INSTRUCTIONS:

- Please write your answers in the exam booklet provided. The use of a pencil is recommended.
- Use of books and notes on paper or in electronic format is not necessary, but allowed. You may consult electronic resources. Use of conferencing or messaging software is not allowed.
- Each problem is worth 10 pts.

Time: 75 minutes
Maximum score: 50 pts for grads, 30 pts for undergrads

PROBLEM 1: Given a graph $G = (V, E)$ with edge length function $l : E \to \mathbb{R}_+$, where set $\mathbb{R}_+$ represents the set of positive real numbers. Denote by $\delta(S)$, for $S \subseteq V$, the set of edges with one endpoint in S and the other in $V - S$. Consider the following LP formulation for the minimum weight spanning tree problem.

$$\min \sum_{e \in E} l(e) x_e$$
$$\text{subject to: } \sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S \subseteq V$$
$$x_e \geq 0.$$

(a) Explain the meaning of the first constraint in the LP formulation.
(b) Write the dual of the LP.
(c) Consider the unit square Q whose vertices are points with coordinates $(0,0), (1,0), (0,1), (1,1)$. Construct the complete graph G_Q on four vertices whose edge lengths are given by the Euclidean distances between all pairs of vertices of the unit square. How many variables and constraints does the LP formulation above for G_Q have?

PROBLEM 2: Consider the LP formulation for the minimum weight spanning tree problem from Problem 1.
(a) Provide the optimal solution of the LP for graph G_Q and prove it is optimal.
(b) Recall that the main steps of the primal-dual algorithm are: start with a feasible dual solution (usually 0); obtain a primal solution that satisfies complementary slackness conditions; if the primal solution is not feasible, update the dual values so that dual feasibility is maintained and repeat, otherwise stop. Describe the outcome of the primal-dual method after running for three iterations on graph G_Q. You must use the given LP formulation. Briefly justify your steps.

PROBLEM 3: In the minimum degree spanning tree problem, the goal is to find a spanning tree of a graph G, not necessarily complete and without edge lengths, that has the minimum degree (the degree of a tree equals the degree of a vertex with the largest degree). Show that there can be no α-approximation for the minimum degree spanning tree problem for $\alpha < 3/2$ unless P=NP.

Problem 4: In the minimum-cost Steiner tree problem, we are given as input a complete, undirected graph $G = (V, E)$ with non-negative costs $c_{ij} \geq 0$ for all edges $(i, j) \in E$. The set of vertices is partitioned into terminals R and non-terminals (or Steiner vertices) $V - R$. The goal is to find a minimum-cost tree containing all terminals. Suppose that the edge costs obey the triangle inequality; that is, $c_{ij} \leq c_{ik} + c_{kj}$ for all $i, j, k \in V$. Let $G[R]$ be the graph induced on the set of terminals; that is, $G[R]$ contains the vertices in R and all edges from G that have both endpoints in R. Consider computing a minimum spanning tree in $G[R]$. Show that this gives a 2-approximation algorithm for the minimum-cost Steiner tree problem.

Problem 5: Recall the 2-approximation algorithm for the metric TSP problem that doubles the edges of a minimum weight spanning tree, and traverses the Eulerian graph obtained while short-cutting the vertices that are already visited. Describe an example where this algorithm does not produce the optimal solution. What is the (a posteriori) performance ratio for the solution returned in your example? Justify your answers.

Problem 6: In the dominating set problem, we are given a graph $G = (V, E)$ and we are asked to produce a subset of vertices $S \subseteq V$ of smallest size so that every vertex in V is either in S or it is adjacent to a vertex in S. Explain how an approximation algorithm for set cover can be used to solve the dominating set problem.