Characterization of Special Points on Orthogonal Shimura Varieties

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The Goal

The goal of this talk to explain the results of my paper:

Andrew Fiori *The Characterization of Special Points on Orthogonal Symmetric Spaces*, Journal of Algebra, Volume 372, 15 December 2012, Pages 397-419.

which as the title explains, gives a characterization of the special fields associated to special point on orthogonal Shimura varieties.

Motivation

The points on the Shimura variety associated to the group $G_{\mathbb{Q}}$ correspond to maps:

$$h : \mathbb{S}_{\mathbb{R}} = Res_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}$$

satisfying several axioms.
These points are called **special** if they factor through an algebraic torus, T defined over $\mathbb{Q}$.

$$h : \mathbb{S}_{\mathbb{R}} \to T_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$$

Motivation (continued)

Given a special point:

$$h : \mathbb{S}_{\mathbb{R}} \to T_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$$

there is an associated **special field** $E/\mathbb{Q}$ which will be a subfield of the splitting field of $T_{\mathbb{Q}}$.

Part of the importance is the following:
For the canonical model of the Shimura variety, the the field of definition of a special point is related to the Hilbert class field of E.

This result in the case of the upper half plane is part of the theorey of complex multiplication and gives us an explicit class field theory for quadratic imaginary fields.

We want to characterize the special fields that can be associated to the Shimura varieties associated to an orthogonal group $G_{\mathbb{Q}} = O_q$ of signature $(2, n)$.

This leads to the problem:

Characterize all the maximal tori:

$$T_{\mathbb{Q}} \hookrightarrow O_q.$$

This is the problem we will now focus on.

Quadaratic Spaces

Definition

A **quadratic space** over a field k (with $char(k) = 0$) is a vector space V and a map $q : V \to k$ where $q(x+y) - q(x) - q(y)$ is bilinear.

For the purpose of this talk you may assume $k = \mathbb{Q}$ or $k = \mathbb{Q}_p$. Fix a basis $e_1, \dots e_n$ in which we may express:

$$q(x_1 e_1 + \dots x_n e_n) = \sum a_i x_i^2.$$

The following invariants determine q.

- $D(q) = \prod_i a_i$ the **discriminant**.
- For ν a place of k, $H_\nu(q) = \prod_{i<j} (a_i, a_j)_\nu$ the **Hasse invariant**.
- For each $\rho : k \to \mathbb{R}$, the **signature**, $(r^+, r^-)_\rho = \left(\left|\{i \mid \rho(a_i) > 0\}\right|, \left|\{i \mid \rho(a_i) < 0\}\right|\right)_\rho$.

Orthogonal Symmetric Spaces

Definition

Let (V, q) be a quadratic space over k. The **orthogonal group** is the algebraic group whose points over a field K/k are:

$$O_q(K) := \{g \in \mathrm{GL}(V \otimes_k K) \mid q(gx) = q(x)\ \forall x \in V \otimes_k K\}$$

There are Shimura varieties associated to orthogonal groups of signature $(2, n)$ over $\mathbb{Q}$. For example the $(2, 1)$ case includes the modular curve and Shimura Curves. The $(2, 2)$ case includes Hilbert modular surfaces and the $(2, 3)$ case includes the Siegel space of dimension 3. The classifiactions of special points are reasonably well known in these cases.

Étale Algebras and Tori

By an **étale algebra** E over k we mean a finite product $E = \prod_i E_i$ of seperable field extensions E_i over k.
By an **algebraic torus** T over k we mean an algebraic group over k such that $T_{\overline{k}} \simeq \mathbb{G}_m^n$ for some n.

Given an étale algebra E we define the algebraic torus T_E whose points over a field K/k are:

$$T_E(K) := (E \otimes_k K)^\times$$

Theorem

For every algebraic torus T over k there exists an étale algebra E over k and a morphism $\rho : T \hookrightarrow T_E$.

Étale Algebras with Involution and Tori

By an **étale algebra with involution** (E, σ) over k we mean an étale algebra E over k together with an involution $\sigma : E \to E$ such that $dim_k(E) = 2dim_k(E^\sigma)$.

Given an étale algebra with involution $(E, \sigma)/k$ we define the algebraic torus $T_{E,\sigma} \subset T_E$ by specifying that its points over a field K/k are:

$$T_{E,\sigma}(K) := \{x \in (E \otimes_k K)^\times \mid x\sigma(x) = 1\}$$

As an example taking with $(E, \sigma) = (\mathbb{C}, \bar{\cdot})$ we obtain:

$$T_{E,\sigma} = \{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) | a^2 + b^2 = 1\} = \mathrm{SO}_{2,\mathbb{R}}$$

Given an étale algebra with involution $(E,\sigma)/k$, and $\lambda \in (E^\sigma)^\times$. We define a quadratic form $q : E \to k$ by:

$$q_{E,\sigma,\lambda}(x) = \frac{1}{2} Tr_{E/k}(\lambda x \sigma(x)).$$

This makes (E, q) a quadratic space over k of dimension $dim(E)$.

Proposition

For all $\lambda \in (E^\sigma)^\times$ there is a natural injective map:

$$T_{E,\sigma} \hookrightarrow O_{q_{E,\sigma,\lambda}}$$

induced by the action of multiplication of $E^\times$ on E.

The First Chacterization

Proposition

Let (V, q) be a quadratic space over k. If a torus

$$T \hookrightarrow O_q$$

as a maximal torus then:

- *There exists (E, σ) over k with $T \simeq T_{E,\sigma}$.*
- *There exists $\lambda \in (E^\sigma)^\times$ with $q \simeq q_{E,\sigma,\lambda}$.*

Taking E to be the span of $T(k)$ in $\mathrm{GL}(V)$ and σ the adjoint with respect to q gives us (E, σ) concretely when V is even dimensional. Finding λ is more subtle.

This gives us our first characterization of the maximal tori in O_q.

The Invariants

Given q and (E, σ) how would we know if this is possible? Does there exist λ such that $q \simeq q_{E,\sigma,\lambda}$?

The invariants of $q_{E,\sigma,\lambda}$ are:

- $D(q_{E,\sigma,\lambda}) = (-1)^{n/2}\delta_{E/k}$.
- $H(q_{E,\sigma,\lambda}) = Cor_{E^\sigma/k}(((-1)^{n/2}\lambda f_z'(z), z)_{E^\sigma})(-1,-1)_k^{n(n-2)/8}$ where $z \in E^\sigma$ is such that $\sqrt{z}$ primitively generates E and f_z is its minimal polynomial.
- Signature is $(2r + 2s + t, n - (2r + 2s + t))_\rho$ where
  - s is the number of real places of E^σ over ρ that become complex in E having $\lambda > 0$,
  - r is the number of complex places of E^σ over ρ, and
  - t is the number of real places of E^σ over ρ that stay real in E.

(Hasse invariant uses results of Brusamarello, Chuard-Koulmann, Morales)

Second Characterization

We conclude $T_{E,\sigma} \hookrightarrow O_q$ if $\delta_{E/k} = (-1)^{n/2} D(q)$ and there exists $\lambda \in E^\sigma$ such that:

- $Cor_{E^\sigma/k}(((-1)^{n/2} f_z'(z)\lambda, z)) = H(q)(-1,-1)_k^{n(n-2)/8}$.
- It gives the right signature.

It is easy to check the discriminant and signature conditions.
It is not totally apparent when a λ satisfying the Hasse invariant conditions exists.

Our next goal will be to rephrase the conditions from $\exists\lambda$ to a condition more intrinsic in (E, σ).

Reflex Algebras

Definition

Given (E, σ) a σ-**type** $\phi \subset Hom(E, \overline{k})$ is a collection such that $\phi \cap \sigma(\phi) = \emptyset$ and $\phi \cup \sigma(\phi) = Hom(E, \overline{k})$
To each σ-type ϕ of (E, σ) we associate a σ-**reflex algebra** (E^{ϕ}, σ) as follows.
The group $\Gamma = Gal(\overline{k}/k)$ acts on the set Φ of all σ-types. Set Γ_{ϕ} to be the stabilizer of ϕ and define:

$$E^{\phi} = \begin{cases} \overline{k}^{\Gamma_{\phi}} & \sigma(\Gamma\phi) = \Gamma\phi \\ \overline{k}^{\Gamma_{\phi}} \oplus \overline{k}^{\Gamma_{\phi}} & \text{otherwise} \end{cases}$$

This definition imitates the definition of CM-types and CM-reflex fields. For the application to special points this is the only case we need.

Associated to a quadratic space (V, q) is the even Clifford algebra:

$$Clif^{+}(V, q) = \oplus_n V^{\otimes 2n}/(x \otimes x - q(x))$$

There exists a canonical involution τ on $Cliff^{+}(V, q)$.

Theorem

If $T_{E,\sigma}$ is a maximal torus in O_q then for all σ-types ϕ of (E, σ) we have $E^{\phi} \hookrightarrow Cliff^{+}(V, q)$ with the canonical involution restricting to σ on E^{ϕ}.

This implies that E^{ϕ} splits the algebra $Cliff^{+}(V, q)$. It is this observation which led us to try to prove the next theorem.

Local Characterization (Non-archimedian)

Suppose k is a finite extension of $\mathbb{Q}_p$.

Theorem

Let (E, σ) be an etale algebra with involution of dimension n and (V, q) be a quadratic space of dimension n then the torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if:

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- *E^ϕ splits $Cliff^+(V, q)$ for all σ-types ϕ.*

The Proof

The idea of the proof is the following:

- $\mathit{Cliff}^+(V, q)$ is a matrix algebra over a quaternion algebra.
- E^ϕ splits it if and only if E^ϕ contains a quadratic subextension (or if the algebra is already split.)
- Every reflex algebra contains a quadratic extension if and only if E has a non trivial factor where 'z' is not a square.
- The Hasse invariant can be modified if and only if E has a non trivial factor where 'z' is not a square.

Local Characterization (Archimedian)

Suppose k is $\mathbb{R}$.

Theorem

Let (E, σ) be an etale algebra with involution of dimension n and (V, Q) be a quadratic space of dimension n then the torus:

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if the signanature of q is of the form:

$$(2r + 2i + t, n - (2r + 2i + t)) \quad 0 \leq i \leq s$$

- *s is the number of real places of E^σ that become complex in E,*
- *r is the number of complex places of E^σ , and*
- *t is the number of real places of E^σ that stay real in E.*

Moreover, if $T_{E,\sigma} \hookrightarrow O_q$ then:

- *$D(q) = (-1)^{n/2}\delta_{E/k}$.*
- *E^ϕ splits $Cliff^+(V, q)$ for all ϕ.*

Now let k be a finite extension of $\mathbb{Q}$. We get locally everywhere embedding from global conditions.

Theorem

Let (E, σ) be an etale algebra with involution of dimension n and (V, Q) be a quadratic space of dimension n then the torus:

$$T_{E,\sigma} \otimes k_{\mathfrak{p}} \hookrightarrow O_q \otimes k_{\mathfrak{p}}$$

for all places (including infinite) $\mathfrak{p}$ of k if and only if

- $D(q) = (-1)^{n/2} \delta_{E/k}$.
- E^{ϕ} *splits* $Cliff^{+}(V, q)$ *for all* ϕ.
- *The signature condition at all real places.*

Unfortunately, an example of Prasad - Rapinchuk shows that the local global principle fails in general.

Local-Global - continued

Theorem (Prasad-Rapinchuk)

If for any two factors E_1, E_2 of E there exists a non-archimedian places $\mathfrak{p}$ of k and $\mathfrak{p}_1, \mathfrak{p}_2 | \mathfrak{p}$ of E_1, E_2 respectively such that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are both non split over E^{σ}. Then the local global principle is satisfied for E.

In particular the local-global principle holds if E is a field and we get the following:

Theorem

Let (E, σ) be a number field with involution with $dim(E) = dim(V)$. The torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if

- $D(q) = (-1)^{n/2} \delta_{E/k}$.
- E^{ϕ} *splits* $Cliff^{+}(V, q)$ *for all* ϕ.
- *The signature condition at all real places.*

Non-maximal Tori ($dim(V) = dim(E) + 1$)

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) = dim(E) + 1$ then the torus

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if

- *E^ϕ splits $Cliff^+(V, q)$ for all ϕ.*
- *The 'signature' condition at all real places.*

Non-maximal Tori ($dim(V) = dim(E) + 2$)

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) = dim(E) + 2 = n + 2$ then the torus

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if

- $(E^\phi(\sqrt{(-1)^n D(q)\delta_{E/k}}))$ *splits* $Cliff^+(V, q)$ *for all* ϕ.
- *The 'signature' conditions at all real places are plausible.*
- *Local-global conditions work out.*

Non-maximal Tori ($dim(V) > dim(E) + 2$)

Theorem

Let (E, σ) be a number field with involution. Suppose $dim(V) > dim(E) + 2$ then

$$T_{E,\sigma} \hookrightarrow O_q$$

if and only if the signature conditions are plausible.

What about special points?

Some remarks:

1. For CM-algebras and (V, q) of signature $(2, n)$ the signature conditions are always plausible.
2. The infinite place lets us sort out the local global obstruction for a pair of 2 CM-fields (at a cost see point (5)).
3. We are thus left with the discriminant condition when $dim(E) = dim(V)$ and splitting conditions when $dim(V) - dim(E) \leq 2$.
4. For an étale algebra the special field is the one which contributes to the (2,0) part of the signature.
5. If $T_{E,\sigma} \hookrightarrow O_q$ there is no obstruction to making E the special field unless $dim(E) = dim(V) - 2$. In which case local global obstructions might exist.

The End

Thank you.