DYNKIN DIAGRAMS AND ALGEBRAIC GROUPS

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1. Root Systems

Let V be a Euclidean vector space with an inner product $(\cdot, \cdot)$.

A root system is a finite collection of vectors Φ which satisfy the following:

(1) The span of Φ is V.
(2) If $\alpha \in \Phi$ then Φ is closed under the reflection σ_α in $\alpha^\perp$.
(3) If $\alpha \in \Phi$ then $\lambda\alpha \in \Phi$ if and only if $\lambda = \pm 1$.
(4) For $\alpha, \beta \in \Phi$ we have $\langle \beta, \alpha \rangle \stackrel{\text{def}}{=} 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Remark. The pairing $\langle \cdot, \cdot \rangle$ is not necissarily symmetric, only linear in the first variable.

The purpose of the integrality condition is to make it so that the root system can generate a lattice in a sensible way. (That it is integral and not rational says that the system is somehow 'reduced')

Using that $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\frac{(\alpha,\beta)^2}{|\alpha|^2|\beta|^2} = 4\cos^2(\theta)$ (where θ is the angle between α, β. The integrality condition immediately imposses that the cosine of the angles are one of $\{0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}, \pm 1\}$.

In particular the product is $0, 1, 2, 3$ they both have the same sign, and at least one of them is ± 1.

The **rank** of the root system is the dimension of V.

Root systems are **reducible** if there exist $V_1, V_2 \subset V$ with $V = V_1 \oplus V_2$ and $\Phi = (\Phi \cap V_1) \cup (\Phi \cap V_2)$.

Two root systems are **isomorphic** if there exists an isomorphism between the spaces taking roots onto roots.

The **Weyl group** of Φ is the group of isometries of V generated by σ_α. This group is finite as it acts faithfully on the finite set of roots.

The **Root lattice** of Φ is the $\mathbb{Z}$ module generated by Φ.

Example.
- $(0, 1), (1, 0)$ $(A_1 \times A_1)$
- $(0, 1), (1, 0), (1, 1)$ (B_2)
- Hexagon (A_2)
- Long+Short Hexagon (G_2)
- E_6 = (12,12off)

2. Simple roots

We say that a subset Φ^+ is a system of positive roots if:

(1) For each $\alpha \in \Phi$ exactly one of $\alpha, -\alpha \in \Phi^+$.
(2) If $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

An element of $\gamma \in \Phi^+$ is **simple** (with respect to Φ^+) if γ can not be written as $\alpha + \beta$ for $\alpha, \beta \in \Phi^+$.

Remark. Picking a collection Φ^+ is equivalent to picking a connected component of $V \setminus (\cup_\alpha \alpha^\perp)$. That is, picking a hyperplane $\gamma^\perp$ which contains none of the α and setting $\Phi^+ = \{\alpha \in \Phi | \langle \gamma, \alpha \rangle > 0\}$.

The collection $\Delta(\gamma)$ of simple roots of $\Phi^+(\gamma)$ forms a basis of V and every $\alpha \in \Phi^+(\gamma)$ is a positive integral linear combination of elements of $\Delta(\gamma)$.

Generating set by induction relative to (γ, α). Next have $\alpha, \beta \in \Delta(\gamma)$ implies $\langle \alpha, \beta \rangle < 0$ otherwise one of $\alpha - \beta$ or $\beta - \alpha$ would be in $\Phi^+(\gamma)$. Now if $X = \sum_\alpha s_\alpha \alpha = \sum_\beta t_\beta \beta$ with all $s_i, t_i > 0$, then by the above $(X, X) = 0$ and so $X = 0$, then we consider $(X, \gamma) = 0$ and so $s_i, t_i = 0$.

We can recover a root system from a collection of simple roots. Indeed, inductively on height if α is a non-simple positive root there exists a $\beta \in \Delta$, such that $\alpha - \beta$ is also positive. (use linear independence of simple roots expressing alpha as a sum of them).

We have the following facts about

- The Weyl-group acts transitively on Weyl-champers and thus on all collections of positive and simple roots.
- Every root can be made to be simple.
- The Weyl-group is generated by the $\sigma_\alpha | \alpha \in \Delta$.
- If an element of the weyl-group maps Δ to Δ it is the identity.

3. Dynkin Diagram

We verify the following properties of root systems:

(1) If Δ is a system of simple roots then $\Delta \setminus S$ generates a roots system in the span$(\Delta \setminus S)$.
(2) there are at most n pairs in Δ with $\langle \alpha, \beta \rangle \neq 0$.
(3) If $\alpha, \beta \in \Delta$ and $\langle \alpha, \beta \rangle \neq 0$ then $\tilde{\Delta}$ generates a root system in $V/(\alpha - \beta)$.
(4) For $\alpha \in \Delta$ we have $\sum_{\beta \neq \alpha} \langle \beta, \alpha \rangle \geq -3$.

These correspond directly to the statements which allow us to define admissible Dynkin diagrams. Via some tedius dot product arguements we can then show the classification.

(1) Every sub-diagram of admissible is admissible
(2) fewer than n pairs of connected verticies (implies acyclic) with above
(3) Can delete verties and glue.
(4) No edge has more than 3 lines attached
(5) A vertex with more than 2 lines imposes length of chain conditions.
$\sum ie_i$ and $\sum jn_j$ then look at (e, n) litterally and via cauchy schwartz.

Example.
- A_n
- B_n (out arrow)
- C_n (in arrow)
- D_n (double head)
- E_6, E_7, E_8 (2,1,n)
- F_4 (double in middle)
- G_2 (triple)

4. 'Maximal Tori' in Lie... Algebras

A **Cartan** sub-algebra $\mathfrak{h}$ in a lie algebra $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ ($[A, B] = 0$ for $A, B \in \mathfrak{h}$) such that $A \in \mathfrak{h}$ act semi-simply through on $\mathfrak{g}$ through the adjoint representation.

These correspond to the lie algebras of maximal tori in G.

Definition 4.1. Let $H \in \mathfrak{g}$ consider:

$$\mathfrak{g}(H) \stackrel{\text{def}}{=} \{X \in \mathfrak{g} | \operatorname{ad}(H)^k X = 0 \text{ for some } k\}$$

H is **regular** if $\dim(\mathfrak{g}(H)) = \min_{X\in\mathfrak{g}}(\dim \mathfrak{g}(X))$.

Theorem 4.2. *Cartan sub-algebras exist! Moreover they are the all of the form $\mathfrak{g}(H)$ for regular elements H.*

(This requires semi-simplicity (non-degenerate killing form))

Let α be a linear functional on the space $\mathfrak{h}$. Denote by $\mathfrak{g}^\alpha$ the linear subspace $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X\}$ on which $\mathfrak{h}$ acts via α.

We have the following:

- $\mathfrak{g}^0 = \mathfrak{h}$ and thus $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\neq 0} \mathfrak{g}^\alpha$.
- $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$
  follows from the Jacobi identity:
  $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$
- g^α, g^β are orthogonal if $\alpha + \beta \neq 0$.
  For X_α, Y_β we have $\operatorname{ad}_{X_\alpha} \operatorname{ad}_{Y_\beta} g^\gamma \subset g^{\gamma+\beta+\alpha}$, Consequently the trace must be 0.
- The above implies that $B_\mathfrak{h}$ is non-degenerate, and that $g^\alpha \neq \{0\}$ implies $g^{-\alpha} \neq 0$.
- Let H_α be such that $B(H_\alpha, H) = \alpha(H)$ then $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathbb{C}H_\alpha$.
- $\dim(\mathfrak{g}^\alpha) = 1$ (or 0).
  For non-trivial spaces there exists $E_\alpha, E_{-\alpha}$ with $B(E_\alpha, E_{-\alpha}) = 1$. Fix D_α such that $B(D_\alpha, E_{-\alpha} = 0$. Set $D_n = (\operatorname{ad} E_\alpha)^n D_\alpha$ then $[E_{-\alpha}, D_n] = \frac{n(n+1)}{2}\alpha(H_\alpha)D_{n-1}$ and $D_n \in \mathfrak{g}^{(n+1)\alpha}$. If D_n is non-zero this is a contradiction.

Let Φ be the set of all $\alpha \neq 0$ such that $\mathfrak{g}^\alpha \neq 0$.

- Let $\alpha \in \Phi$ and β be a root, then $\beta + n\alpha \in \Phi$ for $p \leq n \leq q$ where $p + q = -2\frac{\beta H_\alpha}{\alpha H_\alpha}$.
  Compute the trace of H_α on $\sum g^{\alpha+\beta n}$ for a maximal string of n.
- The only multiples of α such that $\lambda\alpha \in \Phi$ are $\pm\alpha$ and 0.
  Uses the above + tricks to rule out non-integers.
- $[g^\alpha, g^\beta] = g^{\alpha+\beta}$.
  Similar arguements with traces and invariance.
- $[X_{-\alpha}, [X_\alpha, X_\beta]] = \frac{q(1-p)}{2}\alpha(H_\alpha)B(X_\alpha, X_{-\alpha})X_\beta$
  Induction on the α series of β

Let $\mathfrak{h}_\mathbb{R} = \sum_{\alpha\in\Phi} \mathbb{R}H_\alpha$.

Theorem 4.3.
- *B is strictly positive definite on $\mathfrak{h}_\mathbb{R}$.*
  $B(H, H') = \sum_{\beta\in\Phi} \beta(H)\beta(H')$
  $\beta(H_\alpha) = -\alpha(H_\alpha)(p_{\beta,\alpha} + q_{\beta,\alpha})$
  $\alpha(H_\alpha) \in \mathbb{R}$ *and positive.*
- *Φ is a root system in $\mathfrak{h}_\mathbb{R}$ relative to B.*

5. The Root System

Fix $X_\alpha \in \mathfrak{g}^\alpha$ such that $B(X_\alpha, X_{-\alpha}) = 1$. Let $N_{\alpha,\beta}$ be such that $[X_\alpha, X_\beta] = N_{\alpha,\beta}$

Lemma 5.1.
- *if $\alpha + \beta + \gamma = 0$ and all terms defined then $N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}$*
- *if $\alpha + \beta + \gamma + \delta = 0$ and all terms defined then $N_{\alpha,\beta}N_{\gamma,\delta} + N_{\beta,\gamma}N_{\alpha,\delta} + N_{\gamma,\alpha}N_{\beta,\delta} = 0$.*

Theorem 5.2. *Let Φ be a root system. Define a Lie algebra with generators $H_\lambda, X_\lambda, Y_\lambda$ to have the relations:*

(1) $[H_\lambda, H_\mu] = 0$ *for all* $\lambda, \mu \in \Phi$.
(2) $[H_\lambda, X_\mu] = (\lambda, \mu) X_\mu$ *for all* $\lambda, \mu \in \Phi$.
(3) $[H_\lambda, Y_\mu] = -(\lambda, \mu) X_\mu$ *for all* $\lambda, \mu \in \Phi$.
(4) $[X_\lambda, Y_\mu] = \delta_{\mu\lambda} H_\mu$ *for all* $\lambda, \mu \in \Phi$.
(5) $\mathrm{ad}_{X_\lambda}^{-(\mu,\lambda)+1} X_\mu = 0$ *for all* $\lambda \neq \mu \in \Phi$.
(6) $\mathrm{ad}_{Y_\lambda}^{-(\mu,\lambda)+1} Y_\mu = 0$ *for all* $\lambda \neq \mu \in \Phi$.

Then the generated Lie algebra is semi-simple with root system Φ

Theorem 5.3. *If two semi-simple lie algebra's have isomorphic root systems then they are isomorphic*

We can obtain an isomorphism of the cartan algebra for free, it is easy to check it is an isometry. The cartan algebra and the system of roots determines a decomposition of the space.

One uses induction and the above lemma to get the result.

Theorem 5.4. *Any two cartan algebras are conjugate, and thus give the same root system. And thus we have converse to the above.*

6. The Classification Theorem

Theorem 6.1. *Classification of complex semi-simple adjoint algebraic groups*

- A_n - SL_{n+1}
- B_n - SO_{2n+1}
- C_n - Sp_{2n}
- D_n - SO_{2n}

Theorem 6.2. *Classification of non-adjoint groups.*
(this uses things about fundamental groups)

Theorem 6.3. *Classification of forms.*
This is galois descent in general, for $\mathbb{R}$ one can work it out using extended Dynkin diagrams.