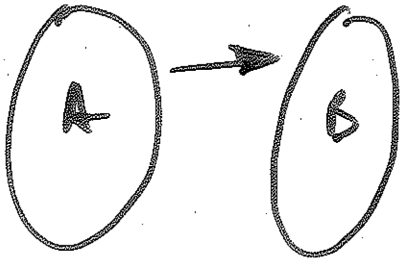
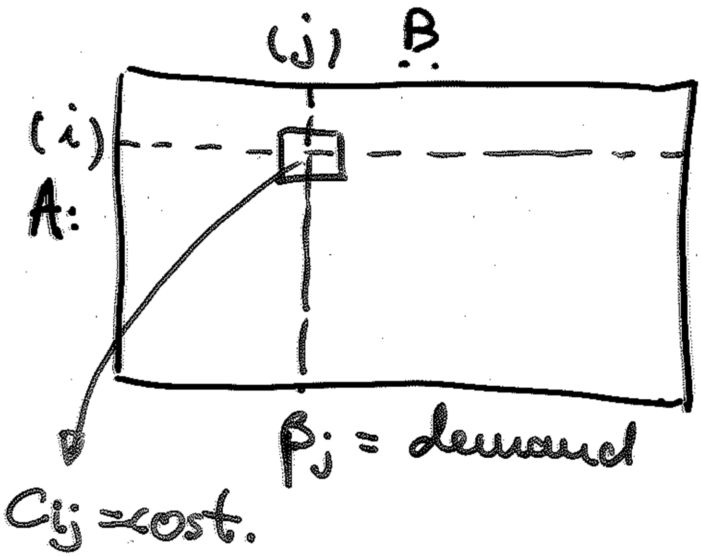
Primal simplex for transportation Pb.



Balanced transportation:
- ship goods from A to B
- min. total cost.



α_i = supply

$$\sum_{i\in A}\alpha_i = \sum_{j\in B}\beta_j$$

x_{ij} = quantity shipped from i to j

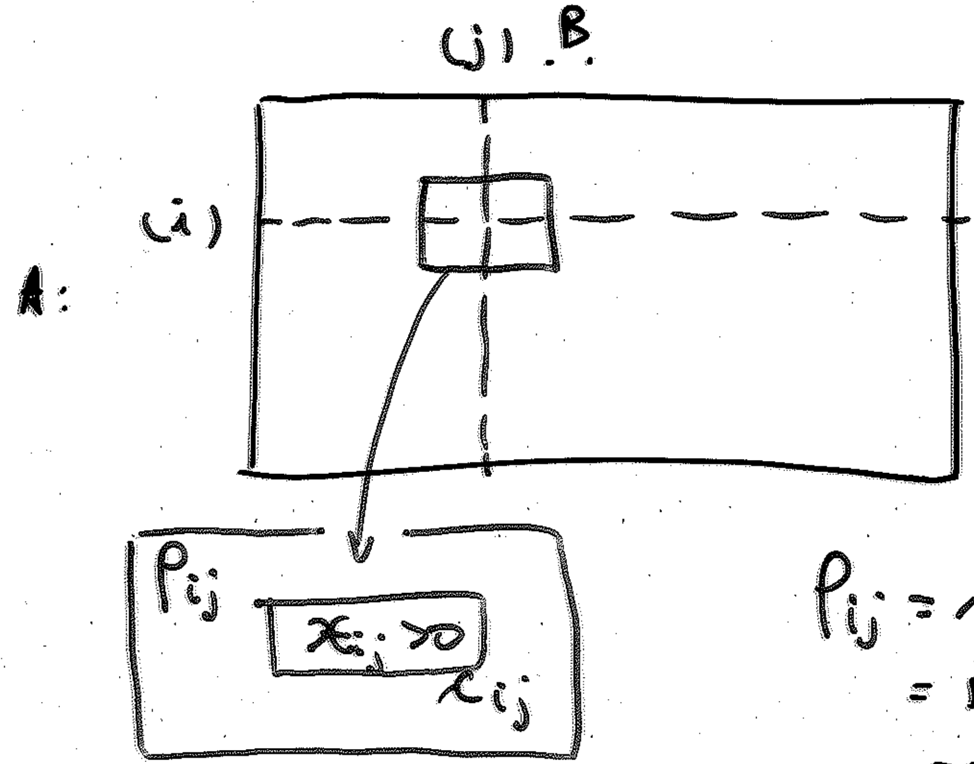
$$(P)\begin{cases} \min \sum_{\substack{i\in A\\ j\in B}} c_{ij}\,x_{ij} \\ \sum_{j\in B} x_{ij} = \alpha_i \quad , \; i\in A \\ \sum_{i\in A} x_{ij} = \beta_j \quad , \; j\in B \\ x_{ij} \geq 0 \end{cases}$$

Transportation (c'ed)

$$(D)\begin{cases} \max \sum_{i\in A} \alpha_i v_i + \sum_{j\in B} \beta_j \cdot z_j \\ v_i + z_j \leq c_{ij} \end{cases}$$

Primal simplex :

- maintain primal feasibility
- maintain C.S. conditions
- "increase" the feasibility of the dual.



$\rho_{ij} = c_{ij} - v_i - z_j$

= relative cost or reduced cost ($\bar{c}_{ij}$ in text)

Transportation (c'td)

Obs : one constraint in (P) is redundant because $\sum_{i \in A} \alpha_i = \sum_{j \in B} \beta_j$. We eliminate one constraint, eg: last.

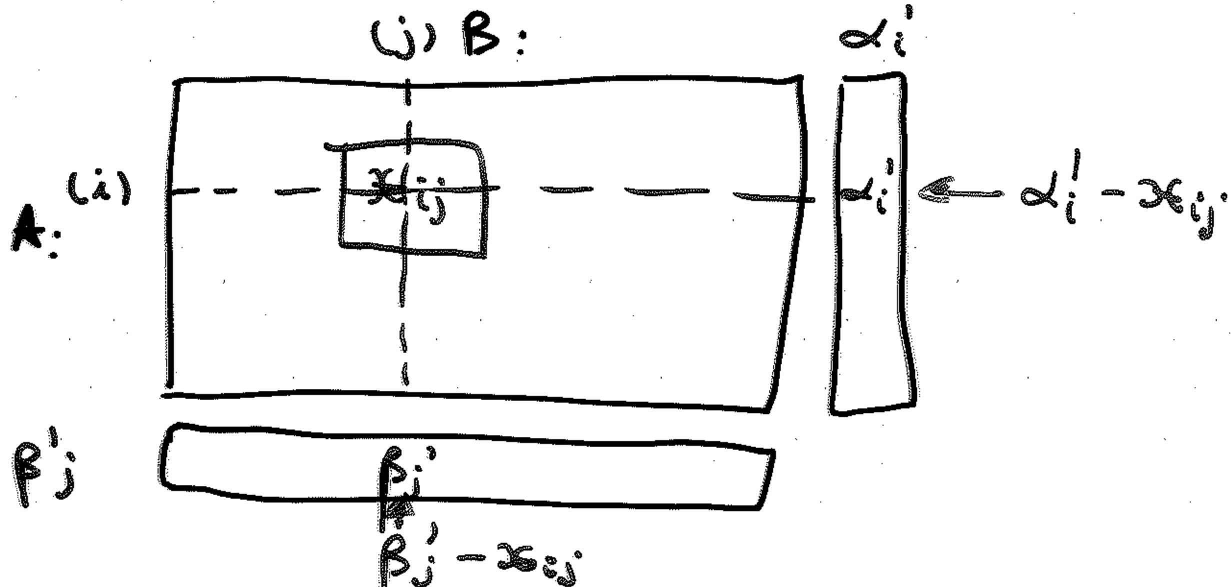
$$
(P) : \begin{cases} \min\ c^T x \\[1em] \underset{n+m-1}{} \ \overbrace{\boxed{\boxed{B} \ A \ \boxed{N}}}^{n \cdot m} \ x = \begin{pmatrix} \vdots \\ \alpha_i \\ \beta_j \\ \vdots \end{pmatrix} \\[1em] x \geq 0 \end{cases}
$$

$n+m-1$ basic variables.

Transportation (c'ed)

a) Compute a primal basic feasible solution.

(simple greedy procedure, $n+m-1$ steps)

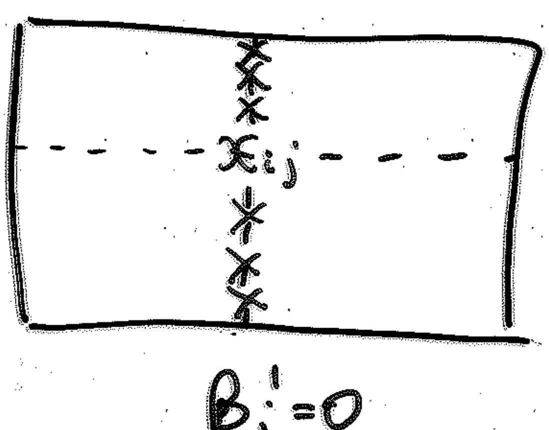


α_i' = remaining supply (α_i initially)

β_j' = remaining demand (β_j initially)

- $x_{ij} = \min\{\alpha_i', \beta_j'\}$
- update α_i', β_j'.

no more goods to ship



$\beta_j' = 0$

- Forbid cols if β_j becomes 0.

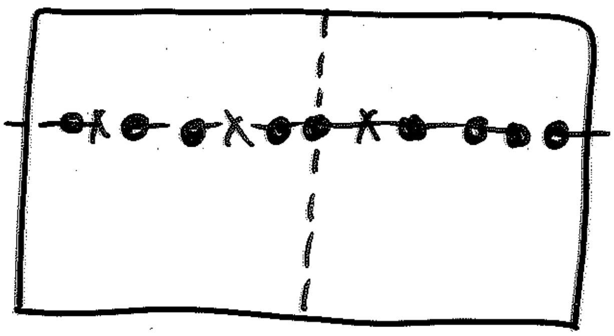
OR

Forbid rows if α_i becomes 0.

a) Primal basic (c'td)

Which cell x_{ij} to select?

- Greedy: not forbidden x_{ij} with smallest c_{ij} (min cost is goal)
- Vogel's rule:



(i) $\theta_i = \min_2(i) - \min(i)$

$\min_2(i)$: second smallest cost c_{ij} in allowed cells of row i.

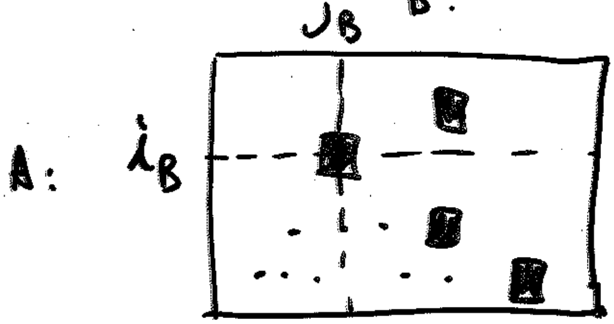
$\min(i)$: smallest cost c_{ij} in allowed cells of row i.

θ_j: same for cols.

- pick row i (col j) with largest θ_i (θ_j).
- choose cell with smallest c_{ij} in chosen row (col). (otherwise we pay θ_i (θ_j) more cost in that row (col.)).

Transportation (c'ed)

b) Compute dual var by satisfying C.S.



Given

- $n+m-1$ basic variables

$(i_B, j_B) \in \mathcal{B}$

Output

$n+m$ dual vars.

C.S. conditions ($(n+m-1)$ equations)

$$-c_{i_B j_B} = \underbrace{v_{i_B} + z_{j_B}}_{\text{unknowns}} \quad , \forall (i_B, j_B) \in \mathcal{B}$$

Obs: we removed last constraint from (P) thus $z_n = 0$.

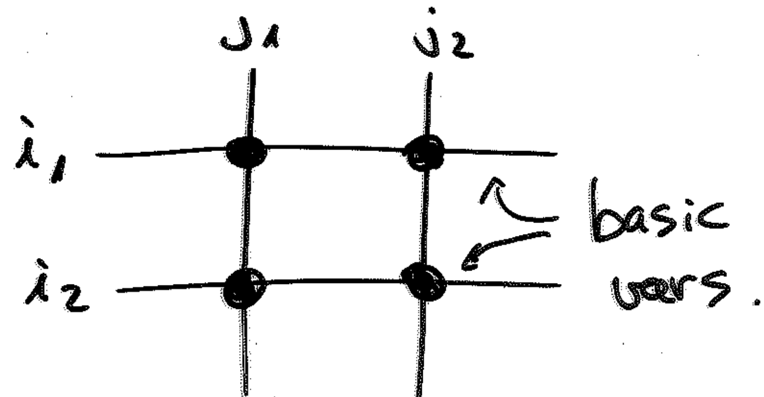
↳ the $(n+m)$-th equation in the system that gives v_i & z_j.

b) Compute duals (c'ed)

Q: Can this approach fail in computing the duals?

Possible reasons?

- τ_i or ζ_j not present in system?
- No solution to the system?



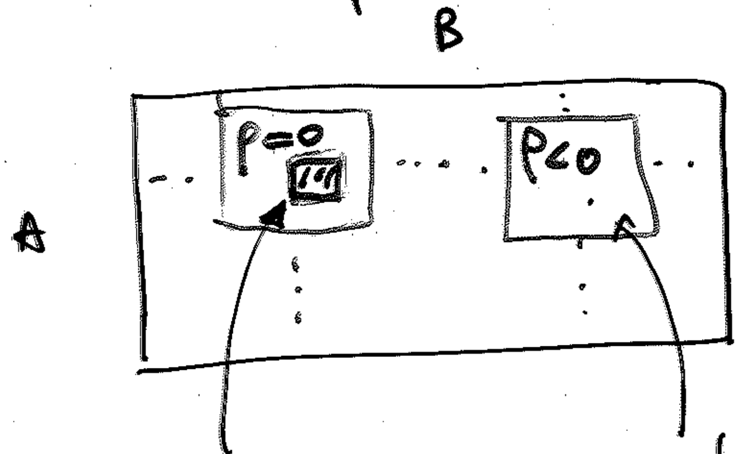
(a configuration for conflicting equations)

↓

cannot occur b.c. "forbidding line rule".

c) <u>Reduce dual infeasibility</u> (pivot)

- we have computed the duals v_i, z_j.



basic cell $\rho=0$ (because of C.S.)

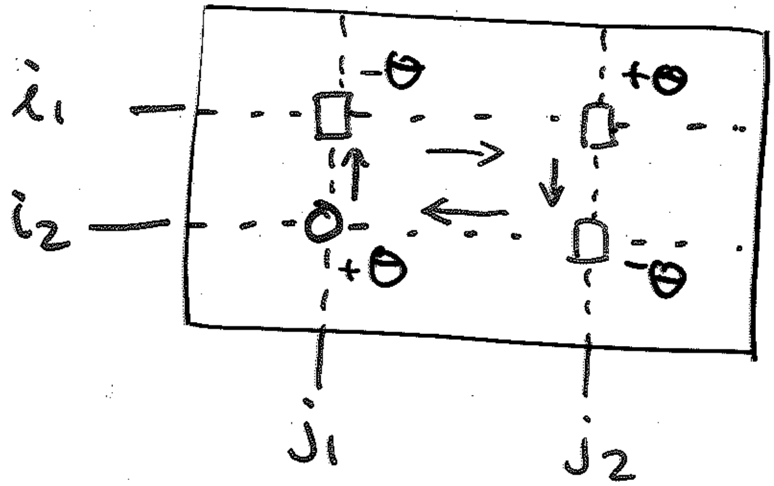
certain non-basic cells have $\rho<0$
$\rho = c_{ij} - v_i - z_j < 0$
(dual infeasible)

- idea:
  - choose a cell (i_0, j_0) with $\rho_{i_0 j_0} < 0$. (ex: choose minimum ρ).
  - make (i_0, j_0) basic (thus dual constraint $c_{i_0 j_0} = v_{i_0} + z_{j_0}$ becomes feasible)

Q: Is this enough to reduce dual infeasibility? We will see shortly that this improves the primal objective.

Primal simplex - transportation

c) Pivot (c'ed) : we are modifying the primal solution



$\circ : \rho < 0$ (entering variable)

$\square : \rho = 0$ (basic variable).

- originally $x_{i_2 j_1} = 0$ (non-basic)
- let $x_{i_2 j_1} = \theta > 0$ (will become basic)

→ j_1 receives θ more shipment

→ compensate @ some basic var on column j_1

$$x_{i_1 j_1} \leftarrow x_{i_1 j_1} - \theta$$

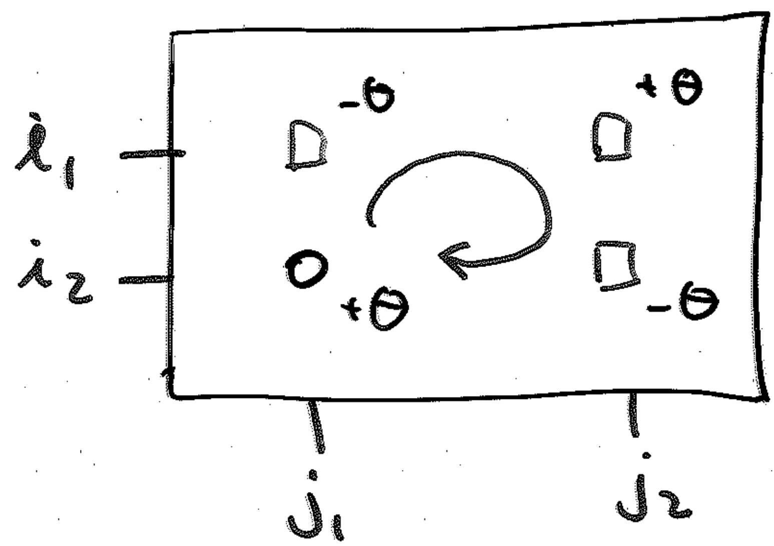
→ i_1 sends θ less shipment

→ compensate @ some basic var on row i_1

⋮ etc...

Primal simplex (transportation)

-c) Pivot (c'ed)



θ-Loop

- $\theta = \min\{x_{i_1 j_1}, x_{i_2 j_2}\}$

  → either $x_{i_1 j_1}$ or $x_{i_2 j_2}$ becomes zero & leaves the basis

  → $x_{i_2 j_1}$ enters the basis

- Is this better?

  value of objective fct:

$$z_{new} = z_{old} + \theta\left(c_{i_2 j_1} + c_{i_1 j_2} - c_{i_1 j_1} - c_{i_2 j_2}\right)$$

$$= z_{old} + \theta\Big(\underbrace{c_{i_2 j_1} + \cancel{v_{i_1}} + \cancel{z_{j_2}} - \cancel{v_{i_1}} - z_{j_1} - \cancel{v_{i_2}} - \cancel{z_{j_2}}}_{\rho_{i_2 j_1} < 0}\Big)$$

Primal simplex (transportation)

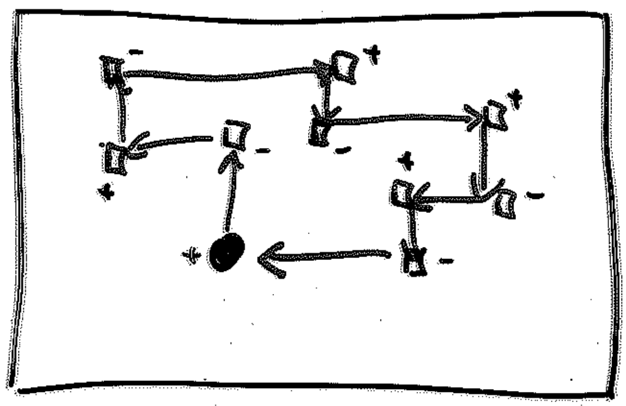
d) End? When no pivot is possible

$\rho_{ij} \geq 0 \quad \forall\, i \in A \;\, j \in B$.

(optimal x_{ij} σ_i τ_j)

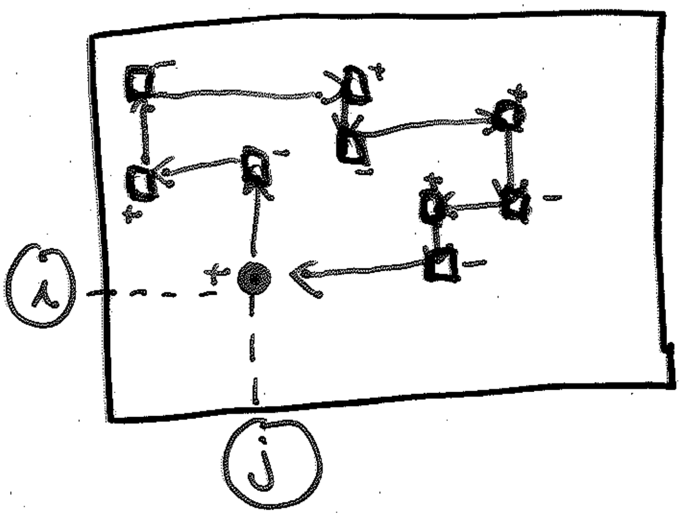
Algorithm for pivoting

- simplest = brute-force, trial & error.



Find a cycle through basic variables, following N-S & E-W directions only.

Primal simplex (transportation) (c'ed)



Q: Is no improvement in the primal objective fct possible?

$$Z_{new} = Z_{old} + \Theta \cdot \rho_{ij}$$

- yes: degeneracy $\Theta = 0$ (at least one basic variable in $\square^-$ is zero).

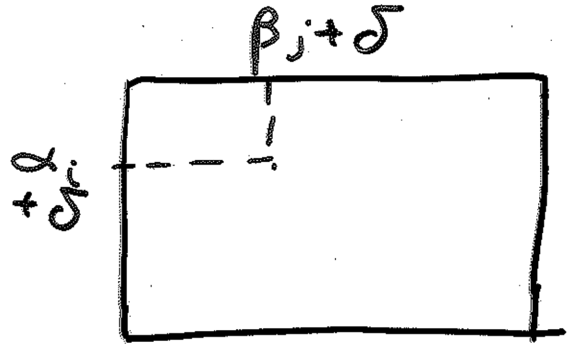
– if degeneracy occurs, we still change basis; eventually we escape degeneracy...

Primal-simplex (transportation)

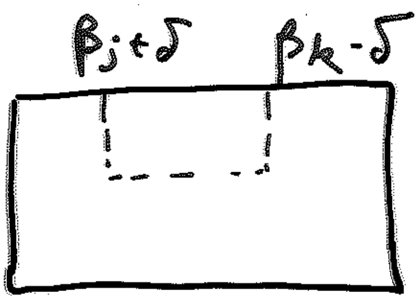
Marginal analysis

→ how do constraints affect optimal cost?

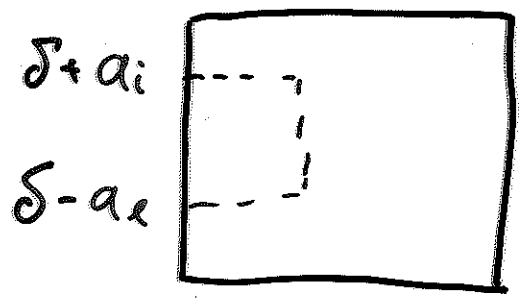
Constraints = supply & demand



(a)



(b)



(c)

Dual objective:

$$\sum_{i \in A} \alpha_i \cdot \sigma_i + \sum_{j \in B} \beta_j \cdot \tau_j = z$$

(@ opt).

a) $\Delta z = (\sigma_i + \tau_j)\,\delta$

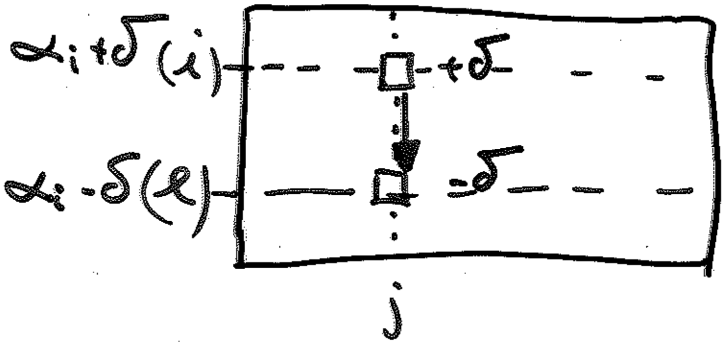
b) $\Delta z = (\tau_j - \tau_k)\,\delta$

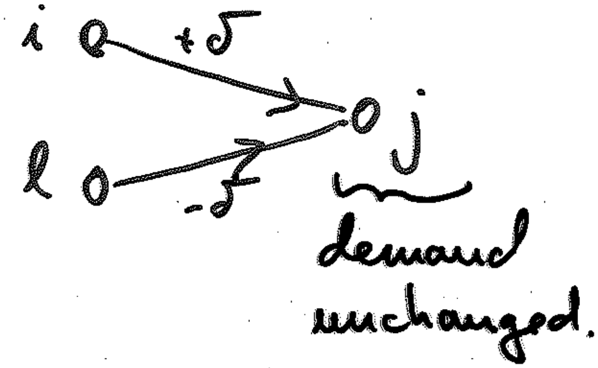
c) $\Delta z = (\sigma_i - \sigma_\ell)\,\delta$

Primal-simplex (transportation)

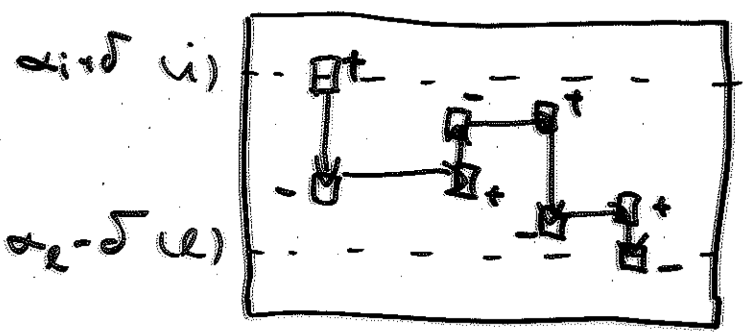
Marginal analysis (c'td)

$\rightarrow$ how large δ to maintain opt. solution?





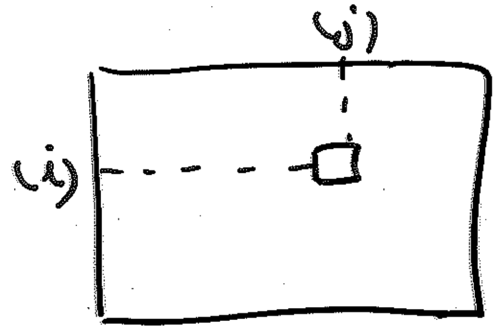
$$x_{lj}\,(\text{new}) = x_{lj} - \delta \geq 0$$



Primal-simplex (transportation)

Sensitivity analysis

→ how sensitive is OPT solution to changes in cost coefficients?



a)

(i,j) - non-basic

$c_{ij} \rightarrow c'_{ij}$

$\rho'_{ij} = c'_{ij} - v_i - \tau_j > 0$

(to maintain dual feasibility; the other 2 conditions remain satisfied)

$c'_{ij} > v_i + \tau_j$

b) (i,j) - basic

→ recompute duals as fct of c'_{ij}

→ impose $\rho_{ij} \geq 0 \quad \forall i \in A \; j \in B$

↓

this gives an interval for c'_{ij}