# BUILDING OF $\mathrm{GL}_n(\mathbb{Q}_p)$

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## 1. What is a building and why do we care?

A building is an a basically a set satisfying a bunch of seemingly random combinatorial relations such that one might well believe that examples simply don't exist or that they would otherwise serve no particular purpose.

It turns out that since their intruduction they have become a topic of keen interest to Geometric group theorists as well as having a number of surprising applications in number theory. As an example they are able to play a role analogous to that of symmetric spaces in the theory of p-adic modular forms. That is one can define modular forms on them, construct Hecke operators, and in many cases recover the same Hecke modules as with classical forms, one can do many of the same modular form things with these spaces. These sorts of things have in fact been worked on by former (and possibly current) students here.

## 2. Abstract Buildings

Now seriously, what is a Building?

An $n$-dimensional building is:

- An $n$-dimensional simplicial complex.
  A set of things with elaborate adjacency conditions satisfying many hypothesis.
  That is there are **facets** (simplicies) that are all attached together.
- The object can be subdivided into **apartments**.
  This subdivision satisfies even more hypotheses.
  Every 2 simplicies in $X$ lie in a common appartment.
  If $x, y \subset A \cap A'$ there is an isomorphism of $X$ taking $A$ to $A'$ fixing $x, y$ pointwise. (This axiom is equivalent to:)
  special case $x$ a chamber.
  If $C \subset A \cap A'$ there is an isomorphism $A \to A'$ fixing $A \cap A'$ pointwise.
  Every appartment is a coxeter complex (This axiom can be weakened if we assume thickness)
  Optional, every $k$-simplex with $k < n$ is in at least 3 $n$-simplicies. (this is thickness, it is an extra assumption)

Basically a building is a whole lot of combinatorial data with even more extra terminology. We summarize some of it as follows:

- We call the $n$-simplicies 'chambers' (rooms).
- We have the notion of types of adjacency for chambers, this is controlled by the type of simplex in which they intersect. But also by the extent to which the automorphism group acts transitively on adjacency.
- We have a notion of gallery, this is a sequence of adjacent chambers (with specified adjacency types within the sequence).

We can create notion of homotopy of galleries and reduced words and use this to define a distance.
- The rank of the building is $n+1$.

It is important to realize that buildings are typically very infinite, but that these seemingly random hypothesis give very huge restrictions on the types of constructions that actually arise. That is to say, buildings can be classified into several families, the structure of families is similar to algebraic groups (though there are many more families for buildings). There are also a lot of things one can prove about all buildings as a consequence of just these random axioms.

## 3. Actions of Groups on Buildings

People actually study abstract buildings because of their combinatorics. We are not interested in this per se, we are interested in using them as an object on which groups act to understand properties of the group.

The definition of a building implies it will typically have many automorphisms. (This is provably very true).

**Theorem 3.1.** *Fixing a chamber $C$, the group $W$ of automorphisms of $X$ mapping $C$ to $C$ is a coxeter group. It is independent of $C$.*

**Definition 3.2.** Coxeter group is a 'reflection group', that is have relations $(r_i r_j)^{m_{ij}} = 1$ and $r_i^2 = 1$.

Note that there is a classification of Coxeter groups by Dynkin diagrams which is largely similar to that for Simple groups. The usual Dynkin classification is classifying Root systems and not their associated groups. Root systems have a few extra axioms so are slightly more restrictive than what occurs with Coxeter groups. Hence the families here are slightly different.

**Theorem 3.3.** *The apartments of $A$ are all isomorphic to the coxeter complexes of $W$, hence are determined by $W$, and hence by $C$ and even more hence by $X$.*

This gives a very strong restriction on the sorts of things that can exist as buildings, and almost implies that the building structure on a building is uniquely determined by simplicial structure.

**Remark.** I am not going to define what the Coxeter complex of a Coxeter group is. The key thing is that it is a simplicial complex on which the group acts.

The best examples are decompositions of spheres with many reflexive symmetries and tessilations of affine spaces. Given a 'reflection group' acting on space you can decompose the sphere into fundamental domains by cutting it with hyperplanes that are fixed by the reflections. This realizes the coxeter complex on the sphere.

## 4. BN-pairs

We wish to study groups by studying their action on buildings. In this context there is an alternate language that is used which focuses on the combinatorics of the groups, that is the terminology of $(B,N)$-pairs. (It arises as an alternate way to capture the same combinatorial data that buildings capture in the context that we are interested in, there is some history in the timing of the development of both theories)

Let $G$ be any group acting simplicially on $X$, we say that action is strongly transitive if:
- The stabilizer of an apartment in $G$ acts transitively on its chambers.
- $G$ acts transitively on ordered pairs of chambers $(x, wx)$ where $w \in W$.

In this case we make the following definitions

**Definition 4.1.** Define a **Borel** to be $B$ the stabilizer in $G$ of a chamber $C$. Define a **Cartan** to be $N$ the stabilizer in $G$ of an apartment $A$ containing $C$. We define the **Parabolics** to be the stabilizers of facets of $X$.

And we find:

**Theorem 4.2.** *If $X$ is a thick building and $G$ acts strongly transitively on $X$ then $B, N$ are a (B,N)-pair or tits system ie:*

- *$G$ is generated by $B, N$.*
- *$B \cap N$ is normal in $N$ and $W = N/B \cap N$.*
  *Let $s_i$ be generators of $W$.*
- *$BsBwB \subset BwB \cup BswB$.*
- *$sBs \neq B$, for $s \neq 1$*

**Proposition 4.3.** *The axioms of a tits system imply:*

- *The generators of $W$ can be taken to have order $2$ and $W$ is a coxeter group.*
- *$B \cup BW'B = \cup_w BwB$ is a group for every special subgroup $W'$ of $W$.*
- *$G = BWB = \cup_w BwB$.*
- *$BwB = BswB$ if $\ell(sw) \geq \ell(w)$ (length of minimal presentation of word)*
- *$BwB = BwB \cup BswB$ if $\ell(sw) \leq \ell(w)$ (length of minimal presentation of word in terms of a system of generators)*

**Remark.** All the language, Borel, Cartan, Parabolic is highly abusive. Alternatives exist in different contexts (Iwahori, ..., Parahoric)

**Theorem 4.4.** *A tits system determines a building.*

Set the chambers to be the cosets $gB$ (or $gBg^{-1}$. Get the $w$-adjacency from $gB \subset BwB$. The faces correspond to parabolics $gBW'B$.

**Remark.** A group $G/$ may have many tits systems, which may determine many different buildings or the same building in different ways.

The group $G$ is not uniquely determined by the building in general, there may exist many groups with strongly transitive actions on a given building. (Eg. the affine buildings for $\mathrm{SL}_2(K)$ where $K$ has residue field $\mathbb{F}_q$ are all the exact same tree).

Most bad counterexamples happen with low rank. For sufficiently large rank (3), think buildings arise from BN-pairs.

## 5. Spherical vs Affine

A building is said to be spherical if $W$ is a finite Coxeter group. (so that we may realize the Coxeter complex as a topological sphere, by viewing the group as a linear reflection group).

A building is said to be affine if $W$ if the coxeter complex comes from the tiling of an affine plane. In this case it has the property that there is an normal abelian subgroup such that the quotient is a finite Coxeter group.

**Remark.** There is a lot that can be said about this phenominon, in fact there is a relation among the Dynkin diagrams, which in turn identifies special (or hyperspecial) elements of the weyl group which in turn gives us special (and hyperspecial) subgroup. Moreover, one can construct a spherical building at infinity.

**Remark.** This is not a complete classification, ie there exist hyperbolic reflection groups and more generally other reflection groups

All the buildings we shall see will be associated to affine or spherical Coxeter groups.

We may be able to associate both Spherical and Affine buildings to the same ‘group scheme’ $G$. In fact there may be many of each.

## 6. Examples

We want to associate buildings to algebraic groups $G/K$ where $K$ is some field. We may of course base change $K$ to any other field to construct the building there, different constructions make sense in different contexts and may be ‘related’.

6.1. **Over algebraically closed.** This is what we call the Spherical building!

Suppose $K$ is algebraically closed, we define the spherical building associated to $G/K$ to be:

*The Chamber system of parabolic subgroups defined over $K$*

The verticies are associated to the maximal parabolics.

That is, for each conjugacy class of maximal parabolic $P_i$ we have verticies:

$$(G/P)(K)$$

Two verticies are on a common face if $gP \cap P$ is parabolic, that is we have faces of different dimension associated to the points of:

$$(G/P)(K)$$

For a system of parabolics $B \subset P$.

An apartment is the set of all parabolics containing a given maximal torus.

It should be apparent that this whole setup comes with a $G$ action. Moreover, if $G$ was actually defined over $k$, you can bootstrap in a $\mathrm{Gal}(K/k)$ action!!!. (This is not totally trivial)

So in this context we have: $B$ is actually Borel, $N$ is the normalizer of a maximal torus contained in $B$ (A Cartan subgroup). $P$ are parabolics, $W$ is the usual Weyl group.

**Example.** For $\mathrm{SL}_n$.

Faces are flags $V_1 \subset \cdots \subset V_\ell$.

Adjacency corresponds to refinement of flags.

The chambers thus correspond to complete flags. (Note the inclusion reversing tendency here).

An apartment is the set of all flags which can be constructed from a basis: $\{e_1, \ldots, e_n\}$. That is any flag whose subspaces have bases consisting of those elements.

6.2. **Over Non-algebraically closed.** This is (also) what we call the Spherical building!

The chambers correspond to maximal parabolics (defined over your field!!).

The faces correspond to parabolics contained in the maximal one.

Two chambers thus intersect if their associated parabolics intersect in a parabolic.

The stabilizer of a face is the corresponding parabolic.

An apartment correspond to maximally split tori, they contain all faces where the parabolics contain the given torus.

Fix $S$ a maximal split torus.

The group $B$ is a minimal parabolic containing $S$ .

The group $N$ is the normalizer of $S$.

(One should notice that this naturally maps into the one above, and indeed it is the Galois invariants in the object over the algebraic closure.) Note that the apartment systems are also compatible with this.

Recall from the theory of algebraic groups that inverse image of a parabolic is parabolic, we thus have a certain functionality of this construction with respects to maps of algebraic groups, one simply must make ‘compatible choices’ along the way.

**Example.** Consider the special unitary group associated to the antidiagonal form:

It has a maximal torus consisting of the diagonal (with trivial middle if $n$ is odd) where top and bottom parts are inverses. The maximal split torus is the diagonal whose entries are in the base field.

In this form, the group $B$ can be the set of upper triangular matricies (which are in the group).

In this case, $SU$ is what is called quasi-split as it contains a maximal torus defined over $\mathbb{Q}$, even though it fails to contain a maximal split torus definied over $\mathbb{Q}$.

Noticing that the elements which interchange diagonal elements are in $SU$ we conclude that the Weyl group for this setting is the semidirect product of the symmetric group on $\lfloor n/2 \rfloor$ elements together with $\lfloor n/2 \rfloor$ many copies of $\mu_2$.

The rational parabolics being stabilizers of rational isotrpic flags.

For the special case $n = 3$, notice there is only $B$!

6.3. **Over Fields with Valuations.** This is what we call the Affine building!

We start first with an example (because everyone prefers to do this than describe the actual construction)

**Example** (The case of $\mathrm{SL}_n(K)$)**.** The verticies correspond to lattices up to homothety.

The dimension $\ell$ faces correspond to (up to homothety):

$$pL_\ell \subset L_1 \subset L_2 \subset \cdots \subset L_\ell$$

Adjacency corresponds to 'refinement'.

Chambers thus correspond to maximal 'flags'

$$pL_n \subset L_1 \subset L_2 \subset \cdots \subset L_n$$

Apartments are the set of all 'flags' constructable with a given basis. (multiplies of $\pi^k e_i$).

In this context the stabilizer of a vertex is:

$$\mathrm{SL}_n(\mathcal{O}_K)$$

The stabilizer of a chamber is 'Borel mod $\pi$'.

The stabilizer of an apartment is the normalizer of a maximal torus.

We thus have that $W$ is a the normalizer of the torus modulo the units of the torus. Thus, it is an extension of the usual Weyl group by the 'value group of the torus' to the rank of the torus.

**In general we have the following:**

(This might be a bit unsatisfying)

The apartments correspond to maximal $K$-split tori $S$.

The group $N$ is the normalizer of the torus.

The group $B$ is a the inverse image of a Borel under reduction modulo $\pi$ (These are called Iwahori subgroups) for some integral model... (which ones are not made particularly clear in general). If $G$ is simply connected and split it is those where $G$ is 'integrally split' (residually split). Alternatively they are the Normalizers of maximal pro-p subgroups...

Fix a maximal compact subgroup and an integral model for which it is the integral points, then $B$ is the inverse image of a borel under the reduction...

Generically... one constructs the building first and defines them to be the stabilizers...

The groups $P$ are then 'parabolics' modulo $\pi$. (These are called parahoric subgroups)

The affine Weyl group is an extension of the ordinary Weyl group by the group $Z/Z_c$ where $Z$ is the centralizer of $S$ and $Z_c$ is the 'group of units'.

The affine space for the apartment is an affine space under $X_*(S) \otimes \mathbb{R}$. The usual Weyl group acting as it usually does... the extension acting by translation (valuation in the direction of some character gets replaced by the action of that character)

Verticies correspond to $P$, the maximal compact subgroups of $G(K)$. (NOTE here the very strong analogy to symmetric spaces!!!)

$B$ maximal compact subgroup of a minimal parabolic containing $S$.

$N$ is the inverse image of a minimal parabolic containing $\overline{S}$ in the reduction.

**Example.** Quasi-split special unitary in dimension $2n+1$ in residue characteristic not 2.

Let $L/K$ be quadratic with $\tau$ the involution.

Pick $\lambda \in \mathcal{O}_L^\times$ with $\mathrm{Tr}(\lambda) = -1$ and $\mathrm{Tr}(\lambda\pi) = 0$.

Let $e_i$ ,$-n \leq i \leq n$ be the basis for $L^{2n+1}$. The Hermitian form is the anti-diagonal one.

For $0 \leq r \leq n$ consider basis: $e_i^{(r)} = \pi^{-1} e_i$ for $i < -r$, $e_i^{(r)} = e_i$ for $-r \leq i \leq 0$ and $e_i^{(r)} = \lambda e_i$ for $i > 0$.

This flag of lattices defines a chamber for the special unitary group.

The stabilizer of this chamber is an iwahori.

The stabilizers of subflags are the parahorics.

**Remark.** Functoriality here is not as nice as we extend the field. One can do things (especially under unramified extensions). This is where terms like quasi-split, residually split come up.

It isn't totally unreasonable as we map between groups.

**Theorem 6.1.** *Buildings coming from rank* 2 *algebraic groups have as automorphisms: The automorphism group of the underlying group. (Its projectivization + field automorphisms + outer automorphisms).*

## 7. Buildings in Buildings

One can observe that the link of an affine building is the spherical building of the group over its residue field.

One can interpret the spherical building as the 'infinite ends' of the affine building.