Average Values of Symmetric Square L-Functions at $Re(s) = 2$

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Abstract

Let $L_{sym^2(f)}(s)$ be the symmetric square L-function associated to a newform of weight 2 and level N. For N prime, we will derive asymptotic formulae for the average values of $L_{sym^2(f)}(s)$ at a general point on the line $Re(s) = 2$ when f varies over the set of all normalized newforms.

RÉSUMÉ: Soit $L_{sym^2(f)}(s)$ la fonction L du carré symétrique d'une forme primitive de poids 2 et niveau N. Pour N premier, on dérive une formule asymptotique pour les valeurs moyennes de $L_{sym^2(f)}(s)$ en un point général de la droite $Re(s) = 2$ et f variant dans l'ensemble des formes primitives normalisées.

1 Introduction

Many important theorems of number theory are intimately connected with the values of various L-functions at the edge of their critical strips. For example, the distribution of prime numbers in arithmetic progressions is related to the non-vanishing of Dirichlet L-functions on the line $Re(s) = 1$. Another famous example is Dirichlet's class-number formula. Here we are interested in a similar situation in the context of modular L-functions.

Let $S_2(N)$ be the space of cusp forms of weight 2 for $\Gamma_0(N)$ with trivial character. The space $S_2(N)$ has an inner product (Petersson inner product)

$$< f, g >= \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}dxdy$$

where $\mathcal{H}$ denotes the upper half plane. For $f \in S_2(N)$ let

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz),\ e(z) = e^{2\pi iz}$$

be the Fourier expansion of f at $i\infty$ and let $\mathcal{F}_N$ be the set of all normalized ($a_f(1) = 1$) newforms in $S_2(N)$.

The *symmetric square L-function* associated to $f \in \mathcal{F}_N$ is defined (for $Re(s) > 2$) by

$$L_{sym^2(f)}(s) = \zeta_N(2s-2)\sum_{n=1}^{\infty}\frac{a_f(n^2)}{n^s} = \sum_{\substack{d,e \\ (d,N)=1}}\frac{a_f(e^2)}{d^{2s-2}e^s} \tag{1}$$

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where $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p|N$ removed. It is known that $L_{sym^2(f)}(s)$ extends to an entire function (see [4]) and for square free N, it satisfies a functional equation of the form

$$R(s) = A^s\ \Gamma(\frac{s}{2})^2\ \Gamma(\frac{s+1}{2})\ L_{sym^2(f)}(s) = R(3-s), \qquad A = \frac{N}{\pi^{\frac{3}{2}}}. \tag{2}$$

Similar to Dirichlet's class number formula the value of $L_{sym^2(f)}(s)$ at the edge of the critical strip (in this case $s=2$) is of interest. One can show that $L_{sym^2(f)}(2)$ is a constant multiple (depending on N) of the Petersson inner product of f and f, more precisely

$$L_{sym^2(f)}(2) = \frac{8\pi^3\phi(N)}{N^2\prod_{p|N}(1-\frac{1}{p})} < f, f > \tag{3}$$

where ϕ is the Euler totient function. Therefore to study the average values of the Petersson inner product when f varies in $\mathcal{F}_N$, it is enough to find an asymptotic formula for the average values of $L_{sym^2(f)}(2)$. In the case that N is prime and $L_{sym^2(f)}(s)$ satisfies the Lindelöf hypothesis, R. Murty [3] has proved:

Theorem: *If we assume that $L_{sym^2(f)}(\frac{3}{2}+it) \ll (N|t|)^\theta$, for some $\theta > 0$, then for N prime*

$$\sum_{f\in\mathcal{F}_N} L_{sym^2(f)}(2) = \frac{N}{12}\zeta^2(2) + O(N^{\frac{7}{10}+\frac{4}{5}\theta}\log^3 N).$$

In this note we develop a similar asymptotic formula which works unconditionally. Also our method enables us to derive asymptotic formulae for average values of symmetric square L-functions at a general point in the line $Re(s) = 2$. The main observation is a modification of Murty's approximate trace formula (Proposition 1). We employ the recent method of Kowalski (see [2], section 3. 5) to obtain this.

2 An approximate trace formula

In this section we will derive an asymptotic formula for $\sum_{f\in\mathcal{F}_N} a_f(n)$ in term of N. Let T and S be positive and non-integer. We start by considering the integral

$$\frac{1}{2\pi i}\int_{(1)} L_{sym^2(f)}(s+2)T^s\frac{ds}{s} = \sum_{\substack{d^2e<T\\(d,N)=1}} \frac{a_f(e^2)}{d^2e^2} = \sum_{n<T}\frac{g_f(n)}{n^2}$$

(see (1)). Upon moving the line of integration from 1 to -2 and using the functional equation (2), this integral is

$$= L_{sym^2(f)}(2) + \frac{1}{A}\frac{1}{2\pi i}\int_{(-2)} \frac{\Gamma(\frac{1-s}{2})^2\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s+2}{2})^2\Gamma(\frac{s+3}{2})} L_{sym^2(f)}(1-s)\left(\frac{T}{A^2}\right)^s\frac{ds}{s}.$$

Since $A = \frac{N}{\pi^{\frac{3}{2}}}$ and $L_{sym^2(f)}(s)$ is absolutely convergent for $Re(s) > 2$, this identity implies that

$$L_{sym^2(f)}(2) = \sum_{\substack{d^2e<T\\(d,N)=1}} \frac{a_f(e^2)}{d^2e^2} + O(\frac{N^3}{T^2}) = \sum_{\substack{d^2e<S\\(d,N)=1}} \frac{a_f(e^2)}{d^2e^2} + \omega(S,T) + O(\frac{N^3}{T^2}) \tag{4}$$

where $\omega(S,T) = \sum_{S\le n<T}\frac{g_f(n)}{n^2}$.

We use the following three lemmas to get some information about $\sum_{f\in\mathcal{F}_N}\frac{L_{sym^2(f)}(2)}{4\pi<f,f>}a_f(n)$.

Lemma 1

$$\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi < f, f >}a_f(m)a_f(n) = \delta_{mn}\sqrt{m}\sqrt{n} + O(N^{-\frac{3}{2}}(m,n)^{\frac{1}{2}}mn).$$

Proof: See [3], Proposition 1. □

Lemma 2

$$\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi<f,f>}\sum_{\substack{d^2e<S\\(d,N)=1}}\frac{a_f(e^2)}{d^2e^2}a_f(n)=\left(\zeta_N(2)+S^{-\frac{1}{2}}n^{\frac{1}{4}}\right)\delta_{n=\square}+O\left(N^{-\frac{3}{2}}n\boldsymbol{d}(n)S\right)$$

where $\boldsymbol{d}(n)$ *is the number of divisors of* n *and* $\delta_{n=\square}=1$ *if* n *is a square and is zero otherwise.*

Proof: This follows from Lemma 1 and familiar estimates of analytic number theory, see [3] p. 272 for details. □

Lemma 3 *For any positive integer* r*, we have*

$$\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi<f,f>}\omega(S,T)a_f(n)\ll\left(\boldsymbol{d}(n)\sqrt{n}(\log N)^{\frac{1}{2r}}N^{-\frac{1}{2r}}\right)\left(\sum_{f\in\mathcal{F}_N}(\omega(S,T))^{2r}\right)^{\frac{1}{2r}}$$

Proof: From the Hölder inequality, for any r and s that $\frac{1}{2r}+\frac{1}{s}=1$, we have

$$\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi<f,f>}\omega(S,T)a_f(n)\leq\left(\sum_{f\in\mathcal{F}_N}(\omega(S,T))^{2r}\right)^{\frac{1}{2r}}\left(\sum_{f\in\mathcal{F}_N}\left(\frac{1}{4\pi<f,f>}|a_f(n)|\right)^{s}\right)^{\frac{1}{s}}.$$

Since $|a_f(n)|\leq\mathbf{d}(n)\sqrt{n}$ (Deligne's bound) and $\frac{1}{4\pi<f,f>}\ll\frac{\log N}{N}$ (see [1] Proposition 4), we have

$$\left(\sum_{f\in\mathcal{F}_N}\left(\frac{1}{4\pi<f,f>}|a_f(n)|\right)^{s}\right)^{\frac{1}{s}}=\left(\sum_{f\in\mathcal{F}_N}\left(\frac{1}{4\pi<f,f>}|a_f(n)|\right)^{s-1}\left(\frac{1}{4\pi<f,f>}|a_f(n)|\right)\right)^{\frac{1}{s}}$$

$$\ll\left(\frac{\mathbf{d}(n)\sqrt{n}\log N}{N}\right)^{\frac{1}{2r}}\left(\mathbf{d}(n)\sqrt{n}\right)^{\frac{1}{s}}=\mathbf{d}(n)\sqrt{n}(\log N)^{\frac{1}{2r}}N^{-\frac{1}{2r}}.\ \square$$

Now we can state and prove the main result of this section.

Proposition 1 *For prime* N*, we have*

$$\sum_{f\in\mathcal{F}_N}a_f(n)=\frac{N-1}{12}\delta_{n=\square}+O\left(N^{-\frac{1}{2}+\delta}n\boldsymbol{d}(n)+\sqrt{n}\boldsymbol{d}(n)N^{1-\frac{1}{2r}}(\log N)^{C}\right)$$

where $0<\delta<1$, $r\geq\frac{11}{\delta}$ *is an integer,* $C>0$ *is a constant depending on* δ *and* r*.*

Proof: From (3) and (4) we get

$$\sum_{f\in\mathcal{F}_N}a_f(n)=\sum_{f\in\mathcal{F}_N}\frac{L_{sym^2(f)}(2)}{L_{sym^2(f)}(2)}a_f(n)$$

$$=\frac{N}{2\pi^2}\left(\sum_{\substack{d^2e<S\\(d,N)=1}}\frac{1}{d^2e^2}\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi<f,f>}a_f(e^2)a_f(n)+\sum_{f\in\mathcal{F}_N}\frac{1}{4\pi<f,f>}\omega(S,T)a_f(n)\right)+O\left(\frac{N^4}{T^2}\mathbf{d}(n)\sqrt{n}\right)$$

Now by applying Lemma 2 and 3 this expression becomes

$$=\left(\frac{N-1}{12}+\frac{N-1}{12N}+\frac{N}{2\pi^2}S^{-\frac{1}{2}}n^{\frac{1}{4}}\right)\delta_{n=\square}+O\left(N^{-\frac{1}{2}}n\mathbf{d}(n)S\right)$$

$$+O\left(\mathbf{d}(n)\sqrt{n}(\log N)^{\frac{1}{2r}}N^{1-\frac{1}{2r}}\right)\left(\sum_{f\in\mathcal{F}_N}(\omega(S,T))^{2r}\right)^{\frac{1}{2r}}+O\left(\frac{N^4}{T^2}\mathbf{d}(n)\sqrt{n}\right). \tag{5}$$

Let $0<\delta<1$ and let $S=N^{\delta}$, choose $r\geq\frac{11}{\delta}$, then from [2] (see Lemma 4, p. 64), we know that for $T<N^{10}$

$$\left(\sum_{f\in\mathcal{F}_N}(\omega(S,T))^{2r}\right)^{\frac{1}{2r}}\ll(\log N)^D$$

where D is a positive number which depends on δ. Applying this inequality in (5) and choosing T a non-integer bigger than N^3 in (5) yields the result. $\square$

3 Mean estimate

In the following lemma we give a representation of $L_{sym^2(f)}(s_0)$ as a sum of two absolutely convergent series.

Lemma 4 *For any $x>0$ and $s_0=\sigma_0+it_0\in\mathbb{C}$ where $\sigma_0\geq\frac{3}{2}$, let $\sigma_0<\eta$ and*

$$W(s_0,x)=\frac{1}{2\pi i}\int_{(\eta)}\pi^{-\frac{3}{2}s}\,\Gamma(\frac{s+s_0}{2})^2\,\Gamma(\frac{s+s_0+1}{2})\,x^s\frac{ds}{s},\quad I_f(s_0,x)=\sum_{\substack{d,e\\(d,N)=1}}\frac{a_f(e^2)}{e^{s_0}d^{2s_0-2}}W(s_0,\frac{x}{d^2e})$$

where $f\in\mathcal{F}_N$. Then we have

$$\Gamma(\frac{s_0}{2})^2\Gamma(\frac{s_0+1}{2})L_{sym^2(f)}(s_0)=I_f(s_0,x)+\left(\frac{\pi^{\frac{3}{2}}}{N}\right)^{2s_0-3}I_f(3-s_0,\frac{N^2}{x}).$$

Proof: It is similar to the proof of Lemma 3 in [1].

Now we evaluate the values of $L_{sym^2(f)}(s_0)$ on average, where f ranges over all newforms of weight 2 and level N. From Lemma 4 with $x=N$ and Proposition 1, we have

$$\sum_{f\in\mathcal{F}_N}L_{sym^2(f)}(s_0)=\frac{1}{\Gamma(\frac{s_0}{2})^2\Gamma(\frac{s_0+1}{2})}\left(\frac{N-1}{12}\sum_{\substack{d,e\\(d,N)=1}}\frac{1}{e^{s_0}d^{2s_0-2}}W(s_0,\frac{N}{d^2e})\right.$$

$$\left.+\left(\frac{\pi^{\frac{3}{2}}}{N}\right)^{2s_0-3}\frac{N-1}{12}\sum_{\substack{d,e\\(d,N)=1}}\frac{1}{e^{3-s_0}d^{4-2s_0}}W(3-s_0,\frac{N}{d^2e})\right)+S_1+S_2 \tag{6}$$

where

$$S_1\ll\frac{1}{\Gamma}\left(N^{-\frac{1}{2}+\delta}\sum_{\substack{d,e\\(d,N)=1}}\frac{e^2\mathbf{d}(e^2)}{e^{\sigma_0}d^{2\sigma_0-2}}\left|W(s_0,\frac{N}{d^2e})\right|+N^{1-\frac{1}{2r}}(\log N)^C\sum_{\substack{d,e\\(d,N)=1}}\frac{e\mathbf{d}(e^2)}{e^{\sigma_0}d^{2\sigma_0-2}}\left|W(s_0,\frac{N}{d^2e})\right|\right) \tag{7}$$

and $S_2\ll\frac{1}{N^{2\sigma_0-3}\Gamma}$

$$\left(N^{-\frac{1}{2}+\delta}\sum_{\substack{d,e\\(d,N)=1}}\frac{e^2\mathbf{d}(e^2)}{e^{3-\sigma_0}d^{4-2\sigma_0}}\left|W(3-s_0,\frac{N}{d^2e})\right|+N^{1-\frac{1}{2r}}(\log N)^C\sum_{\substack{d,e\\(d,N)=1}}\frac{e\mathbf{d}(e^2)}{e^{3-\sigma_0}d^{4-2\sigma_0}}\left|W(3-s_0,\frac{N}{d^2e})\right|\right). \tag{8}$$

Here, $\Gamma=|\Gamma(\frac{s_0}{2})|^2|\Gamma(\frac{s_0+1}{2})|$. Now we apply the following three lemmas to estimate the terms of (6).

Lemma 5 *Let $\sigma_0 > \frac{3}{2}$, then*

$$\sum_{\substack{e,d \\ (d,N)=1}} \frac{1}{e^{s_0} d^{2s_0-2}} W(s_0, \frac{N}{d^2 e}) = \Gamma(\frac{s_0}{2})^2 \Gamma(\frac{s_0+1}{2}) \zeta(s_0) \zeta_N(2s_0-2) + O_{\sigma_0}(N^{\frac{3}{2}-\sigma_0})$$

and

$$\sum_{\substack{e,d \\ (d,N)=1}} \frac{1}{e^{3-s_0} d^{4-2s_0}} W(3-s_0, \frac{N}{d^2 e}) = O_{\sigma_0}\left(N^{\sigma_0-\frac{3}{2}}\right)$$

where $W(s_0, x)$ is defined in Lemma 4.

Proof: From the definition of $W(s_0, x)$ it is clear that

$$\sum_{\substack{e,d \\ (d,N)=1}} \frac{1}{e^{s_0} d^{2s_0-2}} W(s_0, \frac{N}{d^2 e})$$

$$= \sum_{e} \frac{1}{e^{s_0}} \frac{1}{2\pi i} \int_{(\eta)} \pi^{-\frac{3}{2}s}\, \Gamma(\frac{s+s_0}{2})^2 \Gamma(\frac{s+s_0+1}{2}) \zeta_N(2s+2s_0-2) (\frac{N}{e})^s \frac{ds}{s}.$$

By moving the line of integration from (η) to the left of $(\frac{3}{2}-\sigma_0)$, we get the desired result. The second identity proves in a similar way by choosing $\eta > max\{3-\sigma_0, \sigma_0 - \frac{3}{2}\}$ and moving the line of integration of the corresponding integral to the left of $(\sigma_0 - \frac{3}{2})$. □

Lemma 6 $|W(s_0, x)| \leq W(\sigma_0, x)$.

Proof: From the Legendre duplication formula, we have

$$\Gamma(\frac{s+s_0}{2})\Gamma(\frac{s+s_0+1}{2}) = \frac{\sqrt{\pi}}{2^{s+s_0-1}} \Gamma(s+s_0).$$

Now by applying this identity in the definition of $W(s_0, x)$ and writing the Γ functions in terms of integrals, we get

$$W(s_0, x) = \frac{1}{2\pi i} \frac{\sqrt{\pi}}{2^{s_0-1}} \int_{(\eta)} \left(\int_0^\infty \int_0^\infty {t_1}^{\frac{s+s_0}{2}-1} {t_2}^{s+s_0-1} e^{-(t_1+t_2)} dt_1 dt_2 \right) (\frac{\pi^{-\frac{3}{2}} x}{2})^s \frac{ds}{s}.$$

By interchanging the order of integration, we have

$$W(s_0, x) = \frac{\sqrt{\pi}}{2^{s_0-1}} \int_0^\infty {t_1}^{\frac{s_0}{2}-1} e^{-t_1} \left(\int_{\frac{2\pi^{\frac{3}{2}}}{x {t_1}^{\frac{1}{2}}}}^\infty {t_2}^{s_0-1} e^{-t_2} dt_2 \right) dt_1.$$

The result follows by applying the triangle inequality in the above identity. □

Lemma 7 *Let $\alpha < min\{\frac{1+\beta}{2}, \gamma+1\}$, then*

$$\sum_{\substack{d,e \\ (d,N)=1}} \frac{\boldsymbol{d}(e^2)}{e^\alpha d^\beta} W(\gamma, \frac{N}{d^2 e}) \sim \begin{cases} \frac{6}{\pi^2} \frac{\pi^{-\frac{3}{2}(1-\alpha)}}{1-\alpha} \zeta_N(\beta - 2\alpha + 2) \Gamma(\frac{\gamma-\alpha+1}{2})^2 \Gamma(\frac{\gamma-\alpha+2}{2}) N^{1-\alpha} \log^2 N & if \quad \alpha < 1 \\ \frac{6}{\pi^2} \zeta_N(\beta) \Gamma(\frac{\gamma}{2})^2 \Gamma(\frac{\gamma+1}{2}) \log^3 N & if \quad \alpha = 1 \end{cases}.$$

as $N \to \infty$.

Proof: First note that $\sum_{e=1}^{\infty}\frac{\mathbf{d}(e^2)}{e^s} = \frac{\zeta^3(s)}{\zeta(2s)}$ for $Re(s) > 1$. Now by this identity and the definition of $W(.,.)$ the above sum is equal to

$$\sum_{(d,N)=1}\frac{1}{d^{\beta}}\frac{1}{2\pi i}\int_{(\eta)}\pi^{-\frac{3}{2}s}\Gamma(\frac{s+\gamma}{2})^2\Gamma(\frac{s+\gamma+1}{2})\frac{\zeta^3(s+\alpha)}{\zeta(2s+2\alpha)}(\frac{N}{d^2})^s\frac{ds}{s}.$$

Moving the line of integration to the left of $(1-\alpha)$ and calculating the residue at $s = 1-\alpha$ yields the result. □

Now by using Lemma 6 and Lemma 7 in (7) and (8), we get upper bounds for S_1 and S_2. Applying these upper bounds and Lemma 5 in (6) yields

$$\sum_{f\in\mathcal{F}_N} L_{sym^2(f)}(s_0) = \zeta(s_0)\zeta_N(2s_0-2)\frac{N-1}{12}+O_{\sigma_0}\left(N^{\frac{5}{2}-\sigma_0}\right)+O_{\sigma_0}\left(\frac{N^{\frac{5}{2}-\sigma_0+\delta}\log^3 N + N^{3-\sigma_0-\frac{1}{2r}}(\log N)^C}{|\Gamma(\frac{s_0}{2})|^2|\Gamma(\frac{s_0+1}{2})|}\right) \tag{9}$$

where $0<\delta<1$, $r \geq \frac{11}{\delta}$ is an integer and $C > 0$ is a constant depending on δ and r. It is clear that if $\sigma_0 = 2$ the above formula gives us an asymptotic formula, and in this case we can see that the choice of $\delta = \frac{11}{23}$ and $r = 23$ gives the optimal error term, thus we proved the following theorem:

Theorem 1 *Let N be prime, then there exists $B > 0$ such that for any real number t*

$$\sum_{f\in\mathcal{F}_N} L_{sym^2(f)}(2+it) = \zeta(2+it)\zeta_N(2+2it)\frac{N-1}{12} + O\left(\frac{N^{\frac{45}{46}}(\log N)^B}{|\Gamma(\frac{2+it}{2})|^2|\Gamma(\frac{3+it}{2})|}\right).$$

Corollary 1 *Under the assumptions of Theorem 1*

$$\sum_{f\in\mathcal{F}_N} < f,f > = \frac{\pi}{2^7 3^3}N^2 + O\left(N^{\frac{91}{46}}(\log N)^B\right).$$

Proof: In Theorem 1, let $t = 0$ and then use (3) to write $L_{sym^2(f)}(s)$ in terms of $< f,f >$. □

Note: It is worth mentioning that (9) is an asymptotic formula if $\sigma_0 = Re(s_0) > 2 - \frac{1}{46}$.

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