Arithmetic Volumes for Lattices over p-adic Rings

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Abstract

In this paper we develop formulas for computing the arithmetic volume of orthogonal groups for lattices over the maximal orders of finite extensions of $\mathbb{Z}_p$. We specifically develop new explicit formulas for unimodular lattices over 2-adic rings. We also develop a reduction of the general problem to that of unimodular lattices together with the combinatorial problem of computing representatives for all possible Jordan decompositions.

Keywords: local densities, orthogonal group, arithmetic volume, representation densities, lattices

This article contains material originally appearing in my Ph.D. thesis [Fio13].

1. Introduction

The issue of computing local densities goes back decades to when they were first introduced by Siegel [Sie35]. These types of computations have many applications beyond those originally envisioned (see for example [GK93, Kud97, SP04, GHS08, GV12] among others) and formulas for them have been worked out to cover many cases (see for example [Pal65, Wat76, Kit93, CS88, Shi99, Kat99, SH00, GY00, Yan04, Cho12]).

We have in mind several applications of this work. First, these formulas can be used by way of the Siegel mass formula as part of a stopping condition when enumerating the genus of a lattice. This has important applications in the theory of algebraic automorphic forms on orthogonal groups (see [Gro99] and [GV12]). Second, these formulas are related by way of the Hirzebruch-proportionality principle and the Riemann-Roch theorem (see [Mum77, GHS08]), to the dimensions of spaces of modular forms on the associated Shimura varieties. The sections of this paper are organized as follows:

(2) We introduce the general theory of lattices so far as it is needed in the sequel.

(3) We discuss specifically lattices over p-adic rings.

(4) We introduce representation densities and develop formulas for computing them.

(5) We discuss the computational issues associated with our formulas.

Almost none of the introductory content (Sections 2 and 3) is new, however, we present it in the format we intend to use in the sequel. Many results on representation densities are known:

- The work of Pall, Watson and the book of Kitaoka [Pal65, Wat76, Kit93] give formulas for $\beta_{\mathfrak{p}}(L, L)$ over $\mathbb{Z}_p$ for arbitrary L and p.

- The work of Conway and Sloane [CS88] corrected minor errors in the above work verifying their formulas by checking many cases. It is worth mentioning that although [CS88] only gives a sketch of a proof in terms of lattice automorphims and embeddings, their ideas, which carry over to the nondyadic and unramified dyadic settings, have similarities to those involved in some of our present proofs.

- Katsurada [Kat99] computes $\beta_{\mathfrak{p}}(L, M)$ over $\mathbb{Z}_2$.

- Shimura [Shi99] computes formulas for $\beta_{\mathfrak{p}}(L, L)$ when L is maximal, over $\mathcal{O}_{\mathfrak{p}}$ any finite extension of $\mathbb{Z}_p$.

- Hironaka and Sato [SH00] computes $\beta_{\mathfrak{p}}(L, M)$ over $\mathbb{Z}_p$ when $p \neq 2$.

- The work of Gan and Yu [GY00] gives a high level machinery for computing $\beta_{\mathfrak{p}}(L, L)$ when $p \neq 2$ the recent work of Cho [Cho12] extends this to work to cover unramified extensions of $\mathbb{Z}_2$.

However, formulas for all cases do not yet exist. The main results of this paper (Section 4) provide formulas for computing $\beta_{\mathfrak{p}}(L, L)$ for arbitrary L and $\mathfrak{p}$ (including especially $p = 2$). These results are largely the content of Theorems 4.11, 4.18 and 4.26.

Theorem 4.11 reduces the computation of $\beta_{\mathfrak{p}}(L, L)$ for unimodular L to the computation of $\beta_{\mathfrak{p}}(L', L')$ for L' a particular sublattice of L or rank at most 4 (rank at most 1 when $p \neq 2$). The techniques and ideas here are classical, the proofs being only slight modifications of similar well known results over $\mathbb{Q}$. It is worth remarking that the proofs are uniform between the cases $p = 2$ and $p \neq 2$.

Theorem 4.18 is the most technical result of this paper. This theorem gives formulas for $\beta_{\mathfrak{p}}(L', L')$ for all L' arising from Theorem 4.11. The number of apparent cases that need to be considered, as well as the apparently technical nature of the calculations involved is likely the reason these cases have been ignored for so long. Though our results do involve detailed calculations, our approach to the counting problem leads to surprisingly few cases requiring independent analysis and is less technical than one might originally expect. Stated simply, the new idea is that rather than counting in a single step the isometries, we first count the changes of basis that preserve the general shape of the quadratic form, then divide out by the size of the over count. This overcount is precisely the number of quadratic forms which both have the same shape as the original and are isomorphic to it.

Finally, Theorem 4.26 uses a new, yet simple idea, to give a general formula for $\beta_{\mathfrak{p}}(L, L)$ for arbitrary L and $\mathfrak{p}$ in terms of the set of all Jordan decompositions for L and the local densities of the modular blocks. The formula completely describes the dependence of the local density on the set of Jordan decompositions and justifies the general shape of the classical formulas for $\beta_p(L, L)$. A more careful analysis on the combinatorics of the set of all Jordan decompositions explains the role of free and bound Jordan blocks in the formulas over $\mathbb{Z}_2$ (see [CS88, Sec. 13]). It is again worth remarking that the proof is uniform between the cases $p = 2$ and $p \neq 2$.

2. General Notions of Lattices

In this section we introduce the general theory of lattices. Many good references exist which treat this topic in a varying degree of generality. See for example [Kit93] and [O'M00]. We shall initially work quite generally, adding more structure as it is required. We shall eventually be most interested in the theory of lattices over $\mathcal{O}_k$, the maximal order in a number field k. Note that these are not always PIDs, however, their localizations always are.

Definition 2.1. Let R be a Dedekind domain and K be its field of fractions. By a **lattice** Λ over R we mean a projective R-module of finite rank, together with a symmetric R-bilinear pairing:

$$b_\Lambda : \Lambda \times \Lambda \to K,$$

which induces a duality $\operatorname{Hom}_R(\Lambda, K) \cong \Lambda \otimes_R K$. We shall sometimes denote $b_\Lambda(x, y) = (x, y)$ when the pairing b_Λ is understood. A lattice is said to be **integral** if $(x, y) \in R$, **even** if $(x, x) \in 2R$ and **unimodular** if the pairing induces an isomorphism $\operatorname{Hom}_R(\Lambda, R) \cong \Lambda$, or more generally $\mathfrak{a}$-**modular** if the pairing induces an isomorphism $\operatorname{Hom}_R(\Lambda, R) \cong \mathfrak{a}^{-1}\Lambda$ (for $\mathfrak{a}$ a projective R-module of rank 1, that is, an invertible fractional ideal of R). Notice that $\mathfrak{a}$-modular is equivalent to having $\operatorname{Hom}_R(\Lambda, \mathfrak{a}) \cong \Lambda$ by noting that:

$$\operatorname{Hom}_R(\Lambda, \mathfrak{a}) \cong \mathfrak{a} \otimes_R \operatorname{Hom}_R(\Lambda, R) \cong \mathfrak{a} \otimes \mathfrak{a}^{-1}\Lambda \cong \Lambda.$$

We will refer to a lattice as **modular** if there exists some $\mathfrak{a}$ for which it is $\mathfrak{a}$-modular. Note that not all lattices are modular.

We shall sometimes denote the bilinear form as $b_\Lambda(\cdot, \cdot)$ when we need to specify the underlying lattice.

Remark. By requiring $\operatorname{Hom}_R(\Lambda, K) \cong \Lambda \otimes_R K$ we are explicitly requiring that all lattices be non-degenerate with respect to the bilinear form b_Λ. If the pairing on the 'lattice' might not induce an isomorphism the 'lattice' shall be referred to as a module or submodule.

We will at times consider symmetric bilinear forms on an R-module M valued in another R-module M', that is,

$$(\cdot, \cdot) : M \times M \to M'.$$

We may even consider such pairings when R is not an integral domain. These do not fit into our definition of lattices though many notions remain valid. The most common examples of this would be either taking $M' = R/I$, for any ideal I of R, or reducing all of R, M, M' by I.

We will also need the following notion in order to deal with certain complexities in characteristic 2.

Definition 2.2. Let R be a ring and let M' be an R-module. We define a **quadratic module** M over R (or more precisely an M'**-valued quadratic module**) to be a module M over R together with a function $q : M \to M'$ satisfying $q(\lambda x) = \lambda^2 q(x)$ for all $x \in M$ and $\lambda \in R$ and such that

$$B_M(x, y) := q(x + y) - q(x) - q(y)$$

is a bilinear pairing. For a quadratic module M we define:

$$M^\perp := \{x \in M \mid B_M(x, y) = 0 \text{ for all } y \in M\} \text{ and}$$
$$\mathrm{Rad}(M) := \{x \in M^\perp \mid q(x) = 0\}.$$

A quadratic module is said to be **regular** or **non-degenerate** if B_M induces a duality with the dual module.

Remark. In the above, one typically takes $M' = R$ or $M' = K$, the total ring of fractions or $M' = R/I$.

Notation 2.3. Given a lattice Λ, by q_Λ or simply q we shall always mean:

$$q_\Lambda(x) = b_\Lambda(x, x).$$

To a lattice we may also associate another bilinear pairing:

$$B_\Lambda(x, y) := q_\Lambda(x + y) - q_\Lambda(x) - q_\Lambda(y).$$

Note well that $B_\Lambda(x, y) = 2b_\Lambda(x, y)$ and that $q_\Lambda(x) = b_\Lambda(x, x)$ as these conventions vary by author. Notice also that in characteristic 2 one may not recover b_Λ from q_Λ as this would involve dividing by 2 whereas if $2 \in K^\times$ then non-degenerate quadratic modules and lattices are equivalent.

Remark. For both lattices and quadratic modules $L \oplus M$ shall always mean an orthogonal direct sum.

This level of generality is too much for many of our purposes. Having the following additional constraints gives major simplifications to the theory:

1. If Λ is free we may express $(\cdot, \cdot)$ by a matrix.
2. If R is a principal ideal domain, the theory of modules simplifies. In particular, every lattice is free. We may often replace R by its (completed) localizations to attain this.
3. The theory is simpler if 2 is not a zero divisor in R.

Note that some of the results which follow are true without some (or all) of the above constraints, however, for simplicity of presentation we may sometimes assume them. Note that these assumptions hold when we work over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$ for all p, $\mathbb{F}_p$ where $p \neq 2$, or the many finite ring extensions of these. These assumptions may fail for Dedekind domains; however as our study of these is done almost entirely with their localizations this will not be an issue. We will occasionally still need to work in characteristic 2 and it will be apparent when this is happening.

Definition 2.4. Assume that Λ is free and let $X = \{x_1, \ldots, x_n\}$ be a basis for Λ. We write:

$$A = A_X = ((x_i, x_j))_{i,j}$$

for the matrix corresponding to this lattice and choice of basis.

Definition 2.5. Given a lattice Λ we define the **dual** lattice to be:

$$\Lambda^\# = \{x \in \Lambda \otimes K \mid (x, y) \in R \text{ for all } y \in \Lambda\}$$

together with the induced pairing.

Definition 2.6. A submodule $L \subset \Lambda$ is said to be **isotropic** if $(\cdot, \cdot)|_L = 0$. It is said to be **anisotropic** if it has no isotropic submodules. A projective submodule is said to be **metabolic** if it has an isotropic submodule of half its rank. A projective submodule is said to be **hyperbolic** if it is generated by two isotropic submodules.

Definition 2.7. Lattices Λ have the following invariants:

- For Λ projective, the **rank** r_Λ of Λ as an R module.

- For Λ integral, the **discriminant group** $D_\Lambda = \Lambda^\#/\Lambda$ together with the induced pairing mapping into K/R.

- For Λ free, the **discriminant** $\delta_\Lambda = \det(A_X) \in K/(R^\times)^2$ for a choice of basis X.

  If Λ is not free we have at our disposal the discriminant $D(q)$ of $\Lambda \otimes K$ which is an element of $K/(K^\times)^2$, and the **discriminant ideal** which is the R ideal generated by $\det(A_X)$ running over all maximal linearly independent subsets X of Λ. Alternatively, for a projective module over a Dedekind domain, one may take the discriminant ideal to be the product of the local discriminant ideals.

- Supposing $\Lambda \otimes K$ is isomorphic to the diagonal form $(a_i)_i$ and denoting the Hilbert symbol by $(\cdot, \cdot)_K$, the **Hasse invariant** is
$$H(\Lambda) = H(q) = \prod_{i<j} (a_i, a_j)_K \in H^2(K, \{\pm 1\}).$$

  (See [Ser73, Ch. III] and [Ser79, Ch. XIV].)

- The **Witt invariant**, $W(\Lambda) = W(q)$ is the class in $H^2(K, \{\pm 1\})$ of either the Clifford algebra or the even Clifford algebra of Λ when the parity of r_Λ is, respectively, even or odd.

- For each embedding $R \hookrightarrow \mathbb{R}$ we have an associated **signature** (the dimension of any maximal isotropic $\mathbb{R}$-submodule of $\Lambda \otimes_R \mathbb{R}$).

- The **norm ideal** $\mathfrak{N}_\Lambda$ is the R-ideal generated by $\{(x, x) \mid x \in \Lambda\}$.

- The **scale ideal** $\mathfrak{S}_\Lambda$ is the R-ideal generated by $\{(x, y) \mid x, y \in \Lambda\}$.

  Note that $\mathfrak{N}_\Lambda \subset \mathfrak{S}_\Lambda$ and $2\mathfrak{S}_\Lambda \subset \mathfrak{N}_\Lambda$.

- The **norm group** $\mathfrak{n}_\Lambda$ is the group: $\{(x, x) \mid x \in \Lambda\} + 2\mathfrak{S}_\Lambda$, it is an additive subgroup of K.

- If R is Noetherian consider $\mathfrak{m}_\Lambda \subset \mathfrak{n}_\Lambda$ the largest R-ideal contained in $\mathfrak{n}_\Lambda$. Then for π an ideal of R, define the **π-weight ideal** to be the ideal $\mathfrak{w}_{\Lambda,\pi} = \pi \mathfrak{m}_\Lambda + 2\mathfrak{S}_\Lambda$. When we are working over a local ring we shall denote this by $\mathfrak{w}_\Lambda$ as π is understood to be the unique maximal ideal.

Remark. It is clear that the above are all invariants as they are defined naturally. The extent to which these determine a lattice depends largely on the setting. They are typically insufficient to characterize a lattice in the context in which we are working.

Proposition 2.8. *Every unimodular sublattice $L \subset \Lambda$ of an integral lattice is an orthogonal direct summand. More generally, if $\mathfrak{S}_\Lambda = \mathfrak{a}$ then every $\mathfrak{a}$-modular sublattice $L \subset \Lambda$ is an orthogonal direct summand.*

See [O'M00, Thm 82:15a].

3. Lattices over p-adic Rings

Here we enter into the improved setting of having R a (complete) local ring whose maximal ideal is principal, generated by π. More specifically we intend to work with a **p-adic ring**, by which we mean the maximal order of a p-adic field (a finite extension of $\mathbb{Q}_p$). We shall denote by $\nu = \nu_\pi$ the π-adic valuation.

In this context we have the following important result to recall:

Theorem 3.1. *A quadratic module over a p-adic field K is entirely determined by its rank, its discriminant and its Hasse invariant.*

See [O'M00, Thm 63:20].

Notation 3.2. For $a, b \in R$, with $ab \neq 1$, we shall denote by $L_{a,b}$ the binary lattice whose bilinear form has matrix $\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right)$.

For $0 \neq c \in R$ we shall denote by U_c the unary lattice whose bilinear form has matrix (c).

For a lattice L and an element $r \in R$ we shall denote by rL the lattice whose underlying module is L but whose bilinear form is r times that of L, that is, $b_{rL} = rb_L$.

Lemma 3.3.

1. $L_{a,b} = U_{c_1} \oplus U_{c_2}$ if and only if one of a, b or 2 is in $R^\times$.
2. The discriminant of $L_{a,b}$ is $-(1 - ab)$.
3. The Hasse invariant of $L_{a,b}$ is $(a, 1 - ab)_\mathfrak{p} = (b, 1 - ab)_\mathfrak{p}$.

4. *Let M be any integral lattice, suppose $\beta = b_M(x, x)$ for some $x \in M$ and $u \in R^\times$, if $L_{a+u^{-1}\beta, b}$ is unimodular then:*

$$uL_{a,b} \oplus M = uL_{a+u^{-1}\beta, b} \oplus M'$$

for some lattice M'. In the case $b = 0$ then $uL_{a+u^{-1}\beta, b}$ is unimodular and moreover $M' \simeq M$.

Proof. For the first point, in the forward direction use the fact that every unimodular sublattice is a direct summand, together with the determinant of the matrix. For the other direction, use the fact that if none of a,b or 2 is a unit, then $\mathfrak{N}_{L_{a,b}} \neq R$ and is unimodular whereas if $U_{c_1} \oplus U_{c_2}$ is unimodular then $\mathfrak{N}_{U_{c_1} \oplus U_{c_2}} = R$.

The second point is a direct calculation. For the third, notice that over K we have the change of basis:

$$\begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b - a^{-1} \end{pmatrix}.$$

Thus the Hasse invariant is $(a, b - a^{-1})_{\mathfrak{p}} = (a, 1 - ab)_{\mathfrak{p}}$ (using that $(a, -a)_{\mathfrak{p}} = 1$).

The fourth point is [O'M00, 93:12]. □

Theorem 3.4 (Existence of Jordan decompositions). *Every lattice Λ over a p-adic ring R can be expressed as:*

$$\Lambda \simeq \bigoplus_i L_i,$$

*where the L_i are $\mathfrak{a}_i$-modular, with the $\mathfrak{a}_i$ distinct. Such a decomposition is called a **Jordan decomposition**. Note that such decompositions are not in general unique, but see Theorem 3.13.*

See [O'M00, 91C].

It should be remarked that in spite of the following "Witt type theorem," a decomposition $\Lambda = L_1 \oplus K_1 = L_2 \oplus K_2$ with $L_1 \simeq L_2$ does not imply $K_1 \simeq K_2$.

Theorem 3.5 (Kneser). *Let R be a local ring with unique maximal ideal $\mathfrak{p}$. Let $L_1, L_2 \subset \Lambda$ be submodules of Λ and $F \subset \Lambda$ be a subset satisfying:*

1. *$\frac{1}{2}q_\Lambda(F)$ and $b_\Lambda(F, \Lambda)$ are both subsets of R,*
2. *$\mathrm{Hom}(L_1, R), \mathrm{Hom}(L_2, R) \subset \{b_\Lambda(x, \cdot) \mid x \in F\}$, where $b_\Lambda(x, \cdot)$ is viewed as a map from Λ to R, and*
3. *$\sigma : L_1 \to L_2$ an isometry such that $\sigma(x) - x \in F$ for all $x \in L_1$.*

Then σ can be extended to an isometry of Λ which acts trivially on $F^\perp$. Moreover, if F contains an element z such that:

1. *$q_\Lambda(z) \in 2R^\times$ and,*
2. *if the residue field is $\mathbb{F}_2$, then also $(F, z) \subset \mathfrak{p}$,*

then σ is induced by products of reflections in elements of F.

See [Kit93, Thm 1.2.2] or [Kne02, Satz 4.3].

Corollary 3.6. *Suppose R is a p-adic ring. Let M_1, M_2 be integral R lattices and $N_1 = N_2$ unimodular lattices with $\mathfrak{N}_{N_1} \subset (2)$. Then $N_1 \oplus M_1 \simeq N_2 \oplus M_2$ implies that $M_1 \simeq M_2$.*

Proof. Identify $\Lambda := N_1 \oplus M_1$ with $N_2 \oplus M_2$ via any isomorphism. In the notation of the above theorem, take $L_1 = N_1$, $L_2 = N_2$, and $F = \Lambda$. The map which identifies N_1 and N_2 thus extends to an isometry of Λ which necessarily maps $M_1 = N_1^\perp$ to $N_2^\perp = M_2$. □

Lemma 3.7. *For $p \neq 2$ every unimodular lattice Λ over a p-adic ring R with rank at least 3 has a hyperbolic sublattice.*

See [O'M00, 92:1a].

Corollary 3.8. *For $p \neq 2$ and a p-adic ring R, the isomorphism classes of unimodular lattices Λ over R are classified by their rank and discriminant.*

See [O'M00, 92:1].

Lemma 3.9. *Suppose $p = 2$, then the isomorphism classes of unimodular lattices Λ over R are determined by their rank, discriminant, Hasse invariant and norm groups.*

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See [O'M00, 93:16].

Lemma 3.10. *For a lattice L over a 2-adic ring letting $a\pi^t$ be an element of minimal valuation in $\mathfrak{n}_L$ we find:* $\mathfrak{n}_L = a\pi^t R^2 + \mathfrak{w}_L$.

See [O'M00, 93:3].

Theorem 3.11. *Let L be a unimodular lattice over a 2-adic ring R with uniformizer π. Fix $\alpha \in R^\times$ such that $\delta_L = -(1 + \alpha\pi^r)$ modulo $(R^\times)^2$, such that furthermore either r is odd or $r = \nu(4)$. Fix also $a \in R^\times$ such that $a\pi^t \in q_L(L)$ is an element of minimal valuation represented by L. Then $\mathfrak{w}_L = (\pi^s)$, where $r - t \geq s \geq t$ and $s + t$ is odd or $s = \nu(2)$. Let $\rho \in R/\pi R$ be such that $x^2 + x + \rho$ is irreducible mod π.*
Then L is isomorphic to precisely one of:

1. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$,

2. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}(\alpha - 4\rho)\pi^{r-t} \end{pmatrix}$,

3. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus (-\delta_L)$,

4. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus (-(1 - 4\rho)\delta_L)$,

5. $\begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$ *or*

6. $(-(1 - \alpha\pi^r))$.

Proof. This is essentially the content of [O'M00, 93:18].

This is a consequence of Lemma 3.9. One only needs to observe that these examples cover all possible combinations of ranks, discriminants, Hasse invariants and norm groups. Lemma 3.10 allows one to check we have all of the possible norm groups. The observation that $(1 + 4\rho, \pi)_{\mathfrak{p}} = -1$ allows one to check we have all possible Hasse invariants. $\square$

Corollary 3.12. *Every unimodular lattice Λ over a 2-adic ring R with rank at least 5 has a hyperbolic sublattice.*

See also [O'M00, 93:18v].

It should be emphasized before stating the following result that Jordan decompositions over 2-adic rings are not typically unique.

Theorem 3.13 (Uniqueness of Jordan decompositions)**.** *Let $\Lambda = \overset{r_1}{\underset{i=1}{\oplus}} L_i = \overset{r_2}{\underset{j=1}{\oplus}} K_j$ be two Jordan decomposition of a lattice over a p-adic ring with L_i being $\mathfrak{a}_i$-modular and K_j being $\mathfrak{b}_j$-modular, $\mathfrak{a}_{i_1} | \mathfrak{a}_{i_2}$ for $i_1 < i_2$, and $\mathfrak{b}_{j_1} | \mathfrak{b}_{j_2}$ for $j_1 < j_2$. Then:*

1. $r_1 = r_2$,
2. $\mathfrak{a}_i = \mathfrak{b}_i$,
3. $\operatorname{rank} L_i = \operatorname{rank} K_i$,
4. $\mathfrak{N}_{L_i} = \mathfrak{a}_i$ *if and only if* $\mathfrak{N}_{K_i} = \mathfrak{a}_i$, *and*
5. *if $p \neq 2$ then $L_i \simeq K_i$.*

See [O'M00, 91:9].

4. Local Densities

We now move from general theory to a more particular problem, that is, we now focus our attention on what are called interchangeably representation densities, local densities or arithmetic volumes. Throughout this section we shall continue to assume that R is a p-adic ring, with maximal ideal $\mathfrak{p}$. We shall denote by π a uniformizer and $q = |R/\mathfrak{p}R|$ the size of the residue field, which is finite by assumption. We shall fix an additive Haar measure on R, normalized so that the volume of R is 1. In this context we continue to have that all lattices are free.

4.1. Notion of Local Densities

Fundamentally the notion of representation density has to do with assigning a volume to sets of the form:

$$\text{Isom}(\Lambda_1, \Lambda_2) = \{\phi \in \text{Hom}_R(\Lambda_1, \Lambda_2) \mid b_{\Lambda_2}(\phi(x), \phi(y)) = b_{\Lambda_1}(x, y)\},$$

the isometric embeddings from Λ_1 to Λ_2. Such sets are typically infinite, so simply counting elements is insufficient.

This problem can be approached both locally and globally and there are a number of different ways to formulate the notion. The various definitions are typically, up to constants, equivalent. We take the following definition of local density; for some the α definition is more natural.

Definition 4.1. Let L and M be lattices over a p-adic ring R, with bilinear forms b_L, b_M. Consider the map $\mathcal{F}_{b_L} : \text{Hom}_R(M, L) \to \text{Sym}^2(M^\vee)$ which takes the maps from M to L to the space of symmetric bilinear forms on M given by $(\mathcal{F}_{b_L}(\phi))(x, y) = b_L(\phi(x), \phi(y))$. Some references define the **local density** at R to be:

$$\alpha_R(b_M, b_L) = \alpha_R(M, L) = \frac{1}{2} \lim_{U \to b_M} \frac{\int_{\mathcal{F}_{b_L}^{-1}(U)} dX}{\int_U dT}.$$

Here $dX = \prod_{ij} dx_{ij}$ and $dT = \prod_{i \le j} dt_{ij}$ are the standard measures when viewing the spaces as matrix spaces with respect to some chosen basis. The limit is being taken over the directed family of open subset U of $\text{Sym}^2(M^\vee)$ containing b_M. By [Han05, Lemma 2.2] this does not depend on the choice of integral basis.

We define the local density to be:

$$\beta_R(M, L) = (q^{-\text{rank}(M)v_\pi(2)})\alpha_R(M, L).$$

When $R = \mathcal{O}_\mathfrak{p}$ one often denotes the local densities by $\beta_\mathfrak{p}$ rather than β_R.

The above definition may seem quite unwieldy and difficult to compute. The following proposition gives a more concrete interpretation of these values.

Proposition 4.2. *Let R be a p-adic ring with residue field $\mathbb{F}_q$ and uniformizer π. Let M and N be two quadratic modules over R of ranks m and n, respectively. Fix $h \in \mathbb{Z}$ sufficiently large so that $\pi^{h-1}q_M(M^\#) \in (2)$ and $\pi^{h-1}q_N(N) \in (2)$, and let $r, r' \in \mathbb{Z}$ be such that $r, r' - \nu(2) \ge h$. Denote $\xi_r = (q^r)^{m(m+1)/2-mn}$ then define $B_R(M, N, r)$ to be:*

$$\xi_r \cdot |\{\phi \in \text{Hom}_R(M, N/\pi^r N) \mid b_N(\phi(x), \phi(x)) = b_M(x, x) \pmod{2\pi^r}\}|$$

and define $A_R(M, N, r')$ to be:

$$\xi_{r'} \cdot \left|\{\phi \in \text{Hom}_R(M, N/\pi^{r'} N) \mid b_N(\phi(x), \phi(y)) = b_M(x, y) \pmod{\pi^{r'}}\}\right|.$$

These values are independent respectively of r and r'. Moreover,

$$\beta_R(M, N) = B_R(M, N, r) \text{ and } \alpha_R(M, N) = A_R(M, N, r').$$

These results are reasonably well known, and can be deduced from [Han05, Lemma 3.2] and [Kit93, Lemmas 5.6.1 and 5.6.5] or from [Kne02, 15.3-5 and 33.5] or alternatively from the proof of [Kit88, Prop 1].

Remark. It can be useful to think of the local density as counting the number of elements of $\text{Isom}(M, N)$, or of it as being the probability that a linear map is in $\text{Isom}(M, N)$ (even though it is not literally either of those things, it is a rescaling of these numbers when one thinks of L/π^r for large r).

Proposition 4.3. *Suppose that $L = L_1 \oplus L_2$ and the following hypothesis is satisfied:*

$$L_1 \oplus L_2 \simeq M_1 \oplus M_2 \text{ and } L_1 \simeq M_1 \text{ implies } L_2 \simeq M_2.$$

Then for any lattice L_3 we have the following formula:

$$\beta_R(L_1 \oplus L_3, L) = \beta_R(L_1, L)\beta_R(L_3, L_2).$$

Proof. This follows immediately from the description in terms of counting isometries and book-keeping the rescaling constants. □

Remark. This type of 'cancellation law' does not hold in general, nonetheless, one can use cases where it does hold (see for example Corollary 3.6) as a way to inductively prove formulas for representation densities.

Computing local densities is in general considered to be highly technical. The resulting formulas become quite complicated in the general case. In spite of this, in this section we will compute the local densities $\beta_{\mathfrak{p}}(L, L)$ for an arbitrary lattice over an arbitrary p-adic ring. The combinatorics behind actually carrying out the computation in any given case will require detailed understanding of the isomorphism class of the given lattice. In particular one needs to be able to compute the set of all possible Jordan decompositions. We will thus not present complete formulas for this in the most general cases. Instead, we give a reduction formula in terms of these combinatorics and formulas for all the terms that can appear.

The general structure of this section is as follows:

1. Reduce the problem for (π^t)-modular lattices to unimodular lattices. See in particular Proposition 4.4.
2. Reduce the problem for unimodular lattices to the special case of certain lattices of rank at most 4, see Theorem 4.11.
3. Compute the representation density for these special cases. This is done in a series of lemmas culminating in Theorem 4.18.
4. Reduce the general problem for an arbitrary lattice to the combinatorial problem of understanding all the Jordan decompositions together with the problem for modular lattices. See Theorem 4.26.

♦ *Rescaling*

Our first step is an elementary lemma which allows us to compute the local density of rescaled lattices.

Proposition 4.4. *Let R be a p-adic ring with field of fractions K. Let M and L be lattices over R and $c \in K^{\times}$. The following formula holds:*
$$\beta_R(M, L) = |c|_{\pi}^{m(m+1)/2} \beta_R(cM, cL),$$
where $m = \mathrm{rank}(M)$.

Proof. This is an elementary computation, see [Han05, Lemma 3.1]. □

As a consequence of the above proposition, it is possible to compute $\beta_R(L, L)$ in the case of $\mathfrak{a}$-modular lattices simply by treating the case of unimodular lattices.

Remark. There is no reasonable formula for $\beta_R(cM, L)$ or $\beta_R(M, cL)$ in terms of $\beta_R(M, L)$ unless we make further assumptions. In particular some of these could be 0 while the others are not.

♦ *Unimodular Lattices*

We now discuss the problem of computing the local density $\beta_R(L, L)$ for a unimodular lattice.

Lemma 4.5. *Suppose L is any unimodular lattice and $L(e)$ is any even unimodular lattice. The following formula holds:*
$$\beta_R(L(e) \oplus L, L(e) \oplus L) = \beta_R(L(e), L(e) \oplus L) \cdot \beta_R(L, L).$$

Proof. This follows immediately from Corollary 3.6 and Proposition 4.3. □

Lemma 4.6. *Suppose L is a unimodular lattice and $L(e)$ is any even unimodular lattice of rank $2n$. Set $\Lambda = L \oplus L(e)$ then define:*
$$L^{(2)} := \{x \in L \mid (x, x) \in 2R\} \ and \ \Lambda^{(2)} := \{x \in \Lambda \mid (x, x) \in 2R\}.$$

Then $L^{(2)}$ and $\Lambda^{(2)}$ are lattices, $\Lambda^{(2)} = L(e) \oplus L^{(2)}$, and:
$$\beta_R(L(e), \Lambda) = [L : L^{(2)}]^{-2n} \beta_R(L(e), \Lambda^{(2)}).$$

Proof. Denote by $\xi_r = (q^r)^{n-2n^2-2n\ell}$. Now pick r sufficiently large so that $\pi^r L \subset L^{(2)}$. It follows that $\beta_R(L(e), \Lambda)$ is given by:
$$\xi_r \cdot |\{\phi \in \mathrm{Hom}_R(L(e), \Lambda/\pi^r\Lambda) \mid q(x) = q(\phi(x)) \pmod{2\pi^r}\}|,$$
and $\beta_R(L(e), \Lambda^{(2)})$ is given by:
$$\xi_r \cdot \left|\{\phi \in \mathrm{Hom}_R(L(e), \Lambda^{(2)}/\pi^r(\Lambda^{(2)})) \mid q(x) = q(\phi(x)) \pmod{2\pi^r}\}\right|.$$

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Then because $L(e)$ is even, it is clear that $\beta_R(L(e), \Lambda)$ can be computed as:

$$\xi_r \cdot \left| \{ \phi \in \mathrm{Hom}_R(L(e), \Lambda^{(2)}/\pi^r \Lambda) \mid q(x) = q(\phi(x)) \pmod{2\pi^r} \} \right|.$$

For each element $\phi \in \mathrm{Hom}_R(L(e), \Lambda^{(2)}/\pi^r \Lambda)$, there are precisely $[L : L^{(2)}]^{2n}$ many extensions of ϕ to a map in $\mathrm{Hom}_R(L(e), \Lambda^{(2)}/\pi^r \Lambda^{(2)})$, all of which automatically satisfy $q(x) = q(\phi(x)) \pmod{2\pi^r}$ as that condition was already well-defined. Comparing formulas completes the proof. $\square$

Lemma 4.7. *Suppose L is a unimodular lattice of rank ℓ and $L(e)$ is any even unimodular lattice of rank $2n$. Define Λ, $L^{(2)}$ and $\Lambda^{(2)}$ as above. Consider the vector spaces $V_1 = L(e)/\pi L(e)$ and $V_2 = \Lambda^{(2)}/\pi \Lambda^{(2)}$ together with the quadratic form $\tilde{Q}_i(x) = \frac{1}{2}(x,x) \pmod{\pi}$ for their respective pairings valued in $R/\pi R$. Then the local density $\beta_R(L(e), \Lambda^{(2)})$ is:*

$$q^{n-2n^2-2n\ell} \left| \{ \sigma : V_1 \to V_2 \mid \tilde{Q}_1(x) = \tilde{Q}_2(\sigma(x)) \text{ for all } x \} \right|.$$

Proof. Firstly we observe by Proposition 4.2 that $\beta_R(L(e), \Lambda^{(2)})$ is:

$$q^{n-2n^2-2n\ell} \left| \{ \sigma : L(e) \to \Lambda^{(2)}/\pi \Lambda^{(2)} \mid q(x) = q(\sigma(x)) \pmod{2\pi} \} \right|.$$

Secondly, we observe that:

$$\left| \{ \sigma : L(e) \to \Lambda^{(2)}/\pi \Lambda^{(2)} \mid q(x) = q(\sigma(x)) \pmod{2\pi} \} \right| =$$

$$\left| \{ \sigma : V_1 \to V_2 \mid \tilde{Q}_1(x) = \tilde{Q}_2(\sigma(x)) \} \right|.$$

The result then follows immediately. $\square$

Remark. The space V_2 may not be a regular quadratic module.

Definition 4.8. For a regular quadratic module V of dimension $2n$ we define:

$$\chi(V) = \begin{cases} 1 & V \simeq H^n \text{ and } n > 0 \\ -1 & \text{otherwise.} \end{cases}$$

Lemma 4.9. *Every quadratic module W over a field of characteristic 2 decomposes as:*

$$W_0 \oplus W' \oplus \mathrm{Rad}(W)$$

with W_0 a maximal regular sublattice and $W^\perp = W' \oplus \mathrm{Rad}(W)$. Note that the isomorphism class of W_0 is unique if and only if $W^\perp = \mathrm{Rad}(W)$.

See [Kit93, Thm 1.2.1 and Ex. 1.2.2].

Lemma 4.10. *Suppose V is a (non-trivial) regular quadratic module represented by W, that is, for which there exists at least one isometry from V into W. Write $W = W_0 \oplus W^\perp$ as in Lemma 4.9 and set $v = \dim(V)$ and $w = \dim(W_0)$. The number of isometries from V into W is:*

$$q^{v \dim(W) - v(v+1)/2} \left(\prod_{e=(w-v)/2+1}^{w/2-1} (1 - q^{-2e}) \right) (1 - \chi(W_0) q^{-w/2}) \xi,$$

where ξ is given by:

$$\xi = \begin{cases} 1 + \chi(V \oplus -W_0) q^{(v-w)/2} & W^\perp = \mathrm{Rad}(W) \\ 1 + \chi(W_0) q^{-w/2} & W^\perp \neq \mathrm{Rad}(W). \end{cases}$$

See [Kit93, Prop 1.3.3].

Remark. Notice that the above formula, which appears to depend on a choice of W_0 in W, does so only when $W^\perp = \mathrm{Rad}(W)$.

Theorem 4.11. *Consider a unimodular lattice Λ. Then Λ has a decomposition $\Lambda = L(e) \oplus L$, where $L(e)$ is a maximal even dimensional even unimodular sublattice of Λ and L has rank at most 4. Let $\ell = \mathrm{rank}(L)$ and $2n = \mathrm{rank}(L(e))$. Then:*

$$\beta_R(\Lambda, \Lambda) = [L : L^{(2)}]^{-2n} \xi \beta_R(L, L) \prod_{e=1}^{n} (1 - q^{-2e}),$$

where:

$$\xi = \begin{cases} 2(1 + \chi(L(e))q^{-n})^{-1} & L(e) \text{ non-trivial and independent of choices} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Such a decomposition exists by Theorem 3.11. Lemma 4.5 gives us the formula:

$$\beta_R(L(e) \oplus L, L(e) \oplus L) = \beta_R(L(e), L(e) \oplus L) \cdot \beta_R(L, L).$$

Lemma 4.6 allows us to evaluate:

$$\beta_R(L(e), L(e) \oplus L) = [L : L^{(2)}]^{-2n} \beta_R(L(e), L(e) \oplus L^{(2)}).$$

Lemma 4.7 then reduces the computation of $\beta_R(L(e), L(e) \oplus L^{(2)})$ to a computation over the residue field. Finally, Lemma 4.10 gives the precise formula for this computation. Combining the results allows us to conclude the theorem. $\qquad\square$

Remark. If $L(e)$ is as above, then one has $\chi(L(e)) = (\pi, (-1)^{n/2} D(L(e)))_{\mathfrak{p}}$.

Corollary 4.12. *Suppose $p \neq 2$ and maintain the notation of Theorem 4.11, then:*

$$\beta_R(\Lambda, \Lambda) = 2 \prod_{e=1}^{n} (1 - q^{-2e}) \begin{cases} (1 + \chi(L(e))q^{-n})^{-1} & \ell = 0 \\ 1 & \ell = 1. \end{cases}$$

Proof. When $p \neq 2$ all lattices are even and hence we have that L is either 0 or 1-dimensional. The result now follows immediately from the theorem and the observation that for a 1-dimensional lattice the representation density is 2. $\qquad\square$

♦ *Unimodular Lattices of Rank at Most 4*

    We are now left only to consider the case where the residue characteristic is 2. Theorem 4.11 reduces this case to that of computing $\beta_R(L, L)$ and of understanding $L^{(2)}$, in the case of L unimodular of rank at most 4 with no even unimodular factors. Such low rank unimodular lattices with no even unimodular factors are precisely those appearing as L in Theorem 4.11. We first discuss the problems of understanding $L^{(2)}$.

Proposition 4.13. *Consider L a unimodular lattice of rank at most 4 over a 2-adic ring with no nontrivial even unimodular factors. Denote by $W = L^{(2)}/\pi L^{(2)}$ with the induced form $\tilde{Q}(x) = \frac{1}{2}(x, x) \pmod{\pi}$. Then we have the following cases:*

- *Case $n = 4$. Write $L = \begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix} \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4b\pi^{-s} \end{pmatrix}$ with $t < s < r - t$, $t + s$ is odd, and either r odd or $r = \nu(4)$. Then $\mathrm{Rad}(W) \neq W^{\perp}$. Moreover,*

$$\log_q([L : L^{(2)}]) = \nu(2) - (s + t - 1)/2.$$

- *Case $n = 3$. Write $L = \begin{pmatrix} \pi^s & 1 \\ 1 & b\pi^{\nu(4)-s} \end{pmatrix} \oplus (d)$ with $\nu(2) > s > 0$ and s odd. Then $\mathrm{Rad}(W) \neq W^{\perp}$. Moreover,*

$$\log_q([L : L^{(2)}]) = \nu(2) - (s - 1)/2.$$

- *Case $n = 2$, Write L with matrix $\begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix}$ with either $r > t$ odd or $r = \nu(4)$. Then $\mathrm{Rad}(W) = W^{\perp}$ unless $r - t \leq \nu(2)$ or $\nu(2) - t$ is even. Moreover,*

$$\log_q([L : L^{(2)}]) = \begin{cases} \left\lceil \frac{\nu(2)-t}{2} \right\rceil & r - t \geq \nu(2), \\ \nu(2) - (r - 1)/2 & \text{otherwise.} \end{cases}$$

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- *Case $n = 1$* Then $\mathrm{Rad}(W) = W^\perp$ *unless $\nu(2)$ is even. Moreover,*

$$\log_q([L : L^{(2)}]) = \left\lceil \tfrac{\nu(2)}{2} \right\rceil.$$

Proof. In each case we will denote the basis with respect to which the matrix is given by $\{\vec{x}_1, \ldots, \vec{x}_n\}$.

The argument shall use the following observation. If $x, y \in L$ are such that $\nu_\pi(q(x))$ is odd and $\nu_\pi(q(y))$ is even, then since:

$$q(\eta x + \theta y) = \eta^2 q(x) + \theta^2 q(y) \pmod{2},$$

the only way to have $\nu_\pi(q(\eta x + \theta y)) \geq \nu_\pi(2)$ is to have both $2\nu_\pi(\eta) + \nu_\pi(q(x)) \geq \nu_\pi(2)$ and $2\nu_\pi(\theta) + \nu_\pi(q(y)) \geq \nu_\pi(2)$.

The observation allows us to easily compute bases for the following three cases. In the case of $n = 1$ it is clear that a basis for $L^{(2)}$ is:

$$\{\pi^{\lceil \nu_\pi(2)/2 \rceil} \vec{x}_1\}.$$

In the case of $n = 2$ a basis for $L^{(2)}$ is:

$$\{\pi^{\lceil (\nu_\pi(2)-t)/2 \rceil} \vec{x}_1, \pi^{\max(0, \lceil (\nu_\pi(2)-(r-t))/2 \rceil)} \vec{x}_2\}.$$

In the case of $n = 3$ a basis for $L^{(2)}$ is:

$$\{\pi^{\lceil (\nu_\pi(2)-s)/2 \rceil} \vec{x}_1, \vec{x}_2, \pi^{\lceil \nu_\pi(2)/2 \rceil} \vec{x}_3\}.$$

For the case of $n = 4$, we can eliminate some of the conditions by using that $t, s \leq r - t$. We do this by fixing η and θ so that:

$$\eta^2 a \pi^t + \theta^2 \pi^s = c\pi^{r-t} \pmod{2}.$$

Now a basis for $L^{(2)}$ is:

$$\{\pi^{\lceil (\nu_\pi(2)-t)/2 \rceil} \vec{x}_1, \eta \vec{x}_1 + \vec{x}_2 + \theta \vec{x}_3, \pi^{\lceil (\nu_\pi(2)-s)/2 \rceil} \vec{x}_3, \vec{x}_4\}.$$

It is now an easy calculation to determine $[L : L^{(2)}]$. Moreover, it is apparent that $W^\perp = W$ and thus $\mathrm{Rad}(W) = W^\perp$ if and only if $\tilde{Q}$ is trivial. This is easily checked on the bases we have given. $\square$

We now discuss the problem of computing $\beta_R(L, L)$ for unimodular lattices L of rank at most 4 with no even unimodular factors. The general strategy is as follows:

1. Describe a constructive process for enumerating and counting all choices of basis that give a bilinear form that 'looks like' the original.
2. Show that the number of ways of obtaining each possible form that 'looks like' the original is the same.
3. Count the number of possible forms that 'look like' the original.
4. Obtain the result.

The above is made more precise in the following proofs.

Lemma 4.14. *Suppose L is a unimodular lattice of rank 1. Then:*

$$\beta_R(L, L) = 2.$$

This case is a simple check.

Lemma 4.15. *Suppose L is the unimodular lattice of rank 2 over a 2-adic ring represented by $\begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix}$ with $a, c \in R^\times$, $2t < r$ and either $r < \nu(4)$ odd or $r = \nu(4)$. Then:*

$$\beta_R(L, L) = \begin{cases} 4q^{(r-1)/2-\nu(2)} & r - t \leq \nu(2) \\ 2q^{-\lceil (\nu(2)-t)/2 \rceil} & \nu(2) < r - t. \end{cases}$$

Proof. By Proposition 4.2 we need to count the elements in the set:

$$\Phi = \{\phi : L \to L/\pi^{\nu(2)-t+1}L \mid q_L(\phi(x)) = q_L(x) \pmod{\pi^{\nu(4)-t+1}}\}.$$

Consider the following sets:

$$X = \{\vec{x} \in L/\pi^{\nu(2)-t+1}L \mid q_L(\vec{x}) = a\pi^t \pmod{\pi^{\nu(4)-t+1}}\},$$
$$Y_{\vec{x}} = \{\vec{y} \in L/\pi^{\nu(2)-t+1}L \mid (\vec{x}, \vec{y}) = 1 \pmod{\pi^{\nu(2)-t+1}}, \nu(q(\vec{y})) = r - t\}, \text{ and}$$
$$\tilde{Y} = \{q_L(\vec{y}) \pmod{\pi^{\nu(4)-t+1}} \mid \vec{y} \in Y_{\vec{x}}, \ \vec{x} \in X\}.$$

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We claim that $|Y_{\vec{x}}|$ is independent of the choice of $\vec{x} \in X$. Indeed, letting $\vec{x}_0$ and $\vec{y}_0$ be the original basis it is clear that:

$$Y_{\vec{x}} = \{(\vec{x}, (\vec{x}, \vec{y'})^{-1}\vec{y'}) \mid \vec{y'} = (x\pi^{\lceil(r-2t)/2\rceil}\vec{x}_0 + \vec{y}_0)\},$$

where x runs over elements of $R/\pi^{\nu(2)-t+1-\lceil(r-2t)/2\rceil}R$. If follows that:

$$|Y_{\vec{x}}| = q^{\nu(2)+1-\lceil r/2\rceil}.$$

We next compute $\left|\tilde{Y}\right|$. The values of $q_L(\vec{y})$ that can appear are precisely those such that:

$$1 - aq_L(\vec{y})\pi^t = 1 - ac\pi^r \pmod{(R^\times)^2}$$

as these are the values that give isomorphic quadratic forms. This is precisely the same as the number of elements modulo $\pi^{\nu(4)+1}$ that are squares, and congruent to 1 modulo π^r. We thus have:

$$\left|\tilde{Y}\right| = \frac{1}{2}q^{\nu(2)+1-\lceil r/2\rceil}.$$

We now compute $|X|$. We are counting solutions for $x, y \pmod{\pi^{\nu(2)-t+1}}$ of:

$$a\pi^t x^2 + 2xy + c\pi^{r-t}y^2 = a\pi^t \pmod{\pi^{\nu(4)-t+1}}.$$

We make the substitution $x = 1 + x$ and this becomes:

$$a\pi^t x^2 + 2a\pi^t x + 2y + 2xy + c\pi^{r-t}y^2 = 0 \pmod{\pi^{\nu(4)-t+1}}.$$

By inspecting the valuations of monomials that result from such a switch (of $x = x + 1$), in particular the parity of their valuations, it is apparent that we have:

$$x = 0 \pmod{\pi^{\max(\nu(2)-(r-1)/2,\lceil(\nu(2)-t)/2\rceil)}} \text{ and}$$
$$y = 0 \pmod{\pi^{\max(\nu(2)+t-r,\nu(2)+t)}},$$

where the first terms are maximal if and only if $\nu(2) \geq r - t$. If we perform the substitutions:

$$x = \pi^{\max(\nu(2)-(r-1)/2,\lceil(\nu(2)-t)/2\rceil)}x' \text{ and } y = \pi^{\max(\nu(2)+t-r,0)}y'$$

the equation becomes:

$$a\pi^{\nu(2)+\delta}x^2 + 2y + 2\pi P(x,y) = 0 \qquad r - t > \nu(2), \text{ or}$$
$$2y + 2cy^2 + 2\pi P(x,y) = 0 \qquad r - t \leq \nu(2)$$

for some polynomial P and $\delta \in \{0,1\}$. (Notice the only way we could have had both an x^2 and y^2 term was if $r - t = t = \nu(2)$ but we have excluded that case from consideration.) We observe that by dividing by 2 we may solve for y in terms of x. As the equation is non-singular, we may use Hensel's lemma to find solutions and the total number of solutions is equal to the number of solutions modulo π. There are precisely 2 solutions modulo π if $\nu(2) \geq r - t$ and 1 solution otherwise. We thus find:

$$|X| = \begin{cases} 2q^{(r-t-t-1)/2+1} & \nu(2) \geq r - t \\ q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \text{otherwise.} \end{cases}$$

The set Φ corresponds precisely to the fibre of

$$\{(\vec{x}, \vec{y}) \mid \vec{x} \in X, \vec{y} \in Y_{\vec{x}}\}$$

over $c\pi^{r-t} \in \tilde{Y}$. The automorphism group of $L/\pi^{\nu(2)-t+1}L$ acts simply transitively on this fibre. However, noting that the original choice of $c\pi^{r-t}$ is arbitrary, the automorphism group acts simply transitively on each fibre of:

$$\{(\vec{x}, \vec{y}) \mid \vec{x} \in X, \vec{y} \in Y_{\vec{x}}\}$$

over $\tilde{Y}$.

It thus follows that:
$$|\Phi| = \frac{|X|\,|Y_{\vec{x}}|}{\left|\tilde{Y}\right|}.$$

Thus we find:
$$|\Phi| = \begin{cases} 4q^{(r-t-t-1)/2+1} & r - t \le \nu(2) \\ 2q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \nu(2) < r - t. \end{cases}$$

Combining terms completes the result. $\hfill\square$

Lemma 4.16. *Suppose* $L = L_{\pi^t, b\pi^{\nu(4)-t}} \oplus U_{-d}$ *is a unimodular lattice of rank 3 over a 2-adic ring with* $t < \nu(2)$ *odd and* $b, d \in R^\times$, *then:*
$$\beta_R(L, L) = 4q^{(1-t)/2}.$$

Proof. By Proposition 4.2 we need to count elements in the set:
$$\Phi = \{\phi : L \to L/\pi^{\nu(2)+1}L \mid q_L(\phi(x)) = q_L(x) \pmod{\pi^{\nu(4)+1}}\}.$$

As in the previous lemma consider the following sets:
$$X = \{\vec{x} \in L/\pi^{\nu(4)+1}L \mid q_L(\vec{x}) = \pi^t \pmod{\pi^{\nu(4)+1}}\},$$
$$Y_{\vec{x}} = \{\vec{y} \in L/\pi^{\nu(2)+1}L \mid (\vec{x}, \vec{y}) = 1 \pmod{\pi^{\nu(2)+1}}, \nu(q_L(\vec{y})) = \nu(4)\},$$
$$\tilde{Y} = \{q_L(\vec{y}) \pmod{\pi^{\nu(4)+1}} \mid \vec{y} \in Y_{\vec{x}}, \vec{x} \in X\},$$
$$Z_{\vec{x},\vec{y}} = \{\vec{z} \in \langle \vec{x}, \vec{y} \rangle^\perp / \pi^{\nu(2)+1} \mid q_L(\vec{z}) = -d \pmod{\pi^{\nu(4)+1}}\}.$$

We claim that $|Y_{\vec{x}}|$ is independent of $\vec{x} \in X$. Indeed, letting $\vec{x}_0, \vec{y}_0, \vec{z}_0$ be the original basis it is clear for parity reasons that:
$$Y_{\vec{x}} = \{(\vec{x}, (\vec{x}, \vec{y'})\vec{y'}) \mid \vec{y'} = x\pi^{\nu(2)-t}\vec{x}_0 + \vec{y}_0 + z\pi^{\nu(2)-(t-1)/2}\vec{z}_0\},$$
where $x \in R/\pi^{t+1}R$ and $z \in R/\pi^{(t-1)/2+1}R$. We thus find:
$$|Y_{\vec{x}}| = q^{t+(t-1)/2+2}.$$

Next we compute $\left|\tilde{Y}\right| = \frac{1}{2}q$. The argument is identical to the previous lemma, except we note that the discriminant of this block is well-defined modulo squares because it controls the Hasse invariant of the form.

Now $|Z_{\vec{x},\vec{y}}| = 2$ independently of $\vec{x}, \vec{y}$. This follows as the orthogonal complement is isomorphic to U_{-d} by necessity (again because the Hasse invariant controls the discriminant).

We now compute $|X|$. We are counting solutions for $x, y, z \pmod{\pi^{\nu(2)+1}}$ of:
$$\pi^t x^2 + 2xy + \pi^{\nu(4)-t}y^2 + cz^2 = \pi^t \pmod{\pi^{\nu(4)+1}}.$$

It is clear that we may replace z by $\pi^{\lceil\nu(2)/2\rceil}z$ and get:
$$x^2 + \pi^{\nu(2)-t}xy + b\pi^{\nu(4)-2t}y^2 + c\pi^{\nu(2)+2\lceil\nu(2)/2\rceil-t}z^2 = 1 \pmod{\pi^{\nu(4)-t+1}}.$$

We now replace x by $1 + \pi^{\lceil(\nu(2)-t)/2\rceil}x$ and the expression modulo $\pi^{\nu(4)-t+1}$ becomes:
$$2\pi^{\lceil(\nu(2)-t)/2\rceil}x + \pi^{2\lceil(\nu(2)-t)/2\rceil}x^2 + \pi^{\nu(2)-t}y+$$
$$\pi^{\lceil 3(\nu(2)-t)/2\rceil}xy + b\pi^{\nu(4)-2t}y^2 + c\pi^{2\lceil\nu(2)/2\rceil-t}z^2 = 0.$$

This reduces to:
$$2\pi^\delta x + \pi^\delta x^2 + y + \pi^{\delta+\nu(2)-t}xy + b\pi^{\nu(2)-t}y^2 + c\pi^{1-\delta}z^2 = 0 \pmod{\pi^{\nu(2)+1}},$$
where $\delta = \begin{cases} 0 & \nu(2) \text{ odd} \\ 1 & \text{otherwise.} \end{cases}$

As in the previous case, this equation is non-singular in y, hence, for all values of z, x we may find a unique solution for y. It follows that:
$$|X| = q^{\lfloor\nu(2)/2\rfloor+\lfloor(\nu(2)-t)/2\rfloor-t+2} = q^{\nu(2)-(t+1)/2-t+2}.$$

As in the previous lemma it follows that:
$$|\Phi| = 2q^{t+(t-1)/2+1}|X|\,|Y_{\vec{x}}|\,|Z_{\vec{x},\vec{y}}|\left|\tilde{Y}\right|^{-1}.$$

We may thus conclude that $|\Phi| = 4q^{3\nu(2)-3t-(t-1)/2+3}$. Combining terms completes the result. $\hfill\square$

Lemma 4.17. *Suppose $L = L_{\pi^s, b\pi^{\nu(4)-s}} \oplus L_{a\pi^t, c\pi^{r-t}}$ is a unimodular lattice of dimension 4 over a 2-adic ring with $t < s < \nu(2)$, $a, b, c \in R^\times$, $s - t$ odd, and $r < \nu(4)$ odd or $r = \nu(4)$. In this situation:*

$$\beta_R(L, L) = 4q^{-3\nu(2)+2t-2-(r-t-s)/2} \begin{cases} q^{(r-t-t-1)/2+1} & r - t \leq \nu(2) \\ q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \nu(2) \leq r - t. \end{cases}$$

Proof. We make the following definitions:

$$\Phi = \{g \in \mathrm{GL}(L/\pi^{\nu(4)-t+1}L) \mid g^t A g = \begin{pmatrix} \pi^s & 1 \\ 1 & b\pi^{\nu(4)-s} \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix}\},$$

$$X = \{\vec{x} \in L/\pi^{\nu(2)-t+1}L \mid q_L(x) = \pi^s \pmod{\nu(4) - t + 1}\},$$

$$Y_{\vec{x}} = \{\vec{y} \in L/\pi^{\nu(2)-t+1}L \mid (\vec{x}, \vec{y}) = 1 \pmod{\pi^{\nu(2)-t+1}}, \nu(q_L(\vec{y})) \geq \nu(4) - s\},$$

$$\tilde{Y} = \{q_L(\vec{y}) \in R/\pi^{\nu(4)-t+1}R \mid \vec{y} \in Y_{\vec{x}}, \ \vec{x} \in X\},$$

$$Z_{\vec{x},\vec{y}} = \{(\vec{z}, \vec{w}) \in \langle \vec{x}, \vec{y} \rangle^\perp / \pi^{\nu(2)-t+1} \mid (\vec{z}, \vec{w}) = 1 \pmod{\pi^{\nu(2)-t+1}}, \nu(q_L(\vec{z})) = t\},$$

$$\tilde{Z}_{\vec{x},\vec{y}} = \{(q_L(\vec{z}), q_L(\vec{w})) \in (R/\pi^{\nu(4)-t+1}R)^2 \mid (\vec{z}, \vec{w}) \in Z_{\vec{x},\vec{y}}\},$$

$$\hat{Z}_{\tilde{y}} = \{\Lambda \text{ a lattice modulo } \pi^{\nu(4)-t+1} \text{ up to isomorphism} \mid L_{\pi^s, \tilde{y}} \oplus \Lambda \simeq L\}.$$

In the above we are taking $\tilde{y} \in \tilde{Y}$.

Our first claim is that $|Y_{\vec{x}}| = q^{\nu(2)-3t+3+s+(s+t-1)/2}$ and that this is independent of $\vec{x} \in X$. Indeed, we can compute its value as follows:

$$Y_{\vec{x}} = \{(\vec{x}, \vec{y'})^{-1}\vec{y'} \mid \nu(q_L(\vec{y'})) = \nu(4) - s\}.$$

Thus its size is the number of solutions to:

$$\pi^s x^2 + 2x + a\pi^t z^2 + 2zw + c\pi^{r-t}w^2 = 0 \pmod{\pi^{\nu(4)-s}},$$

where x, z, w are taken in $R/\pi^{\nu(2)-t+1}R$. In the event that $r - t > \nu(4) - s$ then for parity reasons we must have:

$$x = 0 \pmod{\pi^{\nu(2)-s}} \text{ and } z = 0 \pmod{\pi^{\nu(2)-(s+t-1)/2}}.$$

One finds then that there are no further conditions and thus counting solutions we find:

$$|Y_{\vec{x}}| = q^{\nu(2)-3t+3+s+(s+t-1)/2}.$$

Otherwise we suppose $r - t \leq \nu(4) - s$. Next we may choose η, ϵ such that:

$$\eta^2 c + \epsilon^2 a\pi = 1.$$

For parity reasons we again find:

$$x = 0 \pmod{\pi^{(r-t-s)/2}} \text{ and } z = 0 \pmod{\pi^{(r+1)/2-t}}.$$

We may thus substitute:

$$x = \pi^{(r-t-s)/2}x' \text{ and } w = \eta x' + w' \text{ and } z = \pi^{(r+1)/2-t}(\epsilon x' + z).$$

The whole expression modulo $\pi^{\nu(4)-s}$ then becomes:

$$2\pi^{(r-t-s)/2}x + \pi^{r-t+1}z^2 + \pi^{r-t}w^2 + \pi^{\nu(2)+(r-t-s)/2+1}P(x, w, z) = 0$$

for some polynomial P. It is now apparent that:

$$z = 0 \pmod{\pi^{\lceil(\nu(2)-(3r-3t-s)/2-1)/2\rceil}} \text{ and } w = 0 \pmod{\pi^{\lceil(\nu(2)-(3r-3t-s)/2)/2\rceil}}$$

and that x is determined modulo $\pi^{\nu(2)-s-(r-t-s)/2}$ by the other parameters. One finds then that there are no further conditions and thus counting solutions we find:

$$|Y_{\vec{x}}| = q^{\nu(2)-3t+3+s+(s+t-1)/2}.$$

Next we compute $\left|\tilde{Y}\right|$. Indeed, so long as there exist values $\alpha, \gamma \in R^\times$ such that:

$$L \simeq L_{\pi^s, \beta\pi^{\nu(4)-s}} \oplus L_{\alpha\pi^t, \gamma\pi^{r-t}}$$

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then $\beta \in \tilde{Y}$. The two conditions:

$$\mathfrak{n}_L = \alpha R^2 + \pi^s, \text{ and}$$
$$H(L) = (\alpha, \delta_L)(\pi^t, \delta_L)(\pi^{s+t}, 1 - \beta\pi^{\nu(4)})$$

can be solved for all β if $r - t \leq \nu(4) - s$. If however, $r - t > \nu(4) - s$ then, since (α, δ_L) cannot depend on α, only half of the potential values for β will work. The other condition:

$$\delta_L = (1 - \alpha\gamma\pi^r)(1 - \beta\pi^{\nu(4)}) \pmod{R^2}$$

can always be solved by γ. It follows that:

$$\left|\tilde{Y}\right| = q^{s-t+1} \begin{cases} \frac{1}{2} & r - t > \nu(4) - s \\ 1 & \text{otherwise.} \end{cases}$$

We now claim that $\left|\tilde{Z}_{\vec{x},\vec{y}}\right|$ is independent of $\vec{x} \in X$ and $\vec{y} \in Y_{\vec{x}}$. Indeed there are three conditions for $(\alpha, \gamma) \in \tilde{Z}_{\vec{x},\vec{y}}$. The first condition is:

$$H(L) = (\alpha, \delta_L)(\pi^t, \delta_L)(\pi^{s+t}, 1 - q_L(\vec{y})\pi^{\nu(4)}).$$

This condition cannot be unsatisfiable. Hence, it is either imposing a condition (independently of $\vec{y}$), or is not imposing a condition (independently of $\vec{y}$). The second condition is:

$$\mathfrak{n}_L = \alpha R^2 + \pi^s.$$

This condition is independent of $\vec{y}$. The final condition is:

$$\delta_L = (1 - \alpha\gamma\pi^r)(1 - q_L(\vec{y})\pi^{\nu(4)}) \pmod{R^2}.$$

For each α satisfying the first two conditions we are imposing a condition on the variable γ. The number of values for γ satisfying the condition is independent of $\vec{y}$.

Now, we claim that $|Z_{\vec{x},\vec{y}}|$ is independent of $\vec{x} \in X$ and $\vec{y} \in Y_{\vec{x}}$. Indeed, the value of $|Z_{\vec{x},\vec{y}}|$ is precisely $\left|\text{Aut}(\langle \vec{x}, \vec{y} \rangle^{\perp}/\pi^{\nu(4)-t+1})\right|\left|\tilde{Z}_{\vec{x},\vec{y}}\right|$. Our computations in Lemma 4.15 show this depends only on t and r. Explicitly, the value is:

$$\left|\text{Aut}(\langle \vec{x}, \vec{y} \rangle^{\perp}/\pi^{\nu(4)-t+1})\right| = \begin{cases} 4q^{(r-t-t-1)/2+1} & r - t \leq \nu(2) \\ 2q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \nu(2) < r - t. \end{cases}$$

Next, we claim that $\left|\hat{Z}_{\tilde{y}}\right|$ is independent of $\tilde{y} \in \tilde{Y}$. Equivalence classes of lattices $\overline{\Lambda} \in \hat{Z}_{\tilde{y}}$ have representatives of the form $L_{\alpha,\gamma}$ where $(\alpha, \gamma) \in \tilde{Z}_{\vec{x},\vec{y}}$ for some $\vec{x} \in X$, $\vec{y} \in Y_{\vec{x}}$. We may thus represent $\overline{\Lambda}$ by $\overline{(\alpha, \gamma)}$. Now, as the Hasse invariant and discriminant of $\Lambda \in \hat{Z}_{\tilde{y}}$ are determined by $\tilde{y}$ and L, the only freedom to modify Λ is picking its norm generator. In terms of $\overline{(\alpha, \gamma)}$ this amount to fixing the square class of α module π^{r-2t}. The first constraint on the square class of α is that it must give the norm generator of L module π^s. This determines the square class of α modulo π^{s-t}. This leaves us with precisely:

$$q^{(r-t-s)/2}$$

many options for such square classes. The only other constraint on α is that it must give the correct Hasse invariant. As above, the Hasse invariant depends on α through (α, δ_L). Thus, it follows that:

$$\left|\hat{Z}_{\tilde{y}}\right| = q^{(r-t-s)/2} \begin{cases} \frac{1}{2} & r - t \leq \nu(4) - s \\ 1 & \text{otherwise.} \end{cases}$$

We now compute $|X|$. We are solving for $x, y, z, w \in R/\pi^{\nu(2)-t+1}R$ in the following equation modulo $\pi^{\nu(4)-t+1}$:

$$\pi^s x^2 + 2xy + b\pi^{\nu(4)-s}y^2 + a\pi^t z^2 + 2zw + c\pi^{r-t}w^2 = \pi^s.$$

Pick η, ϵ such that $\eta^2 + \pi a\epsilon^2 = c \pmod{\pi^{\nu(2)}}$. We may then make the following substitutions:

$$x = 1 + \eta\pi^{\lceil(r-t-s)/2\rceil}w + x \text{ and } z = \epsilon\pi^{\lceil(r+1)/2\rceil-t}w + z.$$

The equation then becomes:

$$\pi^s x^2 + 2y + a\pi^t z^2 + 2zw + \pi^{\nu(2)+1} P(x,y,z,w) = 0$$

for some polynomial P. For parity reasons we now see that:

$$x \equiv 0 \pmod{\pi^{\lceil (\nu(2)-s)/2 \rceil}} \text{ and } z \equiv 0 \pmod{\pi^{\lceil (\nu(2)-t)/2 \rceil}}.$$

This equation is now solvable in y, and determines y modulo $\pi^{\nu(2)-t+1}$. Counting solutions, we find that there are:

$$|X| = q^{\nu(4)-3t+3+(s+t-1)/2}.$$

We now observe that:

$$|\Phi| = |X|\,|Y_{\vec{x}}|\,|Z_{\vec{x},\vec{y}}| \left|\tilde{Y}\right|^{-1} \left|\tilde{Z}_{\vec{x},\vec{y}}\right|^{-1} \left|\hat{Z}_{\tilde{y}}\right|^{-1}.$$

To see this, consider the map:

$$\{(\vec{x},\vec{y},\vec{z},\vec{w}) \mid \vec{x} \in X,\ \vec{y} \in Y_{\vec{x}},\ (\vec{z},\vec{w}) \in Z_{\vec{x},\vec{y}}\} \to (R/\pi^{v(4)-t+1}R)^3$$

given by $(\vec{x},\vec{y},\vec{z},\vec{w}) \mapsto (q_L(\vec{y}), q_L(\vec{z}), q_L(\vec{w}))$ and observe that $|\Phi|$ is precisely the size of each fibre. We thus must show that the size of the image is:

$$\left|\tilde{Y}\right|\left|\tilde{Z}_{\vec{x},\vec{y}}\right|\left|\hat{Z}_{\tilde{y}}\right|.$$

The image of this map is precisely:

$$\{(\tilde{y},\tilde{z},\tilde{w}) \mid \tilde{y} \in \tilde{Y},\ \overline{(\tilde{z},\tilde{w})} \in \hat{Z}_{\tilde{y}}\}.$$

This set is naturally fibred over:

$$\{(\tilde{y},\overline{(\alpha,\gamma)}) \mid \tilde{y} \in \tilde{Y},\ \overline{(\alpha,\gamma)} \in \hat{Z}_{\tilde{y}}\}.$$

Moreover, the size of the fibre over $(\tilde{y},\overline{(\alpha,\gamma)})$ is precisely $\left|\tilde{Z}_{\vec{x},\vec{y}}\right|$ where $\vec{x} \in X$ and $\vec{y} \in Y_{\vec{x}}$ are any vectors such that $(\alpha,\gamma) \in \tilde{Z}_{\vec{x},\vec{y}}$. From this the claim about $|\Phi|$ follows immediately.

We, therefore, have that:

$$\left|\mathrm{Aut}(L/\pi^{\nu(4)-t+1}L)\right| = 4q^{3\nu(2)-4t+4-(r-t-s)/2} \begin{cases} q^{(r-t-t-1)/2+1} & r-t \leq \nu(2) \\ q^{\lfloor (\nu(2)-t)/2 \rfloor +1} & \nu(2) \leq r-t \end{cases}$$

Combining terms gives the desired result. $\qquad \square$

The above lemmas cover the final few cases we needed to completely solve the problem of computing local densities for unimodular lattices over 2-adic rings. By combining the results we get the following theorem:

Theorem 4.18. *Consider a unimodular lattice L of rank at most 4 over a 2-adic ring R with no even unimodular factors. Let π be a uniformizer of R and $q = |R/\pi R|$. Recall that $L^{(2)} = \{x \in L \mid (x,x) \in 2R\}$. Denote by W the quadratic module $L^{(2)}/\pi L^{(2)}$ with the induced form $\tilde{Q}(x) = \frac{1}{2}(x,x) \pmod{\pi}$. Then:*

- *Case $n = 4$. Write $L = \begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix} \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4b\pi^{-s} \end{pmatrix}$ with $t < s < r-t$, $t+s$ is odd, and either r odd or $r = \nu(4)$.*

  Then $\mathrm{Rad}(W) \neq W^\perp$. Moreover, $[L : L^{(2)}] = q^{\nu(2)-(s+t-1)/2}$ and the local density is:

$$\beta_R(L,L) = 4q^{-3\nu(2)+2t-2-(r-t-s)/2} \begin{cases} q^{(r-t-t-1)/2+1} & r-t \leq \nu(2) \\ q^{\lfloor (\nu(2)-t)/2 \rfloor +1} & \nu(2) \leq r-t \end{cases}$$

- *Case $n = 3$. Write $L = \begin{pmatrix} \pi^s & 1 \\ 1 & b\pi^{\nu(4)-s} \end{pmatrix} \oplus (d)$ with $\nu(2) > s > 0$ and s odd.*

  Then $\mathrm{Rad}(W) \neq W^\perp$. Moreover, $[L : L^{(2)}] = q^{\nu(2)-(s-1)/2}$ and the local density is:

$$\beta_R(L,L) = 4q^{(1-t)/2}.$$

- *Case $n = 2$. Write L with matrix $\begin{pmatrix} a\pi^t & 1 \\ 1 & c\pi^{r-t} \end{pmatrix}$ with either $r > t$ odd or $r = \nu(4)$.*

  Then $\mathrm{Rad}(W) = W^\perp$ unless $r - t \le \nu(2)$ or $\nu(2) - t$ is even.

  Moreover, $[L : L^{(2)}] = \begin{cases} q^{\left\lceil \frac{\nu(2)-t}{2} \right\rceil} & r - t \ge \nu(2) \\ q^{\nu(2)-(r-1)/2} & \text{otherwise} \end{cases}$ and the local density is:

$$\beta_R(L, L) = \begin{cases} 4q^{(r-1)/2-\nu(2)} & r - t \le \nu(2) \\ 2q^{-\lceil (\nu(2)-t)/2 \rceil} & \nu(2) < r - t. \end{cases}$$

- *Case $n = 1$. Then $\mathrm{Rad}(W) = W^\perp$ unless $\nu(2)$ is even.*

  Moreover, $[L : L^{(2)}] = q^{\left\lceil \frac{\nu(2)}{2} \right\rceil}$ and the local density is:

$$\beta_R(L, L) = 2.$$

♦ *General Lattices - Jordan Decompositions*

Computing local densities is equivalent to computing $|\mathrm{Aut}(L/\pi^r L)|$ which can be done indirectly by computing the probability that a randomly chosen element of $\mathrm{GL}(L/\pi^r L)$ preserves the quadratic form on L. Once one is working in the realm of probabilities, it is natural to use conditional probabilities that are easier to compute to arrive at a solution. This is the approach we shall take.

We shall use the following notation.

Notation 4.19. Let R be a p-adic ring, with uniformizer π and $|R/\pi| = q$. Suppose L is a lattice over R.

By a Jordan decomposition I of L we mean a decomposition:

$$L = \oplus L_i^I,$$

where the L_i^I are modular and ordered by valuations of their scale ideals. Two Jordan decompositions, I and J, are considered isomorphic if $L_i^I \simeq L_i^J$ for all i. We will denote by JD_L the set of all Jordan decompositions of L up to isomorphism.

We fix r sufficiently large so that the isomorphism classes of all of the L_i^J are determined by their reductions modulo π^r.

We shall say a matrix A which represents the quadratic form on L is in the **Jordan form** $I \in JD_L$ (modulo π^r) if A has a block diagonal decomposition $\oplus A_i$, where the A_i represent modular lattices in ascending order and A_i represents L_i^I for some choice of basis for each i.

Lemma 4.20. *Let A be any matrix representation for L. Then the probability that for $g \in \mathrm{GL}(L/\pi^r L)$ the matrix $g^t A g$ is in Jordan form (modulo π^r) is:*

$$P_{JD,r} = |\mathrm{GL}(L/\pi^r L)|^{-1} \left(\prod_i \left| \mathrm{GL}(L_i^I/\pi^r L_i^I) \right| \right) q^w,$$

where $w = \sum_i (2r - i)n_i \sum_{j>i} n_j$.

Proof. The proof is an inductive exercise in book keeping. We first count the number of ways of finding a minimal modular block. In order to pick a set of vectors which will span a minimally modular block one needs to select a $\mathrm{GL}(L_i^I/\pi^r L_i^I)$ combination of the vectors that were in the original minimally modular block. One can then give an arbitrary contribution from the vectors which were complementary to the minimal modular block. This arbitrary choice contributes a factor of $q^{rn_i \sum_{j>i} n_j}$.

We then must proceed inductively on the space which is orthogonally complementary. The degree of freedom in picking an orthogonally complementary space (modulo π^r) is precisely $q^{(r-i)n_i \sum_{j>i} n_j}$.

Taking products of number of choices at each inductive steps gives us the result. $\qquad\square$

Definition 4.21. Let $I \in JD_L$. Suppose that $g \in \mathrm{GL}(L/\pi^r L)$ is chosen at random. Suppose $g^t A g$ is in Jordan form (modulo π^r). Denote the conditional probability that the Jordan form J of $g^t A g$ is equal to I as Jordan decompositions (modulo π^r) as given that $g^t A g$ is in Jordan form (modulo π^r) as:

$$P_{I=J,r}.$$

Lemma 4.22. *Let A be any matrix representation for L. Let $I \in JD_L$. Fix a matrix A_I representing the Jordan form I. Define $P_{eq,I,r}$ to be the conditional probability that an element $g \in \mathrm{GL}(L/\pi^r L)$, for which the matrix $g^t A g$ is in Jordan form I (modulo π^r), will have $g^t A g = A_I \mod \pi^r$. Then the conditional probability $P_{eq,I,r}$ can be computed as:*

$$P_{eq,I,r} = \prod_i \frac{|\mathrm{Aut}(L_i^I/\pi^r L_i^I)|}{|\mathrm{GL}(L_i^I/\pi^r L_i^I)|}.$$

Proof. The set of possible values of $g^t A g$ is acted upon by $\prod_i \mathrm{GL}(L_i^I/\pi^r L_i^I)$ with the size of the stabilizer being $\left|\prod_i \mathrm{Aut}(L_i^I/L_i^I \pi^r)\right|$. In particular, then the probability that we get any given representative is $\prod_i \frac{|\mathrm{Aut}(L_i^I/L_i^I \pi^r)|}{|\mathrm{GL}(L_i^I/\pi^r L_i^I)|}$. $\square$

Lemma 4.23. *Let A be any matrix representation for L. Let $I \in JD_L$. Fix a matrix A_I representing the Jordan form I. The absolute probability that an element $g \in \mathrm{GL}(L/\pi^r L)$ gives $g^t A g = A_I \mod \pi^r$ is:*

$$P_{Aut,L,r} = P_{JD,r} P_{I=J,r} P_{eq,I,r}.$$

Proof. This is a trivial statement in conditional probabilities. $\square$

Remark. Notice that $P_{Aut,L,r}$ and $P_{JD,r}$ are independent of the choice of I while $P_{I=J,r}$ and $P_{eq,I,r}$ depend on the choice.

Lemma 4.24. *With all the notation as above, we have the formula:*

$$P_{Aut,L,r} = P_{JD,r} \left(\sum_{I \in JD_L} P_{eq,I,r}^{-1} \right)^{-1}.$$

Proof. By observing that $P_{eq,I,r} \neq 0$ for all I we may write:

$$P_{Aut,L,r} P_{eq,I,r}^{-1} = P_{JD,r} P_{I=J,r}.$$

By summing over $I \in JD$ we obtain:

$$P_{Aut,L,r} \sum_{I \in JD} P_{eq,I,r}^{-1} = P_{JD,r} \sum_{I \in JD} P_{I=J,r}.$$

Since $\sum_{I \in JD} P_{I=J,r} = 1$ we obtain the result. $\square$

Lemma 4.25. *Suppose L is a lattice of rank ℓ then:*

$$\beta_R(L,L) = q^{\ell v_\pi(2) + r\ell(1-\ell)/2} |\mathrm{GL}(L/\pi^r L)| P_{Aut,L,r}.$$

Proof. This is immediate from Proposition 4.2 and the definition of the probability. $\square$

Combining the above lemmas we arrive at the following very general theorem.

Theorem 4.26. *With the notation as above we have:*

$$\beta_R(L,L) = q^w \left(\sum_{I \in JD} \prod_i \beta_R(L_i^I, L_i^I)^{-1} \right)^{-1} = q^{\tilde{w}} \left(\sum_{I \in JD} \prod_i \beta_R(\tilde{L}_i^I, \tilde{L}_i^I)^{-1} \right)^{-1},$$

where $\tilde{L}_i^I$ is the unimodular rescaling of L_i^I and $w, \tilde{w}$ are given by:

$$w = \sum_i i n_i (\sum_{j>i} n_j) \text{ and}$$

$$\tilde{w} = w + \sum_i (n_i(n_i+1)/2).$$

Proof. This is a direct calculation. The only tricky part is the book-keeping on the exponents of q. $\square$

Remark. In order to use this theorem to derive specific formulas for a given lattice one must understand the set JD_L. For a non-dyadic ring there is a unique Jordan decomposition. The problem is thus fully solved in this case.

For the dyadic case it is worth remembering that most of the factors involved in the formula of local density for a unimodular lattice do not depend on the isomorphism class. Hence there are many terms which can be factored out of the sum appearing in the formula above. Moreover, whenever there is dependence on the isomorphism class through $\chi(L_i(e))$ it is typically symmetric and cancels out. Both of these phenomena can be seen in the structure of the classical formulas over $\mathbb{Z}_2$, even though the classical proofs did not fully explain why this structure should exist [CS88].

5. Computational Issues

The preceding results give an explicit formula for the problem of computing the representation densities $\beta_{\mathfrak{p}}(L,L)$ for an arbitrary lattice L. Given such formulas, a very natural question which arises is:

Question 1. Are these formulas actually computable?

More concretely the question one might pose is:

Question 2. Given a p-adic ring R and a matrix A representing some bilinear form, what problems arise when trying to use the formulas of the previous section to compute the corresponding representation density?

An encouraging first comment is that the constant r, as in Proposition 4.2, is easily computed using the modularity of the Jordan blocks together with Theorem 3.11. This tells us precisely how accurate an approximation of p-adic numbers we must maintain in order to compute the exact value.

From an implementation point of view, the first hurdle one needs to overcome is:

Question 3. How can one compute the collection of all possible Jordan forms?

An easier question that one must be able to answer first is:

Question 4. How can one find any Jordan form?

This question is answerable and there exists an explicit algorithm:

1. Find matrix entry of minimal valuation.
2. Use index of this entry to determine a basis for a sublattice x or x, y.
3. Find the orthogonal complement of x or x, y.
4. Compute the matrix for the bilinear form on this complement.
5. Proceed inductively.

Before one can use this to answer Question 3, one should first consider:

Question 5. How can you reduce the Jordan blocks to canonical form (as in Theorem 3.11)?

One only needs to know the discriminant, Hasse invariant, and norm group. Finding the discriminant is an explicit and easy computation. Finding the Hasse invariant requires that you diagonalize, but this may be done in the field of fractions where it is easy, otherwise the problem is reduced to computing Hilbert symbols. Computing Hilbert symbols is an explicit problem in local class field theory. Finally, the norm group can be computed easily. Indeed, use the diagonal element of lowest valuation to arrange so that all other diagonal elements have valuation of a different parity, the two diagonal elements of minimal valuation then determine the group. Having these invariants an easy application of Theorem 3.11 lets one express the matrix in a standard form.

Continuing to leave aside Question 3, an encouraging observation is that we can easily answer the question:

Question 6. Given a Jordan decomposition how do we compute its contribution to representation density?

The answer is provided by the fact that the only information that we needed in order to apply Theorems 4.11 and 4.18 were invariants we computed when putting the Jordan form into its standard form.

Now finally back to Question 3. This problem is trivial when $p \neq 2$ as there is only one Jordan decomposition. In the specific case of $\mathbb{Z}_2$ [Nik79, Prop. 1.8.2] gives us a complete description of the relations on the semi-group of quadratic forms under direct sum. Consequently, in this case one can generate all Jordan decompositions by iteratively applying these relations. For unramified extensions of $\mathbb{Z}_2$ relations of a similarly simple sort can be derived; in this case the set of relations should be enumerable. For ramified extensions of $\mathbb{Z}_2$ the relations become increasingly more numerous and complex as the ramification degree grows. It is not clear to me how best to describe them other than by brute force search. The only bright side is that one can bound the size of the relations needed, and they could be precomputed 'once and for all' for any given ring R.

6. Concluding Remarks

In this paper we attacked the problem of computing the local densities for the orthogonal group of an arbitrary lattice over an arbitrary p-adic ring. In spite of the broad scope of the results, there are additional problems worthy of attack:

- One would like to expand the results to the more general setting of finding formulas for $\beta_{\mathfrak{p}}(L, M)$, especially for $\mathfrak{p}$ primes over 2.

- One would like to compute more explicitly the contribution of the structure of distinct Jordan decompositions to $\beta_{\mathfrak{p}}(L, L)$. Specifically, one expects that for unramified extensions of $\mathbb{Q}_2$ the formula should simplify greatly in much the way it does over $\mathbb{Q}_2$.

- One would like to have an effective way of describing the set of all possible Jordan decompositions for a given lattice over a 2-adic ring. Alternatively, one would like at least to have an effective algorithm for enumerating them.

Acknowledgements

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