1. (16 pts) How many solutions are there to the equation

\[ x_1 + x_2 + x_3 + x_4 = 17 \]

where \( x_i \) is an integer and \( x_i \geq 0 \) for \( i \in \{1, 2, 3, 4\} \) so that:

(a) There is no additional restriction on \( x_i \) for \( i \in \{1, 2, 3, 4\} \).

(b) \( x_1 \geq 2 \).

(c) \( x_i \geq 1 \) for all \( i \in \{1, 2, 3, 4\} \).

(d) \( 0 \leq x_1 \leq 7 \). Hint: consider placing 17 indistinguishable balls into 4 distinguishable bins (Example 5 of Section 5.5).

**Solution:**

(a) Suppose we have 4 distinguishable boxes. Let \( x_i \) be the number of objects in the \( i^{th} \) box. Then counting the positive integer solutions to the above equation is the same as counting the number of distributions of 17 indistinguishable objects among 4 distinguishable boxes. By a formula from class, this number is \( C(4 + 17 - 1, 17) = C(20, 17) \).

(b) In terms of boxes, the restriction requires that there be at least 2 objects in the \( 2^{nd} \) box. Therefore we place the required 2 objects in the box and must count the number of ways to distribute the remaining \( 17 - 2 = 15 \) indistinguishable objects among 4 distinguishable boxes. The number of ways to do this is \( C(4 + 15 - 1, 15) = C(18, 15) \).

(c) In terms of boxes, the restrictions require that each box holds at least 1 objects. Thus we place the required object in each box and count the number of ways to distribute the remaining \( 17 - 4 = 13 \) indistinguishable objects among 4 distinguishable boxes. The number of ways to do this is \( C(4 + 13 - 1, 13) = C(16, 13) \).

(d) In part (a), we calculated the number of solutions with no additional restrictions on the \( x_i \). To calculate the answer to this question, we count the number of solutions where \( x_1 \geq 8 \) and subtract this number from the number found in part (a). The restriction \( x_1 \geq 8 \) requires that the first box contain at least 8 objects. We place these objects in the box and count the number of ways to distribute the remaining \( 17 - 8 = 9 \) indistinguishable objects among 4 distinguishable boxes. The number of ways to do this is \( C(4 + 9 - 1, 9) = C(12, 9) \). Thus the number of solutions with \( 0 \leq x_1 \leq 7 \) is \( C(20, 17) - C(12, 9) \).

2. (6 pts) How many different ways are there to travel in \( x, y, z \) space from the origin \((0, 0, 0)\) to the point with coordinates \((3, 5, 4)\) by taking steps only in the positive \( x \) or \( y \) or \( z \) direction, one co-ordinate at a time? Hint: to reach \((3, 5, 4)\) you need to take a total of 3 steps along \( x \), 5 steps along \( y \), and 4 steps along \( z \). The order in which you take these steps defines your path.

**Solution:** We will write \( X \) to denote a move in the \( x \) direction, \( Y \) to denote a move in the \( y \) direction and \( Z \) to denote a move in the \( z \) direction. Then each path is uniquely described by a string of 3 \( X \)s, 5 \( Y \)s and 4 \( Z \)s (in some order), where the first letter listed represents the first move made on the path, and so on. We must count the total number of possible strings of 3 \( X \)s, 5 \( Y \)s and 4 \( Z \)s. Each string will contain \( 3 + 5 + 4 = 12 \) letters where the 3 \( X \)s are indistinguishable.
from one another, as are the 5 Ys and 4 Zs. By a formula from class, the total number of paths is

\[
\frac{12!}{3!5!4!}.
\]

3. (12 pts) Compute the probabilities of the following events. You need not carry out all computations involving large numbers (for example, express your solutions in terms of \(8^8\) and not \(16777216 = 8^8\)). Explain your answers.
(a) Flipping a fair coin six times in a row so that heads appears every time.
(b) Dealing a five card poker hand that contains the queen of spades. How about if the hand contains the ace of hearts instead of the queen of spades?
(c) Rolling at least one 6 when a die is rolled four times.

**Solution:**
(a) The probability of a fair coin coming up heads is \(\frac{1}{2}\). Thus the probability of getting 6 heads in a row is \(\left(\frac{1}{2}\right)^6 = \frac{1}{64}\) (since the \(i\)-th coin flip is independent from the \(j\)-th coin flip and the intersection of the events: “1st coin flip is head”, “2nd coin flip is head”, etc is the product of probabilities).
(b) There are 52 cards in a deck. The total number of five card hands is \(\binom{52}{5}\). We now count the number of five card hands that contain the queen of spades. There are 4 remaining cards in the hand, and they are chosen from a total of 51 cards (the queen of spades has already been accounted for). Thus there are \(\binom{51}{4}\) five card hands that contain the queen of spades. The probability that a five card hand contains the queen of spades is \(\frac{\binom{51}{4}}{\binom{52}{5}} = \frac{\binom{51}{4}}{\binom{52}{5}}\). Of course, the probability of the hand containing any other card is the same. (c) We calculate the probability of rolling no 6s when a die is rolled four times. For each roll, the probability of not rolling a 6 is \(\frac{5}{6}\). Thus the probability of rolling no 6s in four rolls is \(\left(\frac{5}{6}\right)^4\) (for the same reason as in Question (a)). Thus the probability of rolling at least one 6 in four rolls is \(1 - \left(\frac{5}{6}\right)^4\).

4. (6 pts) Suppose that \(E\) and \(F\) are events such that \(p(E) = 0.8\) and \(p(F) = 0.6\). Show that \(p(E \cup F) \geq 0.8\) and \(p(E \cap F) \geq 0.4\).

**Solution:** Recall that

\[
p(E \cup F) = p(E) + p(F) - p(E \cap F).
\]

We let \(p(E) = 0.8\) and \(p(F) = 0.6\) and rearrange to get

\[
p(E \cup F) + p(E \cap F) = 1.4. \tag{1}
\]

Clearly \(p(E \cup F) \leq 1\). Using this in equation (1) above, we have

\[
1.4 = p(E \cup F) + p(E \cap F) \leq 1 + p(E \cap F).
\]

Subtracting 1 from both sides yields the required inequality \(p(E \cap F) \geq 0.4\). To show the second inequality, we note that \(p(E \cap F) \leq p(F) = 0.6\) (since the probability of both \(E\) and \(F\) occurring is less than the probability of just \(F\) occurring). Using this in equation (1) above yields

\[
1.4 = p(E \cup F) + p(E \cap F) \leq p(E \cup F) + 0.6.
\]
Subtracting 0.6 from both sides yields the required inequality $p(E \cup F) \geq 0.8$. (This is not the only way to prove the required inequalities).

5. (4 pts) Find the probability of each outcome if a biased die is rolled, if a 3 is twice as likely to appear as each of the other 5 numbers on the die. Explain your reasoning.

Solution: Let $x$ represent the probability that a 1 is rolled on the die. This is also the probability that a 2 is rolled, a 4 is rolled, a 5 is rolled and a 6 is rolled. The probability that a 3 is rolled is $2x$. Since the sum of the probabilities of all possible outcomes must be 1, we have

$$1 = p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = x + x + 2x + x + x + x = 7x.$$  

Thus $7x = 1$ and so $x = \frac{1}{7}$. Therefore $p(1) = p(2) = p(4) = p(5) = p(6) = \frac{1}{7}$ and $p(3) = \frac{2}{7}$.

6. (6 pts) What is the conditional probability that exactly 4 heads appear when a fair coin is flipped 5 times, given that the first flip came up tails?

Solution: Recall that

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$  

Let $E$ represent getting 4 heads when flipping a fair coin 5 times and let $F$ represent the first of 5 coin flips coming up tails. There are $2^5 = 32$ equally possible outcomes when flipping a coin 5 times. The only way to get 4 heads when the first flip is tails is THHHH, so $p(E \cap F) = \frac{1}{32}$. The probability of the first of 5 coin flips coming up tails is $\frac{1}{2}$, so $p(F) = \frac{1}{2}$. Thus

$$p(E|F) = \frac{1/32}{1/2} = \frac{1}{16}.$$