NLP (II)

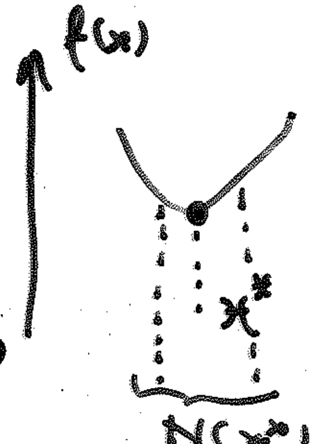
Recap:

- what are NLP
- convexity (important)
  - → we can't actually solve general NLPs
  - → exception: convex fct objective (+ other conditions on constraints).

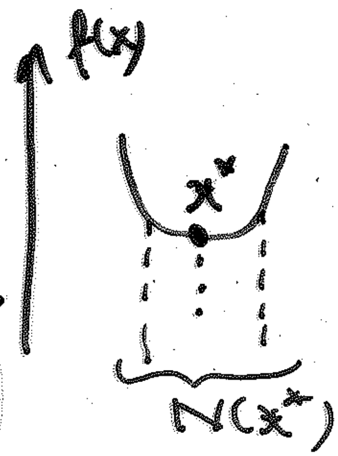
Q: What can we try to obtain for a NLP?

A: a) local minimum solution

→ strong L.M. ($f(x) > f(x^*)$, $x \in N(x^*)$)



→ weak L.M. ($f(x) \geq f(x^*)$, $x \in N(x^*)$)

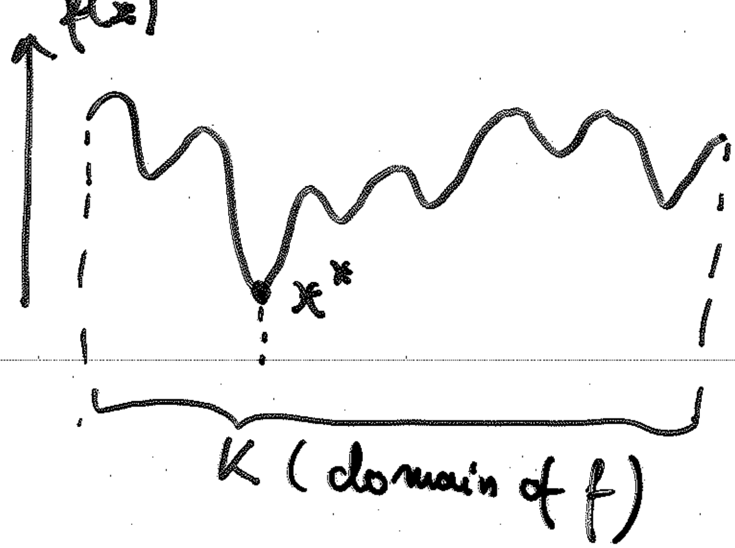


NLP (II) (c'ed)

Solutions to NLP (c'ed)

b) global minimum solution

($f(x) \geq f(x^*)$, $x \in K$)



OBS: if f is convex, then a local min is also a global min.

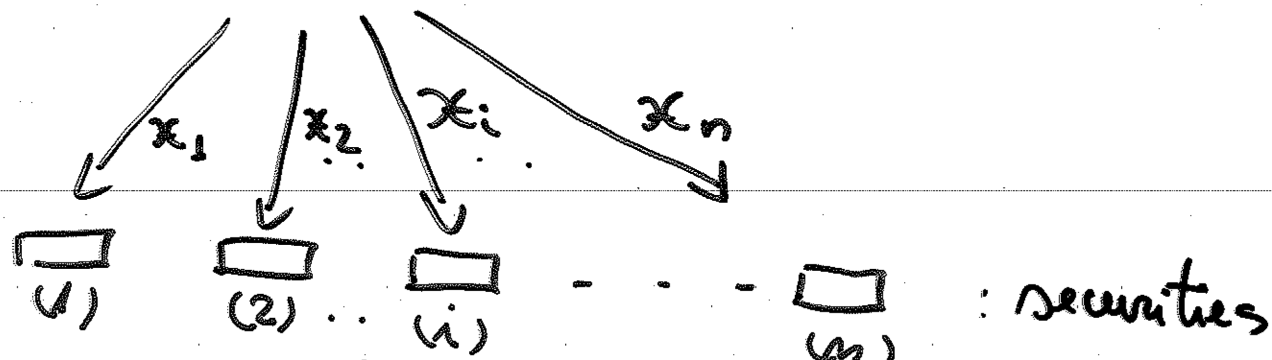
c) local (strong / weak) / global maximum solutions.

OBS: goal in solving NLPs = computing a local minimum.
(for convex programs = global min)

HLP(II) (ctd)

Example: Portfolio optimisation.

\$ 10,000 to invest



μ_i : yield
d_{ii} : variance of yield (risk)

d_{ij}: covariance of yield.

Goal: maximise yield, minimize risk (variance)

stated as:

- L : minimum yield (target)
- objective = minimize risk.

Portfolio optimisation (c'ed)

$x^T = (x_1 \dots x_i \dots x_n)$: distribution of assets

$D = (d_{ij})_{i,j \in \{1..n\}}$: covariance matrix

L : target yield

$$\begin{cases} \min x^T D x \\ \sum_{i=1}^{n} x_i \leq 10{,}000 \\ \sum_{i=1}^{n} \mu_i x_i \geq L \\ x_i \geq 0 \end{cases}$$

(quadratic program)

NLP II (c'ed)

Goal of NLP = find local minima.

→ necessary & sufficient conditions for local optimality (similar to primal-dual for LP)

→ however, even checking for local optimality is a hard problem for general NLP

ex: $\begin{cases} \min x^T D x \\ x \geq 0 \end{cases}$

D - not positive semidefinite.

(if D is PSD, $x = 0$ is local & global min (why))

We do not know of an efficient method to check if $x = 0$ is local min if D is not PSD.

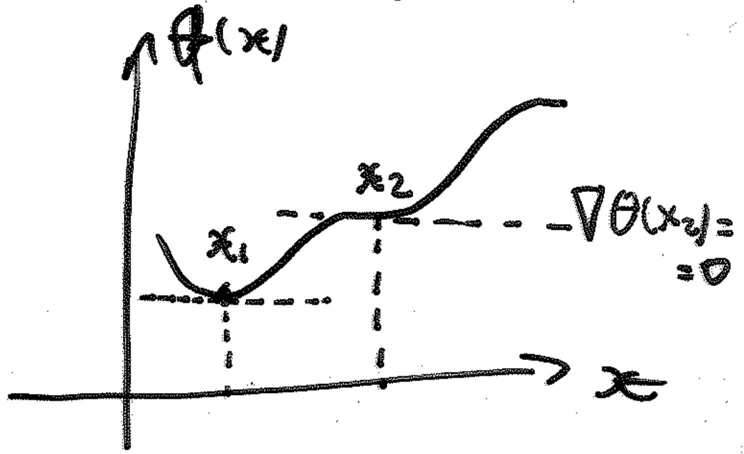
NLP II (c'ed)

$\boxed{\text{Optimality conditions}}$ (for local minima)

A) Smooth NLPs (functions are continuously differentiable, unlike LP!).

A.1) Smooth unconstrained.

$$\min_{x \in \mathbb{R}^n} \theta(x)$$



$\bar{x}$: local min

- Necessary:

$$\nabla \theta(\bar{x}) = 0 \; , \quad \nabla \theta(x) = \begin{pmatrix} \frac{\partial f}{\partial \xi_1} \\ \vdots \\ \frac{\partial f}{\partial \xi_n} \end{pmatrix}$$

- Sufficient:

$$H(\theta(\bar{x})) = \left(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} \right)_{i,j} \text{ is positive definite.}$$

NLP II (c'ed)

A.2) Smooth, linearly constrained

$$\begin{cases} \min \theta(x) \\ Ax = b \\ Dx \geq d \end{cases}$$

any $\xi_i \geq 0$ constraints are included here

$\bar{x}$: local min

- Necessary (KKT conditions)

$$\left.\begin{aligned} \nabla\theta(\bar{x}) - \mu A - \pi D &= 0 \\ \pi &\geq 0 \end{aligned}\right\} \text{like dual feasibility}$$

$$\left.\begin{aligned} A\bar{x} &= b \\ D\bar{x} &\geq d \end{aligned}\right\} \text{" primal feasibility}$$

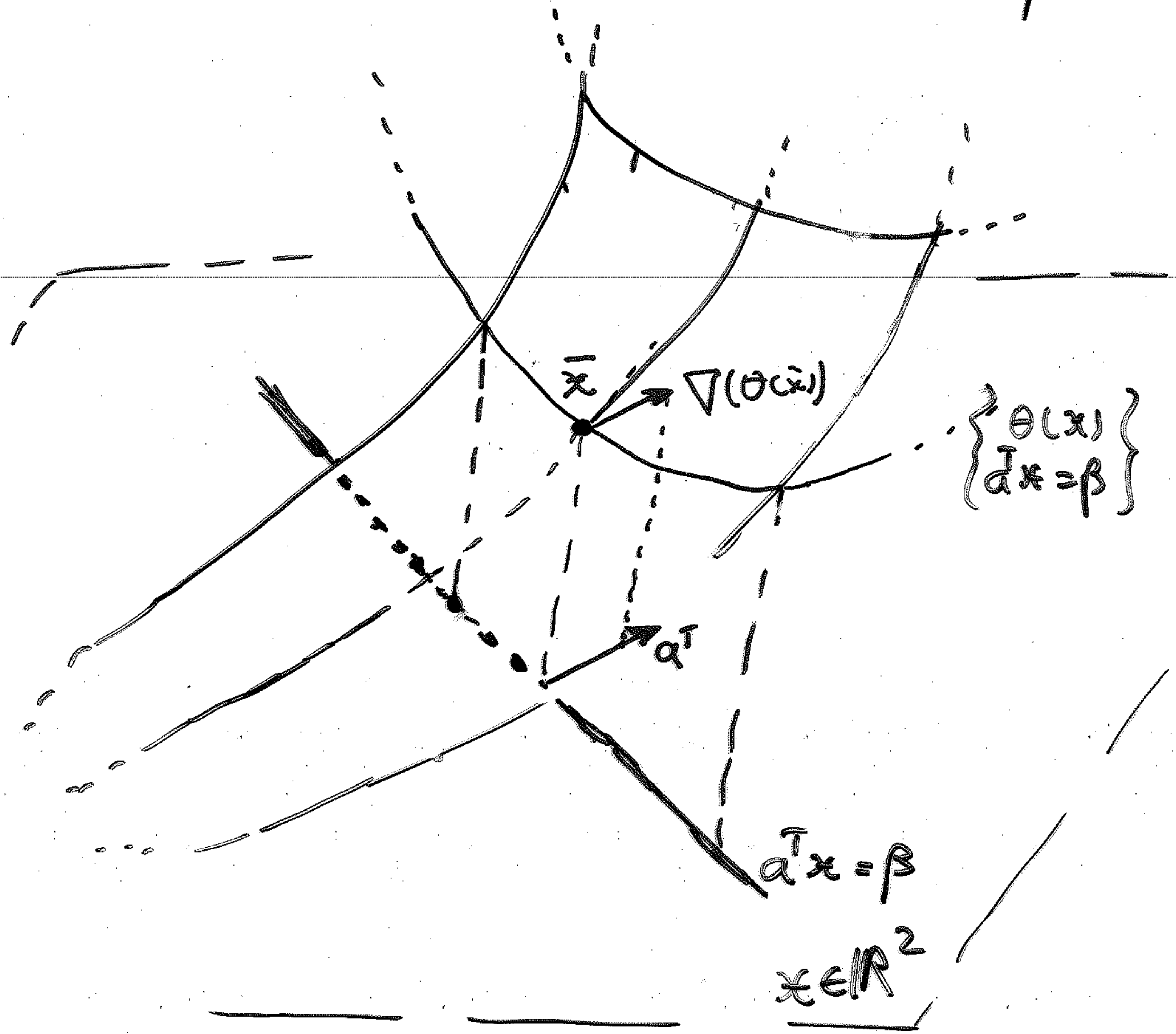
$$\pi^T(D\bar{x} - d) = 0 \; \} \text{" C.S. conditions.}$$

- Sufficient

θ - convex, thus $\bar{x}$ also global min.

NLP II (c'ed)

KKT conditions, smooth, linearly constr (c'ed)



$$\nabla\theta(\bar{x}) = \pi\, a^T$$

π unrestricted ...