Higher rank generalizations of Fomenko’s conjecture

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Abstract

Let $a$ be a natural number greater than 1. For each prime $p$, let $i_a(p)$ denote the index of the group generated by $a$ in $\mathbb{F}_p^*$. Assuming the generalized Riemann hypothesis and Hypothesis A of Hooley, Fomenko proved in 2004

$$\sum_{p \leq x} \log(i_a(p)) = c_a \text{li}(x) + O\left(\frac{x \log \log x}{(\log x)^2}\right),$$

where $c_a$ is a constant dependent on $a$, and where $\text{li}(x)$ is the logarithmic integral. We prove a higher rank version of this result without using Hypothesis A of Hooley. More precisely, let $\{a_1, a_2, \ldots, a_r\} \subset \mathbb{Q}^*$ be a multiplicatively independent set of integers. Let $\Gamma = \langle a_1, a_2, \ldots, a_r \rangle$ be the group generated by $a_1, a_2, \ldots, a_r$ in $\mathbb{Q}^*$. For primes $p$, define $i_{\Gamma}(p)$ to be $[(\mathbb{Z}/p\mathbb{Z})^* : \Gamma \mod p]$, where $\Gamma \mod p$ is the group generated by $a_1, a_2, \ldots, a_r$ inside $\mathbb{F}_p^*$. We show that, for $r \geq 2$, there is a positive constant $c_\Gamma > 0$ such that

$$\sum_{p \leq x} \log(i_{\Gamma}(p)) = c_\Gamma \text{li}(x) + O\left(\frac{x^{\theta}}{(\log x)^2}\right),$$

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1. Introduction

In 1927, Emil Artin made the following conjecture (see [1, Introduction] and [14]): let \( a \) be a fixed integer such that \( a \neq 0, \pm 1 \) or a perfect square. Write \( a = b^h \), where \( b \in \mathbb{Z} \) is not a perfect power and \( h \in \mathbb{N} \). For a group \( G \) with subset \( S \), let \( (S) \) denote the subgroup of \( G \) generated by \( S \). Define \( N_a(x) := \# \{ p \leq x : (\mathbb{Z}/p\mathbb{Z})^* = (a \text{ mod } p) \} \). Then,

\[
N_a(x) \sim A_h \pi(x),
\]

where \( \pi(x) = \# \{ p \leq x : p \text{ prime} \} \) and

\[
A_h = \prod_{q \mid h} \left( 1 - \frac{1}{q - 1} \right) \prod_{q \not\mid h} \left( 1 - \frac{1}{q(q - 1)} \right) > 0. \tag{1}
\]

The heuristic behind this conjecture is based on the following idea. We have \( a \) is primitive root modulo \( p \) if, and only if, \( a^{p-1} \equiv 1 \) (mod \( p \)) for all primes \( q \mid p-1 \). Now \( p \equiv 1 \) (mod \( q \)) occurs with a density of \( 1/\varphi(q) = 1/(q - 1) \) of the primes \( p \) by Dirichlet’s theorem on primes in arithmetic progression, and \( a^{p-1} \equiv 1 \) (mod \( q \)) occurs with a density of \( 1/q \) of the primes since \( a^{p-1} \) is a \( q \)-th root of unity, and there are exactly \( q \) of such elements.

It should be noted the constant was later realized to be incorrect for certain \( a \)'s (see the discussion after Theorem 1).

Artin’s conjecture is still unresolved. However, Hooley [14] provided a conditional resolution. First, we will need to introduce the following notational conventions: let \( f, g \) be functions. By \( f(x) = O(g(x)) \), or equivalently, \( f(x) \ll g(x) \), we mean that there exists a constant \( C > 0 \) such that, for all \( x \) in the domain of \( f/g \), we have \( |f(x)| \leq Cg(x) \). By \( f(x) = O_T(g(x)) \) or \( f(x) \ll_T g(x) \), we mean that the above constant is dependent on \( T \), where \( T \) is allowed to be a set. For example, \( T = \Gamma \), or \( T = \{ r \} \cup \Gamma \). In this latter case, we will write \( O_{r, \Gamma} \) instead of \( O_{\{ r \} \cup \Gamma} \). This notation may be dropped in proofs for convenience. By \( f(x) \sim g(x) \), we mean

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1,
\]

where \( x \) in the above limit is restricted to the domain of \( f \) and \( g \).

The statement “GRH holds for \( a \) on \( A \subset \mathbb{N} \)” will hereafter signify the statement “GRH holds for all Dedekind zeta functions for the fields \( \mathbb{Q}(\zeta_n, a^{1/n}) \), where \( \zeta_n \) is a primitive \( n \)-th root of unity and \( n \) ranges over all values of \( A \subset \mathbb{N} \).”

The statement “GRH holds for \( \Gamma \) on \( A \subset \mathbb{N} \)” will hereafter signify the statement “GRH holds for all Dedekind zeta functions for the fields \( \mathbb{Q}(\zeta_n, a_1^{1/n}, a_2^{1/n}, \ldots, a_r^{1/n}) \), where \( \zeta_n \) is a primitive \( n \)-th root of unity and \( n \) ranges over all values of \( A \subset \mathbb{N} \).”

For a number field \( K \) and a fixed \( \delta \in [1/2, 1) \), the statement “\( \delta \)-GRH holds for \( K \)” will hereafter signify the statement “\( \zeta_K(s) \neq 0 \) for all \( s \) with \( \Re(s) > 1 - \delta \), where \( \zeta_K \) is the Dedekind zeta function of \( K \).” For a fixed \( \delta \in [1/2, 1) \), the statement “\( \delta \)-GRH holds for \( \Gamma \) on \( A \subset \mathbb{N} \)” will hereafter signify the statement “\( \delta \)-GRH holds for all Dedekind zeta functions for the fields \( \mathbb{Q}(\zeta_n, a_1^{1/n}, a_2^{1/n}, \ldots, a_r^{1/n}) \), where \( \zeta_n \) is a primitive \( n \)-th root of unity and \( n \) ranges over all values of \( A \subset \mathbb{N} \).”

Hooley’s theorem [14] is the following result:

**Theorem 1 (Hooley).** Suppose \( a \in \mathbb{Z} \) such that \( a \neq 0, \pm 1 \) or a perfect square. Suppose further that GRH holds for \( a \) on squarefree positive integers. Then,

\[
N_a(x) = A(a)\pi(x) + O_a\left( \frac{x \log \log x}{(\log x)^2} \right).
\]
It should be noted that $A(a)$ in Theorem 1 is different from $A_h$ in (1). It was discovered by Artin and the Lehmers [17] that Artin’s original constant was off by a small factor for some $a \in \mathbb{Z}$ with $a \neq 0, \pm 1$ or a perfect square (see also Stevenhagen [26, §2]). In fact, let $h$ be as above, and let $a = a_1a_2^2$, where $a_1, a_2 \in \mathbb{Z}$ and $a_1$ is squarefree. If $a_1 \not\equiv 1 \pmod{4}$, then $A(a) = A_h$, and if $a_1 \equiv 1 \pmod{4}$, then
$$A(a) = A_h \left( 1 - \frac{1}{q - 2} \prod_{q | h} \frac{1}{q} \prod_{q | \gcd(h, a_1)} \frac{1}{q^2 - q - 1} \right).$$

The best unconditional results are of the following flavor: one of 2, 3, or 5 is a primitive root modulo $p$ for infinitely many primes $p$. In fact, we have
$$\# \{p \leq x: a \text{ is a primitive root modulo } p \} \geq \frac{cx}{(\log x)^2},$$
where $c > 0$ is a constant and $a$ is one of 2, 3, or 5. This result originates in the work of Gupta and Murty [12] and Heath-Brown [13]. It should be noted that 2, 3, and 5 are not the only set of integers for which this result is applicable. In fact, we need three non-zero multiplicatively independent integers $a$, $b$, and $c$ such that none of $a$, $b$, $c$, $-3ab$, $-3ac$, $-3bc$, or $abc$ is a square for the result to be true for one of $a$, $b$, or $c$.

1.1. Generalizing Artin’s conjecture

Let $a$ be as before, and let $p$ be a prime such that $p|a$. We denote by $f_a(p)$ and $i_a(p)$ the order of $a$ modulo $p$ and the index of $a$ modulo $p$, respectively.

We reformulate the quantity $N_a(x)$ in the following manner:
$$N_a(x) = \sum_{p \leq x} \chi_{\{1\}}(i_a(p)),$$
where $\chi_S$ is the characteristic function of the set $S$.

We would like to know what would occur if we change $\chi_{\{1\}}$ to a generic function $F : \mathbb{N} \to \mathbb{C}$. That is, can we obtain the following relation
$$\sum_{p \leq x} F(i_a(p)) \sim A_F(a)\pi(x)$$
where $A_F(a)$ is a constant dependent on $F$ and $a$? This question was first studied by Stephens [25], then by Wagstaff [27], Murata [18], Elliott and Murata [7], Pappalardi [22], Bach, Lukes, Shallit, and Williams [2], and Fomenko [11] among others. It is investigated in detail in [9]. Of course, the functions $F$ will have reasonable restrictions so as to not force an impossibility with the above relation. For example, $F(x) = x$ does not satisfy the above relation. To see this, we note that
$$\sum_{p \leq x} i_a(p) = \sum_{p \leq x} \sum_{d | i_a(p)} \varphi(d) = \sum_{d \leq x} \varphi(d) \sum_{p \leq x \atop d | i_a(p)} 1 \geq \sum_{d \leq (\log x)^{1/7}} \varphi(d) \# \{ p \leq x: d | i_a(p) \}.$$
By [10, Lemma 2.1], $d | i_a(p)$ if, and only if, $p$ splits completely in $\mathbb{Q}(\zeta_d, a^{1/d})$ over $\mathbb{Q}$. So, for $d \leq (\log x)^{1/7}$, the unconditional Chebotarev density theorem (see [16, Theorem 1.4] or [22, p. 376]) gives us

$$
\#\{ p \leq x : d | i_a(p) \} = \frac{\text{li}(x)}{[\mathbb{Q}(\zeta_d : a^{1/d}) : \mathbb{Q}]} + O\left(x \exp\left(-A(\log x)^{5/14}\right)\right)
$$

for any $B > 0$. We also have $[\mathbb{Q}(\zeta_d, a^{1/d}) : \mathbb{Q}] \asymp d \varphi(d)$ by [27, Proposition 4.1]. Hence,

$$
\sum_{p \leq x} i_a(p) \geq \sum_{d \leq (\log x)^{1/7}} \varphi(d)\left(\frac{\text{li}(x)}{[\mathbb{Q}(\zeta_d : a^{1/d}) : \mathbb{Q}]} + O\left(\frac{x}{(\log x)^B}\right)\right)
$$

for any $B \geq 1 + 2/7$.

The case where the function $F(x) = \log x$ and $a = 2$ was first studied by Bach, Lukes, Shallit and Williams [2]. We refer to the following relation as Fomenko’s conjecture since Fomenko [11] (see Theorem 3 below) proved it using GRH and Hypothesis A of Hooley [15, p. 112] (see the hypothesis below):

$$
\sum_{p \leq x} \log(i_a(p)) \sim c_a \text{li}(x),
$$

for some constant $c_a > 0$. The authors of [2] mention heuristics that suggest the above relation is true for $a \geq 2$ and give computational evidence for $a = 2$, $a = 3$, and $a = 5$.

Pappalardi [22] showed the following theorem:

**Theorem 2** (Pappalardi). For $a \in \mathbb{Z}$ different from 0 and $\pm 1$, we have

$$
\frac{x}{\log x} \ll \sum_{p \leq x} \log(i_a(p)) \ll \frac{x \log \log x}{\log x},
$$

where the lower bound is unconditional, and for the upper bound, we suppose GRH holds for $a$ on prime powers.

Recall the following hypothesis of Hooley [15, p. 112]:

**Hypothesis** (Hypothesis A of Hooley). Let $P_b(y; \ell, t)$ be the number of primes $p \leq y$ such that $2^t b$ is an $\ell$th-power residue modulo $p$ and for which $\ell | p - 1$. Then, for $y^{1/2} < \ell < y$, we have

$$
P_b(y; \ell, t) \ll \frac{y}{\varphi(\ell)(\log(2y/\ell))^2},
$$

where the implied constant is absolute.
For any \(a\) which is an integer not equal to 0 or \(\pm 1\), we have the following theorem of Fomenko [11]:

**Theorem 3** (Fomenko). Suppose GRH holds for \(a\) on prime powers. Suppose further Hypothesis A of Hooley holds. Then,

\[
\sum_{p \leq x} \log(i_a(p)) = c_a \log x + O\left(\frac{x \log \log x}{(\log x)^2}\right),
\]

where \(c_a\) is an effectively computable constant dependent on \(a\), and where

\[
\log x = \int \frac{1}{\log t} dt.
\]

In fact, letting \(t = 0\) and restricting \(\ell\) to be a prime in the range \((\sqrt{y}/(\log y)^4, \sqrt{y}(\log y)^2]\) in Hypothesis A of Hooley is all that is needed to prove the above theorem since this gives us

\[
P_b(y; \ell, 0) \ll \frac{y}{\varphi(\ell)(\log y)^2}
\]

in this range.

Hooley’s Hypothesis A is difficult to prove or refute, even numerically. In fact, in order to refute it, one would need to show that

\[
\limsup_{y \to \infty} \frac{P_b(y; \ell, t)}{y/\varphi(\ell)(\log(2y/\ell))^2} = \infty.
\]

That is, for every \(y\), we need to find the \(\ell_y \in (y^{1/4}, y)\) that corresponds to the maximum value for

\[
\frac{P_b(y; \ell, t)}{y/\varphi(\ell)(\log(2y/\ell))^2}
\]

with \(\ell \in (y^{1/4}, y)\). Finding this \(\ell_y\) will become more difficult as \(y \to \infty\). Also, calculating \(P_b(y; \ell, t)\) is highly non-trivial.

We also note that in the above range it is sufficient to assume the Pair Correlation Conjecture instead of Hypothesis A of Hooley. For a formulation of this conjecture, see Murty and Murty [20]. In fact, this conjecture allows us to obtain error terms which are significantly better than in Theorem 1 as well as in the above theorem. It also allows us to solve Hooley’s Hypothesis A in the above range as long as \(t\) is still 0.

**1.2. Higher rank versions**

We let \(\{a_1, a_2, \ldots, a_r\}\) be a multiplicatively independent set of rational numbers. That is, \(a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r} = 1\) with \((n_1, n_2, \ldots, n_r) \in \mathbb{Z}^r\) if, and only if, \((n_1, n_2, \ldots, n_r) = (0, 0, \ldots, 0)\). Throughout \(p\) and \(q\) will denote prime numbers, and \(x, y,\) and \(z\) will denote positive real numbers. Let \(v_p(a)\) be the \(p\)-adic valuation of \(a \in \mathbb{Q}^*\). That is, if \(a = p^n(c/d)\) with \(\gcd(c, d) = \gcd(p, cd) = 1\), then \(v_p(a) = n\). Let \(\Gamma = \langle a_1, a_2, \ldots, a_r \rangle\) be the subgroup of \(\mathbb{Q}^*\) generated by \(a_1, a_2, \ldots, a_r \in \mathbb{Q}^*\). For every prime number \(p\) with \(v_p(a) = 0\) for all \(a \in \Gamma\), we define

\[
\Gamma_p := \Gamma \mod p = \{a \mod p: a \in \Gamma\}.
\]
For each \(i \in \{1, 2, \ldots, r\}\), write \(a_i = A_i / B_i\) with \(A_i, B_i \in \mathbb{Z}\), \(\gcd(A_i, B_i) = 1\), and \(B_i > 0\). There is a unique choice for these \(A_i\)'s and \(B_i\)'s. We note that

\[
\{ p: \nu_p(a) \neq 0 \text{ for some } a \in \Gamma \} = \{ p: p | A_i B_i \text{ for some } i \in \{1, 2, \ldots, r\} \}.
\]

Thus, \(\#\{ p: \nu_p(a) \neq 0 \text{ for some } a \in \Gamma \} < \infty\), and so, we will ignore this set throughout since it only contributes \(O(1)\) to any summation of interest. Notice that \(\Gamma_p\) is a subgroup of \(\mathbb{F}_p^*\). So, we define the index of \(\Gamma\) modulo \(p\) as the index of \(\Gamma_p\) over \(\mathbb{F}_p^*\) and denote it by \(i_{\Gamma}(p)\). That is,

\[
i_{\Gamma}(p) := \left[ \mathbb{F}_p^*: \Gamma_p \right] = \frac{p - 1}{|\Gamma_p|}.
\]

We also define the order of \(\Gamma\) modulo \(p\) and denote it by \(f_{\Gamma}(p) := |\Gamma_p|\).

We are interested in computing

\[
\sum_{p \leq x} \log(i_{\Gamma}(p)).
\]

In Section 2, we will prove a similar result to that of Theorem 3 for \(r \geq 2\). However, we will not need a higher rank version of Hypothesis A of Hooley to do this. We first define the following constants:

\[
\theta := \theta : \begin{cases} 
9/10 & \text{if } r = 2, \\
6/7 & \text{if } r = 3, \\
5/6 & \text{if } r \geq 4
\end{cases}
\]

and

\[
\alpha := \alpha : \begin{cases} 
8/5 & \text{if } r = 2, \\
11/7 & \text{if } r = 3, \\
14/9 & \text{if } r = 4, \\
4/3 & \text{if } r \geq 5.
\end{cases}
\]

**Theorem 4.** Let \(r \geq 2\). Suppose GRH holds for \(\Gamma\) on prime powers. Then, there exists a constant \(c_{\Gamma}\) such that

\[
\sum_{p \leq x} \log(i_{\Gamma}(p)) = c_{\Gamma} \text{li}(x) + O_{\Gamma}(x^{\theta}(\log x)^{\alpha}),
\]

where the implied constant is dependent on \(r\) and \(a_1, a_2, \ldots, a_r\).

In fact, we will be able to show that in Theorem 4, only \(\delta\)-GRH is necessary, where \(\delta \in [1/2, 1)\) for \(r\) sufficiently large. To do this, we first extend the definitions of \(\theta\) and \(\alpha\) as follows:

\[
\theta := \theta(\delta, r) := \begin{cases} 
(1+\delta)(r+1) & \text{if } \frac{\delta}{1-\delta} < r < \frac{1+2\delta}{1-\delta}, \\
\frac{2+\delta}{\delta} & \text{if } r \geq \frac{1+2\delta}{1-\delta}
\end{cases}
\]

and
\[ \alpha := \alpha(\delta, r) := \begin{cases} \frac{3r+2}{2r+1} & \text{if } \frac{\delta}{1-\delta} < r \leq \frac{1+2\delta}{1-\delta}, \\ \frac{4}{3} & \text{if } r > \frac{1+2\delta}{1-\delta}. \end{cases} \]

We note that \( \theta(1/2, r) = \theta(r) \) and \( \alpha(1/2, r) = \alpha(r) \).

In Section 3, we will prove the following theorem:

**Theorem 5.** Let \( \delta \in [1/2, 1) \) be fixed. Suppose \( \delta \)-GRH holds for \( \Gamma \) for some \( \delta \in [1/2, 1) \) fixed. For \( r > \delta/(1-\delta) \), we have

\[
\sum_{p \leq x} \log(i_{\Gamma}(p)) = \text{li}(x) \sum_{d=1}^{\infty} \frac{\Lambda(d)}{n_d} + O_F\left(x^\theta(\log x)^\alpha\right),
\]

where \( n_d = [\mathbb{Q}(\zeta_d, a_1^{1/d}, a_2^{1/d}, \ldots, a_r^{1/d}) : \mathbb{Q}] \) and the implied constant is dependent only on \( r \) and \( a_1, a_2, \ldots, a_r \).

In Theorems 4 and 5, we obtain power-saving results. That is, all of our error terms are \( O(x^\theta(\log x)^\alpha) \) with \( \theta < 1 \) and \( \alpha \in \mathbb{R} \) fixed. This is in stark contrast to Theorems 1 and 3 where error terms are of the form \( x \log \log x \left(\log x\right)^2 \).

The study of these types of questions first began with Gupta and Murty [12] (it is what originally led to the unconditional results about Artin’s conjecture) and continued with Pappalardi [23] and Cangelmi and Pappalardi [3], who studied how often \( \Gamma_p = (\mathbb{Z}/p\mathbb{Z})^* \), the \( r \)-rank analogue of Artin’s conjecture.

We note that these results can be used to prove that the smallest prime \( p_\Gamma \) for which \( i_{\Gamma}(p) \neq 1 \) satisfies \( p_\Gamma \ll (\log a_1 a_2 \cdots a_r)^{6+\epsilon} \). However, by looking at the primes which split in \( \mathbb{Q}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_r}) \) the same can be done.

In Section 4, we will prove a corollary of Theorems 4 and 5.

### 2. Proof of Theorem 4

#### 2.1. The Chebotarev density theorem

The Chebotarev density theorem is one of the main tools we will need in order to prove the results stated within.

Let \( K \) be a finite Galois extension of \( \mathbb{Q} \) with Galois group \( G \), degree \( n_K \), and discriminant \( d_K \). Let \( \pi_K(x) \) denote the number of primes \( p \leq x \) for which \( p \) splits completely in \( K \) over \( \mathbb{Q} \). Then, the following corollary of the Chebotarev density theorem [4,5] states

\[
\pi_K(x) \sim \frac{\text{li}(x)}{|G|},
\]  

as \( x \to \infty \). The original statement of the Chebotarev density theorem is a more general statement about how frequent the conjugacy class of the Frobenius automorphism associated to \( p \) is equal to a fixed conjugacy class of \( G \). In order to use this result, we need error terms. Such a result is due to Lagarias and Odlyzko [16]. It has been improved by Serre [24], Murty, Murty, and Saradha [21], and Murty and Murty [20]. The following version is Serre’s [24] refinement of Lagarias and Odlyzko’s result [16].
Theorem 6. Let $K$ be as above. Assuming GRH for the Dedekind zeta function of $K$, we have

$$\pi_K(x) = \frac{\text{li}(x)}{|\mathcal{G}|} + O\left(\sqrt{x}\left(\frac{\log|d_K|}{n_K} + \log x\right)\right),$$

(3)

where the implied constant is absolute.

The following result, known as Hensel’s inequality, is useful for bounding the error term in Theorems 4 and 5 (see [24, p. 130]):

Lemma 1. Let $K$ be a finite Galois extension with degree $n_K$ and discriminant $d_K$. Then,

$$\log|d_K| \leq n_K \left(\log n_K + \sum_{p \in \mathcal{P}(K/Q)} \log p\right),$$

(4)

where $\mathcal{P}(K/Q)$ is the set of prime numbers $p$ which ramify in $K$ over $Q$.

2.2. The proof of Theorem 4

For $d \in \mathbb{N}$, define

$$\pi_d(x) := \#\{p \leq x : d|i\Gamma(p)\}.$$ 

Then, we have

$$\sum_{p \leq x} \log(i\Gamma(p)) = \sum_{d \leq x} \Lambda(d)\pi_d(x).$$

Let $d \in \mathbb{N}$ be a fixed integer. We note that $d$ divides $i\Gamma(p) = \gcd(i_{a_1}(p), i_{a_2}(p), \ldots, i_{a_r}(p))$ if, and only if, $d|i_{a_j}(p)$ for all $j \in \{1, 2, \ldots, r\}$. This is true if, and only if, $p$ splits completely in $\mathbb{Q}(\zeta_d, a_j^{1/d})$ for all $j \in \{1, 2, \ldots, r\}$. So, $d|i\Gamma(p)$ if, and only if, $p$ splits completely in $\mathbb{Q}(\zeta_d, a_1^{1/d}, a_2^{1/d}, \ldots, a_r^{1/d})$. We note that if a prime ramifies in $\mathbb{Q}(\zeta_d, a_1^{1/d}, a_2^{1/d}, \ldots, a_r^{1/d})$, then it must divide $d\mathcal{P}$, where $\mathcal{P}$ is the product of all primes $p$ with $\nu_p(a) \neq 0$ for some $a \in \Gamma$ by an argument similar to the case when $\Gamma$ is generated by a single element. By Theorem 6 and Lemma 1, we have

$$\pi_d(x) = \frac{\text{li}(x)}{|K_d : \mathbb{Q}|} + O\left(\sqrt{x}\log(xd^{r+2}P)\right),$$

where $P$ is the product of primes $p$ with $\nu_p(a) \neq 0$ for some $a \in \Gamma$, $K_d = \mathbb{Q}(\zeta_d, \Gamma^{1/d})$, and $\Gamma^{1/d} := \{a^{1/d} : a \in \Gamma\}$. Let $n_d = [K_d : \mathbb{Q}]$. Thus, for some $y$ to be chosen later, we have

$$\sum_{d \leq y} \Lambda(d)\pi_d(x) = \sum_{d \leq y} \Lambda(d)\left(\frac{\text{li}(x)}{n_d} + O\left(\sqrt{x}\log(xd^{r+2}P)\right)\right)$$

$$= \text{li}(x) \sum_{d \leq y} \frac{\Lambda(d)}{n_d} + O\left(\sum_{q^a \leq y} \log q \sqrt{x}(\log x + \alpha(r + 2)\log q + \log P)\right).$$
Now,

\[ \sum_{d \leq y} \frac{\Lambda(d)}{n_d} = \sum_{d=1}^{\infty} \frac{\Lambda(d)}{n_d} - \sum_{d > y} \frac{\Lambda(d)}{n_d}. \]

We note that the first summation above is a constant since \( n_d \geq \left[ \mathbb{Q} \left( \zeta_d, a_1^{1/d} \right) : \mathbb{Q} \right] \approx d \varphi(d) \) by \([27, \text{Proposition 4.1}]\). We have

\[ \sum_{d > y} \frac{\Lambda(d)}{n_d} \ll \sum_{d > y} \frac{\Lambda(d)}{\varphi(d)d^\alpha} \ll \frac{1}{y} \]

by partial summation and \([6, \text{Exercise 5.5.3}]\). Thus,

\[ \sum_{d \leq y} \Lambda(d)\pi_d(x) = \frac{\text{li}(x)}{y} \sum_{d=1}^{\infty} \frac{\Lambda(d)}{n_d} + O \left( \frac{\text{li}(x)}{y} \right) \]

\[ + O \left( \sum_{q^\alpha \leq y} \log q \sqrt{x} (\log x + \alpha (r + 2) \log q + \log P) \right). \]

Now,

\[ \sqrt{x} \log x \sum_{q^\alpha \leq y} \log q = \sqrt{x} \log x \sum_{n \leq y} \Lambda(n) \ll y \sqrt{x} \log x. \]

Also,

\[ \sqrt{x} \sum_{q^\alpha \leq y} \alpha(r + 2)(\log q)^2 \ll (r + 2) \sqrt{x} \log y \sum_{n \leq y} \log n \ll y \sqrt{x} (\log y)^2, \]

and

\[ \log P \sum_{q^\alpha \leq y} \log q = \log P \sum_{n \leq y} \Lambda(n) \ll y \log P. \]

We have \( \log P \ll_{\Gamma} 1 \) as \( P \) is the product (of a finite number) of (fixed) primes \( p \) which satisfy \( \nu_p(a) \neq 0 \) for some \( a \in \Gamma' \). Therefore, we have

\[ \sum_{d \leq y} \Lambda(d)\pi_d(x) = \text{li}(x) \sum_{d=1}^{\infty} \frac{\Lambda(d)}{n_d} + O \left( \frac{\text{li}(x)}{y} + y \sqrt{x} \log x + y \sqrt{x} (\log y)^2 \right). \]

Now, we must deal with

\[ \sum_{y < d \leq x} \Lambda(d)\pi_d(x). \]

We have
\[
\sum_{y < d \leq x} \Lambda(d) \pi_q(x) \ll \log x \sum_{y < q^\alpha \leq x} \pi_q(x)
\]
\[= \log x \sum_{y < q^\alpha \leq x} \#\{ p \leq x: q^\alpha | i_p(p) \}.\]

Let us consider
\[
\sum_{y < q^\alpha \leq x} \#\{ p \leq x: q^\alpha | i_p(p) \} = \sum_{y < q \leq x} \#\{ p \leq x: q | i_p(p) \} + \sum_{y < q^\alpha \leq x} \#\{ p \leq x: q^\alpha | i_p(p) \}.
\]

Let
\[
y = \frac{x^{\Theta}}{(\log x)^A},
\]
where
\[
\Theta = \begin{cases} 
 2/5 & \text{if } r = 2, \\
 5/14 & \text{if } r = 3, \\
 1/3 & \text{if } r \geq 4
\end{cases}
\]
and
\[
A = \begin{cases} 
 2/5 & \text{if } r = 2, \\
 3/7 & \text{if } r = 3, \\
 4/9 & \text{if } r = 4, \\
 2/3 & \text{if } r \geq 5.
\end{cases}
\]

We note $\Theta \geq 1/3$ for all $r \geq 2$.

We note that
\[
\sum_{y < q^\alpha \leq x} \#\{ p \leq x: q^\alpha | i_p(p) \} \ll \sum_{y < q^\alpha \leq x} \frac{x}{q^\alpha} \ll \frac{x}{\sqrt{y}}.
\]

We also claim
\[
\sum_{y < q \leq x} \#\{ p \leq x: q | i_p(p) \} \ll \#\left\{ p: |I_p| \leq \frac{x}{y} \right\}.
\]

To see this, suppose the prime number $p$ contributes to the left-hand side. That is, suppose there exist primes $q_1, q_2, \ldots, q_n \in (y, x]$ such that $q_i | i_p(p)$. Since $q_1, q_2, \ldots, q_n \in (y, x]$ are distinct,
\[
\frac{x^{\Theta A}}{(\log x)^A} = y^n < q_1 q_2 \cdots q_n < i_p(p) < x.
\]
Therefore, $n \leq \Theta^{-1} \leq 3$ for $x$ sufficiently large. Thus,

$$
\sum_{y < q \leq x} \# \{ p \leq x : q \mid i \Gamma(p) \} \ll \# \{ p : |\Gamma_p| \leq \frac{x}{y} \}.
$$

However, before we can bound this latter quantity, let us recall the following result of Gupta and Murty [12]:

**Lemma 2** (Gupta and Murty). Suppose $r \geq 2$. Then

$$
\# \{ p : |\Gamma_p| \leq r \} \ll r^{1+\frac{1}{r}}
$$

where the implied constant is dependent on at most $r$ and $a_1, a_2, \ldots, a_r$.

Thus, we have

$$
\sum_{y < q \leq x} \# \{ p \leq x : q \mid i \Gamma(p) \} \ll \left( \frac{x}{y} \right)^{1+\frac{1}{r}}.
$$

Therefore,

$$
\sum_{p \leq x} \log(i \Gamma(p)) = \text{li}(x) \sum_{d=1}^{\infty} \frac{A(d)}{n_d} + O \left( \frac{\text{li}(x)}{y} \right) + O \left( y \sqrt{x \log x} \right)
$$

$$
+ O \left( y \sqrt{x (\log y)^2} \right) + O \left( \frac{x}{\sqrt{y}} \log x \right) + O \left( \left( \frac{x}{y} \right)^{1+\frac{1}{r}} \log x \right)
$$

$$
= \text{li}(x) \sum_{d=1}^{\infty} \frac{A(d)}{n_d} + O \left( x^{1-\Theta} (\log x)^A \right) + O \left( \frac{x^{1+\Theta}}{(\log x)^{A-2}} \right)
$$

$$
+ O \left( x^{1-\Theta} \left( \log x \right)^{\frac{A}{2}+1} \right) + O \left( x^{(1-\Theta)(\frac{1}{2}+1)} \right).
$$

Therefore, Theorem 4 holds.

### 3. Proof of Theorem 5

Recall that $p$ splits completely in $\mathbb{Q}(|a_1^{1/d}, a_2^{1/d}, \ldots, a_r^{1/d})$ if, and only if, $d \mid i \Gamma(p)$. Also, recall $\pi_d(x) := \# \{ p \leq x : d \mid i \Gamma(p) \}$. Therefore, by the $\delta$-GRH and the Chebotarev density theorem (mimicking Lagarias and Odlyzko [16] and Serre [24]), we have

$$
\pi_d(x) = \frac{\text{li}(x)}{|K_d : \mathbb{Q}|} + O \left( x^{\delta} \log(xd^{r+2}) \right),
$$

where $P$ is the product of all primes $p$ satisfying $\nu_p(a) \neq 0$ for some $a \in \Gamma$.

Let

$$
\Theta := \Theta(\delta, r) := \begin{cases} 
\frac{r(1-\delta)+1}{2r+1} & \text{if } \frac{\delta}{1-\delta} < r < \frac{1+2\delta}{1-\delta}, \\
\frac{2(1-\delta)}{r} & \text{if } r \geq \frac{1+2\delta}{1-\delta}
\end{cases}
$$
and

\[ A := A(\delta, r) := \begin{cases} \frac{r}{2r+1} & \text{if } \frac{\delta}{1-\delta} < r \leq \frac{1+2\delta}{1-\delta}, \\ \frac{2}{3} & \text{if } r > \frac{1+2\delta}{1-\delta}. \end{cases} \]

Note that \( \Theta \) and \( A \) in the proof of Theorem 4 are equal to \( \Theta(1/2, r) \) and \( A(1/2, r) \), respectively.

Let \( y = x^{\Theta}/(\log x)^A \). Since \( \Theta > 0 \), all the arguments from the previous section are still true, and we have

\[
\sum_{p \leq x} \log(i_{r}(p)) = \text{li}(x) \sum_{d=1}^{\infty} \frac{A(d)}{n_d} + O \left( \frac{x^{\delta+\Theta}}{(\log x)^{A-2}} \right) + O \left( x^{1-\frac{\Theta}{2}} (\log x)^{\frac{\delta}{2}+1} \right) + O \left( x^{\frac{(1-\delta)(r+1)}{r}} (\log x)^{\frac{\delta}{r}+1} \right).
\]

This completes the proof of Theorem 5 with

\[
c_{r} = \sum_{d=1}^{\infty} \frac{A(d)}{n_d}.
\]

It should be noted that \( \Theta < 1 \) for all \( \delta \in [1/2, 1) \) and \( r > \delta/(1-\delta) \). To see this we note that for case 1,

\[
\frac{(1+\delta)(r+1)}{2r+1} < 1.
\]

This is true if, and only if,

\[
r > \frac{\delta}{1-\delta}.
\]

For case 2, we have

\[
\frac{2+\delta}{3} < 1.
\]

Therefore, as when we assumed GRH, we have obtained power-saving error terms as long as \( r > \delta/(1-\delta) \). We can use the above inequalities on \( r \) to determine corresponding inequalities for \( \delta \) if so desired. Doing this yields the following theorem:

**Theorem 7.** Let \( r \geq 2 \) and let \( \delta < r/(r+1) \). Let \( \theta \) and \( \alpha \) be defined as in Theorem 5. Suppose \( \delta \)-GRH holds for \( \Gamma^* \) on prime powers. Then, we have

\[
\sum_{p \leq x} \log(i_{r}(p)) = \text{li}(x) \sum_{d=1}^{\infty} \frac{A(d)}{n_d} + O \left( x^{\Theta} (\log x)^\alpha \right),
\]

where \( n_d := [\mathbb{Q}(\zeta_d, a_1^{1/d}, a_2^{1/d}, \ldots, a_r^{1/d}) : \mathbb{Q}] \) and the implied constant is dependent on \( r \) and \( a_1, a_2, \ldots, a_r \).
4. An immediate corollary

We have the following corollary of Theorems 4 and 5:

**Corollary 1.** Let $r \geq 2$ and $\delta \in [1/2, 1)$ with $\delta < r/(r+1)$. Suppose $\delta$-GRH holds for $\Gamma$ on prime powers.

$$
\sum_{p \leq x} \log(f_r(p)) = x - c_r \text{li}(x) + O_r(x^\delta (\log x)^\alpha),
$$

where $\theta = \theta(\delta, r)$ and $\alpha = \alpha(\delta, r)$ in Theorem 5.

**Proof.** Note that

$$
\sum_{p \leq x} \log(f_r(p)) = \sum_{p \leq x} \log(p - 1) - \sum_{p \leq x} \log(i_r(p))
$$

$$
= \sum_{p \leq x} \log p + \sum_{p \leq x} \log \left(\frac{p - 1}{p}\right) - \sum_{p \leq x} \log(i_r(p))
$$

$$
= \sum_{p \leq x} \log p - \sum_{p \leq x} \log(i_r(p)) + O(\log \log x).
$$

Also, from Theorem 4 or 5, we have

$$
\sum_{p \leq x} \log(i_r(p)) = c_r \text{li}(x) + O(x^\delta (\log x)^\alpha).
$$

So, we need to evaluate

$$
\sum_{p \leq x} \log p.
$$

We note that by the Aramata–Brauer theorem (see [8, §11.4] or [19, §2.3]), the $\delta$-Riemann hypothesis (that is, the $\delta$-GRH for $K = \mathbb{Q}$) is also true since our extension is Galois. That is, there are no zeros of the Riemann zeta function in the region $\Re(s) > \delta$ and $\delta \in [1/2, 1)$. Therefore, we may assume

$$
\pi(x) = \text{li}(x) + O(x^\delta \log x),
$$

or equivalently,

$$
\theta(x) := \sum_{p \leq x} \log p = x + O(x^\delta (\log x)^2).
$$

Putting all of this together we obtain

$$
\sum_{p \leq x} \log(f_r(p)) = x + O(x^\delta (\log x)^2) - c_r \text{li}(x) + O(x^\delta (\log x)^\alpha)
$$

$$
= x - c_r \text{li}(x) + O(x^\delta (\log x)^\alpha)
$$

since $\theta > \delta$. □
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