Math 2865 Solutions to Assignment # 3

1) $U = 26!$

Let A_1 be arrangements with MATH
A_2 " " " RUNS
A_3 " " " FROM
A_4 " " " JOE

A_1 = all arrangements having MATH = $23!$
A_2 = " " " RUNS = $23!$
A_3 = " " " FROM = $23!$
A_4 = " " " JOE = $24!$

$A_1 \cap A_2 = 20!$ (MATH and RUNS)
$A_1 \cap A_3 = 20!$ (FROMATH)
$A_1 \cap A_4 = 21!$ (MATH JOE)
$A_2 \cap A_3 = 0$ (cannot happen since R is in RUNS and FROM)
$A_2 \cap A_4 = 21!$ (RUNS JOE)
$A_3 \cap A_4 = 0$ (cannot happen since O is in FROM and JOE)

$A_1 \cap A_2 \cap A_3 = 0$ cannot happen
$A_1 \cap A_3 \cap A_4 = 0$ cannot happen
$A_2 \cap A_3 \cap A_4 = 0$ cannot happen
$A_1 \cap A_2 \cap A_4 = 18!$ (MATH, RUNS, JOE)

$A_1 \cap A_2 \cap A_3 \cap A_4 = 0$ cannot happen

So in total there are

$$26! - (3 \cdot 23! + 24!) + (2 \cdot 20! + 2 \cdot 21!) - 18!$$

#21 We can choose the four integers in $\binom{9}{4}$ ways. Then there are D_5 ways of permuting the remaining 5 integers such that none is in its correct position. Hence, the answer is
$\binom{9}{4} D_5 = 126(44) = 5544$

#3) $(n-1)(D_{n-1} + D_{n-2})$

$$= (n-1)\left\{ (n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right] + (n-2)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots (-1)^{n-2}\frac{1}{(n-2)!}\right]\right\}$$

multiply $(n-1)$ through

$$= n(n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right]$$

to this term add $(-1)^n \frac{1}{n!}$ but don't forget to subtract two

$$-(n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right]$$
$$+ (n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^{n-2}\frac{1}{(n-2)!}\right]$$

most of these cancel out

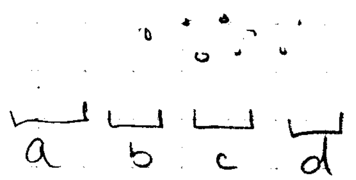
$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right] - \cancel{n!}(-1)^n\frac{1}{\cancel{n!}} - \cancel{(n-1)!}(-1)^{n-1}\frac{1}{\cancel{(n-1)!}}$$

$$= D_n \underbrace{- (-1)^n - (-1)^{n-1}}_{\text{add to } 0} = D_n$$

#4) $S = \{3a, 4b, 5c, 4d\}$ 10 combinations

$T^* = \{\infty a, \infty b, \infty c, \infty d\}$

Use inclusion - exclusion



$A_1 \rightarrow$ set with 4 a's or more

$A_2 \rightarrow$ set with 5 b's or more

$A_3 \rightarrow$ " " 6 c's or more

$A_4 \rightarrow$ " " 5 d's or more

$$|U| = \binom{10+4-1}{10} = \binom{13}{10}$$

$$|A_1| = \binom{6+4-1}{6} = \binom{9}{6}$$ (4 dots in "a" bin, the rest arrange any way)

$$|A_2| = \binom{5+4-1}{5} = \binom{8}{5}$$

$$|A_3| = \binom{4+4-1}{4} = \binom{7}{4}$$

$$|A_4| = \binom{5+4-1}{5} = \binom{8}{5}$$

$$|A_1 \cap A_2| = \binom{1+4-1}{1} = \binom{4}{1}$$ ($4+5=9$ dots in "a" and "b")

$$|A_1 \cap A_3| = 1$$

$$|A_1 \cap A_4| = \binom{1+4-1}{1} = \binom{4}{1}$$

$A_i \cap A_j \cap A_k = 0$

$A_i \cap A_j \cap A_k \cap A_l = 0$

$|A_2 \cap A_3| = 0$

$|A_2 \cap A_4| = 1$

$|A_3 \cap A_4| = 0$

∴ in total

$$\binom{13}{10} - \binom{9}{6} - \binom{8}{5} - \binom{7}{4} - \binom{8}{5} + \binom{4}{1} + 1 + \binom{4}{1} + 1$$

$$= \boxed{65}$$

#5| $a_n = 2a_{n-1} + 3a_{n-2}$, $n \geq 2$, $a_0 = 0$ $a_1 = 8$

the characteristic polynomial is

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0 \Rightarrow$ roots are $x = 3$ and $x = -1$

∴ the solution is of the form

$a_n = c_1(-1)^n + c_2(3^n)$ apply initial conditions.

$$\left.\begin{array}{l} a_0 = 0 = c_1 + c_2 \\ a_1 = 8 = -c_1 + 3c_2 \end{array}\right\} \Rightarrow c_1 = -c_2$$

$c_2 + 3c_2 = 8$

$4c_2 = 8 \Rightarrow \boxed{c_2 = 2}$

$\boxed{c_1 = -2}$

∴ The solution is

$a_n = -2(-1)^n + 2(3^n) = 2\left[(-1)^{n+1} + 3^n\right]$

#6| $a_n = -6a_{n-1} - 9a_{n-2} + n^2 + 3n$, $a_0 = \frac{179}{128}$, $a_1 = \frac{-21}{128}$

<u>General solution</u>

Characteristic polynomial is $x^2 + 6x + 9 = 0$

$(x+3)(x+3) = 0$ ∴ roots are $x = -3$ repeated.

Solution looks like $a_n = c_1(-3)^n + c_2 n(-3)^n$

<u>Particular solution</u> Try $p_n = a + bn + cn^2$

$a + bn + cn^2 = -6[a + b(n-1) + c(n-1)^2] - 9[a + b(n-2) + c(n-2)^2] + n^2 + 3n$

$= -6[a - b + c + (b - 2c)n + cn^2] - 9[a - 2b + 4c + (b - 4c)n + cn^2] + n^2 + 3n$

$= -15a + 24b - 42c + (-15b + 48c + 3)n + (-15c + 1)n^2$

Equate coefficients

$a = -15a + 24b - 42c$
$b = -15b + 48c + 3$

$c = -15c + 1 \Rightarrow 16c = 1 \Rightarrow \boxed{c = 1/16}$

$16b = 48\left(\frac{1}{16}\right) + 3 = 6$

$\Rightarrow \boxed{b = \frac{6}{16} = \frac{3}{8}}$

$16a = 24\left(\frac{3}{8}\right) - 42\left(\frac{1}{16}\right) = \frac{51}{8}$

$\boxed{a = \frac{51}{128}}$

$a_n = p_n + q_n = \frac{51}{128} + \frac{3}{8}n + \frac{1}{16}n^2 + c_1(-3)^n + c_2 n(-3)^n$

Using initial conditions.

$a_0 = \frac{179}{128} = \frac{51}{128} + c_1 \Rightarrow c_1 = \frac{179}{128} - \frac{51}{128} = \boxed{1 = c_1}$

$a_1 = \frac{-21}{128} = \frac{51}{128} + \frac{3}{8} + \frac{1}{16} - 3c_1 - 3c_2$

$\Rightarrow -3c_2 = \frac{-21}{128} - \frac{51}{128} - \frac{3}{8} - \frac{1}{16} + 3 = 2$

$\boxed{c_2 = -\frac{2}{3}}$

$\therefore$ solution is

$a_n = \frac{51}{128} + \frac{3}{8}n + \frac{1}{16}n^2 + (-3)^n - \frac{2}{3}n(-3)^n$