Approximation algorithms: assignment 2 solutions

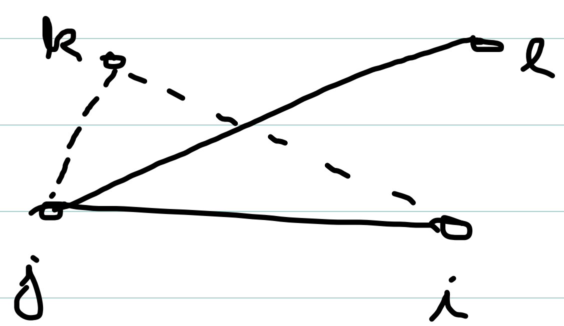
① k-suppliers problem (Exercise 2.1).

Given: set F of suppliers
set D of demand points
cost c_{ij}, $i, j \in F \cup D$ (needed for the proof of approx factor)
$k \in \mathbb{N}$

Output: set of open facilities $S \subseteq F$, $|S| = k$

Measure $\min_{S \subseteq F} \sum_{i \in D} \min_{j \in S} c_{ij}$

Let $F = \{a, b\}$, $k = 1$, $D = \{i\}$, $c_{ia} = 1$, $c_{ib} > \alpha$.
The optimal solution cost is 1, given by opening a facility at a. If the algorithm chooses facility b, the cost is $> \alpha$.

② 

Let $i \in D$ be an arbitrarily chosen customer. Let $j \in F$ be the nearest facility from i.

We can show that $\{i\}$ is a 3-approximation to the 1-suppliers problem.

Let $\ell \in D$ be the customer farthest from i. Then $c_{i\ell}$ is the cost of the solution returned.

Let $k \in F$ be the optimal solution. Let c_{OPT} be the cost of the optimal solution

$\Rightarrow c_{ki} \le c_{OPT}$ (k is optimal) and

$c_{ij} \le c_{ik} \le c_{OPT}$ (j is closest to i)

Using the triangle inequality in triangle kij

$\Rightarrow c_{kj} \le 2\, c_{OPT}$

Using again the triangle inequality in triangle $kj\ell$

$\Rightarrow c_{j\ell} \le c_{kj} + c_{k\ell} \le 3 \cdot c_{OPT}$ because $c_{k\ell} \le c_{OPT}$ by the optimality of k.

2.5) A greedy algorithm for the k-suppliers:

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S ← ∅ ; i ← arbitrary element in D ;
repeat
    j ← nearest facility to i ;
    if j ∈ S return S ; // we are done
    S ← S ∪ {j} ;
    i ← farthest customer from S
until |S| ≥ k.
return S.
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The performance analysis is very similar to that of the greedy k-center algorithm.

Let O_t for $1 \le t \le k$ be a partition of the set of customers determined by the optimal solution to the k-suppliers problem. Let $S_{opt} = \{f^*_1, \ldots, f^*_k\}$ be the optimal set of suppliers. Then,

$$O_t = \{ i \in D : f^*_t \text{ is the closest facility from } S_{opt} \text{ to } i \}$$

Like in k-center analysis, there are two cases. Let $i_1, i_2, \ldots, i_k$ be the set of customers chosen by the greedy algorithm which determines the set of facilities.

Case 1 Without loss of generality, $i_t \in O_t$ for all $1 \le t \le k$.

Case 2 $\exists\, t, u, v$, $t, u, v \in \{1 \ldots k\}$ and $u \neq v$ and $i_u \in O_t$ and $i_v \in O_v$.

Analyse the cases exactly as for the k-centre problem.

(3) Knapsack problem. Show that the greedy algorithm of packing items in the non-increasing order of value per unit size gives an arbitrarily small approx. factor.

Let $0 < \varepsilon < 1$ be given. Let $\varepsilon' < \varepsilon$. We will exhibit an instance for knapsack with performance ratio ε':

Consider the knapsack capacity $B = 1$, item 1 with $v_1 = s_1 = 1$ and item 2 with $v_2 = \varepsilon'$ $s_2 = \varepsilon''$. The optimal solution packs item 1 with total value 1, but greedy packs only item 2 with value ε' because its value per size ratio is $\frac{v_2}{s_2} = \frac{\varepsilon'}{\varepsilon''} > 1$.

(4) Most answers here were correct. Check the online resources.