TRAVAUX DE SHIMURA

DELIGNE (TRANSLATION - ANDREW FIORI)

This translation was performed as part of my attempts to understand the content, consequently, mathematical errors of translation may have occurred.

TODO-Isomorphism arrows are often just arrows TODO-equation labeling

0. INTRODUCTION

Let X/Γ be the quotient of a hermitian symmetric domain X by a discrete arithmetic group Γ (suppose that it is defined by congruence conditions (see 1.7). via Baily and Borel, X/Γ is a complex algebraic varieties. In [Shi64],[Shi67b],[Shi67a] and [Shi70], Shimura shows among other things, that for many cases of these varieties, they can be defined over explicit number fields. This is the subject of the present exposition.

To obtain proper results, we are forced to use an adelic language. This will force us to consider for a given "congruence condition" a scheme over \mathbb{C} , which is the disjoint union of a finite number of varieties of the form X/Γ (see 1 and 2). In a number of cases, these schemes can be defined over a number field E which is independent of the congruence conditions considered. These geometrically connected components, can be defined over the class field of E. This phenomenon, which is essential, will be neglected in the rest of this introduction.

In a small number of cases, X/Γ can be interpreted as the space of isomorphism classes of complex abelian varieties, together with additional algebraic structures (polarization, endomorphism, order n points). The sounding, over \mathbb{Q} , of a moduli problem, gives a scheme M defined over an explicit field F for which X/Γ is thus the complex points. We call M a '**model** of X/Γ . The fundamental case of this is abelian varieties with a principle polarization, together with a level N structure (see 1.6,1.11,4.16,4.17,4.21).

In other cases, X/Γ can be interpreted as a collection of isomorphism classes of complex abelian varieties, with structures as above but also with higher cohomological structures of type (pp) $a_i \in H^{2p}(A \times \cdots \times A, \mathbb{C})$ ([Mum66]). In such cases, the idea is to construct a model M for X/Γ as a subscheme of a model M' constructed for X'/Γ' . We start by constructing in M' the points which correspond to abelian varieties of C.M. type (maximum complex multiplication). We define then M as the gluing together of all these points. This construction relies on the detailed understanding of C.M. abelian varieties from [ST61].

In the cases considered above, we have at our disposal a lot of information about the field of definition of the points of M which correspond to C.M. abelean varieties. This permits us to give a solution to the 12th of Hilbert's problems ([Shi68] and 3.15, 3.16).

By looking at the things these cases have in common, we create a notion of a 'canonical model' for X/Γ (3). We show that there is at most one such, descent techniques permit us to construct several canonical models for certain moduli problems ([Shi67b],[Shi67a],[Shi70],[Miy71]). For the description of these "strange modeles", we refer you to ??.

We will denote by $\mathbb{Z}/n(z)$ the group scheme μ_n of the nth roots of unity in a fixed algebraically closed field \overline{k} . For $\overline{k} = \mathbb{C}$, the exponential map permits us to identify $\mathbb{Z}/n(1)$ and \mathbb{Z}/n . The groups

 $\mathbb{Z}/n(1)$ forms a projective system under the maps $\phi_{n,m}: x \mapsto x^{n/m}$. We define:

$$\hat{\mathbb{Z}} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} \simeq \prod_p \mathbb{Z}_p, \quad \hat{\mathbb{Z}}(1) = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}(1)$$

We also denote by $\mathbb{A}^f = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ the finite adeles, and $\mathbb{A}^f(1) = \mathbb{Q} \otimes \hat{\mathbb{Z}}(1) = \mathbb{A}^f \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}(1)$. We shall denote by \mathbb{A} the complete ring of adeles of \mathbb{Q} .

Remark. There appears to be a typo, in that Z(1) is never defined but is used to define $\mathbb{Z}(1)$.

We will use the following notation:

- $\pi_0(X)$ (for any topological space X) is the connected components of X, we will think of it as having the quotient topology of X and thus be discrete or compact and complete disconnected.
- G^0 (for a topological group G) is the connected component of the identity.
- $G(K), G_K, G \otimes_F K$ for a scheme G over F and an F-algebra K are the points and respectively the scalar extension to K.
- E^* for E a finite dimensional algebra over a field F denotes the algebraic group of units in E. For E also a field $E^*(\mathbb{A})/E^*(\mathbb{Q})$ is the idele class group of E.

We will say that an algebraic group G over \mathbb{Q} satisfies the **Hasse principle** if $H^1(\mathbb{Q}, G) = H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G(\mathbb{Q}))$ injects into the product $\prod_{\nu} H^1(\mathbb{Q}_{\nu}, G_{\mathbb{Q}_{\nu}})$ over all completions of \mathbb{Q} . We will use the following theorems:

Remark. I am not sure when any of the following are true for non-simply connected groups.

Theorem 0.1 (Hasse Principle). A semi-simple simply connected algebraic group with no E_8 factors satisfies the Hasse principle.

[Har66]

Theorem 0.2. A simply connected semi-simple group G over a local non-Archimedean field satisfies $H^1(K,G) = 0$. (See Bruhat-Tits [BT67]).

Theorem 0.3 (Strong approximation). Let G be a simply connected semi-simple group defined over a global field K. If G is K-simple and if ν is a place of K such that $G(K_{\nu})$ is non-compact, then $G(K_{\nu})G(K)$ is dense in $G(\mathbb{A})$. (see Platonov [Pla69])

Theorem 0.4 (Real approximation). Let G be a connected linear algebraic group over \mathbb{Q} . Then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. (This reduces to the case of tori, which is a corollary in [Kne65, 5.1].)

1. Adelic Language

1.0. Summary. X/Γ hermitian symmetric.

G reductive group. (No non-trivial closed connected normal unipotent subgroups). G' the group closure of the image of [G, G], that is the commutator subgroup.

$$T = G/G'$$
.

C = Z(G) the center of G.

 $C\cap G'$ is a finite group.

Fix G and π a faithful representation, a G structure is an element of: $i \in \text{Isom}(V_{\pi}, W)/\pi(G(\mathbb{Q}))$. (If G = Aut(V, Struct) then it is like setting $\overline{i} = \text{Isom}((V, Struct), (W, Struct'))$ where Struct' is the transfer of structure). $G^{\overline{i}}$ is $iGi^{-1} \in GL(W)$.

H are objects $(H, \overline{i}, h, \overline{k})$ where H is a vector space with G structure \overline{i} . $h : \underline{S} \to G_{\mathbb{R}}^{\overline{i}}$ with $i^{-1}hi : \underline{S} \to G_{\mathbb{R}}$. $\overline{k} \in \operatorname{Isom}_{\overline{i}}(H \otimes \mathbb{A}^{f}, V \otimes \mathbb{A}^{f})$.

1.1. Let G be a reductive group over \mathbb{Q} . We denote by G' the derived group of G, by C the center of the identity component of G, and by T = G/G'. We will make use of the following exact sequences of algebraic groups:

$$\begin{array}{c} 0 \rightarrow C \rightarrow G \rightarrow G / C \rightarrow 0 \\ 0 \rightarrow G' \rightarrow G \xrightarrow{\nu} T \rightarrow 0 \end{array}$$

The compositions of maps which maps $C \to T$ and $G' \to G/C$ are isogenies of order $C \cap G'$.

1.2. Let \underline{S} be the real algebraic group of invertible elements in the \mathbb{R} algebra \mathbb{C} . That is the algebraic group which is given by the Weil restriction of scalars from \mathbb{C} to \mathbb{R} of the group \mathbb{G}_m . We have that $\underline{S}(\mathbb{R}) = \mathbb{C}^*$ and $\underline{S}_{\mathbb{C}} \simeq \mathbb{G}_{m\mathbb{C}} \times \mathbb{G}_{m\mathbb{C}}$. The group $\operatorname{Hom}(\underline{S}_{\mathbb{C}}, \mathbb{G}_{m\mathbb{C}})$ of characters of $\underline{S}_{\mathbb{C}}$ has as a basis the characters corresponding to z and \overline{z} in such a way that the maps:

$$\mathbb{C}^* = \underline{S}(\mathbb{R}) \hookrightarrow \underline{S}(\mathbb{C}) \xrightarrow{z,z} \mathbb{C}^*$$

respects complex conjugation.

Let V be a real vector space. For every representation $H : \underline{S} \to \operatorname{GL}(V)$, we denote by V^{pq} the subspace of $V_{\mathbb{C}}$ under which \underline{S} acts by the character $z^{p}\overline{z}^{q}$. The spaces V^{pq} form a bigradation of $V_{\mathbb{C}}$ and the map $h \mapsto V^{pq}$ identifies the representations of \underline{S} in $\operatorname{GL}(V)$ with all of the Hodge structures on $V_{\mathbb{C}}$ (those such that $\overline{V^{pq}} = V^{qp}$. We denote by F(h) the associated hodge filtration of $V_{\mathbb{C}}$ such that:

$$F(h)^p = \bigoplus_{p' > p} v^{p'q'}$$

1.3. We denote by $R : \mathbb{G}_{m\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ the \mathbb{C} -homomorphism such that $(z^p \overline{z}^q) \circ r = (x \mapsto x^p)$ and by $W : \mathbb{C}_{m\mathbb{R}} \to \underline{S}$ the \mathbb{R} -homomorphism such that $(z^p \overline{z}^1 \circ w = (x \mapsto x^{p+q})$. For $h : \underline{S} \to \mathrm{GL}(V)$ and $v^{pq} \in V^{pq}$ we have:

$$h(r(x))(v^{pq}) = x^p v^{pq}$$

and

$$h(w(x))(v^{pq}) = x^{p+q}v^{pq}$$

The composite $w : \mathbb{R}^* = \mathbb{G}_m(\mathbb{R}) \xrightarrow{w} \underline{S}(\mathbb{R}) = \mathbb{C}^a st$ is the natural inclusion.

1.4. Let $h : \underline{S} \to G_{\mathbb{R}}$ be a homomorphism. For all representations $\rho : G \to \operatorname{GL}(V)$ of G on a \mathbb{Q} vector space V, ρh defines a bi-grading on $V_{\mathbb{C}}$ (1.2). If V is a faithful and G is a subgroup of $\operatorname{GL}(V)$ which preserves some tensors s_i , the construction $h \mapsto (V^{pq})$ identifies homomorphisms h with hodge filtrations of $V_{\mathbb{C}}$ such that s_i are of type (0,0).

1.5. Let $h_0 : \underline{S} \to G_{\mathbb{R}}$ be a homomorphism which satisfies the following conditions (see [Mum66] and [?]; the second is motivated by Griffiths transversality)

1.5.1. The image of $h_0 w : \mathbb{G}_m \to G_{\mathbb{R}}$ is central. We call $h_0 w$ the weight of h_0 .

1.5.2. The hodge structure of $\text{Lie}(G)_{\mathbb{C}}$ the complexified lie algebra given by the representation attached to it through the adjoint representation (1.2) is of type (-1,1), (0,0), (1,-1).

1.5.3. The automorphism ad $h_0(i)$ of G (an involution by 1.5.1 induces a cartan involution of G'.

Let K_{∞} be the centralizer of h_0 in $G(\mathbb{R})$; it contains the center of $G(\mathbb{R})$ and $K_{\infty} \cap G'(\mathbb{R})^0$ is a maximal compact subgroup of $G'(\mathbb{R})^0$, which is equal to the centralizer of $h_0(i)$ in $G'(\mathbb{R})^0$. The space $K_{\infty} \setminus G(\mathbb{R})$ is identified with the collection X of conjugates of h_0 by elements of $G(\mathbb{R})$. We verify that it possesses one and only one complex structure such that for all representations $\rho: G \to \operatorname{GL}(V)$ the filtration $F(\rho h)$ of $V_{\mathbb{C}}$ depends holomorphically on $h \in X$. This structure is right invariant under $G(\mathbb{R})$, and the connected components of X are hermitian symmetric domains. We denote by X^0 the connected component containing h_0 . 1.6.

Example. Let Gp(V) be the group of symplectic similitude of a real vector space V which has an non-degenerate alternating form ψ . There exists then a unique conjugacy class X of homomorphisms $h: \underline{S} \to Gp(V)$ satisfying the conditions (1.5 i) which is of weight -1. That is the weight $hw: \mathbb{G}_m \to Gp(V)$ maps $x \mapsto x^{-1}$. This space X is identified with two copies of the Siegel Space.

The construct (1.5) identifies the $h \in X$ with decompositions $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ which are totally isotropic with respect ψ , and that, if C denotes an endomorphism of V which induces multiplication by i^{p-q} on V^{pq} , the symmetric form $\psi(x, Cy) = \psi(x, h(i)y)$ on V will be either positive or negative definite.

1.7. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. We are interested in the case where Γ is defined by congruence conditions. If $V_{\mathbb{Z}}$ is an integral lattice in a faithful representation V of G, this implies that there exists n such that Γ contains as a finite index subgroup $\Gamma_n \subset G(\mathbb{Q})$ the $\gamma \in G(\mathbb{Q})$ such that $\gamma \in G(\mathbb{R})^0$, $\gamma V_{\mathbb{Z}} = V_{\mathbb{Z}}$ and $\gamma \cong \text{Id} \mod n$. By Baily and Borel, the quotient X^0/Γ comes naturally with the structure of a quasi-projective scheme over \mathbb{C} . Let K_n be the open compact subgroup of $G(\mathbb{A}^f)$ formed by the $k = (k_p)$ such that $k_p \cdot V_{\mathbb{Z}} \otimes \mathbb{Z}_p = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ and such that $k_p \cong 1$ mod n when p|n. The group Γ_n is then the intersection in $G(\mathbb{A})$ of $G(\mathbb{Q})$ and $G(\mathbb{R})^0 \times K_n$.

1.8. Let K be a compact open subgroup of $G(\mathbb{A}^f)$. The space:

$$K_{\infty} \times K \setminus G(\mathbb{A}) / G(\mathbb{Q}) = K \setminus X \times G(\mathbb{A}^f) / G(\mathbb{Q}) = X \times K \setminus G(\mathbb{A}^f) / G(\mathbb{Q})$$

is thus the union of a finite collection of spaces like thos considered in 1.7. It is thus a quasiprojective scheme over \mathbb{C} . We denote it by ${}_{K}M_{\mathbb{C}}(G, h_0)$ (or simply ${}_{K}M_{\mathbb{C}}(G)$ or ${}_{K}M_{\mathbb{C}})$.

For K which become smaller and smaller, these schemes form a projective system where the transition maps are finite and surjective:

$$_{K}M_{\mathbb{C}}(G,h_{0}) \qquad (K \to 0)$$

We denote by $M_{\mathbb{C}}(G, h_0)$ (or simply $M_{\mathbb{C}}(G)$ or $M_{\mathbb{C}}$) the quasi-compact and separated (but not of finite type unless $G = \{1\}$) the projective limit of this system:

$$M_{\mathbb{C}}(G, h_0) = \lim K_{\infty} \times K \setminus G(\mathbb{A}) / G(\mathbb{Q})$$

We must consider it like an avatar of the projective system above. The group $G(\mathbb{A}^f)$ acts on $M_{\mathbb{C}}$, or, if we would prefer to say it differently, on the pro-object defined by the projective system, but certainly not on the individual $_K M_{\mathbb{C}}$. We have that:

$$K \backslash M_{\mathbb{C}} = {}_{K} M_{\mathbb{C}}$$

in the sense that:

 $M_{\mathbb{C}} = \lim K \setminus M_{\mathbb{C}}$ K compact open in $G(\mathbb{A}^f)$

This amounts to saying that $G(\mathbb{A}^f)$ acts continuously on $M_{\mathbb{C}}$.

The following construction is useful for interpreting ${}_{K}M_{\mathbb{C}}$ like a moduli scheme.

1.9. Let H be an algebraic group over \mathbb{Q} and V a faithful representation of H. We define an H structure \overline{i} on a vector space W as an isomorphism of V with W given modulo $H(\mathbb{Q}) : i \in \text{Isom}(V, W)/H(\mathbb{Q})$. If $H(\mathbb{Q})$ is the automorphism group of a structure s of the space Σ on V, then an H-structure on W is identified with a structure t of the same space Σ on W, which is isomorphic to s (with $\overline{i} = \text{Isom}((V, s), (W, t))$). Let W be a space with an H-structure \overline{i} . The algebraic group $iHi^{-1} \subset \text{GL}(W)(i \in \overline{i})$ depends only on \overline{i} . We will denote it by $H^{\overline{i}}$. Let A be a \mathbb{Q} algebra. The collection $\text{Isom}(W \otimes A, V \otimes A)$ of **isomorphism permis** fo $W \otimes A$ with $V \otimes A$ is the collection of isomorphisms k such that for $i \in \overline{i}$ we have $ki \in H(A)$.

1.10. Let (G, h_0) be as in (1.5), V be a faithful linear representation of G, K a compact open subgroup of $G(\mathbb{A}^f)$ and consider objects H of the following type: H consists of:

- (1) A vector space $H_{\mathbb{Q}}$ over \mathbb{Q} together with a *G*-structure \overline{i} (1.8).
- (2) A homomorphism $h: \underline{S} \to G_{\mathbb{R}}^{\overline{i}}$ such that for $i \in \overline{i}, i^{-1}hi: \underline{S} \to G_{\mathbb{R}}$ is conjugate to h_0 .
- (3) A class $\overline{k} \in K \setminus \operatorname{Isom}_{\overline{i}}(H \otimes \mathbb{A}^f, V \otimes \mathbb{A}^f)$ (1.9)

1.11.

Example. Following up on 1.6. Let V be a Q-vector space together with a non-degenerate alternating form ψ , G the group of symplectic similitudes of V, $V_{\mathbb{Z}}$ an integral lattice in V on which ψ has discriminant 1 and h_0 like as in 1.6. Take $K = \prod_l Gp(V_{\mathbb{Z}} \otimes \mathbb{Z}_l)$.

An element (a) above, can be interpreted as a vector space H of the same dimension as V, together with an alternating form ψ given up to rational rescaling. An element (b) as above, is like giving a hodge bigrading to $H_{\mathbb{C}}$ as in 1.6. An element (c) is like giving an integral lattice $H_{\mathbb{Z}}$ of H such that a rational multiple of ψ has integral values and discriminant 1: \overline{k} is the collection of k such that for all $l, V_{\mathbb{Z}} \otimes \mathbb{Z}_l = k_l(H_{\mathbb{Z}} \otimes \mathbb{Z}_l)$. we can normalize ψ by conditions of having integral values of discriminant 1 on $H_{\mathbb{Z}}$, and satisfies $\psi(x, Cx) > 0$ (1.6).

Let n be an integer, and K_n the subgroup of K formed by taking the k such that $k \cong 1 \pmod{n}$. (mod n)). For K_n , an element of type (c) is interpreted as $H_{\mathbb{Z}}$ as above, plus a symplectic symilitude $H_{\mathbb{Z}}/nH_{\mathbb{Z}} \to V_{\mathbb{Z}}/nV_{\mathbb{Z}}$.

1.12. Under the hypotheses of 1.10, the points of ${}_{K}M_{\mathbb{C}}(G, h_{0})$ are in bijective correspondence with the isomorphism classes of objects H of the type in 1.10. In effect we have $H = (H_{\mathbb{Q}}, \bar{i}, h, \bar{k})$; to H with $i \in \bar{i}$ we associate the morphism $i^{-1}hi : \underline{S} \to G_{\mathbb{R}}$, the element of $X \simeq K_{\infty} \setminus G(\mathbb{R})$, and $\bar{k}i$ in $K \setminus G(\mathbb{A}^{f})$ and together they make an element of $K \setminus X \times G(\mathbb{A}^{f})$; to H alone we associate only an element of ${}_{K}M_{\mathbb{C}}(G, h_{0}) = K \setminus X \times G(\mathbb{A}^{f})/G(\mathbb{Q}) = K_{\infty} \times K \setminus G(\mathbb{A})/G(\mathbb{Q})$ which determines uniquely the isomorphism class of H.

In certain cases (see 4.11), to give an object H is to give an abelian variety, together with some auxiliary structures. We thus obtain an interpretation of ${}_{K}M_{\mathbb{C}}(G)$ as a moduli scheme coarsely of this type of abelian variety.

1.13. Let $G^i(i = 1, 2)$ be two reductive groups with $h^i : \underline{S} \to G^i_{\mathbb{R}}$ homomorphisms satisfying the conditions of 1.5 and X^i be the collection of conjugates of h^i by elements of $G^i(\mathbb{R})$. Let $G = G^1 \times G^2$, $h : \underline{S} \to G_{\mathbb{R}}$ be the morphism (h^1, h^2) and $X = X^1 \times X^2$ the collection of conjugates of h. We define in the natural way an isomorphism:

$$M_{\mathbb{C}}(G^1, h^1) \times M_{\mathbb{C}}(G^2, h^2) = M_{\mathbb{C}}(G, h)$$

For K^i compact open in $G^i(\mathbb{A}^f)$ and $K = K^1 \times K^2$ we likewise have:

$$_{K^1}M_{\mathbb{C}}(G^1,h^1) \times {}_{K^2}M_{\mathbb{C}}(G^2,h^2) = {}_{K}M_{\mathbb{C}}(G,h)$$

1.14. Let G^1, G^2, h^1, h^2 be as in 1.13. We denote by $u : (G^1, h^1) \to (G^2, h^2)$ a homomorphism $u: G^1 \to G^2$ such that $uX^1 \subset X^2$. Such a homomorphism defines:

$$u: M_{\mathbb{C}}(G^1, h^1) \to M_{\mathbb{C}}(G^2, h^2).$$

For K^i compact open in $G^i(\mathbb{A}^f)$ with $u(K^1) \subset K^2$ it defines:

$$u(K^1, K^2) : {}_{K^1}M_{\mathbb{C}}(G^1, h^1) \to {}_{K^2}M_{\mathbb{C}}(G^2, h^2).$$

1.15.

Proposition. With the notation as in 1.14 suppose that G^1 is a subgroup of G^2 , thus for all compact open subgroups K^1 of $G^1(\mathbb{A}^f)$ there exists a compact open subgroup $K^2 \supset K^1$ of $G^2(\mathbb{A}^f)$ such that $u(K^1, K^2)$ identifies ${}_{K^1}M_{\mathbb{C}}(G^1, h^1)$ with a closed subscheme of ${}_{K^2}M_{\mathbb{C}}(G^2, h^2)$.

If the K^i are sufficiently small so that the image of $X^i \times (K \setminus G^i(\mathbb{A}^f))$ in $_{K^i} M_{\mathbb{C}}(G^i, h^i)$ is etale, then the map $u(K^1, K^2)$ is finite and unramified. It suffices to consider this case and to show that $u(K^1, K^2)$ is injective when $K^2 \supset K^1$ sufficiently small. The graph of the equivalence relation $u(K^1, K^2)(x) = u(K^1, K^2)(y)$ is a closed subscheme of $(_{K^1}M_{\mathbb{C}}(G^1, h^1)^2)$ which decreases with K^2 and is thus fixed for $K^2 \supset K^1$ sufficiently small. It suffices to show:

$$u(K_1): {}_{K^1}M_{\mathbb{C}}(G^1, h^1) \to \lim_{\substack{\leftarrow \\ K^2 \supset K^1}} {}_{K^2}M_{\mathbb{C}}(G^2, h^2) \stackrel{\text{def}}{=} {}_{K^1}M_{\mathbb{C}}(G^2, h^2)$$

is injective. We have:

$$_{K^1}M_{\mathbb{C}}(G^1,h^1) = K^1 \backslash M_{\mathbb{C}}(G^1,h^1)$$
$$_{K^1}M_{\mathbb{C}}(G^1,h^1) = K^1 \backslash M_{\mathbb{C}}(G^2,h^2)$$

so that it suffices to prove:

Claim. $u: M_{\mathbb{C}}(G^1, h^1) \to M_{\mathbb{C}}(G^2, h^2)$ is injective.

This assertion follows easily from the following two lemmas.

Lemma. If $U^i \subset C^i(\mathbb{Q})$ **TODO-le group des unites** in the center of G^i and let $G^i(\mathbb{Q}) = \lim_{i \to \infty} (G^i(\mathbb{Q})/(U^i)^n)$ then:

$$M_{\mathbb{C}}(G^{i},h^{i}) = K^{i}_{\infty} \backslash G^{i}(\mathbb{A}) / G^{i}(\mathbb{Q})$$

This is a corollary to Chevalley's theorem that the $(U^i)^n$ are congruence subgroups of U^i .

Lemma. $u: G^1(\mathbb{A}^f)/G^1(\mathbb{Q})^{\vee} \to G^2(\mathbb{A}^f)/G^2(\mathbb{Q})^{\vee}$ is injective.

TODO- the \lor may be upside down



Where $G^2/G^1(\mathbb{Q})^{\vee} = \lim_{\leftarrow} (G^2/G^1)(\mathbb{Q})/(U^2/U^1)^n$.

2. Connected Components

2.1. Let G be a connected reductive group over \mathbb{Q} . Recalling the notation of 1.1 and suppose that G' has no non-trivial normal subgroups H with $H(\mathbb{R})$ compact. This hypothesis doesn't exclude the possibility that G' has compact factors. We let \tilde{G}' be the simply connected cover of G'.

Proposition. • The group $G(\mathbb{A}^f)$ acts transitively on the connected components of $G(\mathbb{A})/G(\mathbb{Q})$.

• The group $\tilde{G}'(\mathbb{A})$ acts trivially on these.

• $G(\mathbb{A})/\rho(\tilde{G}')$ is commutative

The first assertion is a consequence of 0.4 $G(\mathbb{A}) = G(\mathbb{A}^f)G(\mathbb{R})^0 G(\mathbb{Q}).$

Because \tilde{G} is simply connected, the group $\tilde{G}(\mathbb{R})$ is connected, Thus by 0.3 we have $\tilde{G}(\mathbb{R})\tilde{G}(\mathbb{Q})$ is dense in $\tilde{G}(\mathbb{A})$, in the sense that $\tilde{G}(\mathbb{A})/\tilde{G}(\mathbb{Q})$ is connected. For $x \in G(\mathbb{A})$ we have $\tilde{G}(\mathbb{A})x = x\tilde{G}(\mathbb{A})$ in the sense that $\tilde{G}(\mathbb{A})x$ in $G(\mathbb{A})/G(\mathbb{Q})$ is an image of $\tilde{G}(\mathbb{A})/\tilde{G}(\mathbb{Q})$ it is connected, and the second assertion follows.

Let Z be the center of \tilde{G}' , the third assertion results from the fact that the commutateur $xyx^{-1}y^{-1}: G \times G \to G$ admits a factorization:

$$G\times G\to G/C\times G/C\leftarrow \tilde{G}'/Z\times \tilde{G}'/Z\to \tilde{G}'\to G$$

2.3.

Definition. We denote by $\pi(G)$ the abelian quotient (a further quotient of $G(\mathbb{A}^f)/\rho(\tilde{G}')$) which acts transitively on components.

In other words, $\pi(G)$ is $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$, together with the structure of a commutative group. Recall the following theorem.

2.4.

Theorem. Under the conditions of 2.1, if G' is simply connected then there is a canonical bijective homomorphism:

$$\pi(G) = \pi_0(G(\mathbb{A})/G(\mathbb{Q})) \to \pi_0(T(\mathbb{A})/T(\mathbb{Q}))$$

To prove the surjectivity of 2.4 it suffices after 2.2 applied to T to show that $G(\mathbb{A}^f)$ is sent onto $T(\mathbb{A}^f)$. The kernel G' of $\nu G \to T$ is connected. We know that for almost all p, $\nu(G(\mathbb{Z}_p)) = T(\mathbb{Z}_p)$ (this formula makes sense for almost all p and is a corollary of the theorem of Lang applied to G' reduced modulo p). It suffices thus to prove that for all p, $\nu(G(\mathbb{Q}_p)) = T(\mathbb{Q}_p)$, this is a consequence of 0.2 applied to G'.

The theorem 2.4 is equivalent to the following more concrete formulation:

2.5.

Theorem. For every compact open subgroup $K \subset G(\mathbb{A}^f)$ we have that:

$$\nu : \pi_0(K \setminus G(\mathbb{A})/G(\mathbb{Q})) \to \pi_0(\nu(K) \setminus T(\mathbb{A})/T(\mathbb{Q}))$$

is bijective.

It effect, $\pi_0(G(\mathbb{A})/G(\mathbb{Q})) = \lim_{\leftarrow} \pi_0(K \setminus G(\mathbb{A})/G(\mathbb{Q}))$, and the same for T. By **TODO-ailleurs** = elsewhere $\pi_0(K \setminus G(\mathbb{A})/G(\mathbb{Q})) = G(\mathbb{R})^0 \times K \setminus G(\mathbb{A})/G(\mathbb{Q})$. Shows that if $x, y \in G(\mathbb{A})$ have the same image in the member in the right of 2.5 they also have the same image in the member on the left. A left translation by y^{-1} which replaces K by $y^{-1}Ky$ permits us to suppose that y = e. Modifying x by an element of $G(\mathbb{R})^0 \times K$ we can suppose that $\nu(x) = \tau$ with $\tau \in T(\mathbb{Q})$. By the Hasse principal (0.1) for G' the equation $m(\gamma) = \tau$ has a solution in $G(\mathbb{Q})$. Correcting x by γ , we can suppose that $x \in G(\mathbb{A})$. As a result of 2.22 the image of x in $G(\mathbb{A})/G(\mathbb{Q})$ is in the identity component, thus a fortiori x and y have the same image in the member on the left of 2.5.

This proof seems to use via 0.1 that G' has no E_8 factors. In fact, the class in $H^1(\mathbb{Q}, G')$ which obstructs the equation $m(\gamma) = \tau$ comes from the center of G' and the factors E_8 are adjoint, this accounts for the butter.

2.6.

Proposition. We have that $K_{\infty} = C(\mathbb{R}) \cdot (K_{\infty} \cap G'(\mathbb{R})^0)$. The group $K_{\infty} \cap G'(\mathbb{R})^0$ is connected and:

$$\pi_0(K_\infty) \to Im(\pi_0(C(\mathbb{R}))) \to \pi_0(G(\mathbb{R}))$$

Let $h': \underline{S} \to G_{\mathbb{R}} \to (G/C)_{\mathbb{R}}$. The centralizer of h' in $(G/C)_{\mathbb{R}}$ is connected because it is a compact group and thus the complexification is connected (as the centralizer of a torus). In particular, the image of K_{∞} in $G/C(\mathbb{R})$ lands in the connected component of the identity, the image of $G'(\mathbb{R})^0$ and $K_{\infty} = C(\mathbb{R})(K_{\infty} \cap G'(\mathbb{R})^0)$. The group $K_{\infty} \cap G'(\mathbb{R})^0$ is connected in all maximal compact subgroup of connected group. 2.6 follows.

2.7. The image of $\pi_0(K_\infty)$ in $\pi_0(G(\mathbb{A})/G(\mathbb{Q})$ doesn't depend on the conjugacy class of h, and $\pi_0(M_{\mathbb{C}}(G,h)) = \lim_{\leftarrow} \pi_0(K_\infty \times K \setminus G(\mathbb{A})/G(\mathbb{Q}))$ identifies with $\pi(G)/\pi_0(K_\infty)$.

In particular, if G' is simply connected we have:

$$\pi_0(M_{\mathbb{C}}(G,h)) \to \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$$

More concretely, for all compact open subgroup K of $G(\mathbb{A}^f)$, $\pi_0({}_{K}M_{\mathbb{C}}(G,h))$ is thus a principal homogeneous space under $\nu(K_{\infty} \times K) \setminus T(\mathbb{A})/T(\mathbb{Q})$.

3. MODELS

3.0. Summary. We define the notion of model over an algebraic field.

We introduce how models relate for subvarieties.

We describe what a good model for special points is.

We define the notion of canonical model to be one which has good model for all its special points.

3.1. Let G be a reductive group over \mathbb{Q} , $h : \underline{S} \to G_{\mathbb{R}}$ satisfy the conditions of 1.5 and let E be a field together with a complex embedding $\rho : E \to \mathbb{C}$. We write $M_{\mathbb{C}}(G) = M_{\mathbb{C}}(G, h)$.

Definition. A model over E of $M_{\mathbb{C}}(G)$ consists of:

- (1) A scheme M over E, endowed with a continuous action of $G(\mathbb{A}^f)$;
- (2) An isomorphism m of $M \otimes_{E,\rho} \mathbb{C}$ with $M_{\mathbb{C}}(G)$ compatible with the action of $G(\mathbb{A}^f)$.

Let F be a finite extension of E, together with a complex embedding extending that of E. If $M_E(G,h)$ is a model of $M_{\mathbb{C}}(G,h)$ over E, we denote by $M_F(G,h)$ the model $M_E(G,h) \otimes_E F$ of $M_{\mathbb{C}}(G,h)$ over F.

3.2.

Remark. To give a scheme M over E together with a continuous action of $G(\mathbb{A}^f)$ amounts to giving:

- (1) for every open compact subgroup K of $G(\mathbb{A}^f)$ a scheme _KM over E;
- (2) for every K and L two compact open subgroups of $G(\mathbb{A}^f)$ and for $x \in G(\mathbb{A}^f)$ with $xKx^{-1} \subset L$ a homomorphism $J_{L,K}(x) : {}_{K}M \to {}_{L}M$ such that these homomorphisms satisfy:
 - (a) $J_{M,L}(y)J_{L,K}(x) = K_{M,K}(yx).$
 - (b) $J_{K,K}(x) = \text{Id if } x \in K.$
 - (c) For K normal in L, J defines an action of L/K on $_KM$ and $J_{L,K}(e)$ defines $(L/K)\backslash_KM \to _LM$.

We will have:

$$M = \lim_{\leftarrow} {}_{K}M$$
$$_{K}M = K \backslash M$$

3.3. Let M(G) be a model over E of $M_{\mathbb{C}}(G)$. For K compact open in $G(\mathbb{A}^f)$, we let $_KM(G) = K \setminus M(G)$. The $_KM(G)$ are quasi-projective schemes over E, they are not necessarily geometrically connected. The structural isomorphism m induces an isomorphism:

$$_{K}M(G)\otimes_{E,\rho}\mathbb{C}\to {}_{K}M_{\mathbb{C}}(G).$$

3.4. Suppose that G verifies the hypothesis of 2.1 and that (M, m) is a model of $M_{\mathbb{C}}(G, h)$ over E. Let \overline{E} be the algebraic closure of E in \mathbb{C} . Since M is defined over E, $\operatorname{Gal}(\overline{E}/E)$ acts on the profinite system:

$$\pi_0(M \otimes_E \overline{E} = \lim_{\leftarrow} \pi_0({}_K M \otimes_E \overline{E}) \xrightarrow[m]{} \pi_0(M_{\mathbb{C}}(G,h)).$$

The group $G(\mathbb{A}^f)$ acts on $\pi_0(M \otimes_E \overline{E})$ through its quotient $\pi(G)/\pi_0(K_\infty)$ and $\pi_0(M \otimes_E \overline{E})$ is a principle homogeneous space under the commutative group $\pi(G)/\pi_0(K_\infty)$. The action of $\operatorname{Gal}(\overline{E}/E)$ commutes with that of $G(\mathbb{A}^f)$ and thus with that of $\pi(G)/\pi_0(K_\infty)$. The action of $\operatorname{Gal}(\overline{E}/E)$ can be nothing but the translation defined by an element $\lambda(\sigma)$ of $\pi(G)/\pi_0(K_\infty)$ and λ is a homomorphism:

 $\lambda_M : \operatorname{Gal}(\overline{E}/E) \to \pi(G)/\pi_0(K_\infty).$

Suppose that E is a number field. Class field theory identifies the largest abelian quotient of \overline{E}/E with $\pi_0(E^*(\mathbb{A})/E^*(\mathbb{Q}))$ and the above as:

$$\lambda_M : \operatorname{Gal}(\overline{E}/E)^{ab} = \pi_0(E^*(\mathbb{A}))/E^*(\mathbb{Q})) \to \pi(G)/\pi_0(K_\infty).$$

In the case that G' is simply connected, then using 2.7 the homomorphism above is:

$$\lambda_M : \pi_0(E^*(\mathbb{A})/E^*(\mathbb{Q})) \to \pi_0(T(\mathbb{A})/T(\mathbb{Q}))/\pi_0(K_\infty)$$

3.5.

Definition. The homomorphisms above are called **The law of reciprocity** of the model M. If G' is simply connected and λ_M comes from a homomorphism of algebraic groups over \mathbb{Q} of E^* in T this last one is again called the **reciprocity law** of M.

3.6. For each of the models constructed by Shimura, there exists a rule for determining E and the reciprocity law λ_M from G and h. The field E, for arbitrary G and λ_M when G' is simply connected are described as follows.

3.7. The composition of the homomorphisms $h \circ r$ of 1.3:

$$hr: \mathbb{G}_m \xrightarrow{r} S_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}$$

is a homomorphism over \mathbb{C} between algebraic groups defined over \mathbb{Q} . The subfield E of \mathbb{C} is the field of definition E(G, h) of the conjugacy class of this homomorphism.

For G semi-simple and adjoint, the field E(G, h) can be described as follows. Letting Δ be the Dynkn diagram of $G_{\mathbb{C}}$ and $|\Delta|$ the collection **TODO-sommet**?? of Δ . The Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on Δ .

Let H be a maximal torus of $G_{\mathbb{C}}$, together with a system of simple roots $(\alpha_i)_{i\in|\Delta|}$. The α_i identify H with $\mathbb{G}_m^{|\Delta|}$. For all homomorphisms $u : \mathbb{G}_m \to G_{\mathbb{C}}$. There exists a unique homomorphism $u' : \mathbb{G}_m \to H = \mathbb{G}_m^{\Delta} : x \mapsto (x^{n_i})_{i\in|\Delta|}$ with $n_i \geq 0$, which is conjugate to u. The construction $n \mapsto \underline{n}(u) = (n_i)_{i\in|\Delta|}$ identifies the conjugacy classes of maps of \mathbb{G}_m in $G_{\mathbb{C}}$ with the collection of functions of $|\Delta|$ into \mathbb{N} . We deduce that the field E(G, h) is defined by the subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which stabilizes $\underline{n}(hr)$.

3.8.

Proposition. Let $G/C = \prod_{i=1}^{g} G_i$ be the decomposition of the adjoint group G/C into \mathbb{Q} -simple factors, and let h_i be the composition $h_i : S \xrightarrow{h} G_{\mathbb{R}} \to G_{i\mathbb{R}}$.

- (1) The subfield E(G,h) of \mathbb{C} is the composition of the subfields $E(T, \nu \circ h)$ and $E(G_i, h_i)$ for $(1 \le i \le g)$ of \mathbb{C} .
- (2) If the involution **TODO-d'opposition** of the Dynkin diagram of G_i respects $\underline{n}(h_i r)$ then $E(G_i, h_i)$ is totally real. Otherwise, $E(G_i, h_i)$ is a quadratic imaginary extension of a totally real field.

The first assertion is easy to check, and it suffices to prove the second in the case of G a \mathbb{Q} -simple adjoint group. Let Δ be its Dynkin diagram, as a result of the hypothesis in 1.5 that $G_{\mathbb{R}}$ admits a maximal compact torus; this implies that complex conjugation acts on Δ by the involution **TODO-d'opposition** *i*. This is in the center of the group of automorphisms of Δ .

Let E' be a Galois extension of \mathbb{Q} defined by the subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which acts trivially on Δ . Thus, the image of σ (complex conjugation) is central in $\operatorname{Gal}(E'/\mathbb{Q})$; it follows that E' is totally real or a quadratic imaginary extension of a totally real field. Finally, σ is the identity on $E(G, h) \subset E'$ if and only if *i* respects $\underline{n}(hr)$ which is our assertion.

3.9. We recall the notation of 3.6. We suppose that G' is simply connected, that E contains E(G, h) and that the complex embedding of E extends that of E(G, h). The composite morphism:

$$r''(h): \nu hr: \mathbb{G}_{m\mathbb{C}} \xrightarrow{r} \underline{S}_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}} \xrightarrow{\nu} T_{\mathbb{C}}$$

depends only on the conjugacy class of hr. It is thus defined over E, that is, it comes from the extension of scalars from E to \mathbb{C} of a homomorphism of algebraic groups over E, again denote $r''(h) : \mathbb{G}_m \to T_E$. Applying the restriction of scalars of Weil, et let $r'(h) : E^* \to T$ be the composite:

$$E^* = R_{E/\mathbb{Q}}(\mathbb{G}_{mE}) \xrightarrow{R_{E/\mathbb{Q}}(r''(h))} R_{E/\mathbb{Q}}(T_E) \xrightarrow{N_{E/\mathbb{Q}}} T$$

. The law of reciprocity of M is the inverse r(G,h) of r'(h):

$$r(G,h) = r'(h)^{-1} : E^* \to T$$

TODO-what??? shouldn't inverse go the other way???

3.10.

Example. Let H be a torus and $h : \underline{S} \to H_{\mathbb{R}}$. The conditions of 1.5 are automatically verified. The ${}_{K}M_{\mathbb{C}}(H,h)$ are finite collections. A reduced finite scheme over E(H,h) (3.7) is identified with a Galois set. We deduce that there exists up to unique isomorphism one and only one model of $M_{\mathbb{C}}(H,h)$ over E(H,h), with the reciprocity law given by 3.9. We shall denote it M(H,h).

3.11. Let F be a finite extension of E, together with a complex embedding extending that of E. The diagram:



is commutative. If $M_E(G,h)$ is a model over E of $M_{\mathbb{C}}(G,h)$ satisfying the reciprocity law of 3.9, it follows that $M_F(G,h)$ (3.1) satisfies again the reciprocity law of 3.9.

3.12. Let $u: (G^1, h^1) \to (G^2, h^2)$ be like as in 1.14 and $M_{E^i}(G^i, h^i)$ be models of $M_{\mathbb{C}}(G^i, h^i)$ over E^i . Let E be the composite of E^1 and E^2 in \mathbb{C} . We have (in the notation of 3.1):

$$M_E(G^i, h^i) \otimes_E \mathbb{C} = M_{\mathbb{C}}(G^i, h^i) \quad (i = 1, 2)$$

There is thus a sense in asking if

$$u: M_{\mathbb{C}}(G^1, h^1) \to M_{\mathbb{C}}(G^2, h^2)$$

is defined over E.

3.13.

Definition. Let G be a connected reductive group over \mathbb{Q} satisfying the conditions of 2.1. $h: \underline{S} \to G_{\mathbb{R}}$ a homomorphism satisfying the conditions of 1.5 and E and extension of E(G, h) (3.7) together with an complex embedding extending that of E(G, h). A model $M_E(G, h)$ of $M_{\mathbb{C}}(G, h)$ over E is said to be **weakly canonical** if, for every torus $u: H \hookrightarrow G$ in G, together with $h': \underline{S} \to H_{\mathbb{R}}$ such that uh' is a conjugate of h by an element of $G(\mathbb{R})$, the morphism (1.14):

$$M_{\mathbb{C}}(H,h') \to M_{\mathbb{C}}(G,h)$$

is defined (3.103.12) over the composite $E(H, h') \cdot E \subset \mathbb{C}$. A model $M_E(G, h)$ is said to be **canonical** if it is weakly canonical and E = E(G, h).

3.14. Traduction. Let M be a canonical model of $M_{\mathbb{C}}(G, h)$ with reciprocity law λ_M . If G' is simply connected, and thus λ_M is given by 3.9 (??). For every compact open subgroup K of $G(\mathbb{A}^f)$ denote again by $\nu(K)$ the image of K in $\pi(G)/\pi_0(K_\infty)$ and let E(K) be the extension of E(G, h)defined by the idele classes $\lambda)M^{-1}(\nu(K))$. We verify that E(K) is the field of definition of the connected component of the identity $_K M^0_{\mathbb{C}}$ of $_K M_{\mathbb{C}}$ considered as a subscheme of $_K M_{\mathbb{C}} = _K M \otimes_{E,\rho} \mathbb{C}$.

By definition, ${}_{K}M^{0}_{\mathbb{C}}$ is given by extension of scalars from E(K) to \mathbb{C} of the subscheme ${}_{K}M'$ of ${}_{K}M \otimes_{E} E(K)$. The scheme ${}_{K}M'/E(K)$ is geometrically connected over E(K) and we can again describe the scheme ${}_{K}M'$ like that of the connected components (over E) of ${}_{K}M$ which, after extension of scalars from E to \mathbb{C} contain the origin.



Shimura has the habit of expressing in terms of a system of varieties ${}_{K}M'/E(K)$ and homomorphism like as in 2.2 that we have between these. For a typical such result see [Shi70, p146]. For the traduction [Shi70, 2.7]

3.15. For $u: (H, h') \to (G, h)$ as in 3.13 the image of $M_{\mathbb{C}}(H, h')$ in ${}_{K}M_{\mathbb{C}}(G, h)$ is a finite collection. The hypothesis is that this piece of ${}_{K}M_{\mathbb{C}}(G, h)$ is defined over E(H, h'), that these points are defined over a finite abelian extension of E(H, h') and that $\operatorname{Gal}(\overline{E}(H, h')/E(H, h'))$ permutes the points in a prescribed manner.

We call the points in ${}_{K}M_{\mathbb{C}}(G,h)$ that are in the image of ${}_{K}M_{\mathbb{C}}(H,h')$ special.

Suppose that we have a projective embedding $q: {}_{K}M\mathbb{C}^{0} \to \mathbb{P}^{r}$ that is defined over E(K). We can interpret this as $\tilde{q}: X^{0} \to \mathbb{P}^{r}$ (cf. 1.7,1.8). If $uh' \in X$ is in X^{0} , then the coordinates of $\tilde{q}(uh')$

will be in E(K)E(H, h') an abelian extension of E(H, h') which can be explicitly described as a class field.

3.16. The classical example of a situation of such a situation is the example of the moduli of elliptic curves (corresponding to the case $G = \operatorname{GL}_2$ and $K = \operatorname{GL}_2(\hat{\mathbb{Z}})$, and h as in 1.6), the special points here are elliptic curves with complex multiplication.

Let j be the modular invariant, viewed as a function on the Poincare half plane X^0 . If $\tau \in X^0$ generates a quadratic imaginary field K, let $L(\tau) = Z \oplus Z\tau \subset \mathbb{C}$, and $\theta(\tau) = \{k \in K | kL(\tau) \subset L(\tau)\}$. The field $K(j(\tau))$ is an abelian extension of K whose Galois group is the idele class group over the ring $\theta(\tau)$. If σ in the Galois group corresponds to an invertible ideal L, and if $L(\tau') \simeq L(\tau) \otimes_{\theta(\tau)} L$, we have $j(\tau) = j(\tau')^{\sigma}$.

For more explicit and strange examples see [Shi68, p45-46].

4. Abelian Varieties

4.0. Summary. We describe abelian varieties with enough information to discuss the classification, this is similar in many ways to what dylan talked about.

Construct a canonical model for the symplectic shimura variety

4.1. Let k be an algebraically closed field of characteristic 0. Let A be an abelian variety over k. For $n \in \mathbb{N}^+$ we denote by A_n the kernel of multiplication by n. The A_n form a projective system $(\phi_{nm,n}A_{nm} \to A_n \text{ by } x \mapsto x^m)$ we define:

$$\hat{T}(A) = \lim A_n, \quad \hat{V}(A) = \mathbb{A}^f \otimes_{\hat{\mathbb{Z}}} \hat{T}(A)$$

The $\hat{\mathbb{Z}}$ -module $\hat{T}(A)$ is the product of the Tate modules (the Weil representation) $T_l(A)$. If $k = \mathbb{C}$ we have:

$$\hat{T}(A) = \hat{\mathbb{Z}} \otimes H_1(A, \mathbb{Z})$$

and

$$\hat{V}(A) = \mathbb{A}^f \otimes H_1(A, \mathbb{Q})$$

4.2. The category of abelian varieties up to isogeny over k is the category in which the objects are abelian varieties over k and $\operatorname{Hom}_{cat}(A, B) = \operatorname{Hom}(A, B) \otimes \mathbb{Q}$. We denote by $A \otimes \mathbb{Q}$ the abelian variety up too isogeny underlying the abelian variety A. All additive functors on the category of abelian varieties into an additive category which are \mathbb{Q} -linear factor through $A \mapsto A \otimes \mathbb{Q}$. Thus, the contravariant functor of "dual variety" $A \mapsto A^*$, descends to abelian varieties up to isogeny. The functor $A \mapsto \hat{V}(A)$ descends likewise. Moreover, if A_0 is an abelian variety up to isogeny, it is equivalent to give B and an isomorphism $B \otimes \mathbb{Q} = A_0$ or to give $\hat{T}(B) \subset \hat{V}(A_0)$.

4.3. Let A be an abelian variety over k. We denote by NS(A) the group of algebraic equivalence classes of invertible sheaves on A. We define an effective polarization (resp. polarization) to be an element of NS(A) (resp. $NS(A) \otimes \mathbb{Q}$)) such that a positive multiple is defined by a projective embedding of A. A polarization **TODO-homogene** is an element of $NS(A) \otimes \mathbb{Q}/\mathbb{Q}^*$ which is the class of a polarization.

Recall that $NS(A) \otimes \mathbb{Q}$ is identified by a bijection $p \to p'$ with the elements of $\operatorname{Hom}(A, A^*) \otimes \mathbb{Q}$ which are equal to their **TODO-transpose (adjoint??)**. An isomorphism $u \in \operatorname{Hom}(A \otimes \mathbb{Q}, B \otimes \mathbb{Q})$ induces an isomorphism of $\operatorname{Hom}(A, A^*) \otimes \mathbb{Q}$ with $\operatorname{Hom}(B, B^*) \otimes \mathbb{Q}$, from $NS(A) \otimes \mathbb{Q}$ with $NS(B) \otimes \mathbb{Q}$ which takes polarizations to polarizations. This permits us to speak of the polarization of an abelian variety up to isogeny. **4.4.** A polarization p of A defines in $\operatorname{End}(A) \otimes \mathbb{Q}$ a positive involution $u \mapsto p'^{-1}u^*p'$, which depends only on the **TODO-homogene** polarization $\mathbb{Q}^*.p$.

Let F be a product of totally real fields and let $\rho: F \to \operatorname{End}(A) \otimes \mathbb{Q}$. Let $NS_{\rho}(A) \subset NS(A)$ be the collection of p such that $p'\rho(f) = \rho(f)^*p'$ for $f \in F$. The vector space $NS_{\rho}(A) \otimes \mathbb{Q}$ is equipped for $f \in F$ with an action of F given by $(f \cdot p)' = p' \circ \rho(f) = \rho(f)^* \circ p'$. A weak polarization of A (relative to ρ, F) is an element of $NS_{\rho}(A) \otimes \mathbb{Q}/F^*$ which is the class of a polarization. The collection of weak polarizations of A depends only on $A \otimes \mathbb{Q}$. The restriction to F of the involution on $\operatorname{End}(A) \otimes \mathbb{Q}$, defined by a polarization $p \in NS_{\rho}(A)$, depends only on the weak polarization $F^* \cdot p$.

4.5. If A_0^* is the dual abelian variety up to isogeny of A_0 , then $\hat{V}(A_0)$ and $\hat{V}(A_0^*)$ are in duality with values in $\mathbb{A}^f(1)$. If p is a polarization of A_0 , the **polarization form** $\phi_f(x, y) = \langle x, p'(y) \rangle$ is a non-degenerate alternating form on the free \mathbb{A}^f -module $\hat{V}(A_0)$ with values in $\mathbb{A}^f(1)$.

The results 4.1, 4.5 extend to abelian schemes over any base.

4.6. Let $k = \mathbb{C}$. The additive functor $A \mapsto H_1(A, \mathbb{Q})$ extends to abelian varieties up to isogeny over \mathbb{C} (4.2). For A_0 an abelian variety up to isogeny, $H_1(A_0, \mathbb{Q})$ and $H_1(A_0^*, \mathbb{Q})$ are in duality. Via 4.12 and the isomorphism of $\mathbb{A}^f(1)$ with \mathbb{A}^f defined by the exponential, this duality is compatible with the one considered in 4.5. The alternating form $\psi_p(x, y) = \langle x, p'(y) \rangle$ on $H_1(A_0, \mathbb{Q})$ defined by a polarization p of A_0 is called again the polarization form of the hodge structure of $H_1(A_0, \mathbb{Q})$ and the homomorphism $h : \underline{S} \to \operatorname{GL}(H_1(A_0, \mathbb{Q}))_{\mathbb{R}}$ such that \underline{S} acts on $H_1(A_0, \mathbb{Q}) \otimes \mathbb{C}$ by the characters z^{-1} and \overline{z}^{-1} and such that $F(h)^0$ (1.4) is the kernel of the exponential $H_1(A_0, \mathbb{Q}) \to \operatorname{Lie}(A)$.

4.7.

Theorem. The constructions:

$$(A, p) \mapsto (H_1(A, \mathbb{Q}), \psi_p, H_1(A, \mathbb{Z}), h)$$
$$(V, \psi, V_{\mathbb{Z}}, h) \mapsto F(h)^0 \backslash V \otimes \mathbb{C}/V_{\mathbb{Z}}$$

establishes an equivalence between:

- (1) Polarised abelian varieties over \mathbb{C} ;
- (2) Systems formed of a vector space V over \mathbb{Q} a non-degenerate alternating form ψ on V, a lattice $V_{\mathbb{Z}}$ in V and a homomorphism h of S into the group $Gp(V)_{\mathbb{R}}$ of simplectic similitudes of $V \otimes \mathbb{R}$ of the type in 1.6 and such that:

$$\psi(x, h(i)x) > 0$$
 $(x \in V \otimes \mathbb{R}, x \neq 0)$

4.8. It is equivalent to give either an abelian variety up to isogeny A over \mathbb{C} , equipped with a polarization **TODO-homogene** or to give a vector space over \mathbb{Q} , together with a non-degenerate alternating form ϕ , up to rescaling and a map $h: \underline{S} \to Gp(B)$ of the type in 1.6.

4.9. Let L be a semi-simple algebra with involution over \mathbb{Q} , and V a vector space over \mathbb{Q} , given with a faithful L-module structure and a non-degenerate alternating form ψ such that:

$$\psi(lx,y) = \psi(x,l^*y)$$

We denote by G the algebraic group over \mathbb{Q} of L-linear simplectic similitudes of V. The group $G(\mathbb{Q})$ is the collection of $g \in \operatorname{GL}_L(V)$ such that there exists $\mu(g) \in \mathbb{Q}^*$ such that:

$$\psi(gx,gy) = \mu(g)\psi(x,y)$$

Suppose we give a homomorphism $h_0: \underline{S} \to G_{\mathbb{Q}}$ such that via h_0, \underline{S} acts on $V_{\mathbb{C}}$ by the characters z^{-1} and \overline{z}^{-1} and that the form $\psi(x, h_0(i)y)$ is symmetric and positive definite. The conditions (1.5.1 to 1.5.3) are thus verified by (G, h_0) and the involution of L is positive.

4.10. Apply (1.11) to (G, h_0) and to the representation V of G. A G-structure \overline{i} on a vector space H is interpreted by 1.8, as giving to H the structure of an L-module and an alternating form ψ given up to rescaling, the vector space H, together with these structures, is isomorphic to V. Let \overline{i} be a G structure on H. A given 1.9b $h : \underline{S} \to G_{\mathbb{R}}^{\overline{i}}$ on H thus defines, after 4.8 applied to (H, ψ, h) an abelian variety up to isogeny A, together with a polarization **TODO-homogene** \overline{p} , such that H is $H_1(A, \mathbb{Q})$, h the hodge structure of $H_1(A, \mathbb{Q})$ and ψ the polarization form. Moreover, the L-module structure on H comes from $\rho : L \to \text{End}(A)$ and H, together with \overline{i} and h is determined by (A, \overline{p}, ρ) .

For a triple (A, \overline{p}, ρ) to come from an H like above, it is necessary and sufficient that:

- (1) $H_1(A, \mathbb{Q})$ together with its structure of an *L*-module and the polarization form (up to rescaling) is isomorphic to *V*.
- (2) For $i: V \to H_1(A, \mathbb{Q})$ an isomorphism $i^{-1}h$ is conjugate to h_0 .
- (3) Let *i* be the map from *L* into \mathbb{C} given by:

$$t(l) = \operatorname{Tr}(l; V_{\mathbb{C}}/F^0(h_0))$$

4.11.

.

Theorem (Scholie). Let K be an open compact subgroup of $G(\mathbb{A}^f)$. The points in ${}_{K}M_{\mathbb{C}}(G,h_0)$ correspond bijectively to the isomorphism classes of abelian varieties up to isomorphism A together with:

(1) $\rho L \rightarrow \text{End}(A)$ such that:

$$\operatorname{Tr}(\rho(l), \operatorname{Lie}(A)) = t(l) \quad (l \in L)$$

- (2) A polarization **TODO-homogene** $\overline{\rho}$ which induces the given involution of L.
- (3) A class mod K, \overline{k} of L-linear simplectic similitudes $k : \hat{V}(A) \to V \otimes \mathbb{A}^{f}$.
- (4) The conditions (1)(2) are satisfied.

We apply 1.11 and 4.10 and note that an item 1.9(c) is equivalent to an item (c), and that conditions (b) and (a) result from (1) and (2) respectively.

4.12. Let K be a compact open subgroup of $G(\mathbb{A}^f)$, $V_{\mathbb{Z}}$ a lattice in V such that $V_{\hat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$ is Kinvariant, $L_{\mathbb{Z}}$ an order in L such that $L\mathbb{Z}V_{\mathbb{Z}} \subset V_{\mathbb{Z}}$, $\psi_{\mathbb{Z}}$ a positive rational multiple of the alternating
form on V with integral values on $V_{\mathbb{Z}}$, and $V'_{\mathbb{Z}}$ the largest lattice in V such that $\psi_{\mathbb{Z}}(V_{\mathbb{Z}}, V'_{\mathbb{Z}}) \subset \mathbb{Z}$.
Let n be an integer and $K_n = \{g \in G(\mathbb{A}^f) | (g^{-1}V_{\hat{\mathbb{Z}}} \subset nV_{\hat{\mathbb{Z}}})$. We suppose n is sufficiently large so
that $K \supset K_n$.

If A is as in 4.11, and thus $k^{-1}(V_{\hat{\mathbb{Z}}}) \subset \hat{V}(A)$ does not depend on $k \in \overline{k}$ is defined by an abelian variety V with $V \otimes \mathbb{Q} \simeq A$ and $\hat{T}(B) = k^{-1}(V_{\hat{\mathbb{Z}}}) \subset \hat{V}(A)$. The action of L on A is induced by the action of $L_{\mathbb{Z}}$ on B. There exists a unique effective polarization p of B, with $p \in \overline{p}$, such that $k^{-1}(V_{\hat{\mathbb{Z}}})$ is the largest lattice $\hat{T}'(B) \supset \hat{T}(B)$ with $\psi_p(\hat{T}(B), \hat{T}(B)) \subset \hat{Z}(1)$. Finally, the collection \overline{k} of symplectic isomorphisms which are L-linear of $\hat{T}(B)$ with $V_{\hat{\mathbb{Z}}}$ are the inverse image of its image $\overline{k_n}$ in $\text{Isom}(B_n, V_{\mathbb{Z}}/nV_{\mathbb{Z}})$.

This permits us again to interpret the points of ${}_{K}M\mathbb{C}(G,h_0)$ as corresponding to a system $(B, p, \rho, \overline{k}_n)$ consisting of:

- (1) A polarized abelian variety (B, p) with complex multiplication by $L_{\mathbb{Z}}$, satisfying 4.11 (a)(b),
- (2) A class modulo K/K_n , \overline{k}_n of an isomorphism $k_n : B_n \simeq V_{\mathbb{Z}}/nZ$ with can be lifted to an *L*-linear symplectic isomorphism $k : \hat{T}(B) \to V_{\hat{\mathbb{Z}}}$, which verifies conditions 1 and 2.

4.13. TODO- 4.13 - this looks important... what is it

4.14. TODO- 4.14 - Theorem on level structure

4.15. TODO- 4.15

4.16. TODO- 4.16 - Example

4.17.

Proposition. The \mathbb{Q} -scheme $M(Gp, h_0) = \lim_{\leftarrow K^n} M$ is a model over $\mathbb{Q} - E(Gp, h_0)$ of $M_{\mathbb{C}}(Gp, h_0)$.

4.18. Putting ourselves in the particular case of 4.9 where V is an L-module **TODO-monogene**. The groups G and G_1 are thus contained in the center of L^* .

Let \overline{E} be an algebraic closure of E(G, h) and M^+ the collection of isomorphism classes $[A, \rho, k, \overline{p}]$ of objects $(A, \rho, k, \overline{p})$ consisting of:

(1) An abelian variety up to isogeny A/\overline{E} together with a **TODO-homogene** polarization \overline{p} and of $\rho: L \to \text{End}(A)$ such that in the notation of 3 we have for $l \in L$:

$$\operatorname{Tr}(l, \operatorname{Lie}(A)) = t(l) \in E(G, h)$$

(2) An isomorphism $K \otimes \mathbb{A}^{f}$ -linear $k : \hat{V}(A) \to V \otimes \mathbb{A}^{f}$. The abelian variety A is thus of CM-type.

We refer you to [ST61] for the proof of the following fundamental result:

4.19.

Theorem (Shimura-Taniyama). The Galois group $\operatorname{Gal}(\overline{E}/E(G,h))$ acts on M^+ via its largest abelian quotient $\pi_0(E(G,h)^*(\mathbb{A})/E(G,h)*(\mathbb{Q}))$. For $e \in E(G,h)*(\mathbb{A})$ the image $\phi(e)$ in the abelianization of the Galois group, and of the composition in $E(G,h)*(\mathbb{A}^f)$ we have:

 $\phi(e)([A,\rho,k,\overline{p}] = [A,\rho,r(G,h)(e^f) \cdot k,\overline{p}]$

TODO- 4.19 - Shimura-Taniyama— CFT is correct here

4.20.

Proposition. The E(G,h)-scheme $M(G,h) = \lim_{\leftarrow K} M(G,h)$ is a canonical model of $M_{\mathbb{C}}(G,h)$.

4.21.

Theorem. The model of $M(Gp, h_0)$ constructed in 4.17 is a canonical model.

TODO- 4.21 - details TODO- 4.22 - 'Hilbert modular version'

5. TECHNIQUES OF CONSTRUCTION

5.0. Summary. Prove properties of canonical models in particular uniqueness, and restriction to subvarieties.

Conjecture they always exist.

5.1. TODO-This

6. STRANGE MODEL

I think he looks at twisted forms of symplectic group, haven't really read this.

References

- [BT67] F. Bruhat and J. Tits, Groupes algébriques simples sur un corps local, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 23–36. MR 0230838 (37 #6396)
- [Har66] Günter Harder, Über die Galoiskohomologie halbeinfacher Matrizengruppen. II, Math. Z. 92 (1966), 396– 415. MR 0202918 (34 #2777)
- [Kne65] Martin Kneser, Starke Approximation in algebraischen Gruppen. I, J. Reine Angew. Math. 218 (1965), 190–203. MR 0184945 (32 #2416)
- [Miy71] Katsuya Miyake, Models of certain automorphic function fields, Acta Math. 126 (1971), 245–307. MR 0563067 (58 #27794)
- [Mum66] David Mumford, Families of abelian varieties, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 347–351. MR 0206003 (34 #5828)
- [Pla69] V. P. Platonov, Strong approximation in algebraic groups, and the Kneser-Tits hypothesis, Dokl. Akad. Nauk BSSR 13 (1969), 585–587. MR 0260756 (41 #5379)
- [Shi64] Goro Shimura, On the field of definition for a field of automorphic functions, Ann. of Math. (2) 80 (1964), 160–189. MR 0167466 (29 #4739)
- [Shi67a] _____, Algebraic number fields and symplectic discontinuous groups, Ann. of Math. (2) 86 (1967), 503– 592. MR 0222048 (36 #5100)
- [Shi67b] _____, Construction of class fields and zeta functions of algebraic curves, Ann. of Math. (2) 85 (1967), 58–159. MR 0204426 (34 #4268)
- [Shi68] _____, Automorphic functions and number theory, Lecture Notes in Mathematics, No. 54, Springer-Verlag, Berlin, 1968. MR 0234915 (38 #3229)
- [Shi70] _____, On canonical models of arithmetic quotients of bounded symmetric domains, Ann. of Math. (2) 91 (1970), 144–222. MR 0257031 (41 #1686)
- [ST61] Goro Shimura and Yutaka Taniyama, Complex multiplication of abelian varieties and its applications to number theory, Publications of the Mathematical Society of Japan, vol. 6, The Mathematical Society of Japan, Tokyo, 1961. MR 0125113 (23 #A2419)