Non-linear programming (Ch 14)

General NLP:

$$\boxed{\begin{aligned} &\min \ \theta(x) \\ &\text{s.t.} \quad h_i(x) = 0, \quad i \in \{1, m\} \\ &\qquad\ \ g_p(x) \geq 0, \quad p \in \{1, t\} \\ &x \in \mathbb{R}^n \ (\text{vector of decision variables}) \\ &\theta, h_i, g_p : \mathbb{R}^n \to \mathbb{R}, \text{ continuous} \end{aligned}}$$

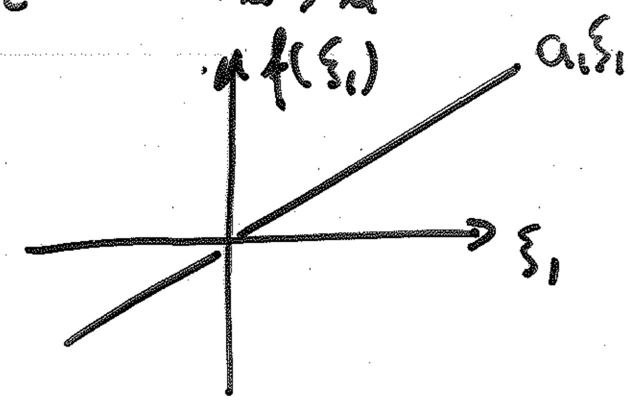
Obs x: continuous variables
(not in $\mathbb{Z}^n$)

NLP (c'ed)

Some examples of typical functions for θ, h_i, g_p.

- linear functions. $x^T = (\xi_1\ \xi_2 \ldots \xi_n)$

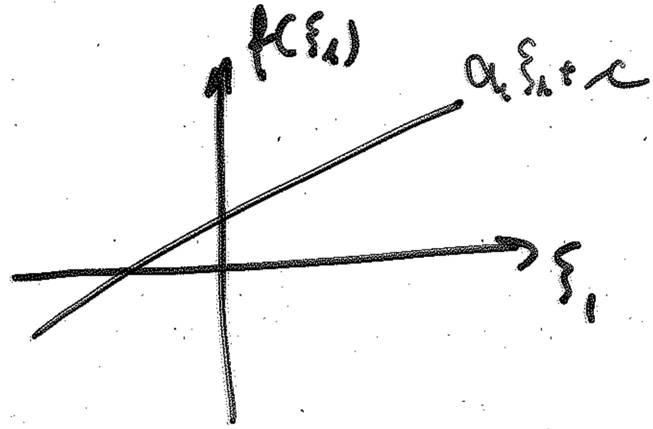
$$f(x) = a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n$$



- affine functions.

$$f(x) = a_1\xi_1 + \ldots + a_n\xi_n + \underline{c}$$

a constant.



MLP (cont'd)

- quadratic form

$$f(x) = \sum_{i,j} a_{ij} \, \xi_i \cdot \xi_j$$

also:

$$f(x) = x^T \cdot D \cdot x$$

$$\begin{array}{cc} & j \\ i & \begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \\ & \vdots & \end{bmatrix} \end{array}$$

<u>Obs</u>: since $\xi_i \cdot \xi_j = \xi_j \cdot \xi_i$

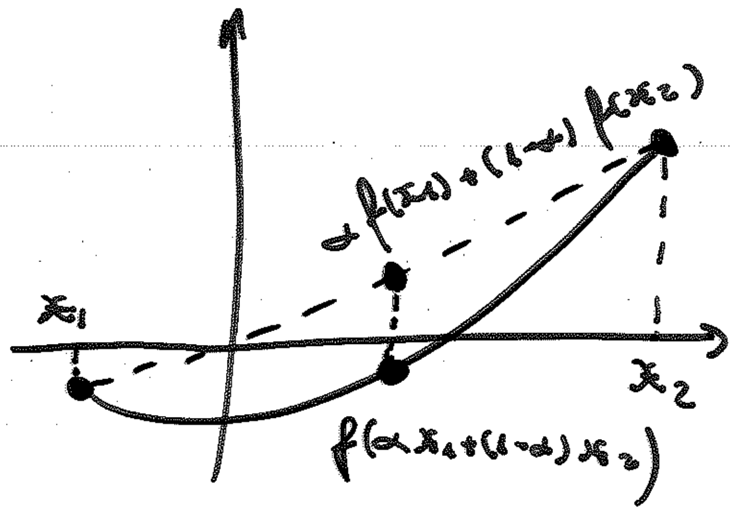
$$\Rightarrow f(x) = x^T \underbrace{(D + D^T) \cdot \tfrac{1}{2}}_{\text{symmetric}} \cdot x$$

- quadratic function
  = quad. form + affine

- convex functions

$\forall\, x_1, x_2 \in \mathbb{R}^n,\ \alpha \in [0,1]$

$$f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$



Convex functions are nice for opt. problems: they have a global min.

Q: When are affine fct & quadratic fct convex?

NLP (c'ed)

A : • affine fct are always convex
(both convex & concave)

$$f(\alpha x_1 + (1-\alpha) x_2) = \alpha f(x_1) + (1-\alpha) f(x_2)$$

• quadratic forms are convex

$f(x) = x^T D x$

iff

D : positive semidefinite

$\boxed{\text{Def}}$ D : positive semidefinite
iff $y^T D y \geq 0 \quad \forall y \in \mathbb{R}^n$

NLP (c'ed)

- $\sum$ of convex functions is a convex function.

  $\Rightarrow f(x) = x^T A x + b^T x + c$ is convex iff A is positive semidefinite.

- in $\mathbb{R}^1$ (one dimension):

  $f(x) = ax^2 + bx + c$ is convex iff $a \geq 0$.

  In general $f(x)$: convex iff $f''(x) \geq 0 \quad \forall x \in \mathbb{R}$.

NLP (c'ed)

- back to $\mathbb{R}^n$.

$f(x)$ is convex iff the Hessian

$$H(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 \xi_1} & \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2} & \cdots & \frac{\partial^2 f}{\partial \xi_1 \partial \xi_n} \\ \frac{\partial^2 f}{\partial \xi_2 \partial \xi_1} & \frac{\partial^2 f}{\partial^2 \xi_2} & \cdots & \frac{\partial^2 f}{\partial \xi_2 \partial \xi_n} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial \xi_n \partial \xi_1} & \frac{\partial^2 f}{\partial \xi_n \partial \xi_2} & \cdots & \frac{\partial^2 f}{\partial^2 \xi_n} \end{pmatrix}$$

... is positive semidefinite $\forall x \in \mathbb{R}^n$

Not easy!

However, we can easily check if f is locally convex @ x^* by checking

$H(x^*) \succeq 0$ (pos. semidef.)

NLP (c'ed)

Checking for $\overbrace{\text{pos (semi) def}}^{P(S)D}$ condition:

M : matrix we check

Let $D = \begin{cases} M & , M \text{ symmetric} \\ M + M^T & , \text{otherwise} \end{cases}$

$$D = \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & & & \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{pmatrix}$$

a) $\exists i \quad \delta_{ii} < 0 \Rightarrow M$ not PSD

$\delta_{ii} \leq 0 \Rightarrow M$ not PD

b)

$$D \xrightarrow[\text{operations}]{\text{row}} \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & & \delta_{1n} \\ 0 & \delta'_{22} & \cdots & & \delta'_{2n} \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \delta'_{ii} & \cdots & \\ \vdots & \vdots & \delta_{i+1,i} & D_i & \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \end{pmatrix}$$

ignore this part

ignore this part

NLP (c'ed)

Procedure for verifying P(S)D :

$\boxed{\text{Fact 1}}$ if D = PD, then $\delta_{ii} > 0$

$\boxed{\text{Fact 2}}$ if D = PSD, then

- $\delta_{ii} \geq 0$
- if $\delta_{ii} = 0$, then rows & col $i = 0$

$\boxed{\text{Fact 3}}$ if D = P(S)D and

$$D \underset{\text{op.}}{\overset{\text{row}}{\rightsquigarrow}} \begin{pmatrix} \delta & d^T \\ 0 & D' \end{pmatrix}, \text{ then } D' \text{ is P(S)D}$$

Algorithm

- check Fact 1 & 2 on D
  (stop if conditions fail. M & D not P(S)D.
- $D \rightsquigarrow \begin{pmatrix} \delta & d^T \\ 0 & D' \end{pmatrix}$

  $D \leftarrow D'$

8bis

HLP (cont'd)

Why does this algorithm work?

The following is NOT a proof, but a hint @ what the proof might look like.

$$\begin{pmatrix} \xi & x^T \end{pmatrix} \begin{pmatrix} \delta & d^T \\ 0 & D' \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix} =$$

$$= \begin{pmatrix} \xi\delta , & \xi d^T + x^T D' \end{pmatrix} \begin{pmatrix} \xi \\ x \end{pmatrix} =$$

$$= \xi^2 \delta + \xi d^T x + x^T D' x \geq 0$$

$$\forall \xi, x.$$

i) $x = 0, \xi \neq 0$

ii) $\xi = 0$, x arbitrary

iii) $\delta = 0$. If $d^T \neq 0$, choose ξ & x to make the quad form < 0.