VARIETES DE SHIMURA: ...

DELIGN (TRANSLATION - ANDREW FIORI)

TODO-math symbol for 'inner' automorphism is *int*
TODO-citations
TODO-equation/enumeration numbering
TODO-references to enumeration numbering are often as text
TODO-some of the math is 'unclear' translationally, things like covers and lifts
I find much of the math here more difficult to follow than his previous paper, in the sense that it is much harder to see what he is trying to talk about. In particular, he spends a lot of time switching between adjoint forms, derived groups and various covers. These all certainly play a role in conjugacy, it is unclear to me now what he is trying to do with it.

TODO-other things like when he in text says something is a restriction of scalars... it is not clear which is the restrictee and which the restriction...

0. Introduction

This article follows up on [**?**], thus we use the esential results of it (those of paragraphs 4 and 5). In the first part, we try to motivate the axioms imposed on the system (G, X) 2.1.1 based on which we define Shimura Varieties. We demonstrate that, generally speaking, they correspond to moduli of hodge structures X^+ of the following type:

(1) X^+ is a connected component of the space of all hodge structures on some fixed vector space V relative to several fixed tensors $t_1, \ldots, t_n$ of type $(0,0)$. The algebraic group G is the subgroup of $\mathrm{GL}(V)$ which fixes the t_i and X is the orbit $G(\mathbb{R}) \cdot X^+$ of X^+ under $G(\mathbb{R})$.
(2) The family of hodge structres on V parametrised by X^+ satisfies certain conditions, which are satisfied by families of hodge structures which appear naturally in algebraic geometry; for the correct complex structure (and uniquely determined) on X^+, it is a polarisable variation of hodge structres.

The space X^+ is automatically a hermitian symmetric domain (with negative curvature). Hermitian symmetric domains can all also be described as moduli spaces of hodge structures (1.1.17), and I believe this description is very useful. For example "the embedding of a hermitian symmetric domain D into its dual $\check{D}$ (a flag variety) corresponds to the mapping (Hodge Structure) $\mapsto$ (Hodge Filtration). Descriptions like "Siegel 3-space" can be interpreted in saing that, under certain hypothesis, if we superimpose on a hodge structure a filtration by weights, we obtain a mixed hodge structure, or a mapping of D into a moduli space of mixed hodge structures (following the construction of [**?**]. This last point will not be meantioned or used in this article.

This point of view, and the description of certain Shimura Varieties as moduli spaces of abelian varieties, are linked by the dictionary: it comes back to (the equivalence of categories $A \mapsto H_1(A, \mathbb{Z})$) which links an abelian variety and a polarizable hodge structure of type $\{(-1,0),(0,-1)\}$ (here we have torsion free $\mathbb{Z}$-hodge structures; by passing to the dual $(A \mapsto H^1(A, \mathbb{Z}))$ we can replace $\{(-1,0),(0,-1)\}$ by $\{(1,0),(0,1)\}$. Polarising an abelian variety is the same as polarising its H_1. **TODO-this sentence is messed up** With some parametrization, it is thus the same to give a polarised abelian variety on a smooth complex variety parametrised by S or a variation of polarised hodge structures, parametrized by S^{an}. An analytic family of abelian varieties, parametrised by S^{an} is automatically algebraic (this follows from [?]). To interpret hodge structures of more complicated types, we would like to replace abelian varieties by the proper "motives", but this remains only a dream.

In section 1.2 we give a convienient description based on the formalism of the classification of hermitan symmetric domains, in terms of Dynkin diagrams and their special **TODO-sommet**. In section 1.3 we classify a certain type of embedding of hermitian symmetric space into a Siegel half-space. The results are parallel to those of Satake [?]. An application of Weyl's unitary trick, for which we refer to [?], reduces the classification to a small part of the table, given for example by Bourbaki [?], gives the expressions for the fundamental weights in terms of linear combinations of simple roots.

The reader who wishes to learn more about the variations of Hodge structures and the ways in which they appear in algebraic geometry, can consult [?] (though we do not follow their sign conventions); some of the facts stated in [?] are proven in [?].

In sections 2.1 and 2.2 we define, in an adelic language, the Shimura varieties ${}_KM_{\mathbb{C}}(G,X)$ (denoted ${}_KM_{\mathbb{C}}(G,h)$ in [?], for h any element of X), their projective limit $M_{\mathbb{C}}(G,X)$ and the notion of **canonical modele**. I refer you to the text for these definitions, and will say only that a canonical model of $M_{\mathbb{C}}(G,X)$ is a model of $M_{\mathbb{C}}(G,X)$ over the **dual field** (2.2.1) $E(G,X)$, That is a scheme $M(G,X)$ over $E(G,X)$ together with an isomorphsim $M(G,X)\otimes_{E(G,X)}\mathbb{C} \xrightarrow{\sim} M_{\mathbb{C}}(G,X)$, which will have good properties ($G(\mathbb{A}^f)$-equivarient, galois behaviour at **special points** 2.2.4). We define also the notion of weak canonical model (same definition as canonical model except with an extension E of $E(G,X)$). These play a technical role in the construction of canonical models. The difference apparent in the definitions of 2.1 and 2.2 and those of [?] come from a different choice of sign convention (left actions vs right actions, reciprocity theory of fields and global classes...).

For a heuristic description, I refer you to the introduction of [?]. For a brief description, with examples, of how to pass from the adelic language to a classical language I refer you to [?, 5,1.6-1.11,3.14-3.16,4.11-4.16].

In [?] we systemified the methods introduced by Shimura for constructing canonical models. In the second part of the the present article, we perfect the results of [?]. In section 2.6 we determine the action of the galois group $\mathrm{Gal}(\overline{Q}/E)$ on the collection of geometrically connected components of a weak canonical model (which we assume exists) of $M_{\mathbb{C}}(G,X)$ over E without supposing (as is done in [?] that the derived group is simply connected, the essential point is the construction given in 2.4 of a morphism of the following type. Let G be a connected reductive group over $\mathbb{Q}$, $\rho : \tilde{G} \to G$ the universal covering of the derived group G^{der} and M a conjugacy class, defined over a number field E, of morphisms from $\mathbb{G}_m$ into G. We construct a morphism q_M from the group of idele classes of E in the abelian quatient $G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A}))\cdot G(\mathbb{Q})$ of $G(\mathbb{A})$. This morphism is functorial in (G,M) and if F is an extension of E the diagram:

$$\begin{tikzcd}
C(F) \arrow[dr] \arrow[dd, "N_{F/E}"] & \\
& G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A})\cdot G(\mathbb{Q}) \\
C(E) \arrow[ur] &
\end{tikzcd}$$

is commutative. If $\tilde{G}$ has no factor G' over $\mathbb{Q}$ such that $G'(\mathbb{R})$ is compact, we deduce from the strong approximation theorem that:

$$\pi_0(G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A})G(\mathbb{Q})) = \pi_0(G(\mathbb{A})/G(\mathbb{Q})),$$

and q_M gives an action on $\pi_0(G(\mathbb{A})/G(\mathbb{Q}))$ of $\pi_0(C(E))$, the abelianization of the galois group $\mathrm{Gal}(\overline{\mathbb{Q}}, E)^{ab}$ from the theory of global class field theory.

The second new idea-in fact a return to the point of view of Shimura-is the following observation: the results of 2.6 permit us to reconstruct a weakly canonical model $M_E(G, X)$ of $M_{\mathbb{C}}(G, X)$ from the connected component of $M^0_{\overline{\mathbb{Q}}}(G, X^+)$ (a geometrically connected component depending on the choice of connected component X^+ of X), together with a semi-linear action of a subgroup H of $G(\mathbb{A}^f) \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ which stabilises it. Letting Z be the center of G and G^{ad} be the adjoint group, $G^{ad}(\mathbb{R})^+$, the connected component of the identity of $G^{ad}(\mathbb{R})$ and $G^{ad}(\mathbb{Q})^+ = G^{ad}(\mathbb{Q}) = G^{ad}(\mathbb{R})^+$. The inclusion of the closure $Z(\mathbb{Q})^-$ of $Z(\mathbb{Q})$ in $G(\mathbb{A}^f)$ acts trivially on $M_{\mathbb{C}}(G, X)$, the action of H on $M^0_{\overline{\mathbb{Q}}}(G, X^+)$ factors through $H/Z(\mathbb{Q})^-$. We can make a slightly larger group act, the extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ by the completion of $G^{ad}(\mathbb{Q})^+$ for the topology of the images of congruence subgroups of $G^{der}(\mathbb{Q})$.

Up to unique isomorphism, this extension depends only on G^{ad}, G^{der} and on the projection X^{+ad} of X^+ in G^{ad} (2.5). We denote it $\mathcal{E}_E(G^{ad}, G^{der}, X^{+ad})$. The connected component of $M^0_{\mathbb{C}}(G, X^+)$ is the projective limit of the quotients of X^{+ad} by arithmetic subgroups of $G^{ad}(\mathbb{Q})^+$, images of congruence subgroups of $G^{der}(\mathbb{Q})$. We verifty that the conditions which must be satisfied by the model $M^0_{\overline{\mathbb{Q}}}(G, X^+)$ over $\overline{\mathbb{Q}}$, and the action of $\mathcal{E}_E(G^{ad}, G^{der}, X^{+ad})$, for corresponding to a weakly canonical model, depend only on G^{ad}, X^{+ad} the image of G^{der} in G^{ad} and the finite extension E (contained in $\mathbb{C}$) of $E(G^{ad}, X^{+ad})$. These conditions define models which are **weakly canonical** (respectively canonical for $E = E(G^{ad}, X^{+ad})$) **connected ??**.

The problem of the existence of a canonical model thus depends, in large, on the derived group. This reduction to the derived group is a much more convienient version than the method of central modification of h used in [**?**, 5.11].

In 2.3 we construct a **TODO-provision** of a canonical model with the help of a syplectic embedding, in invoking [**?**, 4.2,5.7]. The results of 1.3 permits us to obtain the desired symplectic embedding with few calculations. In 2.7 we explain the reduction to the derived group sketched above and we deduce in 2.3 a criteria for the existance of a canonical model which covers all the known cases (Shimura, Miyake and Shih).

In the article, we use the equivalence between weakly canonical models and weak connected canonical models to transport to the later the results of [**?**] (uniqueness, the construction of a canonical model from a family of weakly canonical models). It would have been more natural to write the proofs and the functoriality [**?**, 5.4] and the passage to subgroups [**?**, 5.7] (avoiding by the **TODO-sybilline** proposition [**?**, 1.15]). The lack of time and the wearyness, prevent us.

I recently showed that one can give in a purely algebraic sense the notion of a ration cycle of type (p, p), on an abelian variety A (over a field of characteristic 0). We can recover from this the criteria for the existance (2.3.1) of canonical models, and give a modular description of the models obtained (see [**?**]). This description unfortunately is not ready to be reduced modulo p. This method avoids the use to [**?**, 5.7] (and by a [**?**, 1.15]) and gives partial results on the conjugacy of Shimura varieties.

0.1. **Recall notaion and terminology.**

0.1.1. We will make use of the theorem of strong approximation, the theorem of real appoximation, the Hasse principle and the nullity of $H^1(K,G)$ for G semi-simple and connected over a non-archimedian local field. The bibliographical indications for these results are given in [**?**, (0.1) to (0.4)]. Note, do to another article of G. Prasad (*Strong approximation for semi-simple groups over function fields*, Ann of Math. (2) **105** (1977), 553-572) which proves the theorem of strong approximation over an arbitrary global field. Let G be a semisimple simply connected group, with center Z, over a global fieeld K. We use the Hasse principle for $H^1(K,G)$ only for classes in the image of $H^1(K;Z)$, in particular, E_8 factors are not an issue.

0.1.2. **Reductive Group** will always mean connected reductive group. A cover of a reductive group will always be a connected one. **Adjoint** group means reductive adjoint group. If G is a reductive group, we denote by G^{ad} its adjoint group. G^{der} its derived group and $\rho : \tilde{G} \to G^{der}$ the universal cover of the derived group. We denote by $Z = Z(G)$ the center of G, and $\tilde{Z}$ that of $\tilde{G}$ (note the conflict of notation)

0.1.3. We denote by the exponent ^{0} the algebraic connected component of the identity (for example Z^0 is the connected component of the center of G). The exponent $^+$ indicates the topological connected component of the identity (for example $G(\mathbb{R})^+$ is the connected component of the identity in the real topological group $G(\mathbb{R})$). We denote also by $G(\mathbb{Q})^+$ the **TODO-trace** of $G(\mathbb{R})^+$ under $G(\mathbb{Q})$. For G real reductive we denote by an index $_+$ the inverse image of $G^{ad}(\mathbb{R})^+$ in $G(\mathbb{R})$. The same notation $_+$ for the trace under a group of rational points.

For X a topological space, we denote $\pi_0(X)$ the connected components, given the quotient topology from X. In the arcticle, the space $\pi_0(X)$ will always be discrete and completely disconnected.

0.1.4. A **hermitian symmetric domain** is a hermitian symmetric space of negative curvature. That is without euclidean or compact factors).

0.1.5. Unless we specify otherwise, vector spaces are finite dimensional and a number field is of finite degree over $\mathbb{Q}$. The number fields we shall consider will be most often contained in $\mathbb{C}$. $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

0.1.6. We set $\hat{\mathbb{Z}} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$, $\mathbb{A}^f = \mathbb{Q} \otimes \hat{\mathbb{Z}} = \prod_p \mathbb{Q}_p$ (restricted product) and we denote $\mathbb{A} = \mathbb{R} \times \mathbb{A}^f$ the adele ring of $\mathbb{Q}$. we occasionally denote by $\mathbb{A}$ the ring of adeles of an arbitrary global field.

0.1.7. $G(K)$, $G \otimes_F K$, G_K: for G a scheme over F (for example an algebraic group over F) and K an F algebra, we denote by $G(K)$ the K valued points of G, and by $G_K = G \otimes_F K$ the scheme over K arising from G via extension of scalars.

0.1.8. We normalise the reciprocity isomorphism of global class field theory (choosing it or its inverse):

$$\pi_0 \mathbb{A}_E^*/E^* \xrightarrow{\sim} \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{ab}$$

so that the class of an idele which is a uniformizer at one place and 1 at all others corresponds to geometric frobenius (the inverse of a frobenius substitution) (see 1.1.6 and the justifications cited in **TODO-where...**).

1. Hermitian Symmetric Domains

1.1. Moduli Spaces of Hodge Structures.

1.1.1. Recall that a hodge structre on a real vector space B is a bigrading $V_{\mathbb{C}} = \oplus V^{pq}$ of the complexification of V, such that V^{pq} is the complex conjugate of V^{qp}.

Define an action h of $\mathbb{C}^*$ on $V_{\mathbb{C}}$ by the formula:

$$h(z)v = z^{-p}\overline{z}^{-q}v \qquad \text{for } v \in V^{pq}$$

The $h(z)$ commute with the complex conjugation of $V_{\mathbb{C}}$, thus the action comes from the restriction of scalairs of an action, again denoted h, of $\mathbb{C}^*$ on V. Thinking of $\mathbb{C}$ as an extension of $\mathbb{R}$, and letting $\mathbb{S}$ be its multiplicative group, considered as a real algebraic group (otherwise said, $\mathbb{S} = R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ (Weil restriction)); we have $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, and h is an action of the algebraic group $\mathbb{S}$. We verify that this construction defines an equivalence of categories: (real vector spaces with hodge structures) $\to$ (real vector spaces with actions of the group $\mathbb{S}$) .

To the inclusion $\mathbb{R}^* \subset \mathbb{C}^*$ there corresponds an inclusion of real algebraic groups $\mathbb{G}_m \subset \mathbb{S}$. We denote by w_h (or simply w) the restriction of h^{-1} to $\mathbb{G}_m$, and call it the **weight** $w : \mathbb{G}_m \to \mathrm{GL}(V)$. We say that V is homogenious of weight n if $V^{pq} = 0$ when $p+q \neq n$, that is $w(\lambda)$ is the homothety ratio λ^n.

We denote by μ_h (or simply μ) the action of $\mathbb{G}_m$ on $V_{\mathbb{C}}$ given by $\mu(z)v = z^{-p}v$ for $v \in V^{pq}$. It is a composition $\mathbb{G}_m \to \mathbb{S}_{\mathbb{C}} \to^h \mathrm{GL}(V)$.

The **Hodge Filtration** F_h(or simply F) is defined by $F^p = \oplus_{r \geq p} V^{rs}$ We say that V is of type $\mathcal{E} \subset \mathbb{Z} \times \mathbb{Z}$ if $V^{pq} = 0$ for $(p,q) \notin \mathcal{E}$.

More generally, if A is a subring of $\mathbb{R}$ such that $A \otimes \mathbb{Q}$ is a field (in practice $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$), an A-hodge structure is an A-module of finite type V, together with a hodge structure on $V \otimes_A \mathbb{R}$.

1.1.2.

Example. The fundamental example is that where $V = H^n(X, \mathbb{R})$ for X a compact Kahler variety and where $V^{pq} \subset H^n(X, \mathbb{C})$ is the space of cohomology classes represented by closed forms of type (p,q). Other usefull examples arise from taking tensors, direct factors or duals. In particular the dual $H_n(X, \mathbb{R})$ of $H^n(X, \mathbb{R})$ has a hodge structure of weight $-n$. Integral homology and cohomology thus gives integral hodge structures.

1.1.3.

Example. Hodge structures of type $\{(-1,0),(0,-1)\}$ are those for which the action h of $\mathbb{C}^* = \mathbb{S}(\mathbb{R})$ comes from a complex structure on V; for V of this type, the projection pr of V on $V^{-1,0} \subset V_{\mathbb{C}}$ is bijective and satisfies $pr(h(z)v) = zpr(v)$.

1.1.4.

Example. Let A be a comples torus; it is the quotient L/Γ of its Lie algebra L by the ring Γ. We have $\Gamma \otimes \mathbb{R} \xrightarrow{\sim} L$, or a complex structure on $\Gamma \otimes \mathbb{R}$. Considering this as a hodge structure like 1.1.3. Via the isomorphism $\Gamma = H_1(A, \mathbb{Z})$ it is also that of 1.1.2.

1.1.5.

Example. The hodge structure of tate $\mathbb{Z}(1)$ is the $\mathbb{Z}$-hodge structure of type $(-1,-1)$ of the integer subring $2\pi i\mathbb{Z} \subset \mathbb{C}$. The exponential identifies $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}(1)$, or an isomorphism $Z(1) = H_1(\mathbb{C}^*)$. The hodge structure $\mathbb{Z}(n) = Z(1)^{\otimes n}$ $(n \in \mathbb{Z})$ is the $\mathbb{Z}$-hodge structure of type $(-n,n)$ of the integer lattice $(2\pi i)^n)\mathbb{Z}$. We denote $\ldots(n)$ the tensor product of $\ldots$ by $\mathbb{Z}(n)$ (Tate twist).

1.1.6.

Remark. The rule $h(z)v = z^{-p}\overline{z}^{-q}v$ for $v \in V^{pq}$ is that which I use in (*Les Constanted des equations fonctionelles des fonction L*, Anvers II, Lecture Notes in Math., Vol. 349, pp. 501-597) and the inverse of those of [?] They are justified in part by the example 1.1.4 above, and in part by the desire that $\mathbb{C}^*$ acts on $\mathbb{R}(1)$ as multiplication by the norm (see the end of **??**).

1.1.7. A **Variation of Hodge Structures** on a complex analytic variety S consists of:

(1) A local system V of real vector spaces;
(2) To each point of S, a Hodge structure on the fibre of V at s which varies continuously with s.

We require that the Hodge filtration varies holomorphically with s and verifies the axium we shall call **transversality**: The derivative of a section of F^p is in F^{p-1}.

It will often be given a local system $V_{\mathbb{Z}}$ of $\mathbb{Z}$-modules of finite type, such that, $V = V_{\mathbb{Z}} \otimes \mathbb{R}$. We will thus speak of **the variation of $\mathbb{Z}$-Hodge structures**. Again with $\mathbb{Z}$ replaced by any ring as in 1.1.1.

1.1.8.

Remark. Considering S as a real variety, thus the tangeant space at each point can be given a complex structure, that is, a Hodge structure of type $\{(-1,0),(0,-1)\}$. The integrability of the almost complex structure on S is expressed in saying that the bracket of vector fields is compatible with the hodge filtration of the complexification of the tangean fibres: $[T^{0,-1}, T^{0,-1}] \subset T^{0,-1}$. In the same way, the axiums of variations of Hodge structures express that the derivation (tangeant fibre) $\otimes_{\mathbb{R}}$ (sections C^∞ of V) $\to$ (sections C^∞ of V) (or rather the complexification of this application) is compatible with the Hodge filtration: $\partial_D F^p \subset F^p$ for D in $T^{0,-1}$ (holomorphicity) and $\partial_D F^p \subset F^{p-1}$ for arbitrary D (transversality).

TODO-I do not understand what this is saying

1.1.9. We have the following principle: In algebraic geometry, every time there appears a hodge structure which is dependant on complex parameters, it is a variation of hodge structures on the space of these parameters. The fundamental example is 1.1.2 with parameters $f : X \to S$ a smooth proper morphism, with fibres X_S, kahler, the $H^n(X_s, \mathbb{Z})$ form a local system on S and the Hodge filtration on the complexification $H^n(X_s, \mathbb{C})$ varies holomorphically with s and satisfies the transversality axiom./

1.1.10. A **polarisation** of a real Hodge structure, of weight n, V is a morphism $\Psi : V \otimes V \to \mathbb{R}(-n)$ such that the form $(2\pi i)^n \Psi(x, h(i)y)$ is symmetric and positive definite. We have the same condition for $\mathbb{Z}$-Hodge structures, in replacing $\mathbb{R}(-n)$ by $\mathbb{Z}(-n)$, ... Because $\Psi(h(i)x, y) = \Psi(x, h(-i)y)$ (since $h(i)$ is trivial on $R(-n)$) and $h(-i)y = (-1)^n h(i)y$ the symmetry condition reduces to saying Ψ symmetric for n even and alternating for n odd.

The Hodge structures which appear in algebraic geometry are $\mathbb{Z}$-hodge structrues which are homogeneus and polarisable. The fundamental example: The possitivity theorem of Hodge assures us that $H^n(X, \mathbb{Z})$, for X projective smooth, is polarizable (note that $h(i)$ is the operator which Weil denotes C in his book on kahler varieties).

1.1.11. Let $(V_i)_{i\in I}$ be real vector spaces and $(S_j)_{j\in J}$ a family of tensors in the V_i and their duals. We are interested in families of Hodge structures on the V_i for which the s_j are of type $(0,0)$. To interpret this condition "Type $(0,0)$" in particular cases, note that $f : V \to W$ is a morphism if and only if as an element of $\mathrm{Hom}(V, W) = V^* \otimes W$, it is of type $(0,0)$.

Let G be an algebraic subgroup of $\prod \mathrm{GL}(V_i)$ which fixes the s_j. After 1.1.1 a family of Hodge structures on the V_i is indentified with a morphism $h : \mathbb{S} \to \prod \mathrm{GL}(V_i)$. To have the s_j be of type $(0,0)$ it is necisary and sufficient that h factors through G, we must thus consider algebraic morphisms $h : \mathbb{S} \to G$.

We can regard G, rather than the system V_i,s_j as the primordial object: if G is a real linear algebraic group, it reduces in the same way to giving $h : \mathbb{S} \to G$ or to giving each representation V of G a Hodge structure which is functorial for G-morphisms and compatible with tensors (see Saavedra [**?**, VI.2]). The morphism w_h and μ_h of 1.1.1 come from morphisms of $\mathbb{G}_m$ into G and $\mathbb{G}_m$ into $G_{\mathbb{C}}$ respectively.

1.1.12. The construction 1.1.11 leads to considering spaces of Hodge structures of the following type: we fix a real linear algebraic group G, and we consider a topologically connected component X of the space of morphism (homoprhisms of algraic groups over $\mathbb{R}$) of $\mathbb{S}$ into G.

Let G_1 be the smallest algebraic subgroup of G through which we can factor the $h \in X$: X is again a connected component of the space of morphisms of $\mathbb{S}$ into G_1. Because $\mathbb{S}$ is of multiplicative type, any two elements of X are conjugate: the space X is a class of $G_1(\mathbb{R})^+$ conjugates of morphisms of $\mathbb{S}$ into G. It is also a class of $G(\mathbb{R})^+$-conjugation and G_1 is a normal subgroup of the identity component of G.

1.1.13. In view of 1.1.9 and 1.1.10 we consider only those X that come from faithful representations of G. We have:

(1) For all i, the grading by weight of V_i (for the complexification the grading of $V_{i\mathbb{C}}$ by the $V_{i\mathbb{C}}^n = \oplus_{p+q=n} V_i^{pq}$) is independent of $h \in X$. Equivalently: $h(\mathbb{R}^*)$ is central in $G(\mathbb{R})^0$ or the adjoint representation has weight 0.
(2) For a proper complex structure on X, and all i the family of hodge strucuteres degined by $h \in X$ is a variation of Hodge structures on X.
(3) If V is the homogeneous component of weight n of one V_i, there exists $\Psi : V \otimes V \to \mathbb{R}(-n)$ such that for all $h \in X$, it is a polarization of V.

1.1.14.

Proposition. *Suppose we satisfty the first condition in 1.1.13.*

(1) *There exists a unique complex structure on X such that the hodge filtrations vary holomorphically with $h \in X$.*
(2) *The second condition is satisfied if and only if the adjoint representaion is of type* $\{(-1,1),(0,0),(1,-1)$
(3) *The third condition is satisfied if and only if G_1 (as defined in 1.1.12) is reductive and for $h \in X$, the inner automorphism $inth(i)$ induces a Carton involution on its adjoint group.*

(1) Let V be the sum of the V_i. It is a faithful representation of G. A Hodge structure is determined by the corresponding Hodge filtration (plus the the grading by weight if we are not in a homogenious case): in weight $n = p+q$ we have $V^{pq} = F^p \cap \overline{F}^q$. The mapping ϕ of X into the grassmannian of $V_{\mathbb{C}} : h \mapsto$ the corresponding hodge filtration, is thus injective. We will verify that it identifies X with a complex sub-variety of this grassmannian; this proves the first point: The complex structure on X induced by that of the grassmannian is the only one on which ϕ is holomorphic.

Let L be the Lie algebra of G and $p : L \to \mathrm{End}(V)$ its action on V. The action p is a morphism of G modules, it is injective by hypothesis. For all $h \in X$ it is also a morphism of Hodge structures. The tangeant space of X at h is the quotient of L by the Lie algebra of the stabilizer of h, namely the subspace L^{00} of L for the hodge structure of L defined by

h. The tangeant space of the grassmannian at $\phi(h)$ is $\mathrm{End}(V_{\mathbb{C}})/F^0(\mathrm{End}(V_{\mathbb{C}}))$. Finally, $d\phi$ is the composition:

$$\begin{array}{ccc} L/L^{00} & \xrightarrow{p} & \mathrm{End}(V)/\mathrm{End}(V)^{00} \\ \downarrow l & & \downarrow l \\ L_{\mathbb{C}}/F^0L_{\mathbb{C}} & \xrightarrow{p} & \mathrm{End}(V_{\mathbb{C}})/F^0\,\mathrm{End}(V_{\mathbb{C}}) \end{array}$$

Because p is an injective morphism of hodge structures, $d\phi$ is injective; its image is that of $L_{\mathbb{C}}/F^0L_{\mathbb{C}}$ is a complex subspace, thus we have the assertion.

(2) The axiom of transversality tells us that the image of $d\phi$ is in $F^{-1}\,\mathrm{End})V_{\mathbb{C}})/F^0\,\mathrm{End}(V_{\mathbb{C}})$, that is that $L_{\mathbb{C}} = F^{-1}L_{\mathbb{C}}$.

(3) To prove the last point, we will make use of [**?**, 2.8], which we recall below. Recall that a **Cartan involution** of a linear algebraic group (not necissarily connected) G is an involution σ of G such that the real form G^σ of G (with complex conjugation given by $g \mapsto \sigma\overline{g}$) is **compact**: $G^\sigma(\mathbb{R})$ is compact and intersects all the connected components of $G^\sigma(\mathbb{C}) = G(\mathbb{C})$. For $C \in G(\mathbb{R})$ with central square, a **C-polarisation** of a representation V of G is a bilinear form Ψ G-invariant, such that $\Psi(x, Cy)$ is symmetric and possitive definite. For all $g \in G(\mathbb{R})$, we thus have that $\Psi(x, gCg^{-1}y) = \Psi(g^{-1}x, Cg^{-1}y)$ so that the notion of C polarisation depends only on $G(\mathbb{R})$ conjugacy class of C.

1.1.15.

Proposition. *We recall* **??***2.8]TODO Let G be a real algebraic group and $C \in G(\mathbb{R})$ be a central square, the following are equivalent:*

(1) *$IntC$ is a Cartan involution of G.*
(2) *All real representations of G are C-polarisable.*
(3) *G admits one faithful C-polarisable representation.*

We note that the first condition implies that G^0 is reductive so that it has a compact form. It depends only on the conjugacy class of C. We prove now the third point of 1.1.14. Let G_2 be the smallest algebraic subgroup of G through which factor the restrictions of $h \in X$ to $U^1 \subset \mathbb{C}^*$. For a bilinear form $\Psi : V \otimes V \to \mathbb{R}(-n)$ to satisfy the third condition of 1.1.13 it is necissary and sufficient that $(2\pi i)^n\Psi : V \otimes V \to \mathbb{R}$ is invariant by the $h(U^1)$- thus by G_2 (this expresses the fact that Ψ is a morphism), and a $h(i)$-polarisation. Following 1.1.15 and 1.1.13 this is equivalent to $Inth(i)$ is a Cartan involution of G_2.

We first deduce that G_1 is reductive: G_2 is, to have a compact form, and G_1 is a quotient of a product $\mathbb{G}_m \times G_2$. Because G_2 is generated by compact subgroups, its connected center is compact; it is isogenous to the quotient of G_2 by its derived group. The involution $\theta = Inth(i)$ is thus a Cartan involution of G_2 if and only if it is of its adjoint group, and we conclude in noting that G_1 and G_2 have the same adjoint groups.

TODO-This arguement 1.1.14 is not well translated

1.1.16.

Corollary. *The conditions in 1.1.13 do not depend on the choice of family of representations V_i.*

1.1.17.

Corollary. *The spaces X in 1.1.13 are hermitian symmetric domains.*

Taking X of this type. We successively reduce the cases considered as follows:

(1) (a) That $G = G_1$; replacing G by G_1 does not change X nor the conditions of 1.1.13.
(b) That G is adjoint, by (1) we made it reductive, and its quotient by its finite center is the product of a torus and its adjoint group G^{ad}. The space X again identifies with the connected component of a space of morphisms of $\mathbb{S}$ into G with a fixed projection into T, whose lifting is unique. The conditions of 1.1.14 are verified elsewhere.
(c) That G is simple: we can decompose G into a product of simple groups G_i, this decomposes X into a product of spaces X_i relative to the G_i.

Thus let G be a simple adjoint group, and X a $G(\mathbb{R})^+$-conjugacy class of non-trivial morphisms $H : \mathbb{S}/\mathbb{G}_m \to G$, which satisfies the conditions of 1.1.14(ii) and (iii). The group G is non-compact: otherwise $h(i)$ would be trivial by (iii), since its centralizer is compact; there thus exists on X a $G(\mathbb{R})^+$ equivarient riemannian structure. By (ii) $h(i)$ acts on the tangeant space $\mathrm{Lie}(G)/\mathrm{Lie}(G)^{00}$ of X at h by -1; The space X is riemannien symmetric. We finally check that it is hermitian symmetric for the complex structure of 1.1.14(i). It is thus of the non-compact type (negative curvature) since G is not compact.

(2) Conversely, if X is a hermitien symmetric space, and $x \in X$ we know that the multiplication by u ($|u| = 1$) on the tangeant space T_x of X at x extends to an automorphism $m_x(u)$ of X. Letting A be the group of automorphisms of X and $h(z) = m(z/\overline{z})$ for $z \in \mathbb{C}^*$. The centralizer of x commutes with h, and the condition of 1.1.14(ii) are thus verified: $\mathrm{Lie}(A_x)$ is of type $(0, 0)$ and $T_x = \mathrm{Lie}(A)/\mathrm{Lie}(A_x)$ of type $\{(-1, 1), (1, -1)\}$. Finally we know that A is the connected component of $G(\mathbb{R})$ for G adjoint and that the riemannian symmetric space is of negative curvature if and only if the symmetry $h(i)$ is a Cartan involution of G (see Helgason [**?**]).

1.1.18. We indicate now two variants of 1.1.15 (see [**?**, 2.11]).

(1) We give a reductive real algebraic group G, and a $G(\mathbb{R})$-conjugacy class of morphism $h : \mathbb{S} \to G$. We suppose that w_h- denoted w- is central and thus independant of h (the condition 1.1.13(a)) and that $Inth(i)$ is a Cartan involution of $G/w(\mathbb{G}_m)$.

Because G is reductive, $w(\mathbb{G}_m)$ admits a **TODO-supplement** G_2: connected normal subgroup such that G is a quotient of $w(\mathbb{G}_m) \times G_2$ by a finite central subgroup. It is unique: generated by the derived group and the largest compact torus in the center. It contains the $h(U_1)(h \in X)$ and $h(i)$ is a Cartan involution. If V is a representation of G, its restriction to G_2 admits an $h(i)$-polarisation Φ. If V is of weight n, $w(\mathbb{G}_m)$ acts by similitudes, thus so does G: for a proper representation of G over $\mathbb{R}$, Φ covariant. For this representation $\mathbb{R}$-is of type (n, n); this permits us to make G act on $\mathbb{R}(n)$, in a manner compatible with its Hodge structure, and to vieew $(2\pi i)^{-n}\Phi$ as a G-invariant polarisation form $V \otimes V \to \mathbb{R}(-n)$.

(2) Suppose that G is deduced by extension of scalars to $\mathbb{R}$ of $G_{\mathbb{Q}}$ over $\mathbb{Q}$ and that w is defined over $\mathbb{Q}$. The group G_2 is thus defined over $\mathbb{Q}$, because it is the unique **TODO-supplement** of $w(\mathbb{G}_m)$ and all aracther of G/G_2 are defined over $\mathbb{Q}$, because the group is either trivial or isomorphic to $\mathbb{G}_m$ over $\mathbb{Q}$. If a rational representation of $G_{\mathbb{Q}}$ is of weight n, the G-invariant bi-linear forms $V \otimes V \to \mathbb{Q}(-n)$ form a vector space F over $\mathbb{Q}$. The collection of those which are a polarisation (relative to $h \in X$) is the **TODO-trace** over F of an open $F_{\mathbb{R}}$, and this open is non-empty by ('a'), there thus exist G-invariant polarisation forms $\Psi : V \otimes V \to \mathbb{Q}(n)$.

We take care that the forms in the above two interpretations are not alwas polarisations for all $h' \in X$: if $h' = Int(g)(h)$ the formula $\Psi(x, h'(i)y) = g\Psi(g^{-1}x, h(i)g^{-1}y)$ shows that the form $(2\pi i)^n\Psi(x, h'(i)y)$ is symmetric and definite, but the positive or negativeness of the definiteness depends on the action of g on $\mathbb{R}(-n)$.

1.2. **Classification.** After this paragraph, we will use the relation 1.1.17 between hermitian symmetric domains and moduli spaces of Hodge structres to reformulate certain results of [**?**]...[**?**] and give several extensions.

1.2.1. Consider systems (G, X) of a simple adjoint real algebraic group G, and a $G(\mathbb{R})$ conjugacty class X of morphisms of real algebraic groups $h : \mathbb{S} \to G$ which satisfy (the notations are that of 1.1.1 and 1.1.11):

(1) The adjoint representation Lie(G) is of type $\{(-1, 1), (0, 0), (1, -1)\}$ (in particular h is trivial on $\mathbb{G}_m \subset \mathbb{S}$)
(2) $Inth(i)$ is a Cartan involution.
(3) h is non-trivial, or what amounts to the same thing (1.1.17) G is non-compact.

Following 1.1.17 the connected components of the space X are again irreducible hermitian symmetric domains.

The second hypothesis assures that the Cartan involutions of G are inner automorphisms, and thus that G is an inner form of its compact form (see 1.2.3) in particular, G being simple, it is absolutely simple.

The $G(\mathbb{C})$ conjugacy class of $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$ does not depend on the choice of $h \in X$. We denote it my M_X.

1.2.2.

Proposition. *Let $G_{\mathbb{C}}$ be a simple adjoint complexe algebraic group. To each system (G, X) formed from a real form G of $G_{\mathbb{C}}$ and X satisfying the conditions of 1.2.1 we associate M_X. We obtain a bijection between $G_{\mathbb{C}}(\mathbb{C})$ conjugacy classes of systems (G, X) and $G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of non-trivial morphisms $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ which satisfy the following condition:*

$(*)$ *In the representation* $\mathrm{ad}\,\mu$ *of* $\mathbb{G}_m$ *on* $\mathrm{Lie}(G_{\mathbb{C}})$ *only the characters* $z, 1, z^{-1}$ *appear.*

To check this, we will use the duality between hermitian symmetric domains and compact hermitian symmetric spaces.

1.2.3. Let G be a real form of $G_{\mathbb{C}}$, X a $G(\mathbb{R})$ conjugacy class of morphisms of $\mathbb{S}/\mathbb{G}_m$ into G and $h \in X$. The real form of G corresponds to a choice of complex conjugation σ on $G_{\mathbb{C}}$; define G^* to be the real form corresponding to the complex conjugation coming from $int(h(i))\sigma$:

$$G^*(\mathbb{R}) = \{g \in G_{\mathbb{C}}(\mathbb{C}) | g = int(h(i))\sigma(g)\}$$

The morphism h is again defined over $\mathbb{R}$, of $\mathbb{S}/\mathbb{G}_m$ into G^*: we have $h(\mathbb{C}^*/\mathbb{R}^*) \subset G^*(\mathbb{R})$; define X^* to be the $G^*(\mathbb{R})$ conjugacy class of h. The construction $(G, X) \mapsto (G^*, X^*)$ is an involution on the collection of $G_{\mathbb{C}}(\mathbb{C})$ conjugacy classes of systems of real forms G of $G_{\mathbb{C}}$ and a $G(\mathbb{R})$ conjugacy class of non-trivial morphism of $\mathbb{S}/\mathbb{G}_m$ into G. It exchanges objects (G, X) as in 1.2.2 with those where G is compact and X satisfies the first condition of 1.2.1.

We know that the compact real forms are all conjugate to each other. Because if $g \in G_{\mathbb{C}}$ normalises a real form G, we have $G \in G(\mathbb{R})$ (this is because G is adjoint), the duality reduces 1.2.2 to the following:

1.2.4.

Lemma. *Let G be a compact form of $G_{\mathbb{C}}$. The construction $h \mapsto \mu_h$ induces a bijection between:*

(1) *$G(\mathbb{R})$-conjugacy classes of morphisms $h : \mathbb{S}/\mathbb{G}_m \to G$ satisfying the first condition of 1.2.1.*
(2) *$G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of morphisms $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ satisfying 1.2.2$(*)$.*

Let T be a maximal torus of G, and $T_{\mathbb{C}}$ its complexification. We verify first that the mapping $h \mapsto \mu_h$: $\mathrm{Hom}(\mathbb{S}/\mathbb{G}_m, T) \to \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})$ is bijective. If W is the Weil group of G, we know that:

$$\mathrm{Hom}(U^1, T)/W \xrightarrow{\sim} \mathrm{Hom}(U^1, G)/G(\mathbb{R})$$

and

$$\mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})/W \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, G_{\mathbb{C}})/G_{\mathbb{C}}(\mathbb{C})$$

The mapping $h \mapsto \mu_h$ thus induces a bijection:

$$\mathrm{Hom}(\mathbb{S}/\mathbb{G}_m, G)/G(\mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_m, G)/G_{\mathbb{C}}(\mathbb{C})$$

and to have h satisfy the first condition of 1.2.1 it is necissary and sufficient that μ_h satisfy 1.2.2($*$).

1.2.5. Let G be a simple adjoint complex algebraic group. We will enumerate the conjugacy classes of non-trivial $\mu : \mathbb{G}_m \to G$ which satisfy the condition 1.2.2($*$), in terms of Dynkin diagrams D of G. Recall that this diagram is canonically attached to G, in particular the automorphisms of G act on D. We can identifier the **TODO-Sommet** with conjugacy classes of maximal parabolic subgroups.

Let T be a maximal torus, $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ (The dual of $X(T)$ for the pairing $X(T) \times Y(T) \to^0 \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$), $R \subset X(T)$ the collection of roots, B a system of simple roots, α_0 the **TODO-oppose** of the largest root and $B^+ = B \cup \{\alpha_0\}$. The **TODO-sommet** of D are parametrised by B and those of the extended Dynkin diagram D^+ by B^+.

A conjugacy class of a morphism of $\mathbb{G}_m$ into G has a unique representative $\mu \in Y(T)$ in the fundamental chamber $\langle \alpha, \mu \rangle \geq 0$ for $\alpha \in B$. It is uniquely determined by the positive integers $\langle \alpha, \mu \rangle$ ($\alpha \in B$) and G being adjoint, these can be chosen arbitrarily. The condition 1.2.2($*$) for μ non-trivial can be rewritten as:

$$\langle \alpha_0, \mu \rangle = -1$$

Writting the largest root as a linear combination of simple roots, $\sum_{\alpha \in B^+} n(\alpha)\alpha = 0$ with $n(\alpha) = 1$ and call special the **TODO-sommet** of D^+ such that for the corresponding root $\alpha \in B^+$, we have $n(\alpha) = 1$. We know that the quotient of the group of coweights by the coroots acts on D^+, and in a manner which is simply transitive on the special roots, the special roots are thus the conjugates under $\mathrm{Aut}(D^+)$ of the root corresponding to α_0 and the number of them is the number of components $|\pi_1(G)|$ of G (see Bourbaki [**?**, VI,2ex2 and 5a]).

The condition ($*$) is rewritten as: $(*)''$ for a simple root $\alpha \in B$ corresponding to a special **TODO-sommet** of D we have $\langle \alpha, \mu \rangle = 1$ for the other simple roots we have $\langle \alpha, \mu \rangle = 0$.

1.2.6. In total, the $G_{\mathbb{C}}(\mathbb{C})$-conjugacy classes of systems (G, X) as in 1.2.2 are parametrised by the special **TODO-sommet** of the Dynkin diagram D of $G_{\mathbb{C}}$. In particular, for G a given real form of $G_{\mathbb{C}}$, X is determined by the corresponding special **TODO-sommet** $s(X)$ ($G(\mathbb{R}) \subset G_{\mathbb{C}}(\mathbb{C})$ is in effect its proper normalizer). The **TODO-sommet** corresponding to $X^{-1} = \{h^{-1} | h \in X\}$ is the transformation of $s(X)$ by the involution **TODO-d'opposition**.

In 1.2.3 G and G^* are inner formes of each other, if there exists X satisfying the conditions of 1.2.1, G is thus an inner form of its compact form. In other words, complex conjugation acts on the Dynkin diagram of $G_{\mathbb{C}}$ by its involution **TODO-d'opposition**.

1.2.7.

Proposition. *Let G be a simple adjoint real algebraic group, and suppose that there exist morphisms $h : \mathbb{C}^*/\mathbb{R}^* \to G$ satisfying the conditions of 1.2.1. The collection of these morphisms has two connected components exchanged by $h \mapsto h^{-1}$. Each has for its stabilizer the connected component $G(\mathbb{R})^+$ of G.*

The second hypothesis of 1.2.1 assures that the centralizer K of $h(i)$ is a maximal compact subgroup of $G(\mathbb{R})$. In particular, $\pi_0(K) \simeq \pi_0(G(\mathbb{R}))$. It has the same Lie algebra as the centralizer of h. This, latter group, is a connected algebraic group, as it is the centrelizer of a torus, and compact, as it is a subgroup of the centralizer of $h(i)$. It is thus topologically connected, and $Centr(h) = K^+ = K \cap G(\mathbb{R})^+$. The center of K^+ has dimension 1: The complexification of K^+ is the centralizer of μ_h thus by $(*)''$, it is the Levi subgroup of a maximal parabolic subgroup. We can also deduce that the representation of K^+ on $\mathrm{Lie}(G)/\mathrm{Lie}(K^+)$ is irreducible (see [**?**, proof of V,1.1]. The morphism h is thus an isomorphism of $\mathbb{S}/\mathbb{G}_m$ with connected center K^+, and K^+ determines h up to sign. A fortiori, $h(i)$ determines h up to sign. Thus:

(1) The mapping $h \mapsto h(i)$ is $2:1$.
(2) It maps isomorphically the orbit $G(\mathbb{R})^+/K^+$ of h under $G(\mathbb{R})^+$ under $G(\mathbb{R})/K$ of all Cartan involutions in $G(\mathbb{R})$.

The following proposition results:

1.2.8.

Corollary. *Let (G,X) be as in 1.2.1, and s the corresponding* ***TODO-sommet*** *in the Dynkin diagram of $G_{\mathbb{C}}$.*

(1) *If s is not fixed by the involution* ***TODO-d'opposition****, $G(\mathbb{R})$ and X are connected.*
(2) *If s is fixed by the involution* ***TODO-d'opposition****, $G(\mathbb{R})$ and X have two connected components, the components of X are interchanged by $h \mapsto h^{-1}$ and by the $g \in G(\mathbb{R}) \setminus G(\mathbb{R})^+$.*

We point out that the first condition is equivalent to either of:

- The system of relative roots for G is of type C (rather than BC);
- X is a tube domain.

1.3. **Symplectic Embeddings.**

1.3.1. Let V be a real vector space, with a non-degenerate alternating bilinear form Ψ. The coreseponding **Siegel half space** S^+ has the following description: it is the space of complex structures h on V for which Ψ is of type $(1,1)$ (for the identification of 1.1.3 between complex structures and Hodge structures of type $\{(-1,0),(0,-1)\}$) and for which the form $\Psi(x, h(i)x)$ is symmetric and positive definite.

If we replace 'positive definite' by 'definite' the **Double Seigel half space** $S^\pm$ obtained is a conjugacy class of morphisms $h : \mathbb{S} \to CSp(V)$ (CSp is the group of symplectic similitudes; In [**?**] it is denoted Gp).

1.3.2. Letting G be an adjoint real algebraic group (**??**) and X a conjugacy class of morphisms $h : \mathbb{S} \to G$. We suppose it satisfies the first two conditions of 1.2.1 and we replace the third by:

(iii') G has no compact factors.

The system (G,X) is thus a product of systems (G_i, X_i) as in 1.2.1 and X_i corresponds to a special **sommet** of the Dynkin diagram of G_i (1.2.6).

Consider the diagrams:

$$(G,X) \leftarrow (G_1,X_1) \rightarrow (CSP(V), S^\pm)$$

Where G is the adjoint group of the reductive group G_1, and where X_1 is a $G_1(\mathbb{R})$-conjugacy class of morphisms of $\mathbb{S}$ into G_1. We have a section $\tilde{G} \to G_1$ of the type for which V is a representation of $\tilde{G}$. Our goal is the determination 1.3.8 of the non-trivial complex irreducible representations W of $\tilde{G}$, which is essentially equivalent to that contained in the complexification of the representation obtained. This problem is resolved by Satake in [**?**]

1.3.3.

Lemma. *It is sufficient that there exists $(G_1, X_1) \to (G, X)$ as above, and a linear representation (V, ρ) of type $\{(-1,0),(0,-1)\}$ of G_1 for that W is contained in $V_{\mathbb{C}}$.*

Replacing G_1 by the subgroup generated by the derived group G_1' and the image of h reduces us to supposing that $inth(i)$ is a Cartan involution of $G_1/w(\mathbb{G}_m)$. There then exists on V a polarisation form Ψ (1.1.18(a)) for which ρ is a morphism of (G_1, X_1) into $(CSp(V), S^{\pm})$.

1.3.4. Considering the following projective system $(H_n)_{n\in\mathbb{N}}$: $\mathbb{N}$ is ordered by divisibility, $H_n = \mathbb{G}_m$ and the transition morphism H_{nd} into H_n is $x \mapsto x^d$ (The projective limit of the H_n is the universal cover– in the algebraic sense– of $\mathbb{G}_m$). A **fractional morphism** of $\mathbb{G}_m$ into a group H is an element of $\lim inj Hom(H_n, H)$, the same for the group $\mathbb{S}$. For $\mu : \mathbb{G}_m \to H$ fractional, define by $\mu^n : H_n = \mathbb{G}_m \to H$, and V a linear representation of H, V is the sum of subspaces V_a ($a \in (1/n)\mathbb{Z}$ such that, via μ^n, $\mathbb{G}_m$ acts on V_a by multiplication by x^{na}. The a's such that $V_a \neq 0$ are the **weights** of μ in V. In the same way, a fractional morphism $h : \mathbb{S} \to H$ determines a fractional Hodge decomposition $V^{r,s}$ of V ($r, s \in \mathbb{Q}$).

1.3.5.

Lemma. *For $h \in X$, let $\tilde{\mu}_h$ be the fractional lift of μ_h to $\tilde{G}_{\mathbb{C}}$ the representations W of 1.3.2 are those of $\tilde{\mu}_h$ which have only two weights a and $a+1$.*

The condition is necissary: lifting h to $h_1 \in X_1$, we have $\mu_{h_1} = \tilde{\mu}_h \cdot \nu$ with ν central. On V, μ_{h_1} has weights 0 and 1. If $-a$ is the unique weight of ν on W irreducible in $V_{\mathbb{C}}$, the only weights of $\tilde{\mu}_h$ on W are a and $a+1$. For W non-trivial, the action of $\mathbb{G}_m$ via $\tilde{\mu}_h^n$ (for n sufficiently divisible) is non-trival (because $G_{\mathbb{C}}$ is simple), thus non-central and the two weights a and $a+1$ must both appear.

The condition is sufficient: Take for V the real vector space underlying W, and for G_1 the group generated by the image of $\tilde{G}$ and by the group of homotheties. For $h \in X$, with fractional lifting $\tilde{h}$ to $\tilde{G}$, let $h_1(z) = h(z) z^{-a} \overline{z}^{1-a}$. If W_a and W_{a+1} are the subspaces of weight a and $a+1$ of W, $\tilde{h}$ acts on W_a (resp W_{a+1}) by $(z/\overline{z})^a$ (resp $z/\overline{z}^{1+a}$) and h_1 by $\overline{z}$ (resp. z): So h_1 is a true morphism of $\mathbb{S}$ into G_1, with projection h into G and V is of type $\{(-1,0),(0,-1)\}$ rel. h_1. It remains only to apply 1.3.3.

1.3.6. Translating the conditions in 1.3.5 in terms of roots. Let T be a maximal torus of $G_{\mathbb{C}}$, $\tilde{T}$ its inverse image in $\tilde{G}_{\mathbb{C}}$, B a system of simple roots of T and $\mu \in Y(T)$ the representative of the fundamental champer of the conjugacy class of μ_h ($h \in X$). If α is the dominant weight of W, the smalest weight is $-\tau(\alpha)$ for τ the involution **TODO-d'opposition**. It comes to expressing that $\langle \mu, \beta \rangle$ takes on only two values a and $a+1$, for β a weight of W. These weights are all of the form ($\alpha+$ a $\mathbb{Z}$-linear combination of roots), and the $\langle \mu, r \rangle$ for r a root are integers, the condition is expressed by $\langle \mu, -\tau(\alpha) \rangle = \langle \mu, \alpha \rangle - 1$:

$$(1.3.6.1) \qquad \langle \mu, \alpha + \tau(\alpha) \rangle = 1$$

Determining the solutions of this. For all dominant weights α,$\langle \mu, \alpha + \tau(\alpha) \rangle$ is an integer, because $\alpha + \tau(\alpha)$ is a $\mathbb{Z}$-linear combination of roots. If $\alpha \neq$, it is > 0 otherwise μ annihilates all the weights of the corresponding representation. A dominant weight α satisfying (1.3.6) can only be the sum of two weights.

1.3.7.

Lemma. *Only fundamental weights can satisfy condition 1.3.6.*

1.3.8. After 1.3.7 the representations W we are looking for factor through a simple factor G_i of G and their dominant weight is fundamental; It corresponds to a **TODO-sommet** of the Dynkin diagram D_i of $G_{i\mathbb{C}}$. The necisary and sufficient conditon 1.3.6 depends only on the projection of μ into $G_{i\mathbb{C}}$L this corresponds to a special **TODO-sommet** s of D_i (1.2.6), an s to a simple root α_s. The number $\langle \mu, \omega \rangle$, for ω a weight, is the coefficient of α_s in the expression of ω as a $\mathbb{Q}$ linear combination of simple roots. For ω fundamental these coefficients are given in the tables of Bourbaki [**?**]. They are given in the following table, where we enumerate the Dynkin diagrams with a special **TODO-sommet** which is circled. Each **TODO-sommet** corresponds to a fundamental weight ω, and we indicate the number $\langle \mu, \omega \rangle$. The **TODO-sommet** correspond to weights which satisfy condition 1.3.6 are underlined.

1.3.9. **TODO-the table... sigh...**

1.3.10. Remarks:

(1) For G simple and exceptional, no representations W satisfy 1.3.2.
(2) For G simple and classical, except for the case $D_l^H (l \geq 5)$ the representations W of 1.3.2 form a faithful system of representations of $\tilde{G}$. For D^H we obtain only a faithful representation of a double cover of G (namely, the algebraicly connected component of the group of automorphisms of a vector space H together with an anti-hermitian non-degenerate form–an inner form of $\mathrm{SO}(2n)$)

2. Shimura Varieties

2.0. Preliminaries.

2.0.1. Let G be a group, Γ a subgroup, and $\varphi : \Gamma \to \Delta$ a morphism. Suppose we are given an action r of Δ on G, which stabilizes Γ and such that:

(1) $r(\varphi(\gamma))$ is an inner automorphism int_γ of G;
(2) φ is compatible with action of Δ on Γ by r, and on itself by inner automorphisms: $\varphi(r(\delta)(\gamma)) = int_\delta(\varphi(\gamma))$

Form the direct product $G \rtimes \Delta$. The conditions above, reduce to saying that the collection of $\gamma \cdot \varphi(\gamma)^{-1}$ is a normal subgroup, and we define $G *_\Gamma \Delta$ as the quotient of $G \rtimes \Delta$ by this subgroup.

We note that the hypothesis lead to $Z = \mathrm{Ker}(\varphi)$ being central in G, and that $\Im(\varphi)$ is a normal subgroup of Δ. The lines in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, equal] & \Gamma \arrow[r] \arrow[d, hook] & \Delta \arrow[r] \arrow[d, hook] & \Delta/\Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] & G \arrow[r] & G *_\Gamma \Delta \Delta/\Gamma \arrow[r] & 0 &
\end{tikzcd}$$

are exact, or an isomorphis:

$$\Gamma \backslash G \xrightarrow{\sim} \Delta \backslash G *_\Gamma \Delta$$

and, highlighting a right action of $G *_\Gamma \Delta$ on $\Gamma \backslash G$. For this action, G acts by right translation and Δ acts on the right by r^{-1}.

If G is a topological group, with Δ discrete, and such that the action r is continuous, the group $G *_\Gamma \Delta$ given the quotient topology of that of $G \rtimes \Delta$ is a topological group, $G/\mathrm{Ker}(\varphi)$ is an open subgroup and the mapping above is a homeomorphism.

The construction 2.0.1 maintains meaning in the categorie of algebraic groups over a field. If G is a reductive group over k, we have a canonical isomorphism $G = \tilde{G} *_{Z(\tilde{G})} Z(G)$ (for the trivial action of $Z(G)$ on $\tilde{G}$).

2.0.2. Let G by an algebraic group over a field k and G^{ad} the quotient of G by its center $Z(G)$. The action by inner automorphisms of G on itself $(x,y) \mapsto xyx^{-1} : G \times G \to G$ is invariant under $Z \times \{e\}$ acting by translation, thus factors through an action of G^{ad} on G. Take care that the action of $\gamma \in G^{ad}(k)$ on $G(k)$ is not necissarily an inner automorphism of $G(k)$ (the projection of $G(k)$ into $G^{ad}(k)$ need not be surjective). A typical example of this is the action of $\mathrm{PGL}(n,k)$ on $\mathrm{SL}(n,k)$.

In the same way, the action of the "commutateur" $(x,y) = xyx^{-1}y^{-1}$ is invariant under translation by $Z \times Z$ and thus factors trhough an action of $G^{ad} \times G^{ad} \to G$.

All of this, and the fact that these "commutateurs" and "inner automorphisms" satisfy the usual identities is best seen by descent, that is interpretting G as a **TODO-faisceau** in groups at an appropriate site, and G^{ad} like the quotient of this **TODO-faisceau** in group by its center. In characteristic 0, if we are only interested in points of G in extension of k, it suffices to use galois descent– see 2.4.1.2.4.2.

A variant of this. For G reductive over k, the groups G and $\tilde{G}$ have the same adjoint group, and the proceeding constructions for G and $\tilde{G}$ are compatible. In particular, the application of the commutateur $(\ ,\ ) : G \times G \to G$ has a canonical factorisation:

$$(\ ,\ ) : G \times G \to G^{ad} \times G^{ad} \to \tilde{G} \to G$$

We deduce that the quotient $G(k)$ by the normal subgroup $\rho(\tilde{G}(k))$ is abelian.

2.0.3. Letting k be a global field of characteristic 0, $\mathbb{A}$ its ring of adeles, G a semi-simple group over k and $N = \mathrm{Ker}(\rho : \tilde{G} \to G)$. Let S be a finite collection of places of k, $\mathbb{A}_S$ the ring of S-adeles (restricted product for $\nu \notin S$) and set $\Gamma_S = \rho(\tilde{G}(\mathbb{A}_S)) \cap G(k)$ (the intersection in $G(\mathbb{A}_S)$). It is the group of elements of $G(k)$ which, for all places $\nu \notin S$ can be lifted to $\tilde{G}(k_\nu)$ (recall that $\rho : \tilde{G} \to G$ is proper).

The long exact sequence in cohomology identifies $G(k)/\rho(\tilde{G}(k))$ with a subgroup of $H^1(\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$ and $\Gamma_S/\rho(\tilde{G}(k))$ with elements which are locally trivial in the places $\nu \notin S$ of this subgroup. In particular $\Gamma_S/\rho(\tilde{G}(k))$ is contained in the subgroup $H^1(\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$ of classes whose restriction to all **TODO-monogene** subgroups is trivial (arguement and notations from [**?**]). If $\Im\,\mathrm{Gal}(\overline{k}/k)$ os tje o,age of galois in $\mathrm{Aut}(N(\overline{k}))$, we have $H^1(\mathrm{Gal}(\overline{k}/k), N(\overline{k}) = H^1(\Im\,\mathrm{Gal}(\overline{k}/k), N(\overline{k})$ (loc. cit.); in particular, $\Gamma_S/\rho(\tilde{G}(k))$ is finite.

2.0.4.

Proposition. (1) Γ_S *depends only on the collection of places* $\nu \in S$ *where the decomposition group* $D_\nu \subset \Im\,\mathrm{Gal}(\overline{k}/k)$ *is non-cyclic. In particular, it doesn't change when we adjust* S *with infinite places.*

(2) $\Gamma_S/\rho(\tilde{G}(k))$ *is identified with a subgroup of the finite group* $H^1(\Im\,\mathrm{Gal}(\overline{k}/k), N(\overline{k})$ *formed from classes with trivial restriction to all decomposition groups* D_ν, $\nu \notin S$. *In particular, for* S *large, we have* $\Gamma_S/\rho(\tilde{G}(k)) = H^1\Im\,\mathrm{Gal}(\overline{k}/k), N(\overline{k})$.

The restriction of an element of $H^1(\Im\,\mathrm{Gal}(\overline{k}/k), N(\overline{k}))$ to an element of a cyclic decomposition group is automatically trivial, thus we have the first claim. For the second, we can suppose that S contains the infinite places. The Hasse principle for $\tilde{G}$ (for the classes comming from the center) assures that all the elements of the group are effectively realized as obstruction classes.

2.0.5.

Corollary. *All sufficiently small* S*-congruence subgroups of* $G(k)$ *are in* Γ_S.

If U is an S-congruence subgroup, $U/U \cap \rho(\tilde{G}(k))$ is finite: the obstruction to lifting to $\tilde{G}(k)$ is killed in a galois extenion with bounded ramification, thus is in $H^1(\mathrm{Gal}, N(\overline{k}))$ for Gal a finite quotient of $\mathrm{Gal}(\overline{k}/k)$. The conditions of S-congruence thus permit us to pass from this H^1 to $\Gamma_S/\rho(\tilde{G}(k))$ cf. [**?**].

2.0.6.

Remark. We note from elsewhere that if $\tilde{G}$ satisfies the strong approximation theorem relative to S, all S-congruence subgroups $U \subset \Gamma_S$ of $G(k)$ surject onto $\Gamma_S/\rho(\tilde{G}(k))$.

2.0.7.

Corollary. *For all archimedian places ν, a suffificiently small S-congruence subgroup U of $G(k)$ is in the topologically connected componant $G(k_\nu)^+$ of $G(k_\nu)$.*

Becuase $\tilde{G}(k_\nu)$ is connected, we have $G(k_\nu)^+ = \rho\tilde{G}(k_\nu)$ and $U \subset \Gamma_S = \Gamma_{S\cup\nu} \subset G(k_\nu)^+$ (2.0.4 and 2.0.5).

2.0.8.

Corollary. *The subgroup $G(k)\rho(\tilde{G}(\mathbb{A}_S))$ of $G(\mathbb{A}_S)$ is closed, topologically isomorphic to $\rho(\tilde{G}(\mathbb{A}_S) *_{\Gamma_S} G(k)$ (i.e. $\rho(\tilde{G}(\mathbb{A}_S))$ is an open subgroup).*

It is a subgroup because, in view of 2.0.2, $\rho(\tilde{G}(\mathbb{A}_S))$ is normal in $G(\mathbb{A}_S)$ with a commutative quotient. If $T \supset S$ is sufficiently large so that $\tilde{G}(k)$ is dense in $\tilde{G}(\mathbb{A}_\Gamma)$ (strong approximation). Denote k_{T-S} the product of k_ν for $\nu \in T - S$. For K an open compact subgroup of $G(\mathbb{A}_T)$, we have:

$$G(k)\rho(\tilde{G}(\mathbb{A}_S)) = G(k)\rho(\tilde{G}(k) \cdot \tilde{G}(k_{T-S}) \times \rho^{-1}K) \subset G(k)(\rho(\tilde{G}(k_{T-S})) \times K).$$

From 2.0.5 for K sufficiently small, we have in $G(\mathbb{A}_T) : G(k) \cap K \subset \Gamma_T$, or in $G(\mathbb{A}_S) : G(k) \cap (\rho(\tilde{G}(k_{T-S})) \times K) \subset \Gamma_S \subset \rho(\tilde{G}(\mathbb{A}_S))$. The intersection of $G(k)\rho(\tilde{G}(\mathbb{A}_S))$ with the open subgroup $\rho(\tilde{G}(k_{T-S})) \times K$ is thus contained in $\rho(\tilde{G}(\mathbb{A}_S))$ and the corallary follows.

2.0.9.

Corollary. *If $\tilde{G}$ satisfies the condition of real approximation rel. S. The* ***TODO-adherence*** *of $G(k)$ in $G(\mathbb{A}_S)$ is $G(k) \cdot \rho(\tilde{G}(\mathbb{A}_S))$.*

2.0.10. Let T be a k torus, and S a finite collection of places including the archimean ones. Let $U \subset T(k)$ be the group of S-unites. By a theorem of chevaelly, all subgroups of finite index in U are congruence subgroups (see [**?**] for an elegant proof). It follows that if $T' \to T$ is an isogeny, the image of a congruence subgroup for T' is a congruence subgroup for T.

2.0.11. Let G be reductive over k, $\rho : \tilde{G} \to G^{der}$ the universal covering of its derived group, and Z^0 the connected component of the center Z. Here are several corollaries of 2.0.10 (we suppose that the finite collection S contains the archimedian places).

2.0.12.

Corollary. *For U of finite index in the group of S-unites of $Z(k)$ there exists a compact open subgroup K of $G(\mathbb{A}_S)$ such that:*

$$G(k) \cap (K \cdot G^{der}(\mathbb{A}_S)) \subset G^{der}(k) \cdot U$$

Applying 2.0.10 to the isogeny $Z^0 \to G/G^{der}$: for small K, an element γ of $G(k)$ in $K \cdot G^{der}(\mathbb{A}_S)$ has in $(G/G^{der})(k)$ a small image, for the topology of subgroups of S-congruence, thus can be lifted to a small element z of $Z(k)$ and $\gamma = (\gamma z^{-1}) \cdot z$.

2.0.13.

Corollary. *The product of a congruence subgroup of G^{der} and a finite index subgroup of the S-unites of $Z^0(k)$ is an S-congruence subgroup of $G(k)$.*

2.0.14.

Corollary. *All the sufficiently small S-congruence subgroups of $G(k)$ are contained in the topological connected component $G((\mathbb{R})^+$ of $G(\mathbb{R})$.*

Apply 2.0.13, **??** to G^{der} and 2.0.10 to Z^0.

2.0.15. We know that $G^{der}(k)\rho(\tilde{G}(\mathbb{A}))$ is open in $G(k)\rho(\tilde{G}(\mathbb{A}))$ (because the inverse image of $\{e\} \subset$ the discrete subgroup $G/G^{der}(k)$ of $(G/G^{der})(\mathbb{A})$). From 2.0.8, $G(k)\rho(\tilde{G}(\mathbb{A}))$ is thus a closed subgroup of $G(\mathbb{A})$. We set:

$$\pi(G) = G(\mathbb{A})/G(k)\rho(\tilde{G}(\mathbb{A}))$$

The existance of commutateurs 2.0.2 shows that the action of $G^{ad}(k)$ on $\pi(G)$ deduceed from the action 2.0.2 of G^{ad} on G is trivial.

2.1. **Shimura Varieties.**

2.1.1. Let G be a reductive group, defined over $\mathbb{Q}$ and X a $G(\mathbb{R})$-conjugacy class of morphisms of real algebraic grouups $\mathbb{S}$ into $G_{\mathbb{R}}$. We suppose it satisfies the following axioms (the notation is that of 1.1.1 and 1.1.11.

(1) For $h \in X$, Lie$(G_{\mathbb{R}})$ is of type $\{(-1,1),(0,0),(1,-1)\}$
(2) The involution $inth(i)$ is a cartan involution on the adjoint group $G^{ad}_{\mathbb{R}}$.
(3) The adjoint group admits no factors G' defined over $\mathbb{Q}$ for which the projection of h is trivial.
(4) The morphism $w : \mathbb{G}_m \to G_{\mathbb{R}}$ is defined over $\mathbb{Q}$.
(5) $inth(i)$ is a Cartan involution of the group $(G/w(\mathbb{G}_m))_{\mathbb{R}}$.

From 1.1.14(i), X admits a unique complex structure such that, for all representations V of $G_{\mathbb{R}}$, the Hodge filtration F_h of V varies holomorphically with h. For this complex structure, the connected components of X are hermitian symmetric domains. The proof in 1.1.17 shows also that if we decompose $G^{ad}_{\mathbb{R}}$ into simple factors, h projects trivially onto compact factors, and that each connected component of X is the product of hermitien symmetric domains corresponding to the non-compact factors. The thrid condition can be restated as saying that G^{ad} (respectively $\tilde{G}$, which will give us the same thing) has no factors G' (defined over $\mathbb{Q}$) such that $G'(\mathbb{R})$ is compact, and the theorem of strong approximation assures us that $\tilde{G}(\mathbb{Q})$ is dense in $\tilde{G}(\mathbb{A}^f)$.

2.1.2. **Shimura Varieties** ${}_KM_{\mathbb{C}}(G,X)$ – or simply ${}_KM_{\mathbb{C}}$– are quotients ${}_KM_{\mathbb{C}}(G,X) = G(\mathbb{Q})\backslash X \times (G(\mathbb{A}^f)/K)$ for K a compact open subgroup of $G(\mathbb{A}^f)$. From 1.2.7 and with the notations of **??**, the action of $G(\mathbb{R})$ on X makes $\pi_0(X)$ a principal homogeneous space under $G(\mathbb{R})/G(\mathbb{R})_+$. Because $G(\mathbb{R})$ is dense in $G(\mathbb{R})$ (real approximation), we haev $G(\mathbb{Q})/G(\mathbb{Q})_+ \xrightarrow{\sim} G(\mathbb{R})/G(\mathbb{R})_+$, and if X^+ is a connected component of X we have:

$${}_KM_{\mathbb{C}}(G,X) = G(\mathbb{Q})_+\backslash X^+ \times (G(\mathbb{A}^f)/K)$$

This quotient is a disjoint union, indexed by the finite collection $G(\mathbb{Q})_+\backslash G(\mathbb{A}^f)/K$ of double cosets, of the quotients $\Gamma_f\backslash X^+$ or the hermitian symmetric domain X^+ by the images $\Gamma_g \subset G^{ad}(\mathbb{R})^+$ of the subgroups $\Gamma'_g = gKg^{-1} \cap G(\mathbb{Q})_+$ of $G(\mathbb{Q})$. The Γ_g are arithmetic subgroups, or there is an analytic structure on $\Gamma_g\backslash X^+$. The arcticle [**?**] gives a natural strucutre of quasi-projective algebraic varieties to these quotients, thus on ${}_KM_{\mathbb{C}}(G,X)$. If Γ_g is torsion free (this is the case for K sufficiently small); it is a result of [**?**] that this structure is unique. More precisely, for all reduced schemes Z, an analytic morphism of Z into $\Gamma_g\backslash X^+$ is automatically algebraic.

2.1.3. We have:

$$\pi_{0K}M_{\mathbb{C}} = G(\mathbb{Q})\backslash\pi_0(X) \times (G(\mathbb{A}^f)/K) = G(\mathbb{Q})\backslash G(\mathbb{Q})/G(\mathbb{R})_+ \times K = G(\mathbb{Q})_+\backslash G(\mathbb{A}^f)/K.$$

Because $G(\mathbb{A}^f)/K$ is discrete, we can replace $G(\mathbb{Q})_+$ by its closure in $G(\mathbb{A}^f)$. The connectedness of $\tilde{G}(\mathbb{R})$ assures that $\rho(\tilde{G}(\mathbb{Q})) \subset G(\mathbb{Q})_+$. By the theorem of strong approximation for $\tilde{G}$, $\rho(\tilde{G}(\mathbb{Q}))$ is dense in $\rho(\tilde{G}(\mathbb{A}^f))$ and $G(\mathbb{Q})_+ \supset \rho(\tilde{G}(\mathbb{A}^f))$, Thus,

$$\begin{aligned}\pi_{0K}M_{\mathbb{C}} &= G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A}) \cdot G(\mathbb{Q}) \cdot G(\mathbb{R})_+ \times K \\ &= \pi(G)/G(\mathbb{R})_+ \times K = \overline{\pi}_0\pi(G)/K\end{aligned}$$

In particular $\pi_{0K}M_{\mathbb{C}}$ depends only on the image of K in $G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A}))$.

2.1.4. Letting K vary (becoming smaller and smaller), the ${}_KM_{\mathbb{C}}$ form a projective system. It comes with a right action of $G(\mathbb{A}^f)$: a system of isomorphisms $g : {}_KM_{\mathbb{C}} \xrightarrow{\sim} {}_{g^{-1}Kg}M_{\mathbb{C}}$. It is convienient to consider instead the scheme $M_{\mathbb{C}}(G,X)$–or simply $M_{\mathbb{C}}$– the projective limit of the ${}_KM_{\mathbb{C}}$. This limit exists because the transition morphisms are finite. This scheme comes with a right action of $G(\mathbb{A}^f)$ and it recovers the ${}_KM_{\mathbb{C}} = M_{\mathbb{C}}/K$.

We propose to determine $M_{\mathbb{C}}$ and its decomposition into connected components.

2.1.5.

Definition. Fixing a connected component X^+ of X, the identity component $M^0_{\mathbb{C}}$ of $M_{\mathbb{C}}$ is the component which contains the image of $X^+ \times \{e\} \subset X \times G(\mathbb{A}^f)$.

2.1.6.

Definition. Let G_0 be an adjoint group over $\mathbb{Q}$ with no factors G'_0 defined over $\mathbb{Q}$ such that $G'_0(\mathbb{R})$ is compact and G_1 a covering of G_0. The topology $\tau(G_1)$ on $G_0(\mathbb{Q})$ is that which has for a fundamental system of neighbourhoods of the identity the images of congruence subgroups of $G_1(\mathbb{Q})$.

We denote $^\wedge$ (rel. G_1), or simply $^\wedge$, the completion for this topology. Let $\rho : \tilde{G}_0 \to G_1$ be the natural mapping, denote $^-$ the closure in $G_1(\mathbb{A}^f)$ and set: $\Gamma = \rho(\tilde{G}_0(\mathbb{A}) \cap G_1(\mathbb{Q})$. Because $\tilde{G}_0(\mathbb{R})$ is connected, $\Gamma \subset G_1(\mathbb{Q})^+$, we have (2.0.9,2.0.14)

$$\begin{aligned}G_0(\mathbb{Q})^\wedge(\text{rel}.G_1) &= G_1(\mathbb{Q})^- *_{G_1(\mathbb{Q})} G_0(\mathbb{Q}) = \rho(\tilde{G}_0(\mathbb{A}^f)) *_\Gamma G_0(\mathbb{Q}), \\ G_0(\mathbb{Q})^{+\wedge}(\text{rel}.G_1) &= G_1(\mathbb{Q})^-_+ *_{G_1(\mathbb{Q})_+} G_0(\mathbb{Q})^+ = \rho(\tilde{G}_0(\mathbb{A}^f)) *_\Gamma G_0(\mathbb{Q})^+.\end{aligned}$$

2.1.7.

Proposition. *The identity component $M^0_{\mathbb{C}}$ is the projective limit of the quotients $\Gamma\backslash X^+$ for Γ arithmetic subgroups of $G^{ad}(\mathbb{Q})^+$, open for the $\tau(G^{der})$ topology.*

From 2.1.2 it is the limit of the $\Gamma\backslash X^0$ for Γ the image of a congruence subgroup of $G(\mathbb{Q})_+$. The Corrallary 2.0.13 permits us to replace G by G^{der}.

2.1.8. The projection of G onto G^{ad} induces an isomorphism of X^+ with a $G(\mathbb{R})^+$ conjugacy class of morphisms of $\mathbb{S}$ into $G^{ad}_{\mathbb{R}}$ and from 2.1.7, $M^0_{\mathbb{C}}(G,X)$ depends only on G^{ad}, G^{der} and this class. Formalising this remark. Let G be an adjoint group, X^+ a $G(\mathbb{R})^+$-conjugacy class of morphism of $\mathbb{S}$ into G satisfying (2.1.1,2.1.2,2.1.3) and G_1 a covering of G. The **connected Shimura varieties** (rel. G, G_1, X^+) are the quotients $\Gamma\backslash X^+$, for Γ an arithmetic subgroup of $G(\mathbb{Q})^+$, open for the $\tau(G_1)$ topology. We denote $M^0_{\mathbb{C}}(G,G_1,X^+)$ their projective limit for Γ smaller and smaller. We note that by transport of structures we the action $G(\mathbb{Q})^+$ on $M^0_{\mathbb{C}}(G,G_1,X^+)$ extends by continuity to a complete action $G(\mathbb{Q})^{+\wedge}$(rel. G_1).

With the notations of 2.1.7 and the identification above of X^+ with a $G(\mathbb{R})^+$-conjugacy class of morphisms of $\mathbb{S}$ into $G^{ad}_{\mathbb{R}}$ we have:

$$M^0_{\mathbb{C}}(G,X) = M^0_{\mathbb{C}}(G^{ad}, G^{der}, X^+)$$

2.1.9. Let Z be the center of G, and $Z(\mathbb{Q})^-$ the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}^f)$. By the theorem of Chevalley (2.0.10), it is the completion of $Z(\mathbb{Q})$ for the topology of finite index subgroups of the group of units; it recieves isomorphically the closure of $Z(\mathbb{Q})$ in $\pi_0 Z(\mathbb{R}) \times Z(\mathbb{A}^f)$.

For $K \subset G(\mathbb{A}^f)$ compact open, we have $Z(\mathbb{Q})\cdot K = Z(\mathbb{Q})^- \cdot K$ (in $Z(\mathbb{A}^f)$) and

$$\begin{aligned} {}_K M_{\mathbb{C}} &= G(\mathbb{Q})\backslash X \times (G(\mathbb{A}^f)/K) = \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}^f)/Z(\mathbb{Q})\cdot K) \\ &= \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}^f)/Z(\mathbb{Q})^- \cdot K \end{aligned}$$

The action of $G(\mathbb{Q})/Z(\mathbb{Q})$ on $X \times (G\mathbb{A}^f)/Z(\mathbb{Q})^-$ is proper. This permits us to pass to the limit over K.

2.1.10.

Proposition. *We have*

$$M_{\mathbb{C}}(G,X) = \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}^f)/Z(\mathbb{Q})^-)^-$$

2.1.11.

Corollary. *If the conditions of 2.1.4 and 2.1.5 are satisfied we have* $M_{\mathbb{C}}(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}^f)$.

In this case $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}^f)$ and $Z(\mathbb{Q})^- = Z(\mathbb{Q})$.

2.1.12.

Corollary. *The right action of* $G(\mathbb{A}^f)$ *factors through* $G(\mathbb{A}^f)/Z(\mathbb{Q})^-$.

2.1.13. Let $G^{ad}(\mathbb{R})_1$ be the image of $G(\mathbb{R})$ in $G^{ad}(\mathbb{R})$ and $G^{ad}(\mathbb{Q})_1 = G^{ad}(G)\cap G^{ad}(\mathbb{R})_1$. The action 2.0.2 of G^{ad} on G induces an action (on the left) of $G^{ad}(\mathbb{Q})_1$ on the system of ${}_K M_{\mathbb{C}}$:

$$int(\gamma) : {}_K M_{\mathbb{C}} \xrightarrow{\sim} {}_{\gamma K \gamma^{-1}} M_{\mathbb{C}}$$

and to the limit on $M_{\mathbb{C}}$. For $\gamma \in G^{ad}(\mathbb{Q})^+$, this action stabilises the identity component)thus all connected components, see the following) and induces the action of 2.1.6.

Converting this action to a right action, denoted $\cdot\gamma$. If γ is the image of $\delta \in G(\mathbb{Q})$, the action $\cdot\gamma$ coincidees with the action of δ seen as an element of $G(\mathbb{A}^f)$: for $u \in M_{\mathbb{C}}$ the image of $(x,g) \in X \times G(\mathbb{A}^f)$, $u\cdot\gamma$ is the image of:

$$(\gamma^{-1}(x), int_\gamma^{-1}(g)) = (\delta^{-1}(x), \delta^{-1} g \delta) \sim (x, g\delta) \quad \text{mod } G(\mathbb{Q}) \text{ on the left}$$

In total, we obtain then a right action on $M_{\mathbb{C}}$ of the group:

$$\frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{G(\mathbb{A})/Z(\mathbb{Q})} G^{ad}(\mathbb{Q})_1 = \frac{G(\mathbb{A}^f)}{Z(\mathbb{Q})^-} *_{G(\mathbb{A})_+/Z(\mathbb{Q})} G^{ad}(\mathbb{Q})^+$$

2.1.14.

Proposition. *The right action of $G(\mathbb{A}^f)$ on $\pi_0 M_{\mathbb{C}}$ makes $\pi_0 M_{\mathbb{C}}$ into a principal homogeneous space under ints abelian quotient $G(\mathbb{A}^f)/G(\mathbb{Q})_+^- = \overline{\pi}_0\pi(G)$.*

We will see this soon by passage to the limit of the formulas in 2.1.3

2.1.15. Because $G^{ad}(\mathbb{Q})$ acts trivially on $\pi(G)$ (2.0.15) and because $G^{ad}(\mathbb{Q})^+$ stabilizes at least one connected component (2.1.13) the group $G^{ad}(\mathbb{Q})^+$ stabilizes all of them. For the action 2.1.13 of the group described there on $M_{\mathbb{C}}$, the stabilizer of each connected component is thus:

$$\frac{G(\mathbb{Q})_+^-}{Z(\mathbb{Q})^-} *_{G(\mathbb{Q})_+/Z(\mathbb{Q})} G^{ad}(\mathbb{Q})^+ =_{2.0.13} G^{ad}(\mathbb{Q})^{+\wedge}(\text{rel}, G^{der})$$

2.1.16. Summary: The group $G(\mathbb{A}^f)/Z(\mathbb{Q})^- *_{G(\mathbb{Q})/Z(\mathbb{Q})} G^{ad}(\mathbb{Q})_1$ acts on the right on $M_{\mathbb{C}}$. The profinite collection $\pi_0 M_{\mathbb{C}}$ is a principal homogeneous space under the action of its abelian quotient $G(\mathbb{A}^f)/G(\mathbb{Q})_+^- = \pi_0\pi(G)$ of this group by the closure of $G^{ad}(\mathbb{Q})^+$. This closure is the completion of $G^{ad}(\mathbb{Q})^+$ for the topology of images of congruence subgroups of $G^{der}(\mathbb{Q})$. The action of this completion on the identity component, once converted to a left action, is that of 2.1.8.

2.2. **Canonical Models.**

2.2.1. Let G and X be as in 2.1.1. For $h \in X$, the morphism μ_h (1.1.1, completed by 1.1.11) is a morphism over $\mathbb{C}$ of algebraic groups defined over $\mathbb{Q}$: $\mu_h : \mathbb{G}_m \to G_{\mathbb{C}}$. The **dual field** (reflex field) $E(G,X) \subset \mathbb{C}$ of (G,X) is the field of definition of this conjugacy class. If X^+ is a connected component of X, we will sometimes denote it $E(G,X^+)$.

Let (G',X') and (G'',X'') be as in 2.1.1. If there is a morphism $f : G' \to G''$ which sends X' into X'' we have $E(G',X') \supset E(G'',X'')$.

2.2.2. Let T be a torus, E a number field, and μ a morphism defined over E of $\mathbb{G}_m$ into T_E. The group E^*, viewed as an algebraic group over $\mathbb{Q}$ is the Weil restriction of scalars $R_{E/\mathbb{Q}}(\mathbb{G}_m)$. Applying $R_{E/\mathbb{Q}}$ to μ we obtain $R_{E/\mathbb{Q}}(\mu)$.

We will use also the norm map $N_{E/\mathbb{Q}} : R_{E/\mathbb{Q}}T_E \to T$ (on the rational points it is the norm). And by composition we have a morphism $N_{E/\mathbb{Q}} \circ R_{E/\mathbb{Q}}(\mu) : E^* \to T$. We will denote it simply $NR_E(\mu)$ or $NR(\mu)$. If E' is an extension of E and μ is again defined over E':

$$NR_{E'}(\mu) = NR_E(\mu) \circ N_{E'/E}$$

2.2.3. Let in particular T be a torus, $h : \mathbb{S} \to T_{\mathbb{R}}$ and $X = \{h\}$. If $E \subset \mathbb{C}$ contains $E(T,X)$ the morphism μ_h is defined over E, or the morphism $NR(\mu_h) : E^* \to T$. Passing to the adelic points modulo rational points, we deduce a homomorphism from the group of idele classes $C(E)$ of E in $T(\mathbb{Q})/T(\mathbb{A})$, and, by passage to the connected components, a morphism:

TODO-typo $T(\mathbb{A})/T(\mathbb{Q})$ makes more sense

$$\pi_0 NR(\mu_h) : \pi_0(C(E)) \to \pi_0(T(\mathbb{Q})/T(\mathbb{A}))$$

Global class field theory identifies $\pi_0(C(E)$ with the abelianization of the galois group over E.

The group $\pi_0(T(\mathbb{Q})\backslash T(A))$ is a profinite group, the projective limit of the finite groups $T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R})^+ \times K$ for K compact open in $T(\mathbb{A}^f)$. It is $\pi_0(T(\mathbb{R})) \times T(\mathbb{A}^f)/T(\mathbb{Q})^-$. The Shimura varieties ${}_K M_{\mathbb{C}}(T,X)$

are the finite collections $T(\mathbb{Q})\backslash\{h\}\times T(\mathbb{A}^f)/K = T(\mathbb{A})\backslash T(\mathbb{A}^f)/K$, their projective limit, calculated in 2.1.10 is the quotient of $\pi_0(T(\mathbb{Q})\backslash T(\mathbb{A}))$ by $\pi_0(T(\mathbb{R}))$.

We will the **reciprocity map** to be the map $r_E(T,X) : \mathrm{Gal}(\overline{\mathbb{Q}},E)^{ad} \to T(\mathbb{A}^f)/T(\mathbb{Q})^-$ consisting of the inverse of the compostion of the isomorphism of global class field theory **??** of $\pi_0 NR(\mu_h)$ and the projection of $\pi_0(T(\mathbb{A})/T(\mathbb{Q})$ onto $T(\mathbb{A}^f)/T(\mathbb{Q})^-$. It defines an action r_E of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{ab}$ on the ${}_KM_{\mathbb{C}}(T,X) : \sigma \mapsto$ the right translation by $r_E(T,X)(\sigma)$.

The universal case (in E) is that where $E = E(T,X)$: it results from (**??**) that the action r_E of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ is the restricton to $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(T,X))$ of $r_{E(T,X)}$.

2.2.4. Let G and X be as in 2.1.1. A point $h \in X$ is said to be **special** or of **CM-type** if $h : \mathbb{S} \to G(\mathbb{R})$ factors through a torus $T \subset G$ defined over $\mathbb{Q}$. We note that if T is such a torus, the cartan involution $inth(i)$ is trivial on the image $T(\mathbb{R})$ in the adjoint group, and that this image is thus compact. The field $E(T,\{h\})$ depends only on h, it is the **dual field** $E(h)$ of h.

We transport this terminology to points of ${}_KM_{\mathbb{C}}(G,X)$ and of $M_{\mathbb{C}}(G,X)$: for $x \in {}_KM_{\mathbb{C}}(G,X)$ (respectively $M_{\mathbb{C}}(G,X)$ the class $(h,g) \in X \times G(\mathbb{A}^f)$, the $G(\mathbb{Q})$-conjugacy class of h depends only on x, we say that x is special if h is, that $E(h)$ is the **dual field** $E(x)$ of x and that the $G(\mathbb{Q})$ conjugacy class h is the **type** of x.

On the collection of special points of ${}_KM_{\mathbb{C}}(G,X)$ (resp $M_{\mathbb{C}}(G,X)$) of a given type, corresponding to a dual field E, we will define an action r of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. Let thus $x \in {}_KM_{\mathbb{C}}(G,X)$ (resp $M_{\mathbb{C}}(G,X)$) have class $(h,g) \in X \times G(\mathbb{A}^f)$ $T \subset G$ a torus through which h factors, $\sigma \in \mathrm{Gal}(\overline{Q}/E)$ and $\overline{r}(\sigma)$ a representative in $T(\mathbb{A}^f)$ of $r_E(T,\{h\})(\sigma) \in T(\mathbb{A}^f)/T(\mathbb{Q})^-$. We set $r(\sigma)x =$ the class of $(h,\overline{r}(\sigma)g)$. The reader must verify that this class depends only on x and σ. The action thus defined commutes with the right action of $G(\mathbb{A}^f)$ on $M_{\mathbb{C}}(G,X)$.

2.2.5. A **Canonical Model** $M(G,X)$ of $M_{\mathbb{C}}(G,X)$ is a form over $E(G,X)$ of $M_{\mathbb{C}}(G,X)$ together with a right action of $G(\mathbb{A}^f)$ such that:

(1) The special points are algebraic;
(2) On the collection of special points of type τ given, corresponding to a dual field $E(\tau)$, the gallois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\tau)) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$ acts by the action in 2.2.4.

By "form" we mean a scheme M over $E(G,X)$, with a right action by $G(\mathbb{A}^f)$ and an equivariant isomorphism $M \otimes_{E(G,X)} \mathbb{C} \xrightarrow{\sim} M_{\mathbb{C}}(G,X)$.

Let $E \subset \mathbb{C}$ be a number field containing $E(G,X)$. A **weakly canonical model** of $M_{\mathbb{C}}(G,X)$ over E is a form over E with a right action of $G(\mathbb{A}^f)$ satisfying the first condition, but with the second modified so that $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\tau))$ is replaced by $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\tau) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

2.2.6. In [**?**, 5.4,5.5], inspired by the methods of Shimura, we showed that $M_{\mathbb{C}}(G,X)$ admites at most one weakly canonical model over E for $E(G,X) \subset E \subset \mathbb{C}$ and that if it exists it is fonctorial in (G,X).

2.3. **Construction of Canonical Models.** In this section we determine the cases in which the following criteries demonstrated in [**?**, 4.21,5.7], shows that we can construct canonical models.

2.3.1.

Theorem. *Let (G,X) be as in 2.1.1, V a rational vector space with a non-degenerate alternating form Ψ and $S^{\pm}$ the corresponding double Siegel half space (see 1.3.1). If there exists an embedding $G \hookrightarrow CSp(V)$ which maps X into $S^{\pm}$, then $M_{\mathbb{C}}(G,X)$ has a canonical model $M(G,X)$.*

2.3.2.

Proposition. *Let (G, X) be as in 2.1.1, $w = w_h$ $(h \in X)$ and (V, ρ) a faithful representation of type $\{(-1,0),(0,-1)\}$ of G. If inth(i) is a Cartan involution of $G_{\mathbb{R}}/w(\mathbb{G}_m)$ there exists an alternating form Ψ on V such that ρ induces a map $(G, X) \hookrightarrow (CSp(V), S^{\pm})$.*

By hypothesis, the faithful representation V is homogeneous of weight -1. The weight w is thus defined over $\mathbb{Q}$, and we take for Ψ a polarisation form as in 1.1.18(b).

2.3.3.

Corollary. *Let (G, X) be as in 2.1.1, $w = w_h$ $(h \in X)$ and (V, ρ) a faithful representation of type $\{(-1,0),(0,-1)\}$ of G. If the center Z^0 of G **TODO-deploye** over a CM-field, there exists a subgroup G_2 of G, with the same derived group and through which X factors, and an alternating form Ψ on V, such that ρ induces $(G_2, X) \hookrightarrow (CSp(V), S^{\pm}))$.*

The hypothesis on Z^0 is the same as saying that the largest compact subtorus of $Z^0_{\mathbb{R}}$ is defined over $\mathbb{Q}$. We take G_2 generated by the derived group, this torus, and the image of w and apply 2.3.2.

2.3.4. Let (G, X) be as in 2.1.1, with G $\mathbb{Q}$-simple adjoint. The axiom (2.1.12) assures that $G_{\mathbb{R}})$ is an inner form of its compact form. We exploit this fact.

(1) The simple components of $G_{\mathbb{R}}$ are absolutely simple, if we write G as obtained from restriction of scalars $G = R_{F/\mathbb{Q}} G^s$ with G^s absolutely simple over F, this signifies that F is totally real. Set the notation I = the collection of real embeddings of G and for $v \in I$ $G_v = G^s \otimes_{F,v} \mathbb{R}$, D_v = dynkin diagram of $G_{v\mathbb{C}}$, We have $G_{\mathbb{R}} = \prod G_v$, $G_{\mathbb{C}} = \prod G_{v\mathbb{C}}$ and the dynkin diagram D of $G_{\mathbb{C}}$ is the disjoint union of the D_v. The galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on D and I, in a manner compatible with the projection of D on I.
(2) Complex conjugation acts on D by the involution **TODO-d'opposition**. This is central in $\mathrm{Aut}(D)$. Thus, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on D in a manner which is faithful with that of $\mathrm{Gal}(K_D/\mathbb{Q})$, with K_D totally real if the involution is trivial, CM otherwise.

2.3.5. We have $X = \prod X_\nu$ for X_ν a $G_\nu(\mathbb{R})$-conjugacy class of morphism of $\mathbb{S}$ into G_ν. For G_ν compact, X_ν is trivial. For G_ν noncompact, X_ν is described by **TODO-sommet** s_ν, of the Dynkin diagram D_ν of $G_{\nu\mathbb{C}}$ (1.2.6).

Some notation: I_c = the collection of $v \in I$ such that G_v is compact, $I_{nc} = I - I_c$, D_c (resp D_{nc}) = the union of the D_v for $v \in I_c$ (resp $v \in I_{nc}$), G_c (resp G_{nc}) = the product of the G_v for $v \in I_c$ (resp $v \in I_{nc}$); the same for the universal coverings, $\Sigma(X)$ = the collection of the s_v for $v \in I_{nc}$. The definition 2.2.1 gives:

2.3.6.

Proposition. *The dual field of (G, X) is the sub-field of K_D fixed by the sub group of* $\mathrm{Gal}(K_D/\mathbb{Q})$ *which stabilizes $\Sigma(X)$.*

2.3.7. Suppose we have the diagram:

$$(G, X) \leftarrow (G_1, X_1) \hookrightarrow (CSp(V), S^{\pm})$$

The universal covering $\tilde{G}$ lifts in G_1, this permits us to restrict the representation V to $\tilde{G}$. The quotient of $\tilde{G}$ which acts faithfully is by hypothesis the derived group of G_1. Applying 1.3.2,1.3.8 to the diagram:

$$(G_{nc}, X) \leftarrow (\mathrm{Ker}(G_{1\mathbb{R}} \to G_c)^0, X_1) \to (CSp(V), S^{\pm})$$

We find that the non-trivial irreducible components of the representation $V_{\mathbb{C}}$ of $\tilde{G}_{nc}$ factors through one of the $\tilde{G}_{v\mathbb{C}}$ ($v \in I_{nc}$), and that their dominant weight is fondamental, and is one of those types permitted by table 1.3.9. The collection of dominant weights of the irreducible components of the representation $V_{\mathbb{C}}$ of $\tilde{G}_{\mathbb{C}}$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Because $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on I, and $I_{nc} \neq \emptyset$, we find that:

(1) All the irreducible components W of $V_{\mathbb{C}}$ are of the form $\otimes_{v\in T} W_v$ with W_v a fondamental representation of $G_{v\mathbb{C}}$ ($v \in T \subset I$), corresponding to a **TODO-sommet** $\tau(v)$ of D_v. We denote $\mathcal{L}(V)$ the collection of the $\tau(v) \subset D$ for $W \subset V_{\mathbb{C}}$ irreducible.
(2) If $S \in \mathcal{L}(V)$,$S \cap D_{nc}$ is empty or reduices to a single point $s_S \in D_v$ ($v \in I_{nc}$) and in the table 1.3.9 for (D_v, s_v), S_s is one of the underlined **TODO-sommet**.
(3) $\mathcal{L}$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have $\mathcal{L}$**TODO-what?? not the empty set**.
    If a collection of one of the parts of $\mathcal{L}$ of D satisfies (b) and (c) we denote $\tilde{G}(\mathcal{L})_{\mathbb{C}}$ the quotiend of $\tilde{G}_{\mathbb{C}}$ which acts faithfully in the corresponding representation of $\tilde{G}_{\mathbb{C}}$. The condition (c) assures that it is defined over $\mathbb{Q}$. the most interesting case is that where:
(4) $\mathcal{L}$ is formed from parts made of single elements.
    If $\mathcal{L}$ satisfies (b), (c), the collection $\mathcal{L}'$ of the $\{s\}$ for $s \in S \in \mathcal{L}$ verifies (b),(c),(d) and $\tilde{G}(\mathcal{L}')$ dominates $\tilde{G}(\mathcal{L})$.

In the following table–deduced from 1.3.9– we give

The list of the cases where there exists $\mathcal{L}$ satisfying (b),(c). From 1.3.10 this can be only the case if G is of one of the types A, B, C, D and the cases will be successively reviewed.

The collection $\mathcal{L}$ verfying the (b),(c),(d) maximal, and the corresponding group $\tilde{G}(\mathcal{L})$. (it dominates all the $\tilde{G}(\mathcal{L})$ for $\mathcal{L}$ satisftying (b),(c)).

2.3.8. The Table:

TODO-this

- Types A, B, C
- Type D_l ($l \geq 5$)
- sub-case Type $D_l^{\mathbb{R}}$
- sub-case Type $D_l^{\mathbb{H}}$
- Type D_4
- sub-case Type $D_4^{\mathbb{R}}$
- sub-case Type $D_4^{\mathbb{H}}$

For the remainder of this work, it will be convienient for us to redefine the case $D_4^{\mathbb{H}}$ as to exclude $D_4^{\mathbb{R}}$. With this terminology, there exists $\mathcal{L}$ satisfying (b),(c) if and only if (G, X) is one of the types $A, B, C, D^{\mathbb{R}}, D^{\mathbb{H}}$ and except for the type $D^{\mathbb{H}}$ there exists $\mathcal{L}$ satisfying (b),(c),(d) and such that $\tilde{G}(\mathcal{L})$ is the universal covering of G.

2.3.9. We will consider imaginary quadratic extensions K of F, together a collection T of complex embeddings: one on top of each real embedding $v \in I$ One single T defines a hodge structure $h_T : \mathbb{S} \to K_{\mathbb{R}}^*$ over K (considered as a rational vector space, and on which K^* acts by multiplication): if J is a collection of complex embeddings of K, we have $K \otimes \mathbb{C} = \mathbb{C}^J$, and we define h_T in asserting that the factors indexed by $\sigma \in K$ are of type $(-1, 0)$ for $\sigma \in T$, $(0, -1)$ for $\overline{\sigma} \in T$ and $(0, 0)$ if σ is above one of I_{nc}, the primary result of this section is:

2.3.10.

Proposition. *Let (G, X) be as in 2.1.1, with G $\mathbb{Q}$-simple adjoint and of type $A, B, C, D^{\mathbb{R}}, D^{\mathbb{H}}$. For all totally imaginary quadratic extensions K of F, given with a T as in 2.3.9 there exists a* *diagram:*

$$(G, X) \leftarrow (G_1, X_1) \hookrightarrow (CSp(V), S^{\pm})$$

for which

(1) *$E(G_1, X_1)$ is teh composite $E(G, X)$ and $E(K^*, h_T)$.*
(2) *The derived group G_1' is simply connected for G of type $A, B, C, D^{\mathbb{R}}$ and the universal covering is as described in 2.3.8 for $D^{\mathbb{H}}$.*

TODO-the proof

2.3.11.

2.3.12.

2.3.13.

2.4. **Preliminaries on the Reciprocity Law.** The constructions of this section will permit us, in section 2.6, to cacluate the reciprocity lay of canonical models, that is, the action of the galois group on the collection of geometrically connected components.

Although they are expressed better in the language of *fppf* descent, we express them in the language of Galois descent, we believe this is more familiar to non-geometers. This exposes several repetitions and inconsistencies, and introduces parasitic hypothesis about seperability and characteristic 0.

Let G be a reductive group over a global field k, with the notation of **??** our goal is to construct canonical morphisms of the following two types:

(1) For k' a finite (we shall suppose seperable) extension of k, and G' the extension of scalars of G to k', a norm map:
$$N_{k'/k} : \pi(G') \to \pi(G)$$
(2) For T a torus and M a conjugacy class, defined over k, of morphisms of T into G, a morphism:
$$q_M : \pi(T) \to \pi(G)$$
If $m \in M(k)$, q_M will be a morphism q_m induced by m; the difficulty will be to show that this morphism doesn't depend on the choice of m, and to construct q_M even if M doesn't have representatives defined over k.

The functorial properties of these morphisms will be evident from their constructions.

2.4.1. We use systematically the language of torseurs (which I prefer to that of cocycles), and that of Galois descent, in the form that Grohendieck gives (cf. SGA 1, or SGA 4.5 [Arcata]).

Galois Descent: Let K be a finite seperable extension of a field k. To construct an object X over k (for example a torseur), it suffices to construct (a) for all seperable extensions k' of k such that there exists a morphism of k-algebras of K into k', an object $X_{k'}$, over k; (b) for k'' an extension of k', an isomorphism $\chi_{k'',k'} X_{k'} \otimes k'' \xrightarrow{\sim} X_{k''}$; and satisfy the compatibility $\chi_{k'''k''}\chi_{k''k'} = \chi_{k'''k'}$.

In practice, this signifies, that to construct X, we can suppose the existance of auxilliary objects which exist only in seperable extensions K of k– at the cost of showing that the X constructed does not depend–up to unique isomorphism– on the choice of this auxilliary object.

Remark. Galois descent is a particular case of localisation in the etale topology; a construction like (a),(b),(c) above is often introduced with the adverb "locally"

2.4.2.

Example. Explaining the canonical covering used in 2.0.2 of the commutateur map. The use of galois descent–rather than *fppf*– obliges us to suppose that the projection of G on G^{ad} is smooth, and only considering $(\ ,\ ) : G^{ad}(k) \times G^{ad}(k) \to G(k)$, rather than the morphism $G^{ad} \times G^{ad} \to G$. If $X_1, x_2 \in G^{ad}(k)$, we can, locally, write $x_i = \rho(\tilde{x}_i)z_i$ with z_i in the cnter of G. The element $\tilde{x}_i$ is unique, up to multiplication by an element of the center of $\tilde{G}$. The commutateur of $\tilde{x}_1$ and $\tilde{x}_2$ does not depend on the choise of $\tilde{x}_i$ and we set $(x_1, x_2) = \tilde{x}_1\tilde{x}_2\tilde{x}_1^{-1}\tilde{x}_2^{-1}$.

2.4.3. For G an algebraic group over a field k, a G-torseur is a scheme P over k, with a right action of G which gives it the structure of a principle homogenious space. The trivial G-torseur G_d is G with the action of G by right translation. We identifiy the points $x \in P(k)$ with the **trivializations** of P (isomorphisms $\phi : G_d \xrightarrow{\sim} P$) by $\phi(g) = xg$.

If $f : G_1 \to G_2$ is a morphism, and P a G_1-torseur, there exists a G_2-torseur $f(P)$ together with $f : P \to f(P)$ which satisfies $f(pg) = f(p)f(g)$, and it is unique up to unique isomorphism. We interest ourselves in the category $[G_1 \to G_2]$ of G_1 torseurs P together with a trivialization of $f(P)$. For morphisms, take isomorphisms of G_1 torseurs, compatible with G_2 trivializations. We denote $H^0(G_1 \to G_2)$ the group of automorphisms of (G_{1d}, e) (it is $\ker(G_1(k) \to G_2(k))$) and $H^1(G_1 \to G_2)$ the collection (pointed by (G_{1d}, e)) of isomorphism classes of these objects.

Each $x \in G_2(k)$ defines an object $[x]$ of $[G_1 \to G_2]$" the trivial G_1-torseur G_{1d} together with the trivialization x of $f(G_{1d}) = G_{2d}$. When it doesn't cause confusion, we denote this simply x. The collection of morphisms of $[x]$ into $[y]$ identifies with $\{g \in G_1(k) | f(g)x = y\}$, to g accociate $u \mapsto ug : G_{1d} \to G_{1d}$. An object is of the form $[x]$ if and only if as a G_1-torseur, it is trivial–or there is an exact sequence:

$$1 \to H^0(G_1 \to G_2) \to G_1(k) \to G_2(k) \to H^1(G_1 \to G_2) \to H^1(G_1) \to H^1(G_2)$$

(this does not describe the inverse image of $p \in H^1(G_1)$; to describe it, one must proceed by **TODO-twisting** as in [**?**]).

2.4.4. If g is an epimorhpism, with kernal K, it is the same thing to give a G_1 toresur P which is G_2 trivialized by $x \in f(P)k$ or to give a K-torseur $f^{-1}(x) \subset P$: the natural functor $[K \to \{e\}] \to [G_1 \to G_2]$ is an equivalence.

More generally, if $g : G_2 \to H$ induces an epimorphism of G_1 on H, and $K_i = \ker(G_1 \to H)$, the natural foncteur $[K_1 \to K_2] \to [G_1 \to G_2]$ is an equivalence.

2.4.5. If G is cummutative, the sum $s : G \times G \to G$ is a morphism, and we denote the sum of G-torseurs by $P + Q = s(P \times Q)$, if G_1 and G_2 are commutative, we add in the same way the objects of $[G_1 \to G_2]$ which becomes a **Picard Category** (strictely commutative) (SGA 4, XVIII, 1.4).

All the proceeding works for group schemes over an arbitrary base.

2.4.6. If k' is a finite extension of k (the case where k'/k is seperable is sufficient for us) and G' is an algebraic group over k', the foncteur of restriction of scalaires of Weil $R_{k'/k}$ is an equivalence of categories between G' torseurs and $R_{k'/k}G'$-torseurs. This corresponds to a lemma of Shapiro $H'(k', G') = H^1(k, R_{k'/k}G')$. If G' comes from G by extension of scalars from G commutative–over k, we will make use of the trace map $R_{k'/k}G' \to G$–or a trace functor $Tr_{k'/k}$ of G'-torseurs into G-torseurs. More generally, for $G_1 \to G_2$ a morphism of commutative groups gives an additive functor:

$$Tr_{k'/k} : [G'_1 \to G'_2] \to [G_1 \to G_2]$$

These functors are described in great generatlity in [?, XVII, 6.3]. For k'/k seperable, we can give a more simple definition by descent: locally, k' is the sum of $[k':k]$ copies of k $[G_1' \to G_2']$ is identified with the category of $[k':k]^{uples}$ of objects $G_1 \to G_2$ and $Tr_{k'/k}$ is the sum.

When groups are denoted multiplicatively, we will speak rather of the norme functors $N_{k'/k}$.

2.4.7. **TODO-very long section... not sure what it is doing**

2.4.8.

Proposition. *If k is a local of global field, the morphism deduced from (??) by passing to the isomorphism classes of objects induces a morphism of $G(k')/\rho(\tilde{G}''(k'))$ in $G(k)/\rho(\tilde{G}(k))$:*

$$G(k')/\rho(\tilde{G}(k') \qquad G(k)/\rho(\tilde{G}(k))$$

$$H^1(\tilde{G}' \to G') \qquad H^1(\tilde{G} \to G)$$

TODO-arrows

TODO-the proof

2.4.9. For k non-archimedian local field with ring of integers V, k' unramified over k with ring of integers V', and G reductive over V, the morphism 2.4.8 induces a morphism of $G(V')/\rho(\tilde{G}(V)$: we see this by repeating the proceeding arguements over V, galois descent is replaced by etale localisation (here it is formally identical to galois descent over the residue field).

We can thus make adelic 2.4.8: for k a global field, the restricted product of morphisms 2.4.8 for the completions of k, is a morphism:

$$N_{k'/k}G(\mathbb{A}')/\rho(\tilde{G}(\mathbb{A}')) \to G(\mathbb{A})/\rho(\tilde{G}(\mathbb{A}))$$

Dividing by the global trace map, we obtain in the end the morphism (1)

$$N_{k'/k} : \pi(G') \to \pi(G)$$

The same as the construction of the morphism (1) based on that of the fonctor (??), that of (2) is based on the following:

2.4.10. **TODO-fancy construction of the maps q_M...**

2.4.11.

2.4.12.

2.5. Application: A Canonical Extension.

2.5.1.

2.5.2.

2.5.3.

2.5.4.

2.5.5.

2.5.6.

2.5.7.

2.5.8.

2.5.9.

2.5.10.

2.6. the Reciprocity Law of Canonical Models.

2.6.1. Let (G, X) be like as in 2.1.1 and $E \subset \mathbb{C}$ a number field containing $E(G, X)$. Suppose that $M_{\mathbb{C}}(G, X)$ admits a weakly canonical model $M_E(G, X)$ over E. The galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ then acts on the profinite system $\pi_0(M_{\mathbb{C}}(G, X))$ of the geometrically connected components of $M_E(G, X)$. This action commutes with that of $G(\mathbb{A}^f)$, by hypothesis is defined ovef E. From 2.1.14 the (right) action of $G(\mathbb{A}^f)$ makes $\pi_0 M_{\mathbb{C}}(G, X)$ into a principal homogeneous space under the abelian quotient $\overline{\pi}_0\pi G - G(\mathbb{A}^f)/G(\mathbb{Q})^-_+$. The galois action is thus given by a homomorphism $r_{G,X}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ into $\pi_0\pi(G)$, called the **reciprocity**. Our sign convention: the (right) action of σ coincides with the (right) action of $r_{G,X}(\sigma)$. This morphism factors through the abelianization of the Galois group, identified by global class field theory with $\pi_0\pi(G_{mE})$ where:

$$r_{G,X} : \pi_0\pi(G_{mE}) \to \overline{\pi}_0\pi(G)$$

2.6.2. Let M be the conjugacy class of μ_h, for $h \in X$. Because $E \supset E(G, X)$, it is definied over E. Composing the morphisms **??**, we obtain $N_{E/\mathbb{Q}} q_M : \pi(\mathbb{G}_{mE}) \to \pi(G_E) \to \pi(G)$.

By passage to π_0, we deduce:

$$\pi_0 N_{E/\mathbb{Q}} q_M : \pi_0\pi(\mathbb{G}_{mE}) \to \pi_0\pi(G) \to \overline{\pi}_0\pi(G)$$

2.6.3.

Theorem. *The morphism (2.6.1) gives the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ *on the collection of geometrically connected components of the weakly canonical model $M_E(G, X)$ of $M_{\mathbb{C}}(G, X)$ over E, it is the inverse of the morphism $\pi_0 N_{E/\mathbb{Q}} q_M$ of 2.6.2.*

The idea of the proof is that, for eacy type τ of special point (2.2.4) we know the action of a finite index subgroup of Gal_τ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the special points of this type (this by the definition of weakly canonical models), thus on the collection of connected components via mapping to the connected component which contains each point and using that this action is compatible. That the action of Gal_τ thus obtained is the restriction to Gal_τ of the action defined by the inverse of $\pi_0(N_{E/\mathbb{Q}} q_M)$ is verified in 2.6.4 below, and it remains to show that the Gal_τ generate $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

A type of special point τ is defined by $h \in X$ which factors by a torus $\iota : T \to G$ defined over $\mathbb{Q}$. The subgroup Gal_τ corresponds to $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h)) = \mathrm{Gal}(\overline{\mathbb{Q}}/E \cdot E(T, h))$. From [**?**, 5.1], for every finite extension F of $E(G, X)$ there exists (T, h) such that the extension $E(T, h)$ of $E(G, X)$ is linearly disjoint from F, this is more than enough to ensure that the Gal_τ generate $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

2.6.4. Let T and h be as above and $\mu = \mu_h$, The morphism $\mu : \mathbb{G}_m \to T$ is defined over $E(T, h)$ and the morphism $\pi_0 NR(\mu_h)$ of 2.2.3 is derived, by the application of the fonctor π_0 from $N_{E(T,h)/\mathbb{Q}} \circ q_\mu : \pi(G_{mE(T,h)}) \to \pi(T)$. We deduce that the action of $\mathrm{Gal}(\overline{Q}/E) \cap \mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ on the special points of type τ is compatible with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))^{ab} = \pi_0\pi(G_{mE(T,h)})$ on $\pi_0(M_{\mathbb{C}}(G, X))$ defined, by application of the fonctor π_0 to the inverse of:

$$\iota \circ N_{E(T,h)/\mathbb{Q}} \circ q_\mu : \pi(\mathbb{G}_{mE(T,h)}) \to \pi(T) \to \pi(G)$$

From the fonctoriality of N and q, it results that the composition is $N_{E(T,h)/\mathbb{Q}} \circ q_M$:

TODO-diagrams

is equal to $N_{E(G,X)/\mathbb{Q}} \circ q_M \circ N_{E(T,h)/E(G,X)}$:

Because the norm $N_{E(T,h)/E(G,X)}$ corresponds via class field theory to the inclusion of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T,h))$ into $\mathrm{Gal}(\overline{\mathbb{Q}}/E(G,X))$ we have correctly the promised action.

2.7. Reduction to the Derived Group, and the Existance Theorem.