we have $0 < f(k) < 1$. So $f(k)$ never is an integer and therefore $q \neq f(k)$. Thus Corollary 2.2 implies the result. □

5. GL(2), Real Analytic Case

For $t \ge 0$, let $g(t) = (2\pi)^2 \exp\left(\pi \operatorname{sech}(\pi t)\right) \exp\left(-2\Re\left(\psi(\frac{1}{2} + it)\right)\right)$. From (5) and $\lim_{t\to\infty} \operatorname{sech}(t) = 0$ we have

$$\lim_{t\to\infty} g(t) = 0.$$

By employing Maple one can show that $g(0) = 46,368.09\cdots$, and $0 < g(t) < 1$ for $t \ge 6.29$. The following is the graph of $g(t)$ for $0 \le t \le 0.7$.



From here it is clear that for any integer $1 \le q \le 46,368$ there exists a unique $0 < t_q < 6.29$ such that $g(t_q) = q$.

Proposition 5.1. *If f is an even Maass cuspidal newform of weight zero and level q, nebentypus χ and eigenvalue λ, then under the assumption of the Selberg Eigenvalue Conjecture we have the following.*
(i) *If $q \le 46,368$, then*

$$\Re\left(\frac{L^{(r+1)}(f,1/2)}{L^{(r)}(f,1/2)}\right) = 0 \iff \lambda = \frac{1}{4} + t_q^2.$$

(ii) *If $q \ge 46,369$ or $\lambda \ge 6.54$, then*

$$L^{(r+1)}(f,1/2) \neq 0.$$

Proof. Let λ be the eigenvalue corresponding to f, then $\lambda = \frac{1}{4} + t^2$. In this case for the ∞ factors in the functional equation (1) we have $\mu_1 = it$ and $\mu_2 = -it$. More precisely, we have

$$L(\pi_\infty, s) = L(\tilde{\pi}_\infty, s) = \pi^{-s}\Gamma\left(\frac{s+it}{2}\right)\Gamma\left(\frac{s-it}{2}\right).$$

From the Selberg Eigenvalue Conjecture we know that t is real, so by Corollary 2.3 and the definition of the function $g(t)$ we have

$$\Re\left(\frac{L^{(r+1)}(f,1/2)}{L^{(r)}(f,1/2)}\right) = 0 \iff$$

$$q = (2\pi)^2 \exp\left(\pi \ \mathrm{sech}\,(\pi t)\right) \exp\left(-2\Re\left(\psi(\frac{1}{2}+it)\right)\right) \iff q = g(t).$$

But $q = g(t)$ if and only if $t = t_q$. □

6. Conclusion

Our observations here indicate that in several cases $L^{(r+1)}(\pi, 1/2) \neq 0$. We end this note by raising the following question.

Question Is there an automorphic representation π such that $L^{(r+1)}(\pi, 1/2) = 0$?

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