Example Usage of FFreduce and Modreduce

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This document shows how to use the FFreduce and Modreduce routines implemented in Maple. These routines implement algorithms in:

- Beckermann, B., Cheng, H. and Labahn, G. "Fraction-free Row Reduction of Matrices of Ore Polynomials." Journal of Symbolic Computation, 41(5), pages 513-543, 2006.
- Cheng, H. and Labahn, G. "Modular Computation for Matrices of Ore Polynomials." Computer Algebra 2006: Latest Advances in Symbolic Algorithms: Proceedings of the Waterloo Workshop in Computer Algebra 2006, pages 43-66, 2007.

For both routines, the input consists of five parameters:

- F: a matrix of Ore polynomials over some domain $D[Z;\sigma, \delta]$ (D must be the integers for Modreduce)
- Z: the indeterminate in the domain $D[Z;\sigma, \delta]$
- σ: the automorphism, implemented as a function $(D, n) \rightarrow D$
- δ: the derivation, implemented as a function $(D, n) \rightarrow D$
- n: the variable in the domain D (e.g. Q[n, q^n])

The output consists of four components:

- M: the order basis
- R: the residual
- ω: the final order achieved
- μ: the final degree constraints achieved

The output satisfies the order condition:

$$M \cdot F = R \cdot Z^{\omega}$$

with the trailing coefficient of R having the same rank as F.

Note that the implementation of FFreduce and Modreduce routines return the same result up to sign.

To use these routines, we first define the σ and δ functions. For example, the following defines the Ore algebra for studying usual differential equations:

$$\sigma := (x, t) \rightarrow x$$

$$(x, t) \rightarrow x$$

$$\delta := (x, t) \rightarrow diff(x, t)$$

$$(x, t) \to \frac{\partial}{\partial t}\, x \tag{2}$$

We read in the routine definitions:

read "ffreduce.mpl"

read "modreduce.mpl"

Now we define an input matrix F:

$$F := \begin{bmatrix} 2\,Z^2 + 3\,t \cdot Z + 6\,t^2 & Z^2 - Z + 2 \\ (t-1)\,Z + 3\,t^2 & 3\,t \cdot Z + t \end{bmatrix} :$$

We can call each routine by

FFreduce(F, Z, σ, δ, t) or Modreduce(F, Z, σ, δ, t)

For example:

$$M_1, R_1, \omega_1, \mu_1 := FFreduce(F, Z, \sigma, \delta, t) :$$

$$M_2, R_2, \omega_2, \mu_2 := Modreduce(F, Z, \sigma, \delta, t) :$$

We can verify that the two results are the same up to sign:

$$map(expand, M_1 + M_2) \tag{3}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$map(expand, R_1 + R_2) \tag{4}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The results are:

$$M_1 \tag{5}$$

$$\begin{aligned} &[[-5598720\,t^{10} + 9331200\,t^9 + 65318400\,t^8 + 223948800\,t^7 + 428457600\,t^6 - 375114240\,t^5 \\ &\quad - 1511654400\,t^4 - 3103660800\,t^3 + 700617600\,t^2 + 1467538560\,t + 5322240 \\ &\quad + (4665600\,t^{11} - 10264320\,t^{10} - 32659200\,t^9 - 45100800\,t^8 + 208785600\,t^7 \\ &\quad + 1542680640\,t^6 + 320112000\,t^5 - 1239494400\,t^4 - 9600249600\,t^3 + 535688640\,t^2 \\ &\quad + 4350974400\,t + 29479680)\,Z + (-1866240\,t^{12} + 5132160\,t^{11} + 9331200\,t^{10} \\ &\quad - 66873600\,t^9 - 447897600\,t^8 - 1428412320\,t^7 + 269697600\,t^6 + 2852003520\,t^5 \end{aligned}$$

$$+ 9723758400\,t^4 - 287867520\,t^3 - 4578102720\,t^2 - 419523840\,t + 5425920\big)\,Z^2$$
$$+ \big(466560\,t^{13} - 1555200\,t^{12} - 2877120\,t^{11} + 35536320\,t^{10} + 267040800\,t^9$$
$$+ 735531840\,t^8 - 141620400\,t^7 - 1530046800\,t^6 - 4683515040\,t^5 - 497158560\,t^4$$
$$+ 1848941280\,t^3 + 587217600\,t^2 - 819132480\,t - 109008000\big)\,Z^3 + \big(-77760\,t^{14}$$
$$+ 311040\,t^{13} + 933120\,t^{12} - 5806080\,t^{11} - 64372320\,t^{10} - 251760960\,t^9 - 7819200\,t^8$$
$$+ 407557440\,t^7 + 1240824420\,t^6 + 363896640\,t^5 - 169721280\,t^4 - 103084560\,t^3$$
$$+ 754528320\,t^2 + 48797280\,t + 4714560\big)\,Z^4 + \big(7776\,t^{15} - 38880\,t^{14} - 233280\,t^{13}$$
$$- 803520\,t^{12} - 2728080\,t^{11} - 5102352\,t^{10} + 7534080\,t^9 - 831600\,t^8 - 99637290\,t^7$$
$$+ 127722330\,t^6 + 115996932\,t^5 - 62396640\,t^4 - 318545280\,t^3 - 12751920\,t^2 + 3800880\,t$$
$$+ 6940368\big)\,Z^5,\ 37324800\,t^9 + 167961600\,t^8 + 849139200\,t^7 + 3776958720\,t^6$$
$$+ 8765729280\,t^5 - 8379417600\,t^4 - 16272316800\,t^3 + 5523681600\,t^2 - 25229724480\,t$$
$$- 3375993600 + \big(-26127360\,t^{10} - 186624000\,t^9 - 979776000\,t^8 - 4224856320\,t^7$$
$$- 9622644480\,t^6 + 9129646080\,t^5 + 19295625600\,t^4 + 683640000\,t^3 + 23828489280\,t^2$$
$$+ 440916480\,t - 10644480\big)\,Z + \big(8398080\,t^{11} + 83980800\,t^{10} + 555206400\,t^9$$
$$+ 2386143360\,t^8 + 5386357440\,t^7 - 4607902080\,t^6 - 11280824640\,t^5 - 4393116000\,t^4$$
$$- 13176807840\,t^3 + 1696014720\,t^2 + 790732800\,t - 10851840\big)\,Z^2 + \big(-1555200\,t^{12}$$
$$- 20528640\,t^{11} - 174182400\,t^{10} - 895432320\,t^9 - 2081090880\,t^8 + 1333117440\,t^7$$
$$+ 4038042240\,t^6 + 3060802080\,t^5 + 6103075680\,t^4 - 13829760\,t^3 - 1099431360\,t^2$$
$$- 326376000\,t + 26138880\big)\,Z^3 + \big(155520\,t^{13} + 2643840\,t^{12} + 28304640\,t^{11}$$
$$+ 186494400\,t^{10} + 603313920\,t^9 - 159058080\,t^8 - 891587520\,t^7 - 943399440\,t^6$$
$$- 2143006200\,t^5 - 856464480\,t^4 + 437588640\,t^3 + 390208320\,t^2 - 47108160\,t$$
$$- 10321920\big)\,Z^4\big],$$
$$\big[-2332800\,t^{10} + 4665600\,t^9 + 30326400\,t^8 + 119750400\,t^7 + 215978400\,t^6 - 180092160\,t^5$$
$$- 757900800\,t^4 - 1551830400\,t^3 + 305078400\,t^2 + 733942080\,t + 9270720$$
$$+ \big(1866240\,t^{11} - 4665600\,t^{10} - 13996800\,t^9 - 30326400\,t^8 + 108086400\,t^7$$
$$+ 777288960\,t^6 + 196110720\,t^5 - 617932800\,t^4 - 4790664000\,t^3 + 136321920\,t^2$$
$$+ 2168791200\,t + 35125920\big)\,Z + \big(-699840\,t^{12} + 2099520\,t^{11} + 2916000\,t^{10}$$
$$- 29548800\,t^9 - 224434800\,t^8 - 734773680\,t^7 + 103475880\,t^6 + 1428114240\,t^5$$
$$+ 4879569600\,t^4 - 30382560\,t^3 - 2272181760\,t^2 - 230493600\,t + 1339200\big)\,Z^2$$
$$+ \big(155520\,t^{13} - 544320\,t^{12} - 466560\,t^{11} + 17405280\,t^{10} + 131868000\,t^9 + 376922160\,t^8$$
$$- 38521440\,t^7 - 777763080\,t^6 - 2367325440\,t^5 - 297283680\,t^4 + 945617040\,t^3$$
$$+ 296781840\,t^2 - 403296480\,t - 58733280\big)\,Z^3 + \big(-19440\,t^{14} + 77760\,t^{13} + 77760\,t^{12}$$
$$- 3512160\,t^{11} - 32811480\,t^{10} - 126055440\,t^9 - 12543120\,t^8 + 180021960\,t^7$$
$$+ 654078105\,t^6 + 221021280\,t^5 - 95705640\,t^4 - 121025160\,t^3 + 372124080\,t^2$$
$$+ 28491120\,t + 3705480\big)\,Z^4,\ -933120\,t^{10} + 18662400\,t^9 + 88646400\,t^8 + 427680000\,t^7$$
$$+ 1863984960\,t^6 + 4479909120\,t^5 - 3916512000\,t^4 - 8243467200\,t^3 + 2485080000\,t^2$$

$$
\begin{aligned}
&- 12461502240\,t - 1792078560 + (933120\,t^{11} - 13996800\,t^{10} - 97977600\,t^{9} \\
&- 488332800\,t^{8} - 2103485760\,t^{7} - 4911865920\,t^{6} + 4276696320\,t^{5} + 9759268800\,t^{4} \\
&+ 618580800\,t^{3} + 11851345440\,t^{2} + 324194400\,t - 18541440)\,Z + (-466560\,t^{12} \\
&+ 5132160\,t^{11} + 45489600\,t^{10} + 274492800\,t^{9} + 1188211680\,t^{8} + 2766972960\,t^{7} \\
&- 2159712720\,t^{6} - 5706067680\,t^{5} - 2344010400\,t^{4} - 6566415120\,t^{3} + 769355280\,t^{2} \\
&+ 409276800\,t - 2678400)\,Z^{2} + (155520\,t^{13} - 1244160\,t^{12} - 12208320\,t^{11} \\
&- 87220800\,t^{10} - 443244960\,t^{9} - 1065856320\,t^{8} + 583785360\,t^{7} + 2062871280\,t^{6} \\
&+ 1607528160\,t^{5} + 3051854640\,t^{4} - 34323120\,t^{3} - 540794880\,t^{2} - 167173920\,t \\
&+ 13082400)\,Z^{3} + (-38880\,t^{14} + 233280\,t^{13} + 2099520\,t^{12} + 15448320\,t^{11} \\
&+ 94381200\,t^{10} + 302784480\,t^{9} - 59987520\,t^{8} - 400904640\,t^{7} - 543363390\,t^{6} \\
&- 1130230620\,t^{5} - 410263380\,t^{4} + 352671120\,t^{3} + 201107520\,t^{2} - 23293440\,t \\
&- 8515440)\,Z^{4} + (7776\,t^{15} - 38880\,t^{14} - 233280\,t^{13} - 803520\,t^{12} - 2728080\,t^{11} \\
&- 5102352\,t^{10} + 7534080\,t^{9} - 831600\,t^{8} - 99637290\,t^{7} + 127722330\,t^{6} + 115996932\,t^{5} \\
&- 62396640\,t^{4} - 318545280\,t^{3} - 12751920\,t^{2} + 3800880\,t + 6940368)\,Z^{5}]]
\end{aligned}
$$

$$R_1$$

$$
\begin{aligned}
&[[46656\,t^{17} - 233280\,t^{16} - 1516320\,t^{15} - 4315680\,t^{14} - 13646880\,t^{13} - 37534752\,t^{12} \\
&- 36398160\,t^{11} - 348915600\,t^{10} - 736560540\,t^{9} + 2715066540\,t^{8} + 2588842782\,t^{7} \\
&- 1626555330\,t^{6} - 8760969900\,t^{5} - 1021978800\,t^{4} + 1158713280\,t^{3} + 733873968\,t^{2} \\
&- 1530321840\,t - 103588560 + (23328\,t^{16} - 116640\,t^{15} - 855360\,t^{14} - 1788480\,t^{13} \\
&- 6318000\,t^{12} - 26919216\,t^{11} - 106142400\,t^{10} - 506016720\,t^{9} - 314550270\,t^{8} \\
&+ 1198281870\,t^{7} + 2829639636\,t^{6} + 540603360\,t^{5} - 1295078400\,t^{4} - 244424880\,t^{3} \\
&+ 1520459280\,t^{2} + 118415664\,t + 9429120)\,Z + (15552\,t^{15} - 77760\,t^{14} - 466560\,t^{13} \\
&- 1607040\,t^{12} - 5456160\,t^{11} - 10204704\,t^{10} + 15068160\,t^{9} - 1663200\,t^{8} - 199274580\,t^{7} \\
&+ 255444660\,t^{6} + 231993864\,t^{5} - 124793280\,t^{4} - 637090560\,t^{3} - 25503840\,t^{2} \\
&+ 7601760\,t + 13880736)\,Z^{2}, (7776\,t^{15} - 38880\,t^{14} - 233280\,t^{13} - 803520\,t^{12} \\
&- 2728080\,t^{11} - 5102352\,t^{10} + 7534080\,t^{9} - 831600\,t^{8} - 99637290\,t^{7} + 127722330\,t^{6} \\
&+ 115996932\,t^{5} - 62396640\,t^{4} - 318545280\,t^{3} - 12751920\,t^{2} + 3800880\,t + 6940368)\,Z^{2} \\
&+ (-7776\,t^{15} - 38880\,t^{14} + 544320\,t^{13} + 1736640\,t^{12} - 3078000\,t^{11} - 59269968\,t^{10} \\
&- 259295040\,t^{9} - 6987600\,t^{8} + 507194730\,t^{7} + 1113102090\,t^{6} + 247899708\,t^{5} \\
&- 107324640\,t^{4} + 215460720\,t^{3} + 767280240\,t^{2} + 44996400\,t - 2225808)\,Z + 15552\,t^{15} \\
&+ 466560\,t^{14} + 7620480\,t^{13} + 80818560\,t^{12} + 556956000\,t^{11} + 1899645696\,t^{10} \\
&+ 56695680\,t^{9} - 1933074720\,t^{8} - 3578650740\,t^{7} - 8944445160\,t^{6} - 7384811256\,t^{5} \\
&+ 860535360\,t^{4} + 2485560240\,t^{3} - 334139040\,t^{2} - 891293760\,t - 99841824], \\
&[23328\,t^{17} - 116640\,t^{16} - 758160\,t^{15} - 2099520\,t^{14} - 6940080\,t^{13} - 17600976\,t^{12} \\
&- 11472840\,t^{11} - 162959040\,t^{10} - 397906830\,t^{9} + 1332002070\,t^{8} + 1592537031\,t^{7} \\
&- 1027907820\,t^{6} - 4677451740\,t^{5} - 544947300\,t^{4} + 1274718960\,t^{3} + 411697584\,t^{2}
\end{aligned}
\tag{6}
$$

$$-761694120\,t-74249280+\left(7776\,t^{16}-46656\,t^{15}-233280\,t^{14}-414720\,t^{13}\right.$$
$$-1769040\,t^{12}-9398592\,t^{11}-52986528\,t^{10}-260476560\,t^{9}-123891930\,t^{8}$$
$$+587403540\,t^{7}+1296430812\,t^{6}+263648988\,t^{5}-447559920\,t^{4}+63743040\,t^{3}$$
$$\left.+760800960\,t^{2}+60121728\,t+470592\right)Z,\left(23328\,t^{16}-116640\,t^{15}-719280\,t^{14}\right.$$
$$-2332800\,t^{13}-8106480\,t^{12}-18819216\,t^{11}-10209240\,t^{10}-128550240\,t^{9}$$
$$-311454990\,t^{8}+563188950\,t^{7}+1002068901\,t^{6}+33831360\,t^{5}-1051341480\,t^{4}$$
$$\left.-159280920\,t^{3}+383526720\,t^{2}+49312224\,t+3705480\right)Z+7776\,t^{16}-38880\,t^{15}$$
$$-97200\,t^{14}+2073600\,t^{13}+30942000\,t^{12}+240165648\,t^{11}+889569000\,t^{10}$$
$$+190140480\,t^{9}-925359930\,t^{8}-3215470590\,t^{7}-2790701163\,t^{6}-2141579520\,t^{5}$$
$$-398059560\,t^{4}-3120966360\,t^{3}-332700480\,t^{2}-393380352\,t+41666760]]$$

$$\omega_1$$

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} \tag{7}$$

$$\mu_1$$

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix} \tag{8}$$