Instructions:
- You have 75 minutes. Please write your name and student number on the front page of the answer sheet.
- Please write your answers clearly on the answer sheet. *You may lose marks if your answers are not clearly stated.* You may use the last pages of the answer sheet for rough work.

Problem 1: (total 12 pts)

We are given a group of n people ($n > 1$) so that any two either mutually know each other or do not know each other ("A knows B but B doesn't know A" is not possible).

(6pts) a. Assume first that every person knows at least someone in the group. Show that there are two people with the same number of acquaintances.

Answer:

If everybody has at least one acquaintance in the group, the number of possible acquaintances for each person in the group belongs to the set $\{1, 2, \ldots n-1\}$. There are n persons, so at least two have the same number of acquaintances (by pigeon hole principle).

(4pts) b. The same as in case a, but without any restriction on the number of acquaintances for each person (in other words, it is possible to have people with zero acquaintances in the group).

Answer:

If two or more persons have zero acquaintances, we are done. If only one person has zero acquaintances then n is at least 3 (why?). We can eliminate her from the group because this will not change the number of acquaintances for the others (they mutually know each other), and we are in case a.

(2pts) c. Will there still be two people with the same number of acquaintances if we remove the condition of mutual acquaintance? For example, Alice knows Bob but Bob does not know Alice. Explain your answer!

Answer:

The statement is false in this case. In the group formed by Alice and Bob, Alice has 1 acquaintance and Bob zero.

Problem 2: (total 12 pts)

5 women, 5 men, and one dog come for a photo session.

(2pts) a. In how many different ways can we arrange them on one row for the photo session? Explain briefly!

Answer:

11! ways

(2pts) b. How many photo arrangements are there if one of the men is taking the picture (not necessarily the same man)? Explain briefly!

Answer:

$5 \cdot 10!$

(2pts) c. How many photo arrangements are there if the men and women are grouped together, but the dog can appear anywhere? Explain your answer!

Answer:

$2 \cdot 11 \cdot (5!)^2$

(2pts) d. After taking the photos, the men, women, and the dog are tired and sit at a round table. How many different table arrangements are there? Explain briefly!

Answer:

10!

(4pts) e. What if no two men and no two women wish to stay next to each other at the table? Explain your answer!

Answer:

$2 \cdot (5!)^2$. (we fix the position of the dog at the table, so around the dog we obtain a linear arrangement; there are two possibilities "M W M W ..." and "W M W M ..."; on each of these positions for men (M) and women (W), the persons permute in 5! different ways)

Problem 3: (total 16 pts)

Alice (A), Bob (B), and Carmen (C) visit aunt Molly for Thanksgiving. Aunt Molly is vegan, she only eats fruits, and all her fruits of the same kind are alike.

(4pts) a. On the first day, aunt Molly gives the children 15 apples. In how many different ways can these apples be distributed to the children? Explain your answer!

Answer:

$\binom{17}{2}$

(6pts) b. On the second day, aunt Molly gives the children a banana and 10 oranges. In how many different ways can these fruits be distributed to the children if Bob is guaranteed the banana? What if Bob receives *at least one* fruit? Explain your answer!

Answer:

In the first case, Bob gets the banana, so 10 oranges are to be distributed to 3 kids in:

$$\binom{12}{2} \text{ ways.}$$

In the second case, we can think of Bob getting the banana or not getting it. If he gets it, we obtain the same number as above. If he doesn't, then 9 oranges are distributed to the children in $\binom{11}{2}$ ways (Bob already has an orange), and then the banana can go to Alice or Carmen. The answer is

$$\binom{12}{2} + 2 \cdot \binom{11}{2} = \frac{16}{5}\binom{11}{2}.$$

(6pts) c. On the third day, aunt Molly turns sour and gives the children 7 grapefruits and 6 limes. In how many different ways can these be distributed? Explain your answer!

Answer:

7 grapefruits to 3 children can be distributed in $\binom{9}{2}$ ways and 6 limes in $\binom{8}{2}$ ways, so the total is

$$\binom{8}{2} \cdot \binom{9}{2}.$$