Lecture 11 : RPCD + g=2

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1 Constructing Polyhedral cone decompositions in general

We first introduce the notation we shall be using throughout.

- L a lattice with a positive definite bilinear form $\langle \cdot, \cdot \rangle$.
- $L^{\#}$ the dual of L with respect to $\langle \cdot, \cdot \rangle$.
- Ω is a convex open homogeneous cone in $V = L \otimes \mathbb{R}$ self adjoint with respect to $\langle \cdot, \cdot \rangle$.
- Γ a subgroup of $\mathrm{Aut}_L(\Omega, V)$.

Definition 1. A subset K is said to be a *kernel* of Ω if: $0 \notin \overline{K}$ and $K + \Omega \subset K$.
Say two kernels are *comparable* if $\lambda K' \subset K \subset \lambda^{-1} K'$.
The semi-dual of a set A is $A^\vee = \{h \in \mathrm{Hom}(V, \mathbb{R}) | h(a) \geq 1 \ \forall a \in A\}$.
The *extreme points* of a convex set A are $E(A) = \{x \in \overline{A} | x = \frac{y+z}{2} \Rightarrow y = z = x\}$

Proposition 2. *For a kernel K we have the following:*

- *$K^\vee$ is a kernel*
- *$K = \cup_{e \in E(K)} e + C$.*
- *The closed convex hull of $\Omega \cap L$ is a kernel for Ω.*
  (All such are comparable independant of L).

Proof. SEE ASH □

Definition 3. A kernel is called a *core* if K is comparable to the closed convex hull of $\Omega \cap L$. It is called a *co-core* if $K^\vee$ is a core.

Proposition 4. *We have the following two examples of cores:*

- *K_{cent} the closed convex hull of $\Omega \cap L$ is a core.*
- *$K_{perf} =$ (closed convex hull of $\overline{\Omega} \cap L \setminus 0)^\vee$ is a core.*

Proof. SEE ASH $\square$

DEFINITION 5. A closed convex kernel is called *locally rationally polyhedral* if for any rational polyhedral cone Π whose verticies are in $\overline{\Omega}$ we have:

$$\Pi \cap K = \{y \in \Pi | \langle x_i, y \rangle \geq 1\}$$

(for some finite collection of $x_i \in V_{\mathbb{Q}} \cap \overline{\Omega}$.)

It is said to be Γ-*polyhedral* if it is moreover Γ invariant. (where Γ is a discrete subgroup of $\mathrm{Aut}(V, \Omega)$.

DEFINITION 6. Let $T \subset \frac{1}{N} L \cap \overline{\Omega} \setminus 0$ we define $K_T = \{x \in \overline{\Omega} | \langle x, y \rangle > 0 \ \forall y \in T\}$.

PROPOSITION 7. *If T is stable under the action of $\mathrm{Aut}(L^{\#}, \Omega)$ then K_T is $\mathrm{Aut}(L, \Omega)$-polyhedral. If K is $\mathrm{Aut}(L^{\#}, \Omega)$-polyhedral then $K^{\vee}$ is $\mathrm{Aut}(L, \Omega)$-polyhedral.*

DEFINITION 8. For a convex set $A \subset V$ a hyperplane H is said to *support* A if $A \setminus H$ is connected and $A \cap H \neq \varnothing$.

For $y \in \overline{\Omega}$ denote by $H_y := \{x \in V | \langle x, y \rangle = 1\}$ the associated hyperplane.

Given a kernel K define $\mathcal{Y}_K = \{y \in \overline{\Omega} | H_y \text{ supports K}, H_y \cap E(K) \text{ spans } V\}$.

For $\underline{y} = \{y_1, \ldots, y_m\} \subset \mathcal{Y}_K$ let $\sigma_{\underline{y}}$ be the cone generated by $\cap_i H_{y_i} \cap E(K)$.

PROPOSITION 9. *Let K be a Γ-polyhedral co-core for Ω then*

$$\Sigma := \{\sigma_{\underline{y}} | \underline{y} \subset \mathcal{Y}_K \textit{ finite}\}$$

Then Σ is a Γ admissible decomposition.

One should read the above to say that if we understand how the extreme points of a lattice intersect hyperplanes, then we understand how to construct a cone decomposition. In general what these extreme points look like and how they behave is complicated.

2 Explicit Charts for $g = 2$

We recall that in this case there are three types of boundary components:

- $F_\alpha = \{pt\}$, then $D(F_\alpha) = U_{\alpha,\mathbb{C}} = \{(\tau_1, \tau_3, \tau_2)\}$ and $L_\alpha \backslash D(F_\alpha) = (\mathbb{C}^*)^3 = \{(t_1, t_3, t_2)\}$.
- $F_\beta = \mathcal{H}_1$, then $D(F_\beta) = \mathcal{H}_1 \times V_\beta \times U_{\beta,\mathbb{C}} = \{(\tau_1, \tau_3, \tau_2) | \tau_1 \in \mathcal{H}_1\}$ and $L_\beta \backslash D(F_\beta) = \mathcal{H}_1 \times V_\beta \times T_\beta = \{(\tau_1, \tau_3, t_2)\}$.
- $F_\gamma = \mathcal{H}_2$, then $D(F_\gamma) = \mathcal{H}_2$ with coordinates (τ_1, τ_3, τ_2) and $L_\gamma \backslash D(F_\gamma) = D(F_\gamma)$

We have the following diagram:

$$\begin{tikzcd}
L_\alpha\backslash D(F_\alpha) & = & (t_1, t_3, t_2) \arrow[r] \arrow[dr] & \Gamma_\alpha\backslash(L_\alpha\backslash D(F_\alpha)_{\Sigma_\alpha} = \Gamma_\alpha\backslash(T_\alpha)_{\Sigma_\alpha} \arrow[d, dashed] \\
L_\beta\backslash D(F_\beta) \arrow[u] & = & (\tau_1, \tau_3, t_2) \arrow[u, "{(exp,exp,id)}"] \arrow[r] \arrow[dr] & \Gamma_\beta\backslash(L_\beta\backslash D(F_\beta)_{\Sigma_\beta} = \mathcal{E} \times (T_\beta)_{\Sigma_\beta} \arrow[d, dashed] \\
L_\gamma\backslash D(F_\gamma) \arrow[u] & = & (\tau_1, \tau_3, \tau_2) \arrow[u, "{(id,id,exp)}"] \arrow[r] & \Gamma_\gamma\backslash(L_\gamma\backslash D(F_\gamma)_{\Sigma_\gamma} = X
\end{tikzcd}$$

The vertical dashed maps on the right only being defined in a neighbourhood of the boundary, these exist because the diagonal and horizontal maps are injections near the boundary.

We note that cone decompositions of U_β are trivial, and we always add a copy of $\mathcal{E}$ at the boundary.

We are interested in understanding what happens to U_α so as to see how everything fits together.

We have the following cones: (these correspond to a fundamental domain, they are not necissarily the best choices, consequently some of the descriptions below are not the most natural)

- $\langle(2, 1, 2), (1, 0, 1), (0, 0, 1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1 t_3^{-2}, t_3, t_2 t_1^{-1}]) = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

  This adds the point $(0, 0, 0)$, in this model.

- $\langle(1, 0, 1), (0, 0, 1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1 t_3^{-2}, t_3^{\pm 1}, t_2 t_1^{-1}]) = \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$

  This adds the $(0, \mathbb{C}^*, 0)$ piece of the above,

- $\langle(0, 0, 1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1 t_3^{-2}, t_3^{\pm 1}, (t_2 t_1^{-1})^{\pm 1}]) = \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^*$.

  This adds what is actually added by the F_α boundary, that is the part of $(\mathbb{C}^*, \mathbb{C}^*, 0)$ it adds is that which quotients to $\mathcal{E}(\Gamma) \to X(\Gamma)$.

  The $\mathbb{C}^*$ piece above is what would be $(0, \mathbb{C}^*, 0)$ here, it thus corresponds to something over the cusp $i\infty$ in the upper half plane, that is, we are adding a copy of $\mathbb{C}^*$ (a piece of a generalized elliptic curve) over the cusp.

  The very top piece, which adds the $(0, 0, 0)$ point is adding an 'intersection point' to this generalized elliptic curve.

  In order to see how all three of these pieces fit together, look at them inside the top $\mathbb{C}^3$.

- $\langle(1,r,r^2+1),(0,0,1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1,(t_1^{-r}t_3)^{\pm 1},t_2t_1^{-1}t_3^{-r}]) = \mathbb{C}\times\mathbb{C}^*\times\mathbb{C}$.

  This is simply another translate of $\langle(1,0,1),(0,0,1)\rangle$, both of which contain $(0,0,1)$ (the F_α boundary). This cone adds another $(0,\mathbb{C}^*,0)$ piece over $i\infty$ that is, it is adding more branches to the generlized elliptic curve we are placing at infinity.

  The total structure of the generalized curve we get at infinity depends on the level and on how many of these branches get glued together.

  To summarize what this is telling us, it is that when we are on the cusp F_β (that is with $t_2=0$) we have a compactification of the universal elliptic curve by adding a generalized elliptic curve at the boundary. (the charts $u_r = t_3t_1^{-r}$, $v_r = t_3^{-1}t_1^{r+1}$ are precisely the ones Victoria used to describe this compactification, and this is the same algebra as arrises by speciallizing $t_2=0$).

- $\langle(2,1,2),(1,0,1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1t_3^{-2},t_3,(t_2t_1^{-1})^{\pm 1}]) = \mathbb{C}\times\mathbb{C}\times\mathbb{C}^*$.

  This adds $(0,0,\mathbb{C}^*)$, this gives the 'perpendicular' direction to the F_α piece at $i\infty$ on it.

- $\langle(2,1,2)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[(t_1t_3^{-2})^{\pm 1},t_3,(t_2t_1^{-1})^{\pm 1}]) = \mathbb{C}^*\times\mathbb{C}\times\mathbb{C}^*$.

  This adds $(0,\mathbb{C}^*,\mathbb{C}^*)$, looking again at the cusp $i\infty$ we can think of this as having our generalized elliptic curve extended perpendicularly into the moduli.

- $\langle(1,0,1)\rangle$

  Gives $\mathrm{Spec}(\mathbb{C}[t_1t_3^{-2},t_3^{\pm 1},(t_2t_1^{-1})^{\pm 1}]) = \mathbb{C}\times\mathbb{C}^*\times\mathbb{C}^*$.

  This adds $(\mathbb{C}^*,0,\mathbb{C}^*)$, the first coordinate being 'like' the modular curve, we can think of this as being the zero of the elliptic curves at each point extended into the moduli.