LIMITING DISTRIBUTIONS OF THE CLASSICAL ERROR TERMS OF PRIME NUMBER THEORY

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ABSTRACT. In this article we prove a general theorem which establishes the existence of limiting distributions for a wide class of error terms from prime number theory. As a corollary to our main theorem, we deduce previous results of Wintner (1935), Rubinstein and Sarnak (1994), and of Ng (2004). In addition, we establish limiting distribution results for the error term in the prime number theorem for an automorphic L-function, weighted sums of the Möbius function, weighted sums of the Liouville function, the sum of the Möbius function in an arithmetic progression, and the error term in Chebotarev's density theorem.

1. INTRODUCTION

In recent years, limiting distributions have played a prominent role in many problems in analytic number theory. Indeed it is convenient to study number theoretic questions from a probabilistic point of view. Limiting distributions have been a useful tool in problems concerning summatory functions [18], [35], prime number races [37], [9], [26], and the distribution of values of L-functions [19], [16], [25]. In this article, we shall investigate the limiting distributions associated to some of the classical error terms that occur in prime number theory. In 1935, Wintner [41] proved, assuming the Riemann hypothesis (RH), that the function

$$e^{-y/2}\big(\psi(e^y) - e^y\big) \tag{1.1}$$

possesses a limiting distribution, where $\psi(x) = \sum_{p^m \leq x} \log p$. By his method, one may show that on RH

$$ye^{-y/2}\big(\pi(e^y) - \text{Li}(e^y)\big) \tag{1.2}$$

possesses a limiting distribution, where $\pi(x) = \sharp\{p \leq x \mid p \text{ is a prime}\}$ and $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$. Over the years, other researchers have investigated similar questions for related error terms. Let $q > 2$ and $a_1, \ldots, a_r$ be reduced residues modulo q. Define $\pi(x; q, a)$ to be the number of primes less than or equal to x which are congruent to a modulo q. In 1994, Rubinstein and Sarnak [37] proved, assuming the generalized Riemann hypothesis for Dirichlet L-functions, that the vector-valued function

$$ye^{-y/2}\big(\varphi(q)\pi(e^y; q, a_1) - \pi(e^y), \ldots, \varphi(q)\pi(e^y; q, a_r) - \pi(e^y)\big) \tag{1.3}$$

possesses a limiting distribution. These distributions were employed to give a conditional solution to an old problem known as the Shanks-Rényi prime number race game. In 2004, Ng [35] studied

the sum of the Möbius function. This arithmetic function is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ is not squarefree}, \\ (-1)^k & \text{if } n \text{ is squarefree and } n = p_1 \dots p_k, \end{cases}$$

and its summatory function is $M(x) = \sum_{n \leq x} \mu(n)$. He showed that

$$e^{-y/2} M(e^y) \tag{1.4}$$

possesses a limiting distribution assuming the Riemann hypothesis and the conjectural bound

$$\sum_{0 < |\Im(\rho)| \leq T} |\zeta'(\rho)|^{-2} \ll T, \tag{1.5}$$

where $\zeta(s)$ is the Riemann zeta function and ρ ranges through its non-trivial zeros $\rho = \frac{1}{2} + i\gamma$. The common element in the proofs of the existence of a limiting distribution of (1.1), (1.2), (1.3), and (1.4) is an "explicit formula" for each of these functions. For instance, the truncated explicit formula for $\psi(x)$ is

$$\psi(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ |\Im(\rho)| \leq X}} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2(xX)}{X} + \log x \right),$$

valid for $x \geq 2$ and $X > 1$ (see [7, Chapter 17]). On the Riemann hypothesis, it follows that

$$e^{-y/2}\big(\psi(e^y) - e^y\big) = \Re\left(\sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ 0<\gamma \leq X}} \frac{-2e^{iy\gamma}}{\rho} \right) + O\left(\frac{e^{\frac{y}{2}} \log^2(e^y X)}{X} + y e^{-\frac{y}{2}} \right). \tag{1.6}$$

Based on this formula Wintner deduced that (1.1) possesses a limiting distribution. In this article, we shall prove a general limiting distribution theorem for functions $\phi(y)$, possessing an explicit formula of a particular shape which is modelled on (1.6). Our theorem will include the above results as special cases and we will provide some new examples of functions with limiting distributions.

We now recall the definition of a limiting distribution for a vector-valued function $\vec{\phi} : [0, \infty) \to \mathbb{R}^\ell$, where $\ell \in \mathbb{N}$.

Definition 1.1. We say that a function $\vec{\phi} : [0, \infty) \to \mathbb{R}^\ell$ has a *limiting distribution* μ on $\mathbb{R}^\ell$ if μ is a probability measure on $\mathbb{R}^\ell$ and

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f\big(\vec{\phi}(y)\big) dy = \int_{\mathbb{R}^\ell} f d\mu$$

for all bounded continuous real functions f on $\mathbb{R}^\ell$.

We next describe the functions considered in this article. Let $\phi : [0, \infty) \to \mathbb{R}$ and let y_0 be a non-negative constant such that ϕ is square-integrable on $[0, y_0]$. We shall assume there exists $(\lambda_n)_{n \in \mathbb{N}}$, a non-decreasing sequence of positive numbers which tends to infinity, $(r_n)_{n \in \mathbb{N}}$, a complex sequence, and c a real constant such that for $y \geq y_0$

$$\phi(y) = c + \Re\left(\sum_{\lambda_n \leq X} r_n e^{i\lambda_n y} \right) + \mathcal{E}(y, X), \tag{1.7}$$

for any $X \geq X_0 > 0$ where $\mathcal{E}(y, X)$ satisfies

$$\lim_{Y \to \infty} \frac{1}{Y} \int_{y_0}^{Y} |\mathcal{E}(y, e^Y)|^2 dy = 0. \tag{1.8}$$

There shall be various conditions imposed on the coefficients r_n and the exponents λ_n.

Our approach in proving the limiting distribution of $\phi(y)$ is to show that $\phi(y)$ is a B^2-almost periodic function. We say that the real function $\phi(y)$ is a B^2-*almost periodic* function if for any $\epsilon > 0$ there exists a real-valued trigonometric polynomial

$$P_{N(\epsilon)}(y) = \sum_{n=1}^{N(\epsilon)} r_n(\epsilon) e^{i\lambda_n(\epsilon)y}$$

such that

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_0^Y |\phi(y) - P_{N(\epsilon)}(y)|^2 dy < \epsilon^2.$$

Our main result is the following.

Theorem 1.2. *Let $\phi : [0, \infty) \to \mathbb{R}$ satisfy (1.7) and (1.8). Let $\alpha, \beta > 0$, and $\gamma \geq 0$. Assume either of the following conditions:*
(a) $\beta > 1/2$ and

$$\sum_{T < \lambda_n \leq T+1} |r_n| \ll \frac{(\log T)^\gamma}{T^\beta} \tag{1.9}$$

for $T > 0$.
(b) $\beta \leq \min\{1, \alpha\}$, $\alpha^2 + \alpha/2 < \beta^2 + \beta$, and

$$\sum_{S < \lambda_n \leq T} |r_n| \ll \frac{(T - S)^\alpha (\log T)^\gamma}{S^\beta} \tag{1.10}$$

for $T > S > 0$.
Then $\phi(y)$ is a B^2-almost periodic function and therefore possesses a limiting distribution.

In Theorem 1.2, we prove that the conditions on ϕ imply that it is a B^2-almost periodic function. However, as it is known that B^2-almost periodic functions possess limiting distributions (see [42, Theorem 8.3] and Theorem 2.9 in this article), we also obtain that ϕ possesses a limiting distribution. It would be interesting to determine the weakest conditions on the coefficients $(r_n)_{n \in \mathbb{N}}$ and the exponents $(\lambda_n)_{n \in \mathbb{N}}$ which imply that ϕ is B^2-almost periodic.

Note that in part (b), the conditions $\beta \leq \alpha$ and $\alpha^2 + \alpha/2 < \beta^2 + \beta$ are equivalent to

$$\beta \leq \alpha < \sqrt{\beta^2 + \beta + \frac{1}{16}} - \frac{1}{4}. \tag{1.11}$$

The next corollary provides simpler criteria for which ϕ possesses a limiting distribution.

Corollary 1.3. *Let $\phi : [0, \infty) \to \mathbb{R}$ satisfy (1.7) and (1.8). Assume either of the following conditions:*
(a) $r_n \ll \lambda_n^{-\beta}$ for $\beta > \frac{1}{2}$, and

$$\sum_{T < \lambda_n \leq T+1} 1 \ll \log T. \tag{1.12}$$

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(b) $0 \leq \theta < 3 - \sqrt{3}$, *(1.12), and*

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^\theta. \tag{1.13}$$

Then $\phi(y)$ is a B^2-almost periodic function and therefore possesses a limiting distribution.

Part (a) of this corollary is useful to apply when the r_n's satisfy the nice bound $r_n \ll \lambda_n^{-\beta}$ where $\beta > 1/2$. On the other hand, if the r_n's oscillate significantly, then by part (b) of the above corollary it suffices to have a modest bound for the second moment of $\lambda_n |r_n|$. Indeed, there exist sequences which satisfy the conditions of (b), but not (a). For instance, let $r_n = \lambda_n^{-1/2}$, where $(\lambda_n)_{n \in \mathbb{N}}$ satisfies (1.12) and $\sum_{\lambda_n \leq T} 1 \ll T^{\theta-1}$ for $1 < \theta < 3 - \sqrt{3}$.

The existence of limiting distributions for (1.1) and (1.3) may be deduced from part (a). If we assume RH and the bound

$$|\zeta'(\rho)|^{-1} \ll |\rho|^{\frac{1}{2}-\varepsilon}, \tag{1.14}$$

then part (a) implies that (1.4) possesses a limiting distribution. It has been conjectured by Gonek [15] that $|\zeta'(\rho)|^{-1} \ll |\rho|^{\frac{1}{3}-\varepsilon}$. However, there currently is not a lot of evidence in support of this bound. On the other hand, if we assume RH and the bound (1.5), then part (b) implies that (1.4) possesses a limiting distribution. Currently, there is significantly more evidence supporting the truth of (1.5), including the work of [14], [19], [21], and [29].

More generally, we prove a version of Theorem 1.2 for vector-valued functions whose components are of the type $\phi(y)$. For instance, let $\vec{\phi} : [0, \infty) \to \mathbb{R}^\ell$ be given by

$$\vec{\phi}(y) = \big(\phi_1(y), \ldots, \phi_\ell(y)\big), \tag{1.15}$$

where each $\phi_k(y)$ is of the shape (1.7). Then we have the following.

Theorem 1.4. *Suppose that the conditions of Theorem 1.2 or Corollary 1.3 hold for each $\phi_k(y)$ for $1 \leq k \leq \ell$. Then $\vec{\phi}(y)$ possesses a limiting distribution.*

This theorem contains as special cases the results of Wintner, Rubinstein and Sarnak, and Ng. That is, the functions in equations (1.1), (1.2), (1.3), and (1.4) possess limiting distributions.

We also provide several new examples of functions which have limiting distributions. These functions are now described.

Let π be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$, and let $L(s, \pi)$ be the automorphic L-function attached to π. We have

$$L(s, \pi) = \prod_{p < \infty} L_p(s, \pi_p),$$

where

$$L_p(s, \pi_p) = \prod_{j=1}^d \left(1 - \frac{\alpha_\pi(p, j)}{p^s}\right)^{-1},$$

for $\Re(s) > 1$. The completed L-function $\Phi(s, \pi)$ is defined by

$$\Phi(s, \pi) = L_\infty(s, \pi_\infty) L(s, \pi),$$

where the Archimedean local factor is

$$L_\infty(s, \pi_\infty) = \prod_{j=1}^d \Gamma_\mathbb{R}(s + \mu_\pi(j))$$

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and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ where Γ is the classical gamma function. For $1 \leq j \leq d$, the complex numbers $\alpha_\pi(p, j)$ and $\mu_\pi(j)$ are called the *local parameters*. It is known that $\Phi(s, \pi)$ is entire (except in the case $L(s, \pi) = \zeta(s - i\tau_0)$ for $\tau_0 \in \mathbb{R}$, which in this case $\Phi(s, \pi)$ has two simple poles) and satisfies the functional equation

$$\Phi(s, \pi) = \epsilon(s, \pi)\Phi(1 - s, \tilde{\pi}),$$

with

$$\epsilon(s, \pi) = \epsilon_\pi Q_\pi^{1/2 - s},$$

where $Q_\pi \geq 1$ is an integer called the conductor of π, ϵ_π is the root number satisfying $|\epsilon_\pi| = 1$, and $\tilde{\pi}$ is the representation contragredient to π. It is expected that all non-trivial zeros of $L(s, \pi)$ are located on the line $\Re(s) = 1/2$ and this is known as the generalized Riemann hypothesis for $L(s, \pi)$.

We now consider prime counting functions associated to $L(s, \pi)$. Let

$$a_\pi(p^k) = \sum_{j=1}^{d} \alpha_\pi(p, j)^k, \tag{1.16}$$

and define

$$\psi(x, \pi) = \sum_{n \leq x} \Lambda(n) a_\pi(n),$$

where $\Lambda(n)$ is the classical von Mangoldt function. We have, for $\Re(s) > 1$,

$$-\frac{L'(s, \pi)}{L(s, \pi)} = \sum_{n=1}^{\infty} \frac{\Lambda(n) a_\pi(n)}{n^s}.$$

The prime number theorem for $L(s, \pi)$ (see [28, Theorem 2.3]) is the assertion that

$$\psi(x, \pi) = \delta(x, \pi) + O(x \exp(-c\sqrt{\log x}))$$

for some positive constant c, where

$$\delta(x, \pi) = \begin{cases} \frac{x^{1+i\tau_0}}{1+i\tau_0} & \text{if } L(s, \pi) = \zeta(s - i\tau_0), \\ 0 & \text{otherwise.} \end{cases}$$

From Corollary 1.3(a) we are able to deduce that a scaled version of the above error term possesses a limiting distribution.

Corollary 1.5. *Under the assumption of the generalized Riemann hypothesis for $L(s, \pi)$ the function*

$$E_1(y, \pi) = e^{-y/2} \left(\psi(e^y, \pi) - \delta(e^y, \pi) \right)$$

has a limiting distribution.

Note that Wintner's theorem (1.1) is a special case of the above corollary. In addition, for a modular newform f of weight k and level N, we conclude, under the assumption of the generalized Riemann hypothesis, that $e^{-y/2}\psi(e^y, f)$ has a limiting distribution.

We now introduce several other functions that possess limiting distributions. These functions are related to certain negative moments of the derivative of an L-function evaluated at its zeros. The

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first case to consider is the Riemann zeta function. Gonek [14] and Hejhal [19] studied $J_{-1}(T) = \sum_{0<\gamma\leq T} |\zeta'(\rho)|^{-2}$ and Gonek conjectured that

$$J_{-1}(T) \sim \frac{3}{\pi^3}T.$$

Assuming the Riemann hypothesis and all zeros of $\zeta(s)$ are simple, Milinovich and Ng [29] proved that $J_{-1}(T) \geq (\frac{3}{2\pi^3} - \varepsilon)T$ for every $\varepsilon > 0$ and T sufficiently large. In our work, we make the weaker assumption

$$J_{-1}(T) \ll T^\theta \text{ with } 1 \leq \theta < 3 - \sqrt{3}. \tag{1.17}$$

Currently, assuming the Riemann hypothesis and the simplicity of zeros of $\zeta(s)$, no upper bounds are known for $J_{-1}(T)$. However, the weak Mertens conjecture, the assumption that

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx \ll \log X,$$

implies $|\zeta'(\rho)|^{-1} \ll |\rho|$ and thus $J_{-1}(T) \ll T^{3+\varepsilon}$ (see [40, p. 377, eq. (14.29.4)]).

We also require a version of (1.17) for Dirichlet L-functions $L(s,\chi)$. We assume there exists a positive θ such that

$$\sum_{\chi \bmod q} \sum_{\substack{0<|\Im(\rho_\chi)|\leq T \\ L(\rho_\chi,\chi)=0}} |L'(\rho_\chi,\chi)|^{-2} \ll_q T^\theta \text{ where } 1 \leq \theta < 3 - \sqrt{3}. \tag{1.18}$$

It seems plausible that such a bound holds and it is natural to conjecture there is a positive constant C_q such that

$$\sum_{\chi \bmod q} \sum_{\substack{0<|\Im(\rho_\chi)|\leq T \\ L(\rho_\chi,\chi)=0}} |L'(\rho_\chi,\chi)|^{-2} \sim C_q T.$$

In fact, we can prove that this sum is greater than a positive constant times T, assuming that all zeros of the $L(s,\chi)$ are simple and lie on the critical line. Finally, observe that (1.17) implies that all zeros of $\zeta(s)$ are simple and (1.18) implies that all nonreal zeros of the $L(s,\chi)$ are simple. We make use of these facts in our applications.

We shall introduce several other summatory functions. For $\alpha \in [0,1]$ and $x > 0$, we set

$$M_\alpha(x) = \sum_{n\leq x} \frac{\mu(n)}{n^\alpha}.$$

Over the years, there has been significant interest in these functions. For instance, Landau showed in his Ph.D. thesis that $M_1(x)$ converges to 0. In 1897 Mertens conjectured that $M_0(x) = M(x)$ is bounded in absolute value by $\sqrt{x}$. This conjecture implies the Riemann hypothesis. Many researchers studied the size of $M_0(x)$. Finally, in 1985, Odlyzko and te Riele [36] showed that Mertens' conjecture is false. On the Riemann hypothesis, it is known that $M(x) \ll x^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$. Hence, by partial summation, it follows that $M_\alpha(x)$ converges to $\zeta(\alpha)^{-1}$ for $\alpha > \frac{1}{2}$. Consequently, we observe that the behaviour of $M_\alpha(x)$ changes at $\alpha = 1/2$ and thus define

$$E_2(y,\alpha) = \begin{cases} e^{y(-1/2+\alpha)} M_\alpha(e^y) & \text{if} & 0 \leq \alpha \leq 1/2, \\ e^{y(-1/2+\alpha)}\left(M_\alpha(e^y) - \frac{1}{\zeta(\alpha)}\right) & \text{if} & 1/2 < \alpha \leq 1. \end{cases} \tag{1.19}$$

We now consider weighted sums of the Liouville function. The Liouville function is given by $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime factors of n. We set

$$L_\alpha(x) = \sum_{n \le x} \frac{\lambda(n)}{n^\alpha}.$$

Pólya and Turán studied $L_0(x) = L(x)$ and $L_1(x)$, respectively. Early numerical calculations suggested that the inequalities $L_0(x) \le 0$ and $L_1(x) > 0$ hold for all $x \ge 2$. In 1958, Haselgrove [17] showed that $L_0(x)$ and $L_1(x)$ change sign infinitely often. Tanaka [39] showed that the first value of n for which $L_0(n) > 0$ is 906,105,257. Borwein, Ferguson, and Mossinghoff [4] determined that the smallest value of n for which $L_1(n) < 0$ is 72,185,376,951,205. It would be interesting to know how often $L_\alpha(x)$ is positive or negative. In order to study such questions we define the error terms

$$E_3(y, \alpha) = \begin{cases} e^{y(-1/2+\alpha)} L_\alpha(e^y) & \text{if} \quad 0 \le \alpha < 1/2, \\ e^{y(-1/2+\alpha)} \left(L_\alpha(e^y) - \frac{y}{2\zeta(1/2)} \right) & \text{if} \quad \alpha = 1/2, \\ e^{y(-1/2+\alpha)} \left(L_\alpha(e^y) - \frac{\zeta(2\alpha)}{\zeta(\alpha)} \right) & \text{if} \quad 1/2 < \alpha \le 1. \end{cases} \tag{1.20}$$

In [35] it was mentioned that $E_3(y, 0)$ possesses a limiting distribution under the same hypotheses for which $e^{-y/2} M(e^y)$ possesses a limiting distribution. Recently, Humphries [22] studied these functions in the range $\alpha \in [0, 1/2)$ and showed that, for these α, the Riemann hypothesis and $J_{-1}(T) \ll T$ imply that $E_3(y, \alpha)$ possesses a limiting distribution.

Our next example concerns the Möbius function in arithmetic progressions. For $q \ge 2$ and $(a, q) = 1$, let

$$M(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \mu(n).$$

This is a variant of $M(x)$ with the extra condition $n \equiv a \pmod q$ inserted. Sums like $M(x; q, a)$ reflect the behaviour of primes in arithmetic progressions. In fact, many theorems which can be established for

$$\sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n) - \frac{x}{\phi(q)}$$

have corresponding analogues for $M(x; q, a)$. For a fixed integer $q \ge 2$, we define

$$E_4(y; q, a) = e^{-y/2} M(e^y; q, a). \tag{1.21}$$

The next corollary establishes the existence of limiting distributions for $E_2(y, \alpha)$, $E_3(y, \alpha)$, and $E_4(y; q, a)$.

Corollary 1.6. *Let $\alpha \in [0, 1]$, $q \ge 2$, and $(a, q) = 1$.*
(i) If RH is true and (1.17) holds, then $E_2(y, \alpha)$ possesses a limiting distribution.
(ii) If RH is true and (1.17) holds, then $E_3(y, \alpha)$ possesses a limiting distribution.
(iii) If the generalized Riemann hypothesis is true for all Dirichlet L-functions modulo q and (1.18) holds, then $E_4(y; q, a)$ possesses a limiting distribution.

Part (i) improves and generalizes the main result of [35]. Similarly, part (ii) improves and generalizes the limiting distribution result of [22]. In [35] and [22] the bound $J_{-1}(T) \ll T$ is employed, whereas we use the weaker bound (1.17). It it possible that parts (i) and (ii) may be

extended to hold for all $\alpha \in \mathbb{R}$. In addition, assuming the same conditions as in part (iii), we can show that for $q \geq 2$ and $(a, q) = 1$ that $e^{-y/2}L(e^y; q, a)$ possesses a limiting distribution where

$$L(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \lambda(n).$$

Our final example of error terms which possess limiting distributions is related to number fields. Let K/k be a normal extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. Denote by $\mathcal{O}_k$ and $\mathcal{O}_K$ the corresponding rings of integers of k and K. We define several counting functions. Let

$$\pi_k(x) = \sum_{N\mathfrak{p} \leq x} 1$$

where $N\mathfrak{p}$ denotes the norm of the prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ and for a conjugacy class C of G

$$\pi_C(x) = \sum_{\substack{N\mathfrak{p} \leq x \\ \sigma_\mathfrak{p} = C}} 1$$

where $\sigma_\mathfrak{p}$ is the Frobenius conjugacy class associated to $\mathfrak{p}$. Associated to r distinct conjugacy classes $C_1, \ldots, C_r$ in G, we define

$$\vec{E}_5(y) = ye^{-y/2} \left(\frac{|G|}{|C_1|} \pi_{C_1}(e^y) - \pi_k(e^y), \ldots, \frac{|G|}{|C_r|} \pi_{C_r}(e^y) - \pi_k(e^y) \right).$$

In order to study $\vec{E}_5(y)$, we require information regarding the zeros of Artin L-functions associated to the extension K/k. Let ρ be a representation of G in $\mathrm{GL}_n(\mathbb{C})$ with character $\chi = \mathrm{tr}(\rho)$. The principal character χ_0 is the character attached to the trivial representation $\rho_0 = 1$. For each character χ of G, we associate the Artin L-function $L(s, \chi, K/k)$. It is known that $L(s, \chi, K/k)$ is a meromorphic function on the complex plane. Moreover, there is the following fundamental conjecture.

Conjecture 1.7 (Artin's Holomorphy Conjecture). *If χ is non-trivial then $L(s, \chi, K/k)$ is entire.*

Also it is conjectured that an analogue of the Riemann hypothesis holds for Artin L-functions. For further information regarding Artin L-functions see [5, pp. 218–225] .

In his Ph.D. thesis [34], the second author showed that $\vec{E}_5(y)$ possesses a limiting distribution. This can be deduced as a corollary of Theorem 1.4.

Corollary 1.8. *Under the assumptions of the generalized Riemann hypothesis and Artin's holomorphy conjecture for $L(s, \chi, K/k)$, where χ ranges through the irreducible characters of G, $\vec{E}_5(y)$ possesses a limiting distribution.*

This result contains as special cases the fact that (1.2) and (1.3) possess limiting distributions.

The above corollaries are just a few applications of Theorems 1.2 and 1.4 and there are other interesting examples. For instance, Fiorilli [10] applies our theorems in his work on highly biased prime number races and also in his work [11] on prime number races associated to elliptic curves.

Our next theorem states that under an additional assumption on the exponent set $(\lambda_n)_{n \in \mathbb{N}}$ the Fourier transform of the limiting distribution of Theorem 1.4 can be explicitly calculated. In order to explain our result we require some notation.

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For $1 \le k \le \ell$, let the component function $\phi_k(x)$ of (1.15) be defined by

$$\phi_k(x) = c_k + \Re\left(\sum_{\lambda_{k,n} \le X} r_{k,n} e^{i\lambda_{k,n} y} \right) + \mathcal{E}_k(y, X),$$

where $c_k \in \mathbb{R}$, $(\lambda_{k,n})_{n \in \mathbb{N}} \subset \mathbb{R}^+$ is an increasing sequence, $(r_{k,n})_{n \in \mathbb{N}} \subset \mathbb{C}$, and $\mathcal{E}_k(y, X)$ satisfies (1.8). Note that the collection of $(\lambda_{k,n})_{n \in \mathbb{N}}$ for $1 \le k \le \ell$ is a multiset. We now consider the set $\cup_{k=1}^{\ell} \cup_{n=1}^{\infty} \{\lambda_{k,n}\}$ and reorder its elements to construct the increasing sequence $(\lambda_m)_{m \in \mathbb{N}}$. Also, we define

$$r_k(\lambda_m) = \begin{cases} r_{k,n} & \text{if } \lambda_m = \lambda_{k,n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

With this notation in hand, we now provide a formula for the Fourier transform of the limiting distribution of $\vec{\phi}(y)$.

Theorem 1.9. *Assume that μ is the limiting distribution associated to $\vec{\phi}(y)$ as given in Theorem 1.4. Suppose that the set $\{\lambda_m\}_{m \in \mathbb{N}}$ is linearly independent over $\mathbb{Q}$. Then the Fourier transform*

$$\hat{\mu}(\vec{\xi}) = \int_{\mathbb{R}^\ell} e^{-i \sum_{j=1}^{\ell} \xi_j t_j} d\mu(t_1, \dots, t_\ell)$$

of μ at $\vec{\xi} = (\xi_1, \dots, \xi_\ell) \in \mathbb{R}^\ell$ exists and is equal to

$$\hat{\mu}(\vec{\xi}) = \exp\left(-i \sum_{k=1}^{\ell} c_k \xi_k \right) \times \prod_{m=1}^{\infty} J_0\left(\left| \sum_{k=1}^{\ell} r_k(\lambda_m) \xi_k \right| \right),$$

where $J_0(z)$ is the Bessel function

$$J_0(z) = \int_0^1 e^{-iz \cos(2\pi t)} dt.$$

The above theorem is a useful tool in studying arithmetic applications of our limiting distribution theorems. We now discuss an application. For $q \ge 2$ and $a_1, \dots, a_r$, r distinct reduced residue classes mod q, consider the set

$$S_{q; a_1, \dots, a_r} = \{x > 0 \mid M(x; q, a_1) > M(x; q, a_2) > \cdots > M(x; q, a_r)\}.$$

In analogy to the Shanks-Rényi prime number race, we ask whether this set contains infinitely many natural numbers and if it possesses a density. In this situation it is convenient to consider logarithmic density.

Definition 1.10. For $P \subseteq [0, \infty)$, set

$$\overline{\delta}(P) = \limsup_{X \to \infty} \frac{1}{\log X} \int_{t \in P \cap [2, X]} \frac{dt}{t}$$

and

$$\underline{\delta}(P) = \liminf_{X \to \infty} \frac{1}{\log X} \int_{t \in P \cap [2, X]} \frac{dt}{t}.$$

If $\overline{\delta}(P) = \underline{\delta}(P) = \delta(P)$, we say that the *logarithmic density* of P is $\delta(P)$.

In order to study $S_{q;a_1,\ldots,a_r}$, we consider

$$\vec{E}_6(y) = e^{-y/2}\big(M(e^y;q,a_1),\ldots,M(e^y;q,a_r)\big).$$

Theorem 1.4 implies $\vec{E}_6(y)$ has a limiting distribution $\mu_{q;a_1,\ldots,a_r}$ assuming the generalized Riemann hypothesis for Dirichlet L-functions modulo q and (1.18). If it were known that $\mu_{q;a_1,\ldots,a_r}$ is an absolutely continuous measure, then it would follow that

$$\delta(S_{q;a_1,\ldots,a_r}) = \mu(\{x \in \mathbb{R}^r \mid x_1 > x_2 > \cdots > x_r\}). \tag{1.22}$$

In order to show that $\mu_{q;a_1,\ldots,a_r}$ is absolutely continuous, we require further information on the imaginary parts of zeros of Dirichlet L-functions. We now recall a folklore conjecture concerning the diophantine nature of the imaginary parts.

Conjecture 1.11 (Linear Independence Conjecture). *The multiset of the nonnegative imaginary parts of the nontrivial zeros of Dirichlet L-functions corresponding to primitive characters is linearly independent over the rationals.*

With this conjecture in hand, it follows from Formula (4.23) and Theorem 1.9 that

$$\hat{\mu}_{q;a_1,\ldots,a_r}(\xi_1,\ldots,\xi_r) = \prod_{\chi \bmod q} \prod_{\gamma_\chi > 0} J_0\left(\frac{2\left|\sum_{j=1}^r \chi(a_j)\xi_j\right|}{\varphi(q)|\rho_\chi L'(\rho_\chi,\chi)|}\right). \tag{1.23}$$

Following the arguments in [8, Lemma 2.1] and [22, Lemma 6.4] we can deduce from (1.23) that $\mu_{q;a_1,\ldots,a_r}$ possesses a density function and is absolutely continuous. Thus on the generalized Riemann hypothesis, Conjecture 1.11, and (1.18) it follows from (1.22) that $\delta(S_{q;a_1,\ldots,a_r})$ exists. We can also employ (1.23) to investigate symmetries of the density function of $\mu_{q;a_1,\ldots,a_r}$. The proof of Proposition 3.1 of [37] yields the following.

Proposition 1.12. *Assume the generalized Riemann hypothesis, (1.18), and the linear independence conjecture. Then the density function of $\mu_{q;a_1,\ldots,a_r}$ is symmetric in $(t_1,\ldots,t_\ell)$ if and only if either $r = 2$ or $r = 3$ and there is $\rho \neq 1$ such that $\rho^3 \equiv 1$, $a_2 \equiv a_1\rho$, and $a_3 \equiv a_1\rho^2$ (modulo q).*

As a consequence of the symmetry of the density function of $\mu_{q;a_1,\ldots,a_r}$ we obtain the next corollary.

Corollary 1.13. *Assume the conditions of Proposition 1.12. If either $r = 2$ or $r = 3$ and there is $\rho \neq 1$ such that $\rho^3 \equiv 1$, $a_2 \equiv a_1\rho$, and $a_3 \equiv a_1\rho^2$ (modulo q), then*

$$\delta\big(\{x > 0 \mid M(x;q,a_1) > M(x;q,a_2) > \cdots > M(x;q,a_r)\}\big) = \frac{1}{r!}.$$

In particular, if a_1 and a_2 are distinct residues modulo q,

$$\delta\big(\{x > 0 \mid M(x;q,a_1) > M(x;q,a_2)\}\big) = \delta\big(\{x > 0 \mid M(x;q,a_2) > M(x;q,a_1)\}\big) = \frac{1}{2}.$$

This shows that the race between the summatory functions of the Möbius function on two arithmetic progressions is unbiased.

Our general limiting distribution theorems can be used in proposing and studying many new arithmetic problems. For example, let

$$\vec{E}_7(y) = \big(ye^{-y/2}(\pi(e^y) - \text{Li}(e^y)), e^{-y/2}M(e^y)\big).$$

Then Corollaries 1.8 and 1.6(i) imply that if the Riemann hypothesis and (1.17) hold, then $\vec{E}_7(y)$ possesses a limiting distribution. If in addition, the linear independence conjecture for the zeros of $\zeta(s)$ is true, then

$$\left\{ x > 0 \ \Big| \ \frac{\log x}{\sqrt{x}} \left(\pi(x) - \text{Li}(x) \right) > \frac{M(x)}{\sqrt{x}} \right\}$$

possesses a logarithmic density. It would be interesting to determine the value of this logarithmic density. However, this requires further analysis of the constructed distribution.

As mentioned before, our strategy in the proof of our general limiting distribution theorem will be to prove that $\phi(y)$ is a B^2-almost periodic function. Since Besicovitch [1, Section 2] proved that B^2-almost periodic functions satisfy a Parseval type identity, we deduce the following result.

Theorem 1.14. *Suppose that the function $\phi(y)$ of* (1.7) *satisfies the conditions of Theorem 1.2 or Corollary 1.3. Then we have*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y)^2 dy = c^2 + \tfrac{1}{2} \sum_{n=1}^{\infty} |r_n|^2. \tag{1.24}$$

In fact, it is possible to show following an argument of Fiorilli [10, Lemmas 2.4, 2.5] that

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y)^2 dy = \int_{\mathbb{R}} t^2 d\mu(t)$$

where μ is the limiting distribution associated to ϕ. A similar argument would also establish that

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y) dy = \int_{\mathbb{R}} t \, d\mu(t) = c. \tag{1.25}$$

Observe that (1.24) provides a formula for $c^2 + \text{Var}(\mu)$, where $\text{Var}(\mu)$ is the variance of μ, and (1.25) shows that the expected value of μ is $\mathbf{E}(\mu) = c$.

As a corollary, we deduce Cramer's result [6] and its analogues for the error term of an automorphic L-function, $e^{-y/2}M(e^y)$, and $e^{-y/2}L(e^y)$.

Corollary 1.15. *(i) Let $L(s, \pi)$ be an automorphic L-function. If the generalized Riemann hypothesis is true for $L(s, \pi)$, then*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \left(\frac{\psi(e^y, \pi) - \delta(e^y, \pi)}{e^{y/2}} \right)^2 dy = 4 \left(\text{ord}_{s=1/2} L(s, \pi) \right)^2 + \sum_{\substack{\gamma > 0 \\ L(1/2 + i\gamma, \pi) = 0}} \frac{2m_\gamma^2}{\frac{1}{4} + \gamma^2},$$

where m_γ denotes the multiplicity of the zero $1/2 + i\gamma$.
(ii) If the Riemann hypothesis is true and (1.17) *holds, then*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \left(\frac{M(e^y)}{e^{y/2}} \right)^2 dy = \sum_{\substack{\gamma > 0 \\ \zeta(\frac{1}{2} + i\gamma) = 0}} \frac{2}{|\rho \zeta'(\rho)|^2}.$$

(iii) If the Riemann hypothesis is true and (1.17) *holds, then*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \left(\frac{L(e^y)}{e^{y/2}} \right)^2 dy = \frac{1}{\zeta(\frac{1}{2})^2} + \sum_{\substack{\gamma > 0 \\ \zeta(\frac{1}{2} + i\gamma) = 0}} 2 \left| \frac{\zeta(2\rho)}{\rho \zeta'(\rho)} \right|^2.$$

Note that Theorem 1.15 (ii) improves Theorem 3 of [35] where the stronger condition $J_{-1}(T) \ll T$ is assumed.

The rest of this article is organized as follows. In Section 2 we review background material on B^p-almost periodic functions and show that almost periodic functions possess limiting distributions. In Section 3, we prove Theorem 1.2 and Corollary 1.3. In Section 4, we deduce Corollaries 1.5, 1.6, and 1.8. In Section 5, we prove Theorem 1.9. Finally, we mention some notation used throughout this article. We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ to mean there exists $M > 0$ such that $|f(x)| \le M|g(x)|$ for all sufficiently large x.

2. B^p-ALMOST PERIODIC FUNCTIONS AND LIMITING DISTRIBUTIONS

The main goal in this section is to provide the necessary background on B^p-almost periodic functions needed in the proof of Theorem 1.4. It has been known since the 1930's that any B^p-almost periodic function ϕ[1] possess limiting distributions. Such a result is mentioned in [24, Theorems 25 and 27] and proven in [42, Theorem 8.3]. However, the authors were unable to find a refereed publication from the 1930's which proves this result. The earliest journal publication we are aware of is [3], though it only proves the result for $\ell = 1$. In order to keep our article self-contained, we provide a proof in the general case of a vector-valued function.

We review some facts from the theory of almost periodic functions. Let $L^p_{loc}([0, \infty))$ be the set of locally p-integrable functions on $[0, \infty)$. For $p \ge 1$ and $\phi \in L^p_{loc}([0, \infty))$, define

$$\|\phi\|_{B^p} = \left(\limsup_{Y \to \infty} \frac{1}{Y} \int_0^Y |\phi(y)|^p dy \right)^{1/p}.$$

Denote by $\mathscr{T}$ the class of all real-valued trigonometric polynomials

$$P_N(y) = \sum_{n=1}^{N} r_n e^{i\lambda_n y} \quad (y \in \mathbb{R}),$$

where $r_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$. The B^p-*closure* of $\mathscr{S}$, denoted $\mathcal{H}_{B^p}(\mathscr{S})$, is the set of functions $\phi \in \mathscr{R}$ that satisfy the following property:

For any $\varepsilon > 0$ there is a function $f_\varepsilon(y) \in \mathscr{S}$ such that

$$\|\phi(y) - f_\varepsilon(y)\|_{B^p} < \varepsilon.$$

Definition 2.1. Any $\phi \in \cup_{p \ge 1} \mathcal{H}_{B^p}(\mathscr{T})$ is called an *almost periodic function*. If $\phi \in \mathcal{H}_{B^p}(\mathscr{T})$ we say that ϕ is a B^p-*almost periodic function*.

For $\phi \in \mathcal{H}_{B^p}(\mathscr{T})$ and given $\varepsilon > 0$ there exists

$$P_{N(\varepsilon)}(y) = \sum_{n=1}^{N(\varepsilon)} r_n(\varepsilon) e^{i\lambda_n(\varepsilon)y} \tag{2.1}$$

in $\mathscr{T}$ such that

$$\|\phi(y) - P_{N(\varepsilon)}(y)\|_{B^p} < \varepsilon.$$

It is an important fact of the theory of almost periodic functions that in (2.1), $\lambda_n(\varepsilon)$ can be taken only from a set $\Lambda(\phi) = \{\lambda_n \mid n \in \mathbb{N}\}$ and the corresponding values for r_n are given by

$$r_n = \lim_{Y \to \infty} \frac{1}{Y} \int_0^Y \phi(y) e^{-i\lambda_n y} dy$$

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[1]In this section ϕ denotes a Lebesgue integrable function.

(see [3]).

Definition 2.2. A vector-valued function $\vec{\phi} : [0, \infty) \to \mathbb{R}^\ell$, $\vec{\phi} = (\phi_1, \ldots, \phi_\ell)$, is called *almost periodic*, if there is a $p \geq 1$ such that each component function ϕ_k $(1 \leq k \leq \ell)$ belongs to $\mathcal{H}_{B^p}(\mathscr{T})$. Moreover $\vec{\phi}$ is called B^p-*almost periodic* if each ϕ_k $(1 \leq k \leq \ell)$ is B^p-almost periodic.

It is known that $\mathcal{H}_{B^p}(\mathscr{T}) \subseteq \mathcal{H}_{B^q}(\mathscr{T})$ if $1 \leq q \leq p$ (see [3, p. 476]). So a vector-valued function is almost periodic if and only if each of its component functions is almost periodic.

The following lemma states a version of the Kronecker-Weyl equidistribution theorem.

Lemma 2.3. *Let $t_1, \ldots, t_N$ be arbitrary real numbers. Suppose that A is the topological closure of $\{y(t_1, \ldots, t_N) \mid y \in \mathbb{R}\} / \mathbb{Z}^N$ in the torus $\mathbb{T}^N$. Let $g : \mathbb{R}^N \to \mathbb{R}$ be a continuous function of period 1 in each of its variables. Then we have*

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y g(yt_1, \ldots, yt_N) dy = \int_A g(a) d\omega$$

where ω is the normalized Haar measure on A.

Proof. This may be deduced from the Kronecker-Weyl theorem (see [20, pp. 1–16]), and is also a special case of Ratner's theorem on unipotent flows (see [33]). □

Next we prove that every vector-valued function whose components are real-valued trigonometric polynomials has a limiting distribution.

Proposition 2.4. *For $1 \leq k \leq \ell$, let $(\lambda_{k,n})_{n=1}^{N_k}$ be a real sequence and $(r_{k,n})_{n=1}^{N_k}$ be a complex sequence. Set*

$$P_k(y) = \sum_{n=1}^{N_k} r_{k,n} e^{i\lambda_{k,n} y} \quad (y \in \mathbb{R}).$$

If $P_k(y) \in \mathbb{R}$ for all $y \in \mathbb{R}$, then

$$\vec{P}(y) = \big(P_1(y), \ldots, P_\ell(y)\big)$$

has a limiting distribution.

Proof. We consider the set $\cup_{k=1}^\ell \cup_{n=1}^{N_k} \{\lambda_{k,n}\}$ and write its elements in increasing order as the sequence $(\lambda_m)_{m=1}^N$. For $1 \leq k \leq \ell$, we set

$$r_k(\lambda_m) = \sum_{\substack{1 \leq n \leq N_k \\ \lambda_{k,n} = \lambda_m}} r_{k,n}.$$

Let $f : \mathbb{R}^\ell \to \mathbb{R}$ be a bounded continuous function. Suppose that $X : \mathbb{T}^N \to \mathbb{R}^\ell$ and $g : \mathbb{T}^N \to \mathbb{R}$ are defined by

$$X(\theta_1, \ldots, \theta_N) = \left(\sum_{m=1}^N r_1(\lambda_m) e^{2\pi i \theta_m}, \ldots, \sum_{m=1}^N r_\ell(\lambda_m) e^{2\pi i \theta_m} \right)$$

and $g(\theta_1, \ldots, \theta_N) = f\big(X(\theta_1, \ldots, \theta_N)\big)$. By applying Lemma 2.3 with $t_1 = \frac{\lambda_1}{2\pi}, \ldots, t_N = \frac{\lambda_N}{2\pi}$, we have

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y g\big(\tfrac{y\lambda_1}{2\pi}, \ldots, \tfrac{y\lambda_N}{2\pi}\big) dy = \int_A g(a) d\omega,$$

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where A is the closure of $\left\{y\left(\frac{\lambda_1}{2\pi}, \ldots, \frac{\lambda_N}{2\pi}\right) \mid y \in \mathbb{R}\right\}/\mathbb{Z}^N$ in $\mathbb{T}^N$ and ω is the normalized Haar measure on A. Define a probability measure μ_N on $\mathbb{R}^\ell$ by $\mu_N(B) = \omega\left(X^{-1}(B) \cap A\right)$, where B is any Borel set in $\mathbb{R}^\ell$. By the change of variable formula [2, Theorem 16.12],

$$\int_A g(a)d\omega = \int_{\mathbb{R}^\ell} f d\mu_N \tag{2.2}$$

and thus

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f\left(\vec{P}(y)\right) dy = \int_{\mathbb{R}^\ell} f d\mu_N,$$

for all bounded continuous real-valued functions f on $\mathbb{R}^\ell$. Therefore, $\vec{P}(y)$ has a limiting distribution. $\square$

Our next goal is to show that every almost periodic function possesses a limiting distribution. This requires several concepts from probability.

Definition 2.5. Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of finite measures on a measurable space X. We say that μ_n *converges weakly* to μ if for every bounded real-valued continuous function f we have

$$\int_X f d\mu_n \to \int_X f d\mu \tag{2.3}$$

as $n \to \infty$.

In fact, it is well known that (2.3) only needs to be verified for Lipschitz functions.

Lemma 2.6 (Portmanteau). μ_n *converges weakly to μ if and only if*

$$\int_X f d\mu_n \to \int_X f d\mu$$

for any bounded Lipschitz function f on X.

Proof. See [30, Theorem 3.5]. $\square$

Next we define the tightness of a sequence of probability measures.

Definition 2.7. A sequence $(\mu_n)_{n \in \mathbb{N}}$ of probability measures on $\mathbb{R}^\ell$ is *tight* if for any $\varepsilon > 0$ there is $A_\varepsilon > 0$ such that $\int_{|\mathbf{x}| \geq A_\varepsilon} d\mu_n < \varepsilon$, for all $n \in \mathbb{N}$.

The following lemma illustrates the importance of a tight sequence of measures.

Lemma 2.8 (Helly's Selection Theorem). *Let $(\mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^\ell$. Then $(\mu_n)_{n \in \mathbb{N}}$ is tight if and only if for every subsequence $(\mu_{n_j})_{j \in \mathbb{N}}$ there is a further subsequence $(\mu_{n_{j_k}})_{k \in \mathbb{N}}$ and a probability measure μ such that $\mu_{n_{j_k}}$ converges weakly to μ.*

Proof. See [2, Theorems 25.8 and 25.10]. $\square$

We are ready to prove the main result of this section.

Theorem 2.9. *Every almost periodic function possesses a limiting distribution.*

Proof. Consider an almost periodic function $\vec{\phi} : [0, \infty) \to \mathbb{R}^\ell$. For $Y \geq 1$, let

$$\nu_Y(B) = \frac{1}{Y} \mathrm{meas}\left([0, Y] \cap (\vec{\phi})^{-1}(B)\right)$$

for any Borel set B in $\mathbb{R}^\ell$, where $\mathrm{meas}(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.

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Note that, by Definition 2.5, $\vec{\phi}(y)$ has a limiting distribution if and only if there exists a probability measure μ such that the sequence $(\nu_Y)_{Y \in \mathbb{N}}$ converges weakly to μ. By Lemma 2.6 this is equivalent to

$$\int_{\mathbb{R}^\ell} f d\nu_Y \to \int_{\mathbb{R}^\ell} f d\mu,$$

as $Y \to \infty$, for any bounded Lipschitz function $f : \mathbb{R}^\ell \to \mathbb{R}$.

Now let $\vec{\phi}(y) = \big(\phi_1(y), \dots \phi_\ell(y)\big)$ such that $\phi_k(y)$ belongs to $\mathcal{H}_{B^1}(\mathscr{T})$ for $1 \le k \le \ell$. (Recall that $\mathcal{H}_{B^p}(\mathscr{T}) \subseteq \mathcal{H}_{B^1}(\mathscr{T})$ for any $p \ge 1$.) Then for each component $\phi_k(y)$ and for $M \in \mathbb{N}$, there exists $N_k(M) \in \mathbb{N}$ and sequences $(r_{k,n})_{n=1}^{N_k}$ and $(\lambda_{k,n})_{n=1}^{N_k}$ such that

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_0^Y \left| \phi_k(y) - \sum_{n=1}^{N_k(M)} r_{k,n} e^{iy\lambda_{k,n}} \right| dy < \frac{1}{M}. \tag{2.4}$$

By Proposition 2.4,

$$\vec{P}_M(y) = \left(\sum_{n=1}^{N_1(M)} r_{1,n} e^{iy\lambda_{1,n}}, \dots, \sum_{n=1}^{N_\ell(M)} r_{\ell,n} e^{iy\lambda_{\ell,n}} \right) \tag{2.5}$$

has a limiting distribution μ_M, i.e.

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y f\big(\vec{P}_M(y)\big) dy = \int_{\mathbb{R}^\ell} f(x) d\mu_M(x) := \mu_M(f),$$

for all bounded continuous functions $f : \mathbb{R}^\ell \to \mathbb{R}$. From now on for a probability measure ν on $\mathbb{R}^\ell$ and a function g, we shall make use of the notation

$$\nu(g) = \int_{\mathbb{R}^\ell} g(x) d\nu(x).$$

Let $f : \mathbb{R}^\ell \to \mathbb{R}$ be a bounded Lipschitz function which satisfies

$$|f(x) - f(y)| \le c_f |x - y|$$

for all $x, y \in \mathbb{R}^\ell$ where c_f is the Lipschitz constant. Then we have

$$\frac{1}{Y} \int_0^Y f\big(\vec{\phi}(y)\big) dy \le \frac{1}{Y} \int_0^Y f\big(\vec{P}_M(y)\big) dy + \frac{c_f}{Y} \int_0^Y \big|\vec{\phi}(y) - \vec{P}_M(y)\big| dy \tag{2.6}$$

and

$$\frac{1}{Y} \int_0^Y f\big(\vec{\phi}(y)\big) dy \ge \frac{1}{Y} \int_0^Y f\big(\vec{P}_M(y)\big) dy - \frac{c_f}{Y} \int_0^Y \big|\vec{\phi}(y) - \vec{P}_M(y)\big| dy \tag{2.7}$$

for any $Y > 0$ and $M \in \mathbb{N}$. Moreover,

$$\frac{1}{Y} \int_0^Y \big|\vec{\phi}(y) - \vec{P}_M(y)\big| dy \le \sum_{k=1}^\ell \frac{1}{Y} \int_0^Y \left| \phi_k(y) - \sum_{n=1}^{N_k(M)} r_{k,n} e^{iy\lambda_{k,n}} \right| dy.$$

If we apply the latter inequality in (2.6) and (2.7) and take $\limsup$ and $\liminf$ as $Y \to \infty$, respectively, by employing (2.4) we obtain

$$\mu_M(f) - \ell c_f / M \le \liminf_{Y \to \infty} \frac{1}{Y} \int_0^Y f\big(\vec{\phi}(y)\big) dy \le \limsup_{Y \to \infty} \frac{1}{Y} \int_0^Y f\big(\vec{\phi}(y)\big) dy \le \mu_M(f) + \ell c_f / M. \tag{2.8}$$

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These inequalities imply that $L(f) := \lim_{Y\to\infty} \frac{1}{Y} \int_0^Y f(\vec{\phi}(y)) dy$ exists. Moreover, (2.8) implies that

$$\lim_{Y\to\infty} \nu_Y(f) = \lim_{M\to\infty} \mu_M(f) = L(f) \tag{2.9}$$

exists for every bounded Lipschitz function $f : \mathbb{R}^\ell \to \mathbb{R}$.

We next show that $(\nu_Y)_{Y\in\mathbb{N}}$ is tight, i.e. for any $\varepsilon > 0$, there is $A_\varepsilon > 0$ such that

$$\int_{|\mathbf{x}| \ge A_\varepsilon} d\nu_Y < \varepsilon$$

for all $Y \in \mathbb{N}$. Let $\varepsilon > 0$ be given. We choose a natural number M such that $\ell/M < \varepsilon$. By (2.4) and (2.5), there exists a vector function $\vec{P}_M(y)$ with trigonometric polynomials as its components such that

$$|\vec{\phi}(y) - \vec{P}_M(y)| < \ell/M < \varepsilon, \tag{2.10}$$

where $|.|$ denote the Euclidean norm in $\mathbb{R}^\ell$. Let

$$A_\varepsilon = \sup_{y\in[0,\infty]} |\vec{P}_M(y)| + 1.$$

Now by employing (2.10) we have

$$\int_{|\mathbf{x}| \ge A_\varepsilon} d\nu_Y = \frac{1}{Y} \text{meas}\{0 \le y \le Y, \ |\vec{\phi}(y)| > A_\varepsilon\} \le \frac{1}{Y} \int_0^Y |\vec{\phi}(y) - \vec{P}_M(y)| dy < \varepsilon.$$

Hence $(\nu_Y)_{Y\in\mathbb{N}}$ is tight, as we stated. Thus, by Lemma 2.8, there is a subsequence $(\nu_{Y_j})_{j\in\mathbb{N}}$ of $(\nu_Y)_{Y\in\mathbb{N}}$ and a probability measure μ on $\mathbb{R}^\ell$ such that

$$L(f) = \lim_{j\to\infty} \nu_{Y_j}(f) = \mu(f).$$

This together with (2.9) shows that

$$\lim_{Y\to\infty} \nu_Y(f) = \lim_{M\to\infty} \mu_M(f) = \mu(f), \tag{2.11}$$

for every bounded Lipschitz function $f : \mathbb{R}^\ell \to \mathbb{R}$ and the proof is complete. $\square$

3. PROOF OF THE MAIN THEOREM

The goal in this section is to prove Theorem 1.4. By Theorem 2.9, we know that $\vec{\phi}(y)$ has a limiting distribution if $\vec{\phi}(y)$ is a B^2-almost periodic function. Since $\vec{\phi}(y)$ is a B^2-almost periodic function if and only if each of its component functions $\phi_k(y)$ is B^2-almost periodic, Theorem 1.4 will follow as a consequence of Theorem 1.2.

The proof of Theorem 1.2 under the conditions given in (a) is based on an idea of Gallagher's [13], using the following lemma. The proof under the assumptions given in (b) follows an argument first employed by Cramér [6] and later used by Ng [35].

For a proof of the following lemma see [12, Lemma 1].

Lemma 3.1 (Gallagher). *Let $(\nu_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers and $(c_n)_{n\in\mathbb{N}} \subset \mathbb{C}$. Assume that $f(x) = \sum_{n=1}^\infty c_n e^{2\pi i \nu_n x}$ is absolutely convergent. Let $0 < \theta < 1$. Then, for $U > 0$,*

$$\int_{-U}^U |f(x)|^2 dx \ll_\theta U^2 \int_{-\infty}^\infty \Big| \sum_{t < \nu_n \le t + \theta/U} c_n \Big|^2 dt.$$

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Proof of Theorem 1.2. Let $X > T > 1$ and $V \geq 0$. Assume either of the conditions of Theorem 1.2. We shall begin by showing that there exist $\delta \geq 0$ and $\eta > 0$ such that

$$\int_V^{V+1} \Big| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \Big|^2 dy \ll \frac{(\log T)^\delta}{T^\eta}. \tag{3.1}$$

First assume that Condition (a) in Theorem 1.2 holds. Then, by the change of variable $y = V + 2\pi t$, we have

$$\int_V^{V+1} \Big| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \Big|^2 dy \;=\; 2\pi \int_0^{1/2\pi} \Big| \sum_{T < \lambda_n \leq X} r_n e^{i\lambda_n V} e^{2\pi i \lambda_n t} \Big|^2 dt$$

$$\leq \; 2\pi \int_{-1}^1 \Big| \sum_{T < \lambda_n \leq X} r_n e^{i\lambda_n V} e^{2\pi i \lambda_n t} \Big|^2 dt.$$

Lemma 3.1 implies

$$\int_{-1}^1 \Big| \sum_{T < \lambda_n \leq X} r_n e^{i\lambda_n V} e^{2\pi i \lambda_n t} \Big|^2 dy \ll \int_{-\infty}^\infty \Big| \sum_{\substack{T < \lambda_n \leq X \\ t < \lambda_n \leq t + \frac{1}{2}}} r_n e^{i\lambda_n V} e^{2\pi i \lambda_n t} \Big|^2 dt$$

$$\ll \int_{-\infty}^\infty \Big(\sum_{\substack{T < \lambda_n \leq X \\ t < \lambda_n \leq t+1}} |r_n| \Big)^2 dt.$$

In the last integral, t satisfies $T - 1 \leq t \leq X$. From (1.9) and $\beta > 1/2$, we have

$$\int_{-\infty}^\infty \Big(\sum_{\substack{T < \lambda_n \leq X \\ t < \lambda_n \leq t+1}} |r_n| \Big)^2 dt \leq \int_{T-1}^X \Big(\sum_{t < \lambda_n \leq t+1} |r_n| \Big)^2 dt \ll \int_{T-1}^X \frac{(\log t)^{2\gamma}}{t^{2\beta}} dt \ll \frac{(\log T)^{2\gamma}}{T^{2\beta-1}}.$$

So (3.1) holds for $\delta = 2\gamma$ and $\eta = 2\beta - 1$.

Next assume that (b) holds. Note that, by dyadic summations, (1.10) and $\alpha \geq \beta$ imply

$$\sum_{\lambda_n \leq T} |r_n| \ll T^{\alpha-\beta}(\log T)^{\gamma+1}.$$

Thus, by partial summation, we conclude that if $\kappa > \alpha - \beta$ and $\nu > 0$, then

$$\sum_{\lambda_n > T} \frac{|r_n|(\log \lambda_n)^\nu}{\lambda_n^\kappa} \ll \frac{(\log T)^{\gamma+\nu+1}}{T^{\kappa-\alpha+\beta}}. \tag{3.2}$$

Since $|z|^2 = z\bar{z}$, we have

$$\int_V^{V+1} \Big| \sum_{T < \lambda_n \leq X} r_n e^{iy\lambda_n} \Big|^2 dy = \sum_{T < \lambda_n \leq X} \sum_{T < \lambda_m \leq X} r_n \overline{r_m} \int_V^{V+1} e^{iy(\lambda_n - \lambda_m)} dy$$

$$\ll \sum_{T < \lambda_n \leq X} \sum_{T < \lambda_m \leq X} |r_n r_m| \min\Big(1, \frac{1}{|\lambda_n - \lambda_m|} \Big) = \Sigma_1 + \Sigma_2,$$

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where Σ_1 is the sum of those terms for which we have $|\lambda_n - \lambda_m| < 1$, and Σ_2 is the sum of the rest of the terms. For Σ_1, by employing (1.10) and (3.2) we have

$$\Sigma_1 \leq \sum_{T < \lambda_n \leq X} |r_n| \sum_{\lambda_n - 1 < \lambda_m < \lambda_n + 1} |r_m| \ll \sum_{\lambda_n > T} \frac{|r_n|(\log \lambda_n)^\gamma}{\lambda_n^\beta} \ll \frac{(\log T)^{2\gamma + 1}}{T^{2\beta - \alpha}}. \qquad (3.3)$$

Note that the last inequality is justified since (1.11) implies that $\beta > \alpha - \beta$. To study Σ_2, we define for any $T \geq 1$

$$S_T(U) = \sum_{\substack{\lambda_m > T \\ |U - \lambda_m| \geq 1}} \frac{|r_m|}{|U - \lambda_m|},$$

where $U \geq T$. Then we can write

$$\Sigma_2 = \sum_{T < \lambda_n \leq X} |r_n| \sum_{\substack{T < \lambda_m \leq X \\ |\lambda_n - \lambda_m| \geq 1}} \frac{|r_m|}{|\lambda_n - \lambda_m|} \leq \sum_{T < \lambda_n \leq X} |r_n| S_T(\lambda_n).$$

We determine an upper bound for $S_T(U)$ as follows. Let $0 < c < 1$ and $T \geq 1$ be fixed. For any number $U \geq T$ consider the set of numbers U^c, $U - U^c$, and $U - 1$. Either of the following cases occurs

$$T \leq U^c, \quad U^c < T \leq U - U^c, \quad U - U^c < T \leq U - 1, \quad \text{or} \quad U - 1 < T \leq U.$$

Suppose that the first case happens, i.e. $T \leq U^c$. Then

$$S_T(U) = \left(\sum_{T < \lambda_m \leq U^c} + \sum_{U^c < \lambda_m \leq U - U^c} + \sum_{U - U^c < \lambda_m \leq U - 1} + \sum_{U + 1 \leq \lambda_m \leq U + U^c} + \sum_{U + U^c < \lambda_m \leq 2U} + \sum_{\lambda_m > 2U} \right) \frac{|r_m|}{|U - \lambda_m|}.$$

Denote these six sums by $\sigma_1, \ldots, \sigma_6$. Then, by applying (1.10), we deduce

$$\sigma_1 \leq \frac{1}{U - U^c} \sum_{T < \lambda_m \leq U^c} |r_m| \ll \frac{(U^c - T)^\alpha (\log U)^\gamma}{(U - U^c) T^\beta} \ll \frac{(\log U)^\gamma}{U^{1 - c\alpha}},$$

$$\sigma_2 \leq \frac{1}{U^c} \sum_{U^c < \lambda_m \leq U - U^c} |r_m| \ll \frac{1}{U^c} \frac{(U - 2U^c)^\alpha (\log(U - U^c))^\gamma}{(U^c)^\beta} \ll \frac{(\log U)^\gamma}{U^{c + c\beta - \alpha}},$$

$$\sigma_3 \leq \sum_{U - U^c < \lambda_m \leq U - 1} |r_m| \ll \frac{(U^c)^\alpha (\log U)^\gamma}{(U - U^c)^\beta} \ll \frac{(\log U)^\gamma}{U^{\beta - c\alpha}},$$

$$\sigma_4 \leq \sum_{U + 1 \leq \lambda_m \leq U + U^c} |r_m| \ll \frac{(U^c)^\alpha (\log U)^\gamma}{U^\beta} \ll \frac{(\log U)^\gamma}{U^{\beta - c\alpha}},$$

and

$$\sigma_5 \leq \frac{1}{U^c} \sum_{U + U^c < \lambda_m \leq 2U} |r_m| \ll \frac{1}{U^c} \frac{U^\alpha (\log U)^\gamma}{U^\beta} \ll \frac{(\log U)^\gamma}{U^{c + \beta - \alpha}}.$$

For σ_6, we divide the interval of summation into subintervals $(2^k U, 2^{k+1} U]$ to get

$$\sigma_6 \leq \sum_{k=1}^{\infty} \frac{1}{(2^k - 1)U} \sum_{2^k U < \lambda_m \leq 2^{k+1} U} |r_m| \ll \left(\sum_{k=1}^{\infty} \frac{k^\gamma}{2^{k(\beta + 1 - \alpha)}} \right) \frac{(\log U)^\gamma}{U^{\beta + 1 - \alpha}} \ll \frac{(\log U)^\gamma}{U^{\beta + 1 - \alpha}},$$

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which is justified since $\alpha < \beta + 1$ by (1.11). We observe that $\sigma_6 \ll \sigma_5 \ll \sigma_2$, $\sigma_4 \ll \sigma_3$, and $\sigma_1 \ll \sigma_3$ since $\beta \leq 1$. Thus we have

$$S_T(U) \ll \sigma_2 + \sigma_3 \ll \frac{(\log U)^\gamma}{U^{c+c\beta-\alpha}} + \frac{(\log U)^\gamma}{U^{\beta-c\alpha}}.$$

In the last inequality we choose $c = \frac{\alpha+\beta}{\alpha+\beta+1}$ and hence if $T \leq U^c$, then

$$S_T(U) \ll \frac{(\log U)^\gamma}{U^\xi},$$

where

$$\xi = \frac{\beta^2 - \alpha^2 + \beta}{\alpha + \beta + 1}.$$

By similar arguments, we find the same bound for $S_T(U)$ in the three other cases of (3). Condition $\alpha^2 + \alpha/2 < \beta^2 + \beta$ yields $\xi > \alpha - \beta$. Hence (3.2) implies

$$\Sigma_2 \ll \sum_{\lambda_n > T} |r_n| S_T(\lambda_n) \ll \frac{(\log T)^{2\gamma+1}}{T^{\xi-\alpha+\beta}}, \tag{3.4}$$

where

$$\xi - \alpha + \beta = \frac{2(\beta^2 - \alpha^2) + (2\beta - \alpha)}{\alpha + \beta + 1}.$$

By (3.3) and (3.4), we have

$$\Sigma_1 + \Sigma_2 \ll \frac{(\log T)^{2\gamma+1}}{T^{\xi-\alpha+\beta}},$$

since $\alpha, \beta > 0$ implies that $\xi - \alpha + \beta < 2\beta - \alpha$. Thus (3.1) holds for $\delta = 2\gamma + 1$ and $\eta = \xi - \alpha + \beta$.

Now we show that (3.1) together with (1.8) imply that $\phi(y)$ is a B^2-almost periodic function. It follows from (1.7) that for $e^Y > T \geq X_0$ and $y \geq y_0$,

$$\phi(y) - c - \Re\left(\sum_{\lambda_n \leq T} r_n e^{i\lambda_n y}\right) = \Re\left(\sum_{T < \lambda_n \leq e^Y} r_n e^{i\lambda_n y}\right) + \mathcal{E}(y, e^Y).$$

Then, by employing (3.1) and (1.8), we obtain

$$\limsup_{Y \to \infty} \frac{1}{Y} \int_{y_0}^Y \left|\phi(y) - c - \Re\left(\sum_{\lambda_n \leq T} r_n e^{i\lambda_n y}\right)\right|^2 dy$$

$$\ll \limsup_{Y \to \infty} \frac{1}{Y} \int_{y_0}^Y \left|\sum_{T < \lambda_n \leq e^Y} r_n e^{iy\lambda_n}\right|^2 dy + \lim_{Y \to \infty} \frac{1}{Y} \int_{y_0}^Y |\mathcal{E}(y, e^Y)|^2 dy$$

$$\ll \limsup_{Y \to \infty} \frac{1}{Y} \sum_{j=0}^{\lfloor Y-y_0 \rfloor} \int_{y_0+j}^{y_0+j+1} \left|\sum_{T < \lambda_n \leq e^Y} r_n e^{iy\lambda_n}\right|^2 dy \ll \frac{(\log T)^\delta}{T^\eta}.$$

This inequality together with

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^{y_0} \left|\phi(y) - c - \Re\left(\sum_{\lambda_n \leq T} r_n e^{i\lambda_n y}\right)\right|^2 dy = 0$$

imply that $\phi(y)$ is B^2-almost periodic. Hence, the theorem follows from Theorem 2.9. $\qquad\square$

We next prove Corollary 1.3.

Proof of Corollary 1.3. (a) Since $r_n \ll \lambda_n^{-\beta}$ then (1.12) implies that

$$\sum_{T < \lambda_n \leq T+1} |r_n| \ll \frac{\log T}{T^\beta}.$$

Now Theorem 1.2(a) implies the result.
(b) By partial summation, using (1.13) and $\theta < 2$, we have

$$\sum_{\lambda_n \geq S} |r_n|^2 = 2 \int_S^\infty \left(\sum_{\lambda_n \leq t} \lambda_n^2 |r_n|^2 \right) \frac{dt}{t^3} + \lim_{X \to \infty} X^{-2} \left(\sum_{\lambda_n \leq X} \lambda_n^2 |r_n|^2 \right) - S^{-2} \left(\sum_{\lambda_n \leq S} \lambda_n^2 |r_n|^2 \right)$$

$$\ll \int_S^\infty t^{\theta-3} dt + S^{\theta-2} \ll S^{\theta-2}.$$

By employing this bound, (1.12), and Cauchy's inequality, we have

$$\sum_{S < \lambda_n \leq T} |r_n| \ll \frac{(T-S)^{\frac{1}{2}} (\log T)^{\frac{1}{2}}}{S^{1-\frac{\theta}{2}}}.$$

Now we choose $\alpha = 1/2$, $\beta = 1 - \theta/2$, and $\gamma = 1/2$, and employ Theorem 1.2. If $1 \leq \theta < 3 - \sqrt{3}$ the conditions given in (b) in Theorem 1.2 are satisfied. Note that this also implies the result for $0 \leq \theta < 1$ since in this case $\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^\theta \leq T$. $\qquad\square$

4. APPLICATIONS OF THE MAIN THEOREM

In this section, by applying Theorem 1.4, we prove Corollaries 1.5, 1.6, and 1.8.

3.1. Proof of Corollary 1.5.

Error term of the prime number theorem for automorphic L-functions. Let π be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_\mathbb{Q})$ and let $L(s,\pi)$ be the automorphic L-function attached to π. We follow the notation in the introduction. For $\delta > 0$ let

$$\mathbb{C}(\delta) = \mathbb{C} \setminus \{z \in \mathbb{C} \mid |z + \mu_\pi(j) + 2k| \leq \delta, \ 1 \leq j \leq d, \ k \geq 0\}.$$

We need the following lemma.

Lemma 4.1. *(i) Let $\sigma \leq -1/2$. Then for all $s = \sigma + it \in \mathbb{C}(\delta)$,*

$$\frac{L'(s,\pi)}{L(s,\pi)} \ll \log |s|.$$

(ii) For any integer $m \geq 2$, there is T_m with $m \leq T_m \leq m+1$ such that

$$\frac{L'(\sigma \pm iT_m, \pi)}{L(\sigma \pm iT_m, \pi)} \ll \log^2 T_m$$

uniformly for $-2 \leq \sigma \leq 2$.
(iii) For $T \geq 2$, the number $N(T, \pi)$ of the zeros of $L(s, \pi)$ in the region $0 \leq \Re(s) \leq 1$, $|\Im(s)| \leq T$ satisfies

$$N(T+1, \pi) - N(T, \pi) \ll \log T$$

and

$$N(T, \pi) \ll T \log T.$$

(iv) There is a constant $0 \leq \theta < 1/2$ such that for all $1 \leq j \leq d$,

$$|\alpha_\pi(p,j)| \leq p^\theta \text{ and } |\Re(\mu_\pi(j))| \leq \theta. \tag{4.1}$$

Proof. For (i) see [32, p. 177] for GL_2 automorphic L-functions. The general case is similar. See [27, Lemma 4.3(a)(d)] for (ii) and (iii). For (iv) see [38, p. 275]. Note that in (ii), and (iii) the implied constants depend on π, and in (i) the implied constant depends on δ and π. $\qquad\square$

We now establish an explicit formula for $\psi(x, \pi)$.

Proposition 4.2. *Let θ be the constant given in (4.1). For all $x > 1$ and $T \geq 2$ we have*

$$\psi(x,\pi) - \delta(x,\pi) = R_\pi - \sum_{|\Im(\rho)| \leq T} \frac{x^\rho}{\rho} + O\Big(\frac{x^{\theta+1}\log^2 x}{T} + x^\theta \log x + \frac{x\log^2 T}{T\log x} + \frac{x\log T}{T}\Big), \tag{4.2}$$

where ρ runs over the nontrivial zeros of $L(s,\pi)$ with $|\Im(\rho)| \leq T$, and

$$R_\pi = \begin{cases} -\dfrac{L'(0,\pi)}{L(0,\pi)} & \text{if } L(0,\pi) \neq 0, \\[2ex] -\log x - \dfrac{L''(0,\pi)}{2L'(0,\pi)} & \text{if } L(0,\pi) = 0. \end{cases}$$

The implied constant in (4.2) depends on δ in Lemma 4.1 and π.

Proof. Recall that for $\Re(s) > 1$, we have

$$-\frac{L'(s,\pi)}{L(s,\pi)} = \sum_{n=1}^\infty \frac{\Lambda(n)a_\pi(n)}{n^s}.$$

From (1.16) and Lemma 4.1(iv) we conclude that $|a_\pi(n)| \leq dn^\theta$. Let $c = 1 + 1/\log x$, and T_m be as in Lemma 4.1(ii). By Perron's formula [40, p. 70, Lemma 3.19] we obtain

$$\psi(x,\pi) = \frac{1}{2\pi i}\int_{c-iT_m}^{c+iT_m}\Big(-\frac{L'(s,\pi)}{L(s,\pi)}\Big)\frac{x^s}{s}ds + O\Big(\frac{x^{\theta+1}\log^2 x}{T} + x^\theta\log x\Big). \tag{4.3}$$

Let $U < -1/2$ and $\delta > 0$ be such that $U \pm it \in \mathbb{C}(\delta)$ for $t \in [-T_m, T_m]$. Consider the contour which consists of the rectangle $\mathcal{C}$ with vertices $c + iT_m, c - iT_m, U + iT_m, U - iT_m$. By the residue theorem, we have

$$\frac{1}{2\pi i}\int_{c-iT_m}^{c+iT_m}\Big(-\frac{L'(s,\pi)}{L(s,\pi)}\Big)\frac{x^s}{s}ds = \delta(x,\pi) + R_\pi - \sum_{\substack{U \leq \Re(\mu) \leq \theta \\ |\Im(\mu)| \leq T}} \frac{x^\mu}{\mu} - \sum_{\substack{0 \leq \Re(\rho) \leq 1 \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho}$$

$$+ \frac{1}{2\pi i}\Big(\int_{c-iT_m}^{U-iT_m} + \int_{U-iT_m}^{U+iT_m} + \int_{U+iT_m}^{c+iT_m}\Big)\Big(-\frac{L'(s,\pi)}{L(s,\pi)}\Big)\frac{x^s}{s}ds, \tag{4.4}$$

where the first and the second sums run over the trivial and the non-trivial zeros of $L(s,\pi)$ inside the rectangle $\mathcal{C}$, respectively. If we follow the argument in [32, pp. 174–178] and employ Lemma 4.1(i) and (ii), we get the following estimates for the integrals on the right-hand side of (4.4). We have

$$\frac{1}{2\pi i}\Big(\int_{c-iT_m}^{U-iT_m} + \int_{U+iT_m}^{c+iT_m}\Big)\Big(-\frac{L'(s,\pi)}{L(s,\pi)}\Big)\frac{x^s}{s}ds \ll \frac{x\log^2 T_m}{T_m\log x},$$

and

$$\frac{1}{2\pi i}\int_{U-iT_m}^{U+iT_m}\left(-\frac{L'(s,\pi)}{L(s,\pi)}\right)\frac{x^s}{s}ds \ll \frac{T_m x^U \log|U+iT_m|}{|U|}.$$

Now we let $U \to -\infty$ through admissible values and note that $(T_m x^U \log|U+iT_m|)/|U| \to 0$. Moreover, Lemma 4.1(iv) implies

$$\sum_{\substack{-\infty \leq \Re(\mu) \leq \theta \\ |\Im(\mu)| \leq T}} \frac{x^\mu}{\mu} \ll x^\theta \left(1 + \sum_{\substack{1 \leq j \leq d \\ k>0}} \frac{x^{-2k}}{|\Re(\mu_\pi(j))-2k|}\right) \ll x^\theta.$$

Inserting the above estimates in (4.4) together with (4.3) establishes (4.2) in the case $T = T_m$. Now note that if we change T_m by an arbitrary $T \in [m, m+1]$, then we have the same estimate as in (4.2), since by Lemma 4.1(iii), we have

$$\sum_{\substack{0 \leq \Re(\rho) \leq 1 \\ T_m \leq |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} + \sum_{\substack{0 \leq \Re(\rho) \leq 1 \\ T \leq |\Im(\rho)| \leq T_m}} \frac{x^\rho}{\rho} \ll \frac{x \log T}{T}.$$

This completes the proof. $\qquad\square$

We now show that, under the assumption of the generalized Riemann hypothesis,

$$E_1(y,\pi) = \frac{\psi(e^y,\pi) - \delta(e^y,\pi)}{e^{y/2}}$$

has a limiting distribution. By pairing the conjugate zeros $\rho = 1/2 + i\gamma$ and $\bar\rho = 1/2 - i\gamma$ in (4.2), for $y > 0$ and $X \geq 2$, we get

$$E_1(y,\pi) = -2\mathrm{ord}_{s=1/2}L(s,\pi) + \Re\left(\sum_{0<\gamma\leq X}\frac{-2e^{i\gamma y}}{\rho}\right) + \mathcal{E}_\pi(y,X),$$

where $\mathrm{ord}_{s=1/2}L(s,\pi)$ is equal to the multiplicity of the zero of $L(s,\pi)$ at $s = 1/2$ if $L(1/2,\pi) = 0$ and $\mathrm{ord}_{s=1/2}L(s,\pi) = 0$ otherwise, and $\mathcal{E}_\pi(y,X)$ satisfies

$$\mathcal{E}_\pi(y,X) = O\left(\frac{y^2 e^{y(1/2+\theta)}}{X} + ye^{y(\theta-1/2)} + \frac{e^{y/2}\log^2 X}{yX} + \frac{e^{y/2}\log X}{X}\right).$$

Note that Condition (1.8) for $y_0 > 0$ is satisfied for $\mathcal{E}_\pi(e^y, e^Y)$. Setting $r_n = -2/\rho_n$ and $\lambda_n = \Im(\rho_n)$ where the non-trivial zeros of $L(s,\pi)$ are labelled $(\rho_n)_{n\in\mathbb{N}}$, we obtain from Lemma 4.1(iii)

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 = \sum_{0<\gamma\leq T}\frac{4\gamma^2}{|\rho|^2} \ll \sum_{0<\gamma\leq T} 1 \ll T\log T.$$

Hence, assuming the generalized Riemann hypothesis for $L(s,\pi)$, Corollary 1.3(b) implies that $E_1(y,\pi)$ has a limiting distribution.

3.2. Proof of Corollary 1.6.

Before proceeding we require the following two lemmas. The first lemma derives an explicit formula for sums of the shape $\sum_{n \leq x} a_n n^{-\alpha}$.

Lemma 4.3. *Let $(a_n)_{n \in \mathbb{N}}$ be a bounded sequence. Assume there exist complex functions $F(w)$ and $G(w)$ such that $\sum_{n=1}^{\infty} a_n n^{-w} = \frac{F(w)}{G(w)}$ for $\Re(w) > 1$. Let $x > 1$, $\alpha \in [0,1]$, $\beta \in \mathbb{R}$, $c = 1 - \alpha + 1/\log x$, $b \neq \alpha$, and $b < c + \alpha$.*
Assume the following three conditions hold:
(i) For any $t > 0$, within and on the box $\mathcal{B}_t$ with vertices $c + \alpha + it, b + it, b - it, c + \alpha - it$, $F(s)$ is either holomorphic or it has a simple pole of residue d_0 at $s_0 \in (b, c + \alpha)$, and $G(s)$ is holomorphic with simple zeros at $\rho_1, \ldots, \rho_J$ inside $\mathcal{B}_t$ and each different from s_0. In the case $F(s)$ has a simple pole at $s = s_0$, let d_1 be the value of $F(s) - d_0/(s - s_0)$ at $s = s_0$.
(ii) There exists $\delta = \delta(b) \in \mathbb{R} - \{0\}$ such that

$$\frac{F(b + it)}{G(b + it)} = O((|t| + 1)^{\delta}).$$

(iii) There exists an increasing sequence of positive numbers $(T_m)_{m \in \mathbb{N}}$ tending to infinity such that

$$\frac{F(\sigma \pm iT_m)}{G(\sigma \pm iT_m)} = O(T_m^{\beta}),$$

uniformly on $b \leq \sigma \leq c + \alpha$.
Then for $\alpha \in [0,1]$ and $x > 1$,

$$\sum_{n \leq x} \frac{a_n}{n^{\alpha}} = R_{\alpha, s_0}(x) + \sum_{\substack{j=1 \\ \rho_j \in \mathcal{B}_t}}^{J} \frac{F(\rho_j) x^{\rho_j - \alpha}}{(\rho_j - \alpha) G'(\rho_j)} + O\left(\frac{x^{1-\alpha} \log x}{T_m} + \frac{x^c T_m^{\beta - 1}}{\log x} + x^{b-\alpha}(T_m^{\delta} + 1) + x^{-\alpha}\right),$$

where

$$R_{\alpha, s_0}(x) = \begin{cases} 0 & \text{if} \quad \alpha, s_0 \notin (b, c + \alpha), \\ \dfrac{F(\alpha)}{G(\alpha)} & \text{if} \quad \alpha \in (b, c + \alpha), \ s_0 \notin (b, c + \alpha), \\ \dfrac{d_0 x^{s_0 - \alpha}}{(s_0 - \alpha) G(s_0)} & \text{if} \quad \alpha \notin (b, c + \alpha), \ s_0 \in (b, c + \alpha), \\ \dfrac{d_0 x^{s_0 - \alpha}}{(s_0 - \alpha) G(s_0)} + \dfrac{F(\alpha)}{G(\alpha)} & \text{if} \quad \alpha, s_0 \in (b, c + \alpha), \ \alpha \neq s_0, \\ \dfrac{d_0 \log x}{G(s_0)} + \dfrac{d_1}{G(s_0)} - \dfrac{d_0 G'(s_0)}{G^2(s_0)} & \text{if} \quad \alpha, s_0 \in (b, c + \alpha), \ \alpha = s_0. \end{cases} \quad (4.5)$$

Proof. Applying Perron's formula [40, p. 70, Lemma 3.19] with $c = 1 - \alpha + 1/\log x$ gives

$$\sum_{n \leq x} \frac{a_n}{n^{\alpha}} = \frac{1}{2\pi i} \int_{c - iT_m}^{c + iT_m} \frac{F(s + \alpha)}{G(s + \alpha)} \frac{x^s}{s} ds + O\left(\frac{x^{1-\alpha} \log x}{T_m} + x^{-\alpha}\right). \quad (4.6)$$

If we replace s by $s - \alpha$ in the integral, we obtain

$$\frac{1}{2\pi i} \int_{c - iT_m}^{c + iT_m} \frac{F(s + \alpha)}{G(s + \alpha)} \frac{x^s}{s} ds = \frac{1}{2\pi i} \int_{c + \alpha - iT_m}^{c + \alpha + iT_m} \frac{F(s)}{G(s)} \frac{x^{s - \alpha}}{s - \alpha} ds.$$

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Cauchy's residue theorem and (i) imply

$$\frac{1}{2\pi i}\int_{c+\alpha-iT_m}^{c+\alpha+iT_m}\frac{F(s)}{G(s)}\frac{x^{s-\alpha}}{s-\alpha}ds = R_{\alpha,s_0}(x) + \sum_{\substack{j=1 \\ \rho_j\in\mathcal{B}_{T_m}}}^{J}\frac{F(\rho_j)x^{\rho_j-\alpha}}{(\rho_j-\alpha)G'(\rho_j)}$$

$$+ \frac{1}{2\pi i}\left(\int_{c+\alpha-iT_m}^{b-iT_m} + \int_{b-iT_m}^{b+iT_m} + \int_{b+iT_m}^{c+\alpha+iT_m}\right)\frac{F(s)}{G(s)}\frac{x^{s-\alpha}}{s-\alpha}ds, \quad (4.7)$$

where $R_{\alpha,s_0}(x)$ equals the sum of the residues at $s = s_0$ and $s = \alpha$, and the sum appears from the residues at the zeros of $G(s)$. Taking into account the various cases for α and s_0, a simple residue calculation yields (4.5). From assumptions (ii) and (iii) we obtain

$$\frac{1}{2\pi i}\int_{b-iT_m}^{b+iT_m}\frac{F(s)}{G(s)}\frac{x^{s-\alpha}}{s-\alpha}ds \ll x^{b-\alpha}(T_m^\delta + 1), \quad (4.8)$$

$$\frac{1}{2\pi i}\int_{b+iT_m}^{c+\alpha+iT_m}\frac{F(s)}{G(s)}\frac{x^{s-\alpha}}{s-\alpha}ds \ll \frac{x^c T_m^{\beta-1}}{\log x}, \quad (4.9)$$

and similarly

$$\frac{1}{2\pi i}\int_{c+\alpha-iT_m}^{b-iT_m}\frac{F(s)}{G(s)}\frac{x^{s-\alpha}}{s-\alpha}ds \ll \frac{x^c T_m^{\beta-1}}{\log x}. \quad (4.10)$$

The result follows by combining (4.6), (4.7), (4.8), (4.9), and (4.10). $\qquad\square$

In the previous lemma, a convenient sequence $(T_m)_{m\in\mathbb{N}}$ of reals is chosen so that $F(s)/G(s)$ is not too large on the contour $\Im(s) = T_m$. Consequently, in the explicit formula for $\sum_{n\le x} a_n n^{-\alpha}$, the sum over ρ_j is constrained by the condition $|\Im(\rho_j)| \le T_m$. The next lemma allows us to replace this condition by $|\Im(\rho_j)| \le T$ for any $T \ge 1$.

Lemma 4.4. *Let $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be sequences and let $x, c_1,$ and c_2 be positive reals. Let $T, T' \in [1,\infty)$ such that $|T - T'| \le 1$. Assume that for $t \ge 1$ we have*

$$\sum_{\lambda_n\le t}|z_n|^2 \ll t^{c_1} \quad (4.11)$$

and

$$\sum_{t<\lambda_n\le t+1} 1 \ll (\log t)^{c_2}. \quad (4.12)$$

Then

$$\sum_{\lambda_n\le T'}\frac{z_n x^{\frac{1}{2}+i\lambda_n}}{\frac{1}{2}+i\lambda_n} = \sum_{\lambda_n\le T}\frac{z_n x^{\frac{1}{2}+i\lambda_n}}{\frac{1}{2}+i\lambda_n} + O\left(x^{\frac{1}{2}}T^{(c_1-2)/2}(\log T)^{c_2/2}\right).$$

Proof. We begin by assuming $T - 1 \le T' \le T$. By the Cauchy-Schwarz inequality

$$\left|\sum_{T'<\lambda_n\le T}\frac{z_n x^{\frac{1}{2}+i\lambda_n}}{\frac{1}{2}+i\lambda_n}\right| \le x^{\frac{1}{2}}\left(\sum_{T'<\lambda_n\le T}\left|\frac{z_n}{\frac{1}{2}+i\lambda_n}\right|^2\right)^{1/2}\left(\sum_{T'<\lambda_n\le T}1\right)^{1/2} \ll x^{\frac{1}{2}}T^{\frac{c_1-2}{2}}(\log T)^{c_2/2}$$

$$(4.13)$$

by (4.11) and (4.12). In the case $T < T' \le T + 1$, we obtain the same bound as (4.13). $\qquad\square$

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We now prove Corollary 1.6. In each part of this corollary, we shall apply Corollary 1.3(b) to establish the existence of the limiting distribution.

(i) Weighted Sums of the Möbius Function. In this proof we assume the Riemann hypothesis and assumption (1.17). We shall show that $E_2(y, \alpha)$, defined in (1.19), possesses a limiting distribution. We start by establishing an explicit formula for

$$M_\alpha(x) = \sum_{n \leq x} \frac{\mu(n)}{n^\alpha}.$$

We first consider the case $\alpha \neq 0$. Let $0 < b < \min(1/2, \alpha)$ and $0 < \varepsilon < 1/2 - b$. Under the assumption of the Riemann hypothesis, there exists a sequence $(T_m)_{m \in \mathbb{N}}$, where $T_m \in [m-1, m]$, such that

$$|\zeta(\sigma + iT_m)|^{-1} \ll T_m^\varepsilon \tag{4.14}$$

uniformly for $-1 \leq \sigma \leq 2$ (see [40, pp. 357–358]). Moreover, for any $\varepsilon > 0$, we have

$$|\zeta(b + it)|^{-1} \ll |t|^{-1/2 + b + \varepsilon}$$

for $|t| \geq 1$ (see [31, Corollary 10.5 and Theorems 13.18 and 13.23]). By taking $F(s) = 1$, $G(s) = \zeta(s)$, $a_n = \mu(n)$, $\beta = \varepsilon$, and $\delta = -1/2 + b + \varepsilon$ in Lemma 4.3 we derive

$$M_\alpha(x) = \frac{1}{\zeta(\alpha)} + \sum_{|\gamma| \leq T_m} \frac{x^{\rho - \alpha}}{(\rho - \alpha)\zeta'(\rho)} + O\left(\frac{x^{1-\alpha} \log x}{T_m} + \frac{x^{1-\alpha}}{T_m^{1-\varepsilon} \log x} + x^{b-\alpha} \right), \tag{4.15}$$

where ρ ranges over the non-trivial zeros of $\zeta(s)$. Let $T \geq 1$ and $m \geq 1$ be the natural number such that $T \in [m-1, m]$. Label the non-trivial zeros of $\zeta(s)$ with positive imaginary part in non-decreasing order by $(\rho_n)_{n \in \mathbb{N}}$. An application of Lemma 4.4 with $\lambda_n = \Im(\rho_n)$, $z_n = \zeta'(\rho_n)^{-1}$, $c_1 = \theta$, $c_2 = 1$, $\beta = 1/2$, T, and $T' = T_m$ implies that

$$\sum_{|\gamma| \leq T_m} \frac{x^{\rho - \alpha}}{(\rho - \alpha)\zeta'(\rho)} = \sum_{|\gamma| \leq T} \frac{x^{\rho - \alpha}}{(\rho - \alpha)\zeta'(\rho)} + O\left(x^{1/2 - \alpha} T^{(\theta - 2)/2} (\log T)^{1/2} \right). \tag{4.16}$$

Substituting (4.16) in (4.15), for $\alpha \neq 0$, we have

$$M_\alpha(x) = \frac{1}{\zeta(\alpha)} + \sum_{|\gamma| \leq T} \frac{x^{\rho - \alpha}}{(\rho - \alpha)\zeta'(\rho)}$$
$$+ O\left(\frac{x^{1-\alpha} \log x}{T} + \frac{x^{1-\alpha}}{T^{1-\varepsilon} \log x} + x^{1/2 - \alpha} \left(T^{\theta - 2} \log T \right)^{1/2} + x^{b-\alpha} \right), \tag{4.17}$$

valid for $x > 1$ and $T \geq 1$. If $\alpha = 0$, we let $0 < b < 1/2$. Then similarly we have

$$M_0(x) = \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho \zeta'(\rho)} + O\left(\frac{x \log x}{T} + \frac{x}{T^{1-\varepsilon} \log x} + x^{1/2} \left(T^{\theta - 2} \log T \right)^{1/2} + x^b \right). \tag{4.18}$$

We now analyze $E_2(y, \alpha)$ in the cases $\alpha \in (0, 1/2)$, $\alpha \in (1/2, 1]$, $\alpha = 0$, and $\alpha = 1/2$.

For $0 < \alpha < 1/2$, by (4.17), for $X \geq 1$ and $y > 0$, we have

$$E_2(y, \alpha) = \frac{1}{e^{y(1/2-\alpha)}\zeta(\alpha)} + e^{y(-1/2+\alpha)} \sum_{|\gamma| \leq X} \frac{e^{y(\rho-\alpha)}}{(\rho-\alpha)\zeta'(\rho)}$$

$$+ O\left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\varepsilon}} + \left(X^{\theta-2}\log X\right)^{1/2} + \frac{1}{e^{y(1/2-b)}}\right).$$

Thus

$$E_2(y, \alpha) = \Re\left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)}\right) + \mathcal{E}_{\mu,\alpha}(y, X),$$

where

$$\mathcal{E}_{\mu,\alpha}(y, X) = O\left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\varepsilon}} + \left(X^{\theta-2}\log X\right)^{1/2} + \frac{1}{e^{y(1/2-\alpha)}}\right).$$

Note that in this case the term $e^{y(\alpha-1/2)}$ in $\mathcal{E}_{\mu,\alpha}(y, X)$ comes from the term $e^{y(\alpha-1/2)}/\zeta(\alpha)$ in $E_2(y, \alpha)$, since we chose $b < \alpha$.

For $\frac{1}{2} < \alpha \leq 1$, we recall that $E_2(y, \alpha) = e^{y(-1/2+\alpha)}\left(M_\alpha(e^y) - 1/\zeta(\alpha)\right)$. By (4.17) and by pairing conjugate zeros, we obtain

$$E_2(y, \alpha) = \Re\left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)}\right) + \mathcal{E}_{\mu,\alpha}(y, X),$$

for $X \geq 1$ and $y > 0$, where

$$\mathcal{E}_{\mu,\alpha}(y, X) = O\left(\frac{ye^{y/2}}{X} + \frac{e^{y/2}}{yX^{1-\varepsilon}} + \left(X^{\theta-2}\log X\right)^{1/2} + \frac{1}{e^{y(1/2-b)}}\right). \tag{4.19}$$

For $\alpha = 0$, from (4.18) we have

$$E_2(y, 0) = \Re\left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{\rho\zeta'(\rho)}\right) + \mathcal{E}_{\mu,0}(y, X),$$

where $\mathcal{E}_{\mu,0}(y, X)$ satisfies (4.19).

Finally, for $\alpha = 1/2$, (4.17) implies

$$E_2(y, 1/2) = \frac{1}{\zeta(\frac{1}{2})} + \Re\left(\sum_{0 < \gamma \leq X} \frac{2e^{iy\gamma}}{(\rho-1/2)\zeta'(\rho)}\right) + \mathcal{E}_{\mu,1/2}(y, X),$$

where $\mathcal{E}_{\mu,1/2}(y, X)$ is bounded as (4.19).

Note that $\mathcal{E}_{\mu,\alpha}(y, e^Y)$ satisfies (1.8) for $y_0 > 0$, for any $\alpha \in [0, 1]$. Setting $r_n = 2/(\rho_n - \alpha)\zeta'(\rho)$ and $\lambda_n = \Im(\rho_n)$, it follows from (1.17) that

$$\sum_{\lambda_n \leq T} \lambda_n^2|r_n|^2 = \sum_{0 < \gamma \leq T} \frac{4\gamma^2}{|(\rho-\alpha)\zeta'(\rho)|^2} \ll T^\theta.$$

Thus Corollary 1.3(b) implies that, under the assumptions of the Riemann hypothesis for $\zeta(s)$ and (1.17), $E_2(y, \alpha)$ has a limiting distribution.

(ii) Weighted Sums of the Liouville Function. In this part, we show that $E_3(y, \alpha)$, defined by (1.20), possesses a limiting distribution. We begin by establishing an explicit formula for $L_\alpha(x) =$

$\sum_{n \leq x} \lambda(n) n^{-\alpha}$. Assume the Riemann hypothesis for $\zeta(s)$ and (1.17). For $\alpha \in (0,1]$ and $x > 1$, let

$$R_{\alpha,s_0}(x) = \begin{cases} \frac{x^{1/2-\alpha}}{(1-2\alpha)\zeta(1/2)} + \frac{\zeta(2\alpha)}{\zeta(\alpha)} & \text{if} \quad \alpha \neq 1/2, \\ \frac{\log x}{2\zeta(1/2)} + \frac{\gamma_0}{\zeta(1/2)} - \frac{\zeta'(1/2)}{2\zeta(1/2)^2} & \text{if} \quad \alpha = 1/2, \end{cases}$$

where γ_0 is Euler's constant. Let $0 < \epsilon < b < \min(1/4, \alpha)$. Then we have

$$\left| \frac{\zeta(2(b+it))}{\zeta(b+it)} \right| \ll |t|^{-b+\varepsilon}$$

for all $|t| \geq 1$, and

$$\left| \frac{\zeta(2(\sigma + iT_m))}{\zeta(\sigma + iT_m)} \right| \ll T_m^{1/2-2b+\varepsilon}$$

uniformly for $b \leq \sigma \leq c + \alpha$ (see [31, Corollary 10.5 and Theorems 13.18 and 13.23]), where $(T_m)_{m \in \mathbb{N}}$ is the sequence introduced in (4.14). Set $F(s) = \zeta(2s)$, $G(s) = \zeta(s)$, $z_m = \lambda(m)$, $\beta = 1/2 - 2b + \varepsilon$, and $\delta = -b + \varepsilon$. Then if $\alpha \neq 0$, Lemmas 4.3 and 4.4 imply that, for $x > 1$ and $T \geq 1$,

$$L_\alpha(x) = R_{\alpha,s_0}(x) + \sum_{|\gamma| \leq T} \frac{x^{\rho-\alpha}}{\rho - \alpha} \frac{\zeta(2\rho)}{\zeta'(\rho)}$$

$$+ O\left(\frac{x^{1-\alpha} \log x}{T} + \frac{x^{1-\alpha} T^{-1/2-2b+\varepsilon}}{\log x} + x^{1/2-\alpha} \left(T^{\theta-2} \log T \right)^{1/2} + x^{b-\alpha} \right). \quad (4.20)$$

If $\alpha = 0$, we let $0 < \epsilon < b < 1/4$. Similarly, we have

$$L_0(x) = \frac{x^{1/2}}{\zeta(1/2)} + \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \frac{\zeta(2\rho)}{\zeta'(\rho)} + O\left(\frac{x \log x}{T} + \frac{x T^{-1/2-2b+\varepsilon}}{\log x} + x^{1/2} \left(T^{\theta-2} \log T \right)^{1/2} + x^b \right).$$

$$(4.21)$$

For $\alpha \in [0,1]$, let

$$C_\alpha = \begin{cases} \frac{1}{(1-2\alpha)\zeta(1/2)} & \text{if} \quad 0 \leq \alpha < 1/2 \text{ or } 1/2 < \alpha \leq 1, \\ \frac{\gamma_0}{\zeta(1/2)} - \frac{\zeta'(1/2)}{2\zeta(1/2)^2} & \text{if} \quad \alpha = 1/2. \end{cases}$$

Then (4.20) and (4.21) imply that, for $y > 0$ and $X \geq 1$,

$$\begin{aligned} E_3(y, \alpha) &= C_\alpha + e^{y(-1/2+\alpha)} \sum_{|\gamma| \leq X} \frac{\zeta(2\rho) e^{y(\rho-\alpha)}}{(\rho-\alpha)\zeta'(\rho)} + \mathcal{E}_{\lambda,\alpha}(y, X) \\ &= C_\alpha + \sum_{|\gamma| \leq X} \frac{\zeta(2\rho) e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} + \mathcal{E}_{\lambda,\alpha}(y, X) \\ &= C_\alpha + \Re\left(\sum_{0 < \gamma \leq X} \frac{2\zeta(2\rho) e^{iy\gamma}}{(\rho-\alpha)\zeta'(\rho)} \right) + \mathcal{E}_{\lambda,\alpha}(y, X), \end{aligned}$$

where

$$\mathcal{E}_{\lambda,\alpha}(y, X) \ll \frac{y e^{y/2}}{X} + \frac{e^{y/2} X^{-1/2-2b+\varepsilon}}{y} + \left(X^{\theta-2} \log X \right)^{1/2} + \frac{1}{e^{y(1/2-b)}}.$$

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Observe that (1.8) for $y_0 > 0$ holds for $\mathcal{E}_{\lambda,\alpha}(y, e^Y)$. Since $r_n = 2\zeta(2\rho_n)/(\rho_n - \alpha)\zeta'(\rho)$ and $\lambda_n = \Im(\rho_n)$, it follows from (1.17) that

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 = \sum_{0 < \gamma \leq T} \frac{4\gamma^2 |\zeta(2\rho)|^2}{|(\rho - \alpha)\zeta'(\rho)|^2} \ll \sum_{0 < \gamma \leq T} \frac{4\gamma^2 (\log \gamma)^{3/2+\varepsilon}}{|(\rho - \alpha)\zeta'(\rho)|^2} \ll T^\theta (\log T)^{3/2+\varepsilon}.$$

Note that, in the previous inequalities we have used the fact that $\zeta(1 + it) = O\big((\log t)^{3/4+\varepsilon}\big)$ (see [40, Theorem 6.14]). Hence by Corollary 1.3(b), under the assumptions of the Riemann hypothesis for $\zeta(s)$ and (1.17), $E_3(y, \alpha)$ has a limiting distribution.

(iii) The Summatory Function of the Möbius Function in Arithmetic Progressions. In this part we prove the existence of a limiting distribution for $E_4(y; q, a)$ defined in (1.21). We first establish an explicit formula for

$$M(x; q, a) := \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \mu(n),$$

where $q \geq 2$ and $(a, q) = 1$. Let $0 < b < 1/2$ and $0 < \varepsilon < 1/2 - b$. Assume the generalized Riemann hypothesis for Dirichlet L-functions modulo q and (1.18). An argument analogous to the proof of the existence of the sequence $(T_m)_{m \in \mathbb{N}}$ introduced in (4.14) may be carried out for Dirichlet L-functions. Following the proof of [31, Theorem 13.22], we are able to show that the generalized Riemann hypothesis for Dirichlet L-functions implies that there is a sequence $(T_{m,\chi})_{m \in \mathbb{N}}$, where $T_{m,\chi} \in [m - 1, m]$, such that

$$|L(\sigma + iT_{m,\chi}, \chi)|^{-1} \ll T_{m,\chi}^\varepsilon$$

uniformly for $-1 \leq \sigma \leq 2$. Moreover, for any $\varepsilon > 0$, we have

$$|L(b + it, \chi)|^{-1} \ll |t|^{-1/2+b+\varepsilon}$$

(see [31, Corollary 10.10 and p. 445, Exercises 8 and 10]). The orthogonality relation for characters asserts that

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)}\chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q}, \\ 0 & \text{otherwise}, \end{cases}$$

(see [31, p. 122]). Thus

$$M(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{n \leq x} \mu(n)\chi(n). \tag{4.22}$$

Let $F(s) = 1$, $G(s) = L(s, \chi)$, $z_n = \mu(n)\chi(n)$, $\beta = \varepsilon$, $\delta = -1/2 + b + \varepsilon$, and $\alpha = 0$. Then by applying a slight variant of Lemma 4.3, which takes into the consideration the potential pole of $1/L(s, \chi)$ at $s = 1/2$, and Lemma 4.4, we obtain, for $x > 1$ and $T \geq 1$,

$$\sum_{n \leq x} \mu(n)\chi(n) = \operatorname{Res}_{s=\frac{1}{2}}\left(\frac{x^s}{L(s, \chi)s}\right) + \sum_{\substack{|\gamma_\chi| \leq T \\ \gamma_\chi \neq 0}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)}$$

$$+ O\left(\frac{x \log x}{T} + \frac{x}{T^{1-\varepsilon} \log x} + x^{1/2}(T^{\theta-2} \log T)^{1/2} + x^b\right),$$

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where $\mathrm{Res}_{s=\frac{1}{2}}(.)$ denotes the residue at $s = 1/2$. Substituting this in (4.22) implies that, for $x > 1$ and $T \geq 1$,

$$M(x; q, a) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ L(1/2, \chi) = 0}} \overline{\chi(a)} \mathrm{Res}_{s=\frac{1}{2}} \left(\frac{x^s}{L(s, \chi)s} \right) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{\substack{|\gamma_\chi| \leq T \\ \gamma_\chi \neq 0}} \frac{x^{\rho_\chi}}{\rho_\chi L'(\rho_\chi, \chi)}$$

$$+ O \left(\frac{x \log x}{T} + \frac{x}{T^{1-\varepsilon} \log x} + x^{1/2} (T^{\theta-2} \log T)^{1/2} + x^b \right).$$

Assuming the generalized Riemann hypothesis for Dirichlet L-functions modulo q and (1.18), it follows that, for $y > 0$ and $X \geq 1$,

$$E_4(y; q, a) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ L(1/2, \chi) = 0}} \overline{\chi(a)} \mathrm{Res}_{s=\frac{1}{2}} \left(\frac{e^{ys}}{L(s, \chi)s} \right)$$

$$+ \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{\substack{|\gamma_\chi| \leq X \\ \gamma_\chi \neq 0}} \frac{e^{iy\gamma_\chi}}{\rho_\chi L'(\rho_\chi, \chi)} + \mathcal{E}_{\mu,q,a}(y, X), \tag{4.23}$$

where

$$\mathcal{E}_{\mu,q,a}(y, X) \ll \frac{y e^{y/2}}{X} + \frac{e^{y/2}}{y X^{1-\varepsilon}} + \frac{(\log X)^{1/2}}{X^{1-\theta/2}} + \frac{1}{e^{y(1/2-b)}}.$$

Let $(\lambda_n)_{n \in \mathbb{N}}$ be the non-decreasing sequence that consists of all the numbers $\gamma_\chi > 0$ satisfying $L(1/2 + i\gamma_\chi, \chi) = 0$, for some Dirichlet character $\chi \bmod q$, and let $(r_n)_{n \in \mathbb{N}}$ be defined as

$$r_n = \frac{2 \, \overline{\chi_{\lambda_n}(a)}}{\varphi(q)(1/2 + i\lambda_n)L'(1/2 + i\lambda_n, \chi_{\lambda_n})},$$

where χ_{λ_n} is the character which corresponds to λ_n. We can rewrite (4.23) in the form of

$$E_4(y; q, a) = \frac{1}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ L(1/2, \chi) = 0}} \overline{\chi(a)} \mathrm{Res}_{s=\frac{1}{2}} \left(\frac{e^{ys}}{L(s, \chi)s} \right) + \Re \left(\sum_{\lambda_n < X} r_n e^{iy\lambda_n} \right) + \mathcal{E}_{\mu,q,a}(y, X).$$

Observe that (1.8) for $y_0 > 0$ holds for $\mathcal{E}_{\mu,q,a}(y, e^Y)$ and (1.18) implies

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll T^\theta,$$

for $1 \leq \theta < 3 - \sqrt{3}$. Hence Corollary 1.3(b) implies that, under the assumptions of the generalized Riemann hypothesis for Dirichlet L-functions modulo q and (1.18), $E_4(y; q, a)$ has a limiting distribution.

3.2. Proof of Corollary 1.8.

Chebotarev's Density Theorem. Let K/k be a normal extension of number fields with corresponding Galois group G. We shall consider the squaring function $\mathrm{sq} : G \to G$ given by $\mathrm{sq}(x) = x^2$. For a conjugacy class C of G, let $A_1, \ldots, A_t$ be the conjugacy classes which satisfy $A_i^2 \subseteq C$. We observe that

$$\mathrm{sq}^{-1}(C) = \bigcup_{i=1}^{t} A_i$$

and define

$$c(G, C) = -1 + \frac{|\mathrm{sq}^{-1}(C)|}{|C|} + 2 \sum_{\chi \neq \chi_0} \overline{\chi(C)} \mathrm{ord}_{s=1/2} L(s, \chi, K/k)$$

where χ ranges over the irreducible characters of G and χ_0 denotes the trivial character. It was proven in [34, pp. 71–73] that the generalized Riemann hypothesis and Artin's holomorphy conjecture imply that, for $x > 1$, $T \geq 1$ and $1 \leq j \leq r$,

$$\frac{\log x}{\sqrt{x}} \left(\frac{|G|}{|C_j|} \pi_{C_j}(x) - \pi_k(x) \right) =$$

$$- c(G, C_j) - \sum_{\chi \neq \chi_0} \overline{\chi(C_j)} \left(\sum_{0 < |\gamma_\chi| \leq T} \frac{x^{i\gamma_\chi}}{1/2 + i\gamma_\chi} \right) + O\left(\frac{x^{1/2} \log^2(xT)}{T} + \frac{1}{\log x} \right), \quad (4.24)$$

where for each χ, $\rho_\chi = 1/2 + i\gamma_\chi$ runs over the non-trivial zeros of $L(s, \chi, K/k)$. In this formula, the term $c(G, C_j)$ is the number field analogue of the constant term $c(q, a)$ which appears in the Chebyshev bias phenomenon. Let $(\lambda_n)_{n \in \mathbb{N}}$ be the non-decreasing sequence that consists of all the numbers $\gamma_\chi > 0$ which satisfy $L(1/2 + i\gamma_\chi, \chi, K/k) = 0$ for some $\chi \neq \chi_0$. Suppose that χ_n is the character which corresponds to λ_n, and for $1 \leq j \leq r$ set $r_{j,n} = -2\overline{\chi_n(C_j)}/(1/2 + i\lambda_n)$. Then (4.24) implies that

$$E_5^{(j)}(y) := \left(\frac{|G|}{|C_j|} \pi_{C_j}(e^y) - \pi_k(e^y) \right) y e^{-y/2} = -c(G, C_j) + \Re\left(\sum_{0 < \lambda_n \leq X} r_{j,n} e^{iy\lambda_n} \right) + \mathcal{E}_{G;C_j}(y, X),$$

where

$$\mathcal{E}_{G;C_j}(y, X) = O\left(\frac{e^{y/2} \log^2(e^y X)}{X} + \frac{1}{y} \right).$$

Observe that Condition (1.8) for $y_0 > 0$ holds for $\mathcal{E}_{G;C_j}(y, e^Y)$ and by [23, Theorem 5.8] we have

$$\sum_{\lambda_n \leq T} \lambda_n^2 |r_n|^2 \ll \sum_{\lambda_n \leq T} 1 \ll T \log T.$$

Therefore, Theorem 1.4 implies that, under the assumptions of generalized Riemann hypothesis and Artin's holomorphy conjecture, $\vec{E}_5(y) = \left(E_5^{(1)}(y), \ldots, E_5^{(r)}(y) \right)$ has a limiting distribution.

5. CALCULATION OF THE FOURIER TRANSFORM $\widehat{\mu}$

Proof of Theorem 1.9. Let $\vec{r}_m = \left(r_1(\lambda_m), \ldots, r_\ell(\lambda_m) \right)$ and $N \in \mathbb{N}$. By Proposition 2.4, the vector-valued function

$$\vec{P}(y) = \left(c_1 + \Re\left(\sum_{m=1}^N r_1(\lambda_m) e^{iy\lambda_m} \right), \ldots, c_\ell + \Re\left(\sum_{m=1}^N r_\ell(\lambda_m) e^{iy\lambda_m} \right) \right)$$

has a limiting distribution μ_N. Since $\{\lambda_1, \cdots, \lambda_N\}$ is linearly independent then by the Kronecker-Weyl theorem [20, Chapter 1] we have

$$\lim_{Y \to \infty} \frac{1}{Y} \int_0^Y g\left(\frac{y\lambda_1}{2\pi}, \cdots, \frac{y\lambda_N}{2\pi} \right) dy = \int_{\mathbb{T}^N} g(a) d\omega,$$

where $g : \mathbb{R}^N \to \mathbb{R}$ is any continuous function of period 1 in each of its variables and $d\omega(\theta_1, \ldots, \theta_N)$ is the normalized Haar measure on $\mathbb{T}^N$ which is equal to the Lebesgue measure $d\theta_1 \ldots d\theta_N$ on $\mathbb{T}^N$. Hence, by taking $f(t_1, \ldots, t_\ell) = \exp(-i\sum_{k=1}^{\ell} \xi_k t_k)$ and $A = \mathbb{T}^N$ in (2.2), we obtain

$$\int_{\mathbb{R}^\ell} e^{-i\sum_{k=1}^{\ell} \xi_k t_k} d\mu_N(t_1, \ldots, t_\ell)$$

$$= \int_{\mathbb{T}^N} \exp\left(-i\sum_{k=1}^{\ell}\left[c_k + \Re\left(\sum_{m=1}^N r_k(\lambda_m)e^{2\pi i\theta_m}\right)\right]\xi_k\right) d\omega(\theta_1, \ldots, \theta_N)$$

$$= e^{-i\sum_{k=1}^{\ell} c_k \xi_k} \int_{\mathbb{T}^N} \exp\left(-i\Re\left(\sum_{m=1}^N (\vec{r}_m \cdot \vec{\xi})e^{2\pi i\theta_m}\right)\right) d\theta_1 \ldots d\theta_N$$

$$= e^{-i\sum_{k=1}^{\ell} c_k \xi_k} \times \prod_{m=1}^N \int_0^1 \exp\left(-i\Re\left((\vec{r}_m \cdot \vec{\xi})e^{2\pi i\theta}\right)\right) d\theta. \tag{5.1}$$

Thus, in view of (2.11) and (5.1) we deduce that

$$\hat{\mu}(\vec{\xi}) = \int_{\mathbb{R}^\ell} e^{-i\sum_{k=1}^{\ell} \xi_k t_k} d\mu(t_1, \ldots, t_\ell) = \lim_{N \to \infty} \int_{\mathbb{R}^\ell} e^{-i\sum_{k=1}^{\ell} \xi_k t_k} d\mu_N(t_1, \ldots, t_\ell)$$

$$= e^{-i\sum_{k=1}^{\ell} c_k \xi_k} \times \prod_{m=1}^\infty \int_0^1 \exp\left(-i\Re\left((\vec{r}_m \cdot \vec{\xi})e^{2\pi i\theta}\right)\right) d\theta.$$

If $\vec{r}_m \cdot \vec{\xi} \neq 0$, then

$$\int_0^1 \exp\left(-i\Re\left((\vec{r}_m \cdot \vec{\xi})e^{2\pi i\theta}\right)\right) d\theta = \int_0^1 \exp\left(-i\Re\left(|\vec{r}_m \cdot \vec{\xi}|e^{i(2\pi\theta + \arg(\vec{r}_m \cdot \vec{\xi}))}\right)\right) d\theta$$

$$= \int_0^1 \exp\left(-i|\vec{r}_m \cdot \vec{\xi}| \cos\left(2\pi\theta + \arg(\vec{r}_m \cdot \vec{\xi})\right)\right) d\theta$$

$$= \int_{\arg(\vec{r}_m \cdot \vec{\xi})/2\pi}^{1+\arg(\vec{r}_m \cdot \vec{\xi})/2\pi} \exp\left(-i|\vec{r}_m \cdot \vec{\xi}| \cos(2\pi t)\right) dt$$

$$= \int_0^1 \exp\left(-i|\vec{r}_m \cdot \vec{\xi}| \cos(2\pi t)\right) dt$$

$$= J_0\left(\left|\sum_{k=1}^{\ell} r_k(\lambda_m)\xi_k\right|\right). \tag{5.2}$$

If $\vec{r}_m \cdot \vec{\xi} = 0$, then (5.2) holds trivially. Hence

$$\hat{\mu}(\vec{\xi}) = e^{-i\sum_{k=1}^{\ell} c_k \xi_k} \times \prod_{m=1}^\infty J_0\left(\left|\sum_{k=1}^{\ell} r_k(\lambda_m)\xi_k\right|\right).$$

$\square$

REFERENCES

[1] A. S. Besicovitch, *On generalized almost periodic functions*, Proc. London Math. Soc. **S2-25** no. 1, 1926, 495-512.

[2] P. Billingsley, *Probability and Measure*, John Wiley, New York, 1986.

[3] P. Bleher, *On the distribution of the number of lattice points inside a family of convex ovals*, Duke Math. J. **67** no. 3, 1992, 461-481.

[4] P. Borwein, R. Ferguson, and M. J. Mossinghoff, *Sign changes in sums of the Liouville function*, Math. Comp. **77** no. 263, 2008, 1681-1694.

[5] J. W. S. Cassels and A. Fröhlich, *Algebraic Number Theory*, London Mathematical Society, 2010.

[6] H. Cramér, *Ein Mittlewertsatz in der Primzahltheorie*, Math. Z. **12**, 1922, 147-153.

[7] H. Davenport, *Multiplicative number theory, Second Edition*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1980.

[8] A. Feuerverger and G. Martin, *Biases in the Shanks-Rényi prime number race*, Experiment. Math. **9** no. 4, 2000, 535-570.

[9] D. Fiorilli and G. Martin, *Inequities in the Shanks-Rényi prime number race: an asymptotic formula for the densities*, J. Reine Angew. Math. **676**, 2013, 121-212.

[10] D. Fiorilli, *Highly biased prime number races*, arXiv:1210.6946, 28 pages.

[11] D. Fiorilli, *Elliptic curves of unbounded rank and Chebyshev's bias*, to appear in Int. Math. Res. Not. IMRN, 22 pages.

[12] P. X. Gallagher, *A large sieve density estimate near $\sigma = 1$*, Invent. Math. **11**, 1970, 329-339.

[13] P.X. Gallagher, *Some consequences of the Riemann hypothesis*, Acta Arithmetica **37**, 1980, 339-343.

[14] S. M. Gonek, *The negative moments of the Riemann zeta-function*, Mathematika **36**, 1989, 71–88.

[15] S.M. Gonek, *The second moment of the reciprocal of the Riemann zeta-function and its derivative*, Notes from a talk at MSRI, summer 1999.

[16] A. Granville and K. Soundararajan, *The distribution of values of $L(1, \chi_d)$*, Geom. Funct. Anal. **13** no. 5, 2003, 992-1028.

[17] C. B. Haselgrove, *A disproof of a conjecture of Pólya*, Mathematika **5**, 1985, 141–145.

[18] R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. **60**, 1992, 389–415.

[19] D. Hejhal, *On the distribution of $\log |\zeta'(\frac{1}{2} + it)|$*, Number Theory, trace formula and discrete groups (ed. K. E. Aubert, E. Bombieri, and D. Goldfeld, Academic Press, San Diego), 1989, 343–370.

[20] E. Hlawka, *The Theory of Uniform Distribution*, AB Academic Publishers, Berkhemsted, 1984.

[21] C. P. Hughes, J. P. Keating, and N. O'Connell, *Random matrix theory and the derivative of the Riemann zeta function*, Proc. Roy. Soc. London Ser. A **456**, 2000, 2611–2627.

[22] P. Humphries, *The distribution of weighted sums of the Liouville function and Pólya's conjecture*, Journal of Number Theory **133**, 2013, 545–582.

[23] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society, Colloquium Publications, v. 53, 2004.

[24] B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. **38** no. 1, 1935, 48–88.

[25] Y. Lamzouri, *Distribution of values of L-functions at the edge of the critical strip*, Proc. Lond. Math. Soc. (3) **100** no. 3, 2010, 835–863.

[26] Y. Lamzouri, *Prime number races with three or more competitors*, Math. Ann. **356** no. 3, 2013, 1117–1162.

[27] J. Liu and Y. Ye, *Superposition of zeros of distinct L-functions*, Forum Math. **14**, 2002, 419–455.

[28] J. Liu and Y. Ye, *Perron's formula and the prime number theorem for automorphic L-functions*, Pure Appl. Math. Q. **3** no. 2, 2007, 481–497.

[29] M. B. Milinovich and N. Ng, *A note on a conjecture of Gonek*, Funct. Approx. Comment. Math. **46** no. 2, 2012, 177–187.

[30] I. Molchanov and S. Zuyev, *Advanced Course in Probability: Weak Convergence and Asymptotics*, Spring 2011.

[31] H. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, Cambridge, 2007.

[32] C. J. Moreno, *Explicit formulas in the theory of automorphic forms*, in: Lecture Notes in Math. **626**, Springer, Berlin, 1977, 73–216.

[33] D. W. Morris, *Ratner's Theorem on Unipotent Flows*, Chicago Lectures in Mathematics Series, 2005.

[34] N. Ng, *Limiting distributions and zeros of Artin L-functions*, Ph.D. Thesis, University of British Colombia, 2000.

[35] N. Ng, *The summatory function of the Möbius function*, Proc. London Math. Soc. (3) **89**, 2004, 361–389.

[36] A. M. Odlyzko and A. J. J. te Riele, *Disproof of the Mertens conjecture*, J. reine angew Math. **357**, 1985, 138–160.

[37] M. Rubinstein and P. Sarnak, *Chebyshev's bias*, Experiment. Math. **3** no. 3, 1994, 173–197.

[38] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** no.2, 1996, 269–322.

[39] M. Tanaka, *A numerical investigation on cumulative sum of the Liouville function*, Tokyo J. Math. **3** no. 1, 1980, 187–189.

[40] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1986.

[41] A. Wintner, *On the asymptotic distribution of the remainder term of the prime number theorem*, Amer. J. Math. **57** no. 3, 1935, 534–538.

[42] A. Wintner, *Asymptotic distributions and infinite convolutions*, Notes distributed in the Institute for Advanced Study (Princeton), 1938.

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