Math 666: Toroidal Compactifications

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Lecture 9 : Toroidal Compactifications due to Mumford

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1 Constructing a toroidal compactification

What will we cover $\Gamma \setminus \mathcal{D}$ by?

Consider:

We define $T_{\alpha} = \mathcal{U}_{\alpha,\mathbb{C}}/L_{\alpha}$ an algebraic torus (over \mathbb{C})

Heuristically if you think of Ω_{α} as $(\mathbb{R}^+)^n$ then

$$\mathcal{U}_{\alpha} \setminus \mathcal{D} = \{ (\tau_1, \tau_2, \tau_3) \mid 0 < |\overline{\tau}_2|, f(\tau_1, \tau_2) \}.$$

It seems natural to add points for $\overline{\tau}_2 = 0$. Baily-Borel effectively adds a single point $(\tau_3, \tau_2) = (0, 0)$ which adds F.

What points do we want to add? It depends on a choice of cone decomposition.

A $\rho_l(\Gamma_\alpha)$ -admissible polyhedral decomposition of Ω_α is $\Sigma_\alpha = \{\sigma_\nu\}$ (relative to the L_α rational structure on \mathcal{U}_α) such that

- 1. Σ_{α} is a r.p.p. decomposition of \mathcal{U}_{σ} ,
- 2. Σ_{α} is closed under the action of $\rho_l(\Gamma_{\alpha})$,
- 3. only finitely many $\rho_l(\Gamma_{\alpha})$ orbits in Σ_{α} , and
- 4. $\Omega_{\alpha} \subset \bigcup_{\sigma} \sigma_{\nu} = (\Omega_{\alpha} \text{ rational closure}).$

A Γ -admissible family of polyhedral decompositions is $\Sigma = {\Sigma_{\alpha}}_{\alpha}$ Rational

- 1. Σ_{α} is a $\rho_l(\Gamma_{\alpha})$ -admissible polyhedral decomposition,
- 2. for $\gamma \in \Gamma$ if $\gamma F_{\alpha} = F_{\beta}$ then $\gamma \Sigma_{\alpha} \gamma^{-1} = \Sigma_{\beta}$, and
- 3. for $F_{\alpha} < F_{\beta}$ then $\Sigma_{\beta} = \Sigma_{\alpha} \cap \mathcal{U}_{\beta}$.

1.1 The points being added

Now given such a Σ_{α} we may construct:

$$(L_{\alpha} \setminus \mathcal{D}(F_{\alpha}))_{\Sigma_{\alpha}} = \mathcal{D}(F_{\alpha})' \times (T_{\alpha})_{\Sigma_{\alpha}}$$
$$= \sqcup_{\sigma \in \Sigma_{\alpha}} O(\sigma).$$

where $O(\sigma) = \mathcal{D}(F_{\alpha})' \times O'(\sigma)$. We may view the $O'(\sigma)$ as being the points added with respect to σ to T_{α} . They have the following properties:

1. $O(\sigma)$ a torus bundle over $\mathcal{D}(F_{\alpha})'$,

2. for
$$\sigma < \tau, O(\tau) \subset \overline{O(\sigma)}$$
 (for $\sigma = \{0\}$ we have $O(\sigma) = L_{\alpha} \setminus \mathcal{D}(F_{\alpha})$, and

3. dim σ + dim $O(\sigma)$ = dim \mathcal{D} .

We also have the map introduced by Dylan:

Im :
$$F_{\alpha} \times V_{\alpha} \times \mathcal{U}_{\alpha,\mathbb{C}} \to \mathcal{U}_{\alpha}$$

 $(\tau_1, \tau_3, \tau_2) \mapsto \operatorname{Im} (\tau_2)$

extends to

Im :
$$(L_{\alpha} \setminus \mathcal{D}(F_{\alpha}))_{\Sigma_{\alpha}} \to (\mathcal{U}_{\alpha})_{\Sigma_{\alpha}}.$$

The map is compatible with T_{α} action hence

Im :
$$O(\sigma) \to \overline{O(\sigma)}$$
.

Using the fact that Φ is a translation of Im we can then define Φ as follows:

$$\Phi: (L_{\alpha} \setminus \mathcal{D}(F_{\alpha}))_{\Sigma_{\alpha}} \to (L_{\alpha})_{\Sigma_{\alpha}}(\tau_1, \tau_2, \tau_3) \qquad \mapsto \operatorname{Im} (\tau_2) - (\operatorname{Im} \tau_3)(\operatorname{Im} \tau_1)^{-1}(\operatorname{Im} \tau_3)$$

With all this in hand, we make the following definition:

$$(L_{\alpha} \setminus \mathcal{D}(F_{\alpha}))_{\Sigma_{\alpha}} :=$$
 Interior of closure of $L_{\alpha} \setminus \mathcal{D}$ in $(L_{\alpha} \setminus \mathcal{D}(F_{\alpha})_{\Sigma_{\alpha}})_{\Sigma_{\alpha}}$
 $(\Omega_{\alpha})_{\Sigma_{\alpha}} :=$ Interior of closure of Ω_{α} in $(L_{\alpha})_{\Sigma_{\alpha}}$.

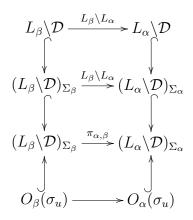
By continuity if follows that $\Phi^{-1}((\Omega_{\alpha})_{\Sigma_{\alpha}}) = (L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}}$. This observation allows one to check the following claim:

CLAIM 1. If $\sigma_u \cap \Sigma_\alpha \neq \emptyset$ then $\overline{O(u)} \subset (\Omega_\alpha)_{\Sigma_\alpha}$.

As a consequence we define $O(F_{\alpha}) := \prod_{\sigma_u \cup \Sigma_{\alpha} \neq 0} O(u)$. We call these $O(F_{\alpha})$ the points added with respect to F_{α} .

Note that the above is not an if and only if. So, what about the other $O(\sigma)$ not part of $O(F_{\alpha})$? These relate to $O(F_{\beta})$ when $F_{\alpha} \leq F_{\beta}$.

The condition $F_{\alpha} \leq F_{\beta}$ implies $\mathcal{U}_{\beta} \subset \mathcal{U}_{\alpha}$ and Ω_{β} boundary of Ω_{α} , $\Sigma_{\beta} = \Sigma_{\alpha} \cup \mathcal{U}_{\beta}$ get maps



This bottom part happening whenever $\sigma_u \in \Sigma_\beta$

1.2 Gluing

Having defined the points we wish to add, we must describe how these all fit together.

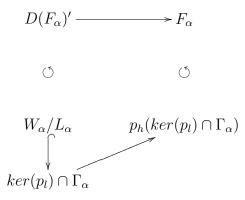
Define $p_{\sigma}: (L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}} \to \overline{\Gamma \setminus \mathcal{D}}^{\text{Sat}}$ via the natural projection $O(F_{\alpha}) \to F_{\alpha}$.

We assert that this map is holomorphic (but remark that this requires more understanding of the holomorphic structure on the Satake compactification than we have).

PROPOSITION 2. $\Gamma_{\alpha}/L_{\alpha}$ acts properly discontinuously on $(L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}}$.

LEMMA 3. Let $f: X \to Y$ be a continuous G-equivariant map of Hausdorff space (suppose G acts on Y via H).

If H acts properly discontinuously on Y and ker : $G \to H$ acts properly discontinuously on X then G acts properly discontinuously on X. The following diagram



 $V_{\alpha} = W_{\alpha}/L_{\alpha}$ acts as $m\tau' + n$ whereas $p_h(ker(p_l) \cap \Gamma_{\alpha})$ is an arithmetic subgroup acting on a symmetric space. Both actions being properly discontinuous, this implies $\rho_l(\Gamma_{\alpha})$ acts properly discontinuously on $\mathcal{D}(F_{\alpha})$.

2) $\rho_l(\Gamma_\alpha)$ acts properly discontinuously on $(\Omega_\alpha)_{\Sigma_\alpha}$

$$(\Omega_{\alpha})_{\Sigma_{\alpha}} = \bigcup_{u} \bigcup_{y \in \Omega_{\alpha}} \{y + \infty \sigma_u\}.$$

Set

$$I = \{ \sigma_v \mid \sigma_v \cap \langle y, \sigma_u \rangle \neq \emptyset \}$$
$$\sigma = \bigcup_{v \in I} \sigma_v.$$

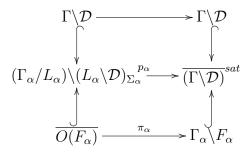
Claim I is finite, $y \in \sigma^o$ (interior) $\rightarrow \bar{\sigma}$ neighborhood of $y + \infty \sigma_u$ so is $\bar{\sigma}^o$. Use I finite to get result.

Apply the Lemma with respect to Φ .

TODO MORE CAN BE SAID ABOUT THIS

THEOREM 4. $(\Gamma_{\alpha}/L_{\alpha})\setminus (L_{\alpha}\setminus\mathcal{D})_{\Sigma_{\alpha}}$. has the structure of a normal analytic space. $\overline{O(F_{\alpha})} := (\Gamma_{\alpha}/U_{\alpha})\setminus O(F_{\alpha})$ is a closed analytic subspace.

We have



We want these $(\Gamma_{\alpha}/L_{\alpha})\setminus (L_{\alpha}\setminus\mathcal{D})_{\Sigma_{\alpha}}$ to give us an "open covering" of $\overline{(\Gamma\setminus\mathcal{D})}_{\Sigma_{\alpha}}$ in the sense that the maps from them give an open covering. Firstly we consider the $(\Gamma_{\alpha}/L_{\alpha})\setminus (L_{\alpha}\setminus\mathcal{D})_{\Sigma_{\alpha}}$ modulo Γ , that is, the collection of all modulo Γ . This is a finite collection.

Now if $F_{\bar{\alpha}}, F_{\bar{\beta}} \leq F_{\bar{\omega}}$, then we glue along the image of $\pi_{\beta,\omega}, \pi_{\alpha,\omega}$ of $(L_{\omega} \setminus \mathcal{D})_{\Sigma_{\omega}}$ in each factor.

- 1. There is no map $(\Gamma_{\omega}/L_{\omega})\setminus (L_{\omega}\setminus\mathcal{D})_{\Sigma_{\alpha}} \to (\Gamma_{\alpha}/L_{\alpha})\setminus (L_{\alpha}\setminus\mathcal{D})_{\Sigma_{\alpha}}.$
- 2. There is a neighbourhood of $\mathcal{D}(F_{\alpha})$ on which $(L_w \setminus \mathcal{D})_{\Sigma_w}$ injects so that the map descends.

There are two ways to think about it.

- 1. Open neighborhoods of $\overline{O(F_{\alpha})}$ glued along intersections from $F_w \geq F_{\alpha}$.
- 2. $\coprod_{F_{\alpha}} (L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}} / \sim \text{where } x_{\alpha} \in (U_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}}, X_{\beta} \in (L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\beta}} \text{ have } x_{\alpha} \sim x_{\beta} \text{ if there exists} F_{w}, \gamma \in \Gamma, x_{w} \in (L_{\omega} \setminus \mathcal{D})_{\Sigma_{\omega}} \text{ such that}$

$$\pi_{\alpha,\omega}(x_{\omega}) = x_{\alpha}$$
$$\pi_{\alpha,\omega}(x_{\omega}) = \gamma x_{\beta}.$$

Define $\overline{\pi}_{F_{\alpha}}$ to be the map

$$(\Gamma_{\alpha}/L_{\alpha})\setminus(L_{\alpha}\setminus\mathcal{D})_{\Sigma_{\alpha}}\to\overline{\Gamma\setminus\mathcal{D}}^{\mathrm{tor}}$$

 $\overline{\pi}_{F_{\alpha}}$ injective near $\overline{O(F_{\alpha})}$ makes $\overline{(\Gamma \setminus \mathcal{D})}^{\text{tor}}$. Near $\overline{O(F_{\alpha})}$ looks like $\overline{(p_h(\Gamma) \setminus F_{\alpha})}^{\text{tor}} \times V/ "V_{\mathbb{Z}} + \tau V_{\mathbb{Z}}" \times T_{\alpha}^{\Sigma_{\alpha}}$.

2 Properties of Toroidal Compactifications

- 1. The boundary has codimension 1, $(O(\sigma_u) \text{ for } \sigma_u \text{ minimal})$.
- 2. The boundary only has toroidal singularities. (It is "Cohen-Macaulay.")
- 3. There is a map $\overline{(\Gamma \setminus \mathcal{D})}^{\text{tor}} \to \overline{(\Gamma \Gamma \setminus \mathcal{D})}^{\text{Sat}}$.
- 4. It is not a unique but it is functorial in (Σ_{α}) compatible with the level structure.
- 5. It is compact.

2.1 Smoothness

DEFINITION 5. A subgroup $\Gamma \in \operatorname{GL}_n$ is *neat* if for all $G \subset \operatorname{GL}_n$, and algebraic maps $\Phi : G \to H$ we have $\Phi(\Gamma \cup G)$ torsion-free.

THEOREM 6 (Borel). Suppose $p \nmid \Phi_{\ell}(1)$ and for all ℓ , we have $deg(\Phi_{\ell}) \leq n$, then $\Gamma(p) \subset GL_n(\mathbb{Z})$ is neat.

Neat subgroups act without fixed points.

PROPOSITION 7. The singularities of $\overline{\Gamma \setminus \mathcal{D}}^{tor}$ are

- 1. finite quotient singularities from non-neat elements of Γ .
- 2. toroidal singularities in $\pi_{F_{\alpha}}(O(u))$ for irregular cones σ_u .

For the proof look at $F_{\alpha} \times$ (Abelian variety) $\times T_{\alpha}^{\Sigma_{\alpha}}$.

THEOREM 8. There exist regular Γ -admissible refinements.

The proof is by induction and the "finite mod Γ property."

2.2 Projectivity of $\overline{(\Gamma \setminus D)}^{tor}$.

DEFINITION 9. A Γ -admissible decomposition Σ_{α} is called *projective* if there exists $f : \Omega \to \mathbb{R}^+$ which is:

- 1. convex, piecewise linear, Γ -invariant, $f(\Gamma \cup \Omega) \subset \mathbb{Z}$, and
- 2. for all $\sigma_u \in \Sigma_\alpha$ there exists ℓ_u a linear functional on \mathcal{U}_α such that

(a)
$$\ell_u \geq f$$
 on Ω_α

(b)
$$\sigma_u = \{x \in \mathcal{U}_\alpha | \ell_u(x) = f(x)\}$$

(equiv σ_u the maximal subsets on which f is linear)

Denote over $\varphi_U = f_{\sigma_U}$ (extended linearly to U_{α}) $\varphi_U^* = \min p_i(\lambda)$ vertices of $p_i = \sigma \cap \{\varphi = 1\}$.

PROPOSITION 10. Every holomorphic function on $\overline{(\Gamma \setminus D)}^{sat}$ has a Fourier expansion of the form

$$\sum_{\rho \in \Omega \cap \mathcal{U}(F)} \theta_p(\tau_1, \tau_3) \underline{e}(\rho(\tau_2)).$$

Definition 11.

$$J_{m,k} = \{ f \in \mathcal{O}_x | \theta_p \neq 0 \text{ only if } \rho \in L_\alpha \cap \Omega, \varphi_u^*(\rho) \ge m \}.$$

 $I_{\alpha,x}$ on $(L_{\alpha} \setminus \mathcal{D})_{\Sigma_{\alpha}}$ be generated by $\underline{e}(\varphi_u(\tau_2))$.

Let $\partial_m := f_*(I_\alpha)^m$.

PROPOSITION 12. $\partial_{m,x} = J_{m,x}$.

Note that one has: \tilde{I} on $(\Gamma \setminus \mathcal{D})$ is $\oplus L_{F^*_{\alpha}} I_{\alpha}$ I on $\overline{(\Gamma \setminus \mathcal{D})}^{\text{tor}}$ is $\Gamma(U, I) = \{s \in \Gamma(p^{-1}U, \tilde{I}) \mid \text{glue}\}.$ $J_m = f_*(I^m).$ The proof of the Prop. is 324 Tai.

THEOREM 13. $\overline{(\mathcal{D}/\Gamma)}^{tor}$ is the normalization of the blow up of $\overline{(\mathcal{D}/\Gamma)}^{rat}$ along J_m . Moreover, $f^*J_m = I^m$.

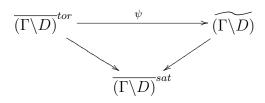
Let $\operatorname{tr}(A_{\rho} \cdot \lambda) = \rho(\lambda)$. Concretely define

$$\begin{aligned} \theta_{\rho}(\tau_{1},\tau_{3}) &:= \sum_{x \in V_{z}} e^{2\pi i (\operatorname{tr}(A_{p}(x^{T}\tau_{1}x++2x^{T}\tau_{3}))} \\ &:= \sum_{x \in V_{z}} \underline{e}(xA_{p}x^{T}\tau_{1}++2xA-p\tau_{3}^{T})) \\ \theta_{\rho}(\tau_{1},\tau_{3}+\tau_{1}m_{1}+m_{2}) &= \theta_{p}(\tau_{1},\tau_{3})e^{-2\pi i (\operatorname{tr}(m_{1}A_{p}m_{1}^{T}\tau_{1}+2m_{1}A_{p}\tau_{3}))} \\ \theta_{\rho}(M\tau_{1},((c\tau_{1}+d)^{T})^{-1}T\tau_{3}) &= \theta_{p}(\tau_{1},\tau_{2})e^{2\pi i (\operatorname{tr}(\tau_{3}A_{p}m_{1}^{T}\tau_{3}^{T}(c\tau_{1}+d)^{-1}c))} \\ \theta_{u^{T}\rho u}(\tau_{1},\tau_{3}) &= \theta_{\rho}(\tau_{1},\tau_{3}u^{T}). \end{aligned}$$

The point is that "things like" $H_{\sigma}(\tau_1, \tau_3, \tau_2) = \Sigma_u(\tau_1, \tau_3 u^T) \underline{e}(\operatorname{tr}(U^T A_p U \tau_2))$ generate ∂_m . Steps of proof

- 1. Reduce to the case where Γ is neat (or finite normal subgroups).
- 2. Check what happens in the limit at the cusps. (Effectively that after the pullback these look like $g \cdot e^{\rho(\dots)}$). This implies $f^* \partial_m < \partial^m$.
- 3. For a large enough m we can get $\theta_{m_p}(\tau_1, \tau_3) \neq 0$. (For a high enough weight ∂_m is ample.) This implies $f^*\partial_m = \partial^m$ rather than just containment.

4. Use the universal properties of blowups:



5. Show that ψ is a local isomorphism since $\Gamma \setminus \mathcal{D}$ is open in both and ψ^+ is injective. Reduce to the strata $\ldots \setminus F_{\alpha} \times V \times O(\alpha)$ use $\mathcal{L}_{m\lambda_a}|_{\pi^{-1}}(t)$ very ample.