Math 666: Toroidal Compactifications — Winter 2012

Lecture 9 : Toroidal Compactifications due to Mumford

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1 Constructing a toroidal compactification

What will we cover $\Gamma\backslash\mathcal{D}$ by?

Consider:

$$\begin{array}{ccc}
\Omega_\alpha & \subset & iU_\alpha = iX_*(T)\otimes\mathbb{R} \\
 & & \uparrow \Phi \\
U_\alpha\backslash\mathcal{D} \subset U_\alpha\backslash D(F_\alpha) & = & F_\alpha \times V_\alpha \times \mathcal{U}_{\alpha,\mathbb{C}}/L_\alpha \\
\downarrow & & \downarrow \pi^i \\
D(F_\alpha)' & = & F_\alpha \times V_\alpha
\end{array}$$

We define $T_\alpha = \mathcal{U}_{\alpha,\mathbb{C}}/L_\alpha$ an algebraic torus (over $\mathbb{C}$)

Heuristically if you think of Ω_α as $(\mathbb{R}^+)^n$ then

$$\mathcal{U}_\alpha\backslash\mathcal{D} = \{(\tau_1,\tau_2,\tau_3) \mid 0 < |\overline{\tau}_2|, f(\tau_1,\tau_2)\}.$$

It seems natural to add points for $\overline{\tau}_2 = 0$. Baily-Borel effectively adds a single point $(\tau_3,\tau_2) = (0,0)$ which adds F.

What points do we want to add? It depends on a choice of cone decomposition.

A $\rho_l(\Gamma_\alpha)$-*admissible polyhedral decomposition of* Ω_α is $\Sigma_\alpha = \{\sigma_\nu\}$ (relative to the L_α rational structure on $\mathcal{U}_\alpha$) such that

1. Σ_α is a r.p.p. decomposition of $\mathcal{U}_\sigma$,
2. Σ_α is closed under the action of $\rho_l(\Gamma_\alpha)$,
3. only finitely many $\rho_l(\Gamma_\alpha)$ orbits in Σ_α, and
4. $\Omega_\alpha \subset \bigcup_\sigma \sigma_\nu =$ (Ω_α rational closure).

A Γ-*admissible family of polyhedral decompositions* is $\Sigma = \{\Sigma_\alpha\}_{\alpha \text{ Rational}}$

1. Σ_α is a $\rho_l(\Gamma_\alpha)$-admissible polyhedral decomposition,
2. for $\gamma \in \Gamma$ if $\gamma F_\alpha = F_\beta$ then $\gamma\Sigma_\alpha\gamma^{-1} = \Sigma_\beta$, and
3. for $F_\alpha < F_\beta$ then $\Sigma_\beta = \Sigma_\alpha \cap \mathcal{U}_\beta$.

1.1 The points being added

Now given such a Σ_α we may construct:

$$\begin{aligned}(L_\alpha\backslash\mathcal{D}(F_\alpha))_{\Sigma_\alpha} &= \mathcal{D}(F_\alpha)' \times (T_\alpha)_{\Sigma_\alpha} \\ &= \sqcup_{\sigma\in\Sigma_\alpha} O(\sigma).\end{aligned}$$

where $O(\sigma) = \mathcal{D}(F_\alpha)' \times O'(\sigma)$. We may view the $O'(\sigma)$ as being the points added with respect to σ to T_α. They have the following properties:

1. $O(\sigma)$ a torus bundle over $\mathcal{D}(F_\alpha)'$,
2. for $\sigma < \tau, O(\tau) \subset \overline{O(\sigma)}$ (for $\sigma = \{0\}$ we have $O(\sigma) = L_\alpha\backslash\mathcal{D}(F_\alpha)$, and
3. dim $\sigma+$ dim $O(\sigma) =$ dim $\mathcal{D}$.

We also have the map introduced by Dylan:

$$\begin{aligned}\text{Im } : F_\alpha \times V_\alpha \times \mathcal{U}_{\alpha,\mathbb{C}} &\to \mathcal{U}_\alpha \\ (\tau_1, \tau_3, \tau_2) &\mapsto \text{Im } (\tau_2).\end{aligned}$$

extends to

$$\text{Im } : (L_\alpha\backslash\mathcal{D}(F_\alpha))_{\Sigma_\alpha} \to (\mathcal{U}_\alpha)_{\Sigma_\alpha}.$$

The map is compatible with T_α action hence

$$\text{Im } : O(\sigma) \to \overline{O(\sigma)}.$$

Using the fact that Φ is a translation of Im we can then define Φ as follows:

$$\Phi : (L_\alpha\backslash\mathcal{D}(F_\alpha))_{\Sigma_\alpha} \to (L_\alpha)_{\Sigma_\alpha}(\tau_1,\tau_2,\tau_3) \qquad \mapsto \text{Im } (\tau_2) - (\text{Im } \tau_3)(\text{Im } \tau_1)^{-1}(\text{Im } \tau_3)$$

With all this in hand, we make the following definition:

$$\begin{aligned}(L_\alpha\backslash\mathcal{D}(F_\alpha))_{\Sigma_\alpha} &:= \text{ Interior of closure of } L_\alpha\backslash\mathcal{D} \text{ in } (L_\alpha\backslash\mathcal{D}(F_\alpha)_{\Sigma_\alpha}. \\ (\Omega_\alpha)_{\Sigma_\alpha} &:= \text{ Interior of closure of } \Omega_\alpha \text{ in } (L_\alpha)_{\Sigma_\alpha}.\end{aligned}$$

By continuity if follows that $\Phi^{-1}((\Omega_\alpha)_{\Sigma_\alpha}) = (L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}$. This observation allows one to check the following claim:

Claim 1. *If $\sigma_u \cap \Sigma_\alpha \neq \varnothing$ then $\overline{O(u)} \subset (\Omega_\alpha)_{\Sigma_\alpha}$.*

As a consequence we define $O(F_\alpha) := \coprod_{\sigma_u \cup \Sigma_\alpha \neq 0} O(u)$. We call these $O(F_\alpha)$ the points added with respect to F_α.

Note that the above is not an if and only if. So, what about the other $O(\sigma)$ not part of $O(F_\alpha)$? These relate to $O(F_\beta)$ when $F_\alpha \leq F_\beta$.

The condition $F_\alpha \leq F_\beta$ implies $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$ and Ω_β boundary of Ω_α, $\Sigma_\beta = \Sigma_\alpha \cup \mathcal{U}_\beta$ get maps

$$\begin{array}{ccc}
L_\beta \backslash \mathcal{D} & \xrightarrow{L_\beta \backslash L_\alpha} & L_\alpha \backslash \mathcal{D} \\
\big\downarrow & & \big\downarrow \\
(L_\beta \backslash \mathcal{D})_{\Sigma_\beta} & \xrightarrow{L_\beta \backslash L_\alpha} & (L_\alpha \backslash \mathcal{D})_{\Sigma_\alpha} \\
\big\downarrow & & \big\downarrow \\
(L_\beta \backslash \mathcal{D})_{\Sigma_\beta} & \xrightarrow{\pi_{\alpha,\beta}} & (L_\alpha \backslash \mathcal{D})_{\Sigma_\alpha} \\
\big\uparrow & & \big\uparrow \\
O_\beta(\sigma_u) & \longrightarrow & O_\alpha(\sigma_u)
\end{array}$$

This bottom part happening whenever $\sigma_u \in \Sigma_\beta$

1.2 Gluing

Having defined the points we wish to add, we must describe how these all fit together.

Define $p_\sigma : (L_\alpha \backslash \mathcal{D})_{\Sigma_\alpha} \to \overline{\Gamma \backslash \mathcal{D}}^{\text{Sat}}$ via the natural projection $O(F_\alpha) \to F_\alpha$.

We assert that this map is holomorphic (but remark that this requires more understanding of the holomorphic structure on the Satake compactification than we have).

PROPOSITION 2. *Γ_α / L_α acts properly discontinuously on $(L_\alpha \backslash \mathcal{D})_{\Sigma_\alpha}$.*

LEMMA 3. *Let $f : X \to Y$ be a continuous G-equivariant map of Hausdorff space (suppose G acts on Y via H).*

If H acts properly discontinuously on Y and $\ker : G \to H$ acts properly discontinuously on X then G acts properly discontinuously on X.

The following diagram

$$
\begin{array}{ccc}
D(F_\alpha)' & \longrightarrow & F_\alpha \\
\circlearrowleft & & \circlearrowleft \\
W_\alpha/L_\alpha & & p_h(ker(p_l) \cap \Gamma_\alpha) \\
\downarrow & \nearrow & \\
ker(p_l) \cap \Gamma_\alpha & &
\end{array}
$$

$V_\alpha = W_\alpha/L_\alpha$ acts as $m\tau' + n$ whereas $p_h(ker(p_l) \cap \Gamma_\alpha)$ is an arithmetic subgroup acting on a symmetric space. Both actions being properly discontinuous, this implies $\rho_l(\Gamma_\alpha)$ acts properly discontinuously on $\mathcal{D}(F_\alpha)$.

2) $\rho_l(\Gamma_\alpha)$ acts properly discontinuously on $(\Omega_\alpha)_{\Sigma_\alpha}$

$$(\Omega_\alpha)_{\Sigma_\alpha} = \bigcup_u \bigcup_{y \in \Omega_\alpha} \{y + \infty\sigma_u\}.$$

Set

$$
\begin{aligned}
I &= \{\sigma_v \mid \sigma_v \cap \langle y, \sigma_u \rangle \neq \emptyset\} \\
\sigma &= \bigcup_{v \in I} \sigma_v.
\end{aligned}
$$

Claim I is finite, $y \in \sigma^o$ (interior) $\to \bar{\sigma}$ neighborhood of $y + \infty\sigma_u$ so is $\bar{\sigma}^o$. Use I finite to get result.

Apply the Lemma with respect to Φ.

TODO MORE CAN BE SAID ABOUT THIS

Theorem 4. *$(\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}$. has the structure of a normal analytic space. $\overline{O(F_\alpha)} := (\Gamma_\alpha/U_\alpha)\backslash O(F_\alpha)$ is a closed analytic subspace.*

We have

$$
\begin{array}{ccc}
\Gamma\backslash\mathcal{D} & \longrightarrow & \Gamma\backslash\mathcal{D} \\
\downarrow & & \downarrow \\
(\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha} & \xrightarrow{p_\alpha} & \overline{(\Gamma\backslash\mathcal{D})}^{sat} \\
\uparrow & & \uparrow \\
\overline{O(F_\alpha)} & \xrightarrow{\pi_\alpha} & \Gamma_\alpha\backslash F_\alpha
\end{array}
$$

We want these $(\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}$ to give us an "open covering" of $\overline{(\Gamma\backslash\mathcal{D})}_{\Sigma_\alpha}$ in the sense that the maps from them give an open covering. Firstly we consider the $(\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}$ modulo Γ, that is, the collection of all modulo Γ. This is a finite collection.

Now if $F_{\bar{\alpha}}, F_{\bar{\beta}} \leq F_{\bar{\omega}}$, then we glue along the image of $\pi_{\beta,\omega}, \pi_{\alpha,\omega}$ of $(L_\omega\backslash\mathcal{D})_{\Sigma_\omega}$ in each factor.

1. There is no map $(\Gamma_\omega/L_\omega)\backslash(L_\omega\backslash\mathcal{D})_{\Sigma_\alpha} \to (\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}$.
2. There is a neighbourhood of $\mathcal{D}(F_\alpha)$ on which $(L_w\backslash\mathcal{D})_{\Sigma_w}$ injects so that the map descends.

There are two ways to think about it.

1. Open neighborhoods of $\overline{O(F_\alpha)}$ glued along intersections from $F_w \geq F_\alpha$.
2. $\coprod_{F_\alpha}(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}/\sim$ where $x_\alpha \in (U_\alpha\backslash\mathcal{D})_{\Sigma_\alpha}, X_\beta \in (L_\alpha\backslash\mathcal{D})_{\Sigma_\beta}$ have $x_\alpha \sim x_\beta$ if there exists $F_w, \gamma \in \Gamma, x_w \in (L_\omega\backslash\mathcal{D})_{\Sigma_\omega}$ such that
$$\pi_{\alpha,\omega}(x_\omega) = x_\alpha$$
$$\pi_{\alpha,\omega}(x_\omega) = \gamma x_\beta.$$

Define $\overline{\pi}_{F_\alpha}$ to be the map
$$(\Gamma_\alpha/L_\alpha)\backslash(L_\alpha\backslash\mathcal{D})_{\Sigma_\alpha} \to \overline{\Gamma\backslash\mathcal{D}}^{\text{tor}}.$$

$\overline{\pi}_{F_\alpha}$ injective near $\overline{O(F_\alpha)}$ makes $\overline{(\Gamma\backslash\mathcal{D})}^{\text{tor}}$.
Near $\overline{O(F_\alpha)}$ looks like $\overline{(p_h(\Gamma)\backslash F_\alpha)}^{\text{tor}} \times V/\text{“}V_{\mathbb{Z}} + \tau V_{\mathbb{Z}}\text{”} \times T_\alpha^{\Sigma_\alpha}$.

2 Properties of Toroidal Compactifications

1. The boundary has codimension 1, ($O(\sigma_u)$ for σ_u minimal).
2. The boundary only has toroidal singularities. (It is "Cohen-Macaulay.")
3. There is a map $\overline{(\Gamma\backslash\mathcal{D})}^{\text{tor}} \to \overline{(\Gamma\Gamma\backslash\mathcal{D})}^{\text{Sat}}$.
4. It is not a unique but it is functorial in (Σ_α) compatible with the level structure.
5. It is compact.

2.1 Smoothness

DEFINITION 5. A subgroup $\Gamma \in \mathrm{GL}_n$ is *neat* if for all $G \subset \mathrm{GL}_n$, and algebraic maps $\Phi : G \to H$ we have $\Phi(\Gamma \cup G)$ torsion-free.

THEOREM 6 (Borel). *Suppose $p \nmid \Phi_\ell(1)$ and for all ℓ, we have deg$(\Phi_\ell) \leq n$, then $\Gamma(p) \subset \mathrm{GL}_n(\mathbb{Z})$ is neat.*

Neat subgroups act without fixed points.

PROPOSITION 7. *The singularities of $\overline{\Gamma \backslash \mathcal{D}}^{tor}$ are*

1. *finite quotient singularities from non-neat elements of Γ.*
2. *toroidal singularities in $\pi_{F_\alpha}(O(u))$ for irregular cones σ_u.*

For the proof look at $F_\alpha \times$ (Abelian variety) $\times T_\alpha^{\Sigma_\alpha}$.

THEOREM 8. *There exist regular Γ-admissible refinements.*

The proof is by induction and the "finite mod Γ property."

2.2 Projectivity of $\overline{(\Gamma \backslash \mathcal{D})}^{\mathbf{tor}}$.

DEFINITION 9. A Γ-admissible decomposition Σ_α is called *projective* if there exists $f : \Omega \to \mathbb{R}^+$ which is:

1. convex, piecewise linear, Γ-invariant, $f(\Gamma \cup \Omega) \subset \mathbb{Z}$, and
2. for all $\sigma_u \in \Sigma_\alpha$ there exists ℓ_u a linear functional on $\mathcal{U}_\alpha$ such that
   (a) $\ell_u \geq f$ on Ω_α
   (b) $\sigma_u = \{x \in \mathcal{U}_\alpha | \ell_u(x) = f(x)\}$

   (equiv σ_u the maximal subsets on which f is linear)

Denote over $\varphi_U = f_{\sigma_U}$ (extended linearly to U_α)
$\varphi_U^* = \min\, p_i(\lambda)$ vertices of $p_i = \sigma \cap \{\varphi = 1\}$.

PROPOSITION 10. *Every holomorphic function on $\overline{(\Gamma \backslash \mathcal{D})}^{sat}$ has a Fourier expansion of the form*

$$\sum_{\rho \in \Omega \cap \mathcal{U}(F)} \theta_p(\tau_1, \tau_3) \underline{e}(\rho(\tau_2)).$$

Definition 11.

$$J_{m,k} = \{f \in \mathcal{O}_x | \theta_p \neq 0 \text{ only if } \rho \in L_\alpha \cap \Omega, \varphi_u^*(\rho) \geq m\}.$$

$$I_{\alpha,x} \text{ on } (L_\alpha \backslash \mathcal{D})_{\Sigma_\alpha} \text{ be generated by } \underline{e}(\varphi_u(\tau_2)).$$

Let $\partial_m := f_*(I_\alpha)^m$.

Proposition 12. $\partial_{m,x} = J_{m,x}$.

Note that one has:
$\tilde{I}$ on $\widetilde{(\Gamma\backslash\mathcal{D})}$ is $\oplus L_{F_\alpha^*} I_\alpha$
I on $\overline{(\Gamma\backslash\mathcal{D})}^{\mathrm{tor}}$ is $\Gamma(U, I) = \{s \in \Gamma(p^{-1}U, \tilde{I}) \mid \text{glue}\}$.
$J_m = f_*(I^m)$.
The proof of the Prop. is 324 Tai.

Theorem 13. *$\overline{(\mathcal{D}/\Gamma)}^{tor}$ is the normalization of the blow up of $\overline{(\mathcal{D}/\Gamma)}^{rat}$ along J_m. Moreover, $f^* J_m = I^m$.*

Let $\mathrm{tr}(A_\rho \cdot \lambda) = \rho(\lambda)$.
Concretely define

$$\begin{aligned}
\theta_\rho(\tau_1, \tau_3) &:= \sum_{x \in V_z} e^{2\pi i(\mathrm{tr}(A_p(x^T \tau_1 x + +2x^T \tau_3))} \\
&:= \sum_{x \in V_z} \underline{e}(x A_p x^T \tau_1 + +2xA - p\tau_3^T)) \\
\theta_\rho(\tau_1, \tau_3 + \tau_1 m_1 + m_2) &= \theta_p(\tau_1, \tau_3) e^{-2\pi i(\mathrm{tr}(m_1 A_p m_1^T \tau_1 + 2m_1 A_p \tau_3))} \\
\theta_\rho(M\tau_1, ((c\tau_1 + d)^T)^{-1} T \tau_3) &= \theta_p(\tau_1, \tau_2) e^{2\pi i(\mathrm{tr}(\tau_3 A_p m_1^T \tau_3^T (c\tau_1 + d)^{-1} c))} \\
\theta_{u^T \rho u}(\tau_1, \tau_3) &= \theta_\rho(\tau_1, \tau_3 u^T).
\end{aligned}$$

The point is that "things like" $H_\sigma(\tau_1, \tau_3, \tau_2) = \Sigma_u(\tau_1, \tau_3 u^T)\underline{e}(\mathrm{tr}(U^T A_p U \tau_2))$ generate ∂_m.
Steps of proof

1. Reduce to the case where Γ is neat (or finite normal subgroups).
2. Check what happens in the limit at the cusps. (Effectively that after the pullback these look like $g \cdot e^{\rho(\cdots)}$). This implies $f^* \partial_m < \partial^m$.
3. For a large enough m we can get $\theta_{m_p}(\tau_1, \tau_3) \neq 0$. (For a high enough weight ∂_m is ample.) This implies $f^* \partial_m = \partial^m$ rather than just containment.

4. Use the universal properties of blowups:

$$\begin{array}{ccc} \overline{(\Gamma\backslash D)}^{tor} & \xrightarrow{\psi} & \widetilde{(\Gamma\backslash D)} \\ & \searrow \quad \swarrow & \\ & \overline{(\Gamma\backslash D)}^{sat} & \end{array}$$

5. Show that ψ is a local isomorphism since $\Gamma\backslash\mathcal{D}$ is open in both and ψ^+ is injective. Reduce to the strata $\ldots\backslash F_\alpha \times V \times O(\alpha)$ use $\mathcal{L}_{m\lambda_a}|_{\pi^{-1}}(t)$ very ample.