Noname manuscript No.
(will be inserted by the editor)

Transfer and Local Density for Hermitian Lattices

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the date of receipt and acceptance should be inserted later

Abstract In this paper we study the integral structure of lattices over finite extensions of $\mathbb{Z}_p$ which arise from restriction or transfer from a lattice over a finite extension. We describe explicitly the structure of the resulting lattices. Special attention is given to the case of lattices whose quadratic forms arise from Hermitian forms. Then, in the case of Hermitian lattices where the final lattice is over $\mathbb{Z}_p$ we focus on the problem of computing the local densities.

Keywords lattices, transfer, restriction, local densities, representation densities

Mathematics Subject Classification (2000) Primary 11H56; Secondary 11H55

This article contains material originally appearing in my Ph.D. thesis [Fio13].

1 Introduction

Throughout this paper k shall be either a number field or one of its non-archimedean completions and $\mathcal{O}_k$ shall be its ring of integers. By a **Hermitian form** we mean a quadratic form of the shape:

$$q_{E,\lambda}(x) = \mathrm{Tr}_{E/k}(\lambda x \sigma(x)),$$

where E is an étale k-algebra with involution σ and λ is a unit of E^σ, the subalgebra of elements fixed by σ. By a **Hermitian lattice** we mean a fractional ideal Λ of $\mathcal{O}_E$ in E. This is a special case of what we shall call transfer, where a lattice on one ring, becomes a lattice over a subring by taking the trace of the original form.

Such Hermitian lattices are natural to study, the quadratic forms have a natural connection to special points on Shimura varieties (see [Fio12]). Understanding the structure of lattices that arise by this process is a natural first step in many computational problems, in particular that of computing local densities.

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The primary application we have in mind in the present work is for computing the arithmetic volumes of the orthogonal groups that arise from Hermitian lattices. These lattices arise in the study of special points on orthogonal Shimura varieties and these arithmetic volumes relate, by way of the Hirzebruch-Mumford-proportionality principle and the Riemann-Roch theorem (see [Mum77, GHS08]), to the dimensions of spaces of modular forms on the associated Shimura varieties.

Another important application is their use, by way of the Siegel mass formula, as part of a stopping condition when enumerating the genus of a lattice. This has important applications in the theory of algebraic automorphic forms on orthogonal groups (see [Gro99] and [GV12]). Formulas for representation densities have been worked out to cover many cases (see for example [Pal65, Wat76, Kit93, CS88, Shi99, Kat99, SH00, GY00, Yan04, Cho12, Fio14]). In the present work we develop formulas more tuned towards the input data of a Hermitian lattice rather than a general abstract lattice. The sections of this paper are organized as follows:

(2) We introduce the general theory of lattices so far as it is needed in the sequel.
(3) We discuss specifically lattices over p-adic rings.
(4) We introduce local densities and formulas for computing them.
(5) We obtain results about the structure of lattices under transfer.
(6) We develop formulas for the representation densities of Hermitian lattices in terms of the invariants of the fields involved.
(7) We discuss the concrete example of $\mathbb{Q}(\mu_p)$.

The main results of this paper are contained in Sections 5 and 6. The major new results of Section 5 are Theorems 31, 34, and 35 which give explicit information about Jordan decompositions for lattices that arise through transfer over p-adic rings. Prior to this work, most of the information about these lattices concerned invariants defined after base change to the field of fractions, such as the discriminant and Hasse invariant (see Theorems 27 - 30) and very little could be said about the lattice structures in the dyadic case.

The main result of Section 6 is Theorem 40 which gives an explicit formula for the local density of a Hermitian lattice in terms of natural invariants of the rings. Prior to this work, explicitly computing the local densities could only be done directly using general formulas expressed in terms of Jordan decompositions (see Theorem 17).

The results of Section 6 make heavy use of the results of Section 5 as well as the strategy for developing formulas for local densities as in Section 4.

2 General Notions of Lattices

In this section we introduce the general theory of lattices. Many good references exist which treat this topic in a varying degree of generality. See for example [Kit93] and [O'M00]. We shall eventually be most interested in the theory of lattices over $\mathcal{O}_k$, the maximal order in a number field k. Note that these are not always PIDs, however, their localizations always are.

Definition 1 *Let R be a Dedekind domain and K be its field of fractions. By a* ***lattice*** *Λ over R we mean a projective R-module of finite rank, together with a symmetric R-bilinear pairing:*

$$b_\Lambda : \Lambda \times \Lambda \to K,$$

which induces a duality $\mathrm{Hom}_R(\Lambda, K) \cong \Lambda \otimes_R K$. *A lattice is said to be* ***integral*** *if* $b_\Lambda(x, y) \in R$. *We will call the lattice* ***even*** *if* $b_\Lambda(x, x) \in 2R$ *and* ***odd*** *otherwise. We shall call an integral lattice* ***unimodular*** *if the pairing induces an isomorphism* $\mathrm{Hom}_R(\Lambda, R) \cong \Lambda$, *or more generally* $\mathfrak{a}$***-modular*** *if the pairing induces an isomorphism* $\mathrm{Hom}_R(\Lambda, R) \cong \mathfrak{a}^{-1}\Lambda$ *(for* $\mathfrak{a}$ *a projective* R*-module of rank* 1*, that is, an invertible fractional ideal of* R*). Notice that* $\mathfrak{a}$*-modular is equivalent to having* $\mathrm{Hom}_R(\Lambda, \mathfrak{a}) \cong \Lambda$ *by noting that:*

$$\mathrm{Hom}_R(\Lambda, \mathfrak{a}) \cong \mathfrak{a} \otimes_R \mathrm{Hom}_R(\Lambda, R) \cong \mathfrak{a} \otimes \mathfrak{a}^{-1}\Lambda \cong \Lambda.$$

We will refer to a lattice as ***modular*** *if there exists some* $\mathfrak{a}$ *for which it is* $\mathfrak{a}$*-modular. Note that not all lattices are modular.*

Remark 2 *By requiring* $\mathrm{Hom}_R(\Lambda, K) \cong \Lambda \otimes_R K$ *we are explicitly requiring that all lattices be non-degenerate with respect to the bilinear form* b_Λ*. If the pairing on the 'lattice' might not induce an isomorphism the 'lattice' shall be referred to as a module or submodule.*

Notation 3 *Given a lattice* Λ*, by* q_Λ *or simply* q *we shall always mean:*

$$q_\Lambda(x) = b_\Lambda(x, x).$$

To a lattice we may also associate another bilinear pairing:

$$B_\Lambda(x, y) := q_\Lambda(x + y) - q_\Lambda(x) - q_\Lambda(y).$$

Note well that $2q_\Lambda(x, x) = B_\Lambda(x, x)$ *and that* $q_\Lambda(x) = b_\Lambda(x, x)$ *as these conventions vary by author. Notice also that in characteristic* 2 *one may not recover* b_Λ *from* q_Λ *as this would involve dividing by* 2*.*

Remark 4 *For lattices* $L \oplus M$ *shall always mean an orthogonal direct sum, so that:*

$$b_{L\oplus M}(\ell_1 \oplus m_1, \ell_2 \oplus m_2) = b_L(\ell_1, \ell_2) + b_M(m_1, m_2).$$

For a lattice L *and an element* $r \in R$ *we shall denote by* rL *the lattice whose underlying module is* L *but whose bilinear form is* r *times that of* L*, that is,* $b_{rL} = rb_L$*.*

This level of generality is too much for many of our purposes. Having the following additional constraints gives major simplifications to the theory:

1. If Λ is free we may express $b_\Lambda(\cdot, \cdot)$ by a matrix.
2. If R is a principal ideal domain, the theory of modules simplifies. In particular, every lattice is free. We may often replace R by its (completed) localizations to attain this.
3. The theory is simpler if 2 is not a zero divisor in R.

Though some of the results which follow are true without the above constraints, for simplicity of presentation we will typically assume them. These assumptions hold when we work over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$ for all p, $\mathbb{F}_p$ where $p \neq 2$, or the many finite ring extensions of these. These assumptions may fail for Dedekind domains; however as our study of these is done almost entirely with their localizations this will not be an issue.

Definition 5 *Assume that Λ is free and let $X = \{x_1, \ldots, x_n\}$ be a basis for Λ. We write:*

$$A = A_X = (b_\Lambda(x_i, x_j))_{i,j}$$

for the matrix corresponding to this lattice and choice of basis.

Definition 6 *Given a lattice Λ we define the **dual** lattice to be:*

$$\Lambda^{\#} = \{x \in \Lambda \otimes K \mid b_\Lambda(x, y) \in R \ \text{ for all } y \in \Lambda\}$$

together with the induced pairing.

Definition 7 *A submodule $L \subset \Lambda$ is said to be **isotropic** if $b_\Lambda(\cdot,\cdot)|_L = 0$. It is said to be **anisotropic** if it has no non-zero isotropic submodules. A projective submodule is said to be **metabolic** if it has an isotropic submodule of half its rank. A projective submodule is said to be **hyperbolic** if it is generated by two isotropic submodules.*

Definition 8 *Lattices Λ have the following invariants:*

- *For Λ projective, the **rank** r_Λ of Λ as an R module.*
- *For Λ free, the **discriminant** $\delta_\Lambda = \det(A_X) \in K/(R^\times)^2$ for a choice of basis X. If Λ is not free we have at our disposal the discriminant $D(q)$ of $\Lambda \otimes K$ which is an element of $K/(K^\times)^2$, and the **discriminant ideal** which is the R ideal generated by $\det(A_X)$ running over all maximal linearly independent subsets X of Λ. Alternatively, for a projective module over a Dedekind domain, one may take the discriminant ideal to be the product of the local discriminant ideals.*
- *Supposing $\Lambda \otimes K$ is isomorphic to the diagonal form $(a_i)_i$ and denoting the Hilbert symbol by $(\cdot,\cdot)_K$, the **Hasse invariant** is*

  $$H(\Lambda) = H(q) = \prod_{i<j} (a_i, a_j)_K \in H^2(K, \{\pm 1\}).$$

  (See [Ser73, Ch. III] and [Ser79, Ch. XIV].)
- *For each embedding $R \hookrightarrow \mathbb{R}$ we have an associated **signature** (the dimension of any maximal isotropic $\mathbb{R}$-submodule of $\Lambda \otimes_R \mathbb{R}$).*
- *The **norm ideal** $\mathfrak{N}_\Lambda$ is the R-ideal generated by $\{b_\Lambda(x, x) \mid x \in \Lambda\}$.*
- *The **scale ideal** $\mathfrak{S}_\Lambda$ is the R-ideal generated by $\{b_\Lambda(x, y) \mid x, y \in \Lambda\}$. Note that $\mathfrak{N}_\Lambda \subset \mathfrak{S}_\Lambda$ and $2\mathfrak{S}_\Lambda \subset \mathfrak{N}_\Lambda$.*
- *The **norm group** $\mathfrak{n}_\Lambda$ is the group: $\{b_\Lambda(x, x) \mid x \in \Lambda\} + 2\mathfrak{S}_\Lambda$, it is an additive subgroup of K.*

Remark 9 *The extent to which these invariants determine a lattice depends largely on the setting. They are typically insufficient to characterize a lattice in the context in which we are working.*

3 Lattices over p-adic Rings

Here we enter into the improved setting of having R a (complete) local ring whose maximal ideal is principal, generated by π. More specifically we intend to work with a **p-adic ring**, by which we mean the maximal order of a p-adic field (a finite extension of $\mathbb{Q}_p$). We shall denote by $\nu = \nu_\pi$ the π-adic valuation.

In this context we have the following important results to recall:

Theorem 10 *A quadratic module over a p-adic field K is entirely determined by its rank, its discriminant and its Hasse invariant.*

See [O'M00, Thm 63:20].

Theorem 11 (Existence of Jordan decompositions) *Every lattice Λ over a p-adic ring R can be expressed as:*

$$\Lambda \simeq \bigoplus_i L_i,$$

*where the L_i are $\mathfrak{a}_i$-modular, with the $\mathfrak{a}_i$ distinct. Such a decomposition is called a **Jordan decomposition**. Note that such decompositions are not in general unique, but see Theorem 12.*

See [O'M00, 91C].

It should be emphasized before stating the following result that Jordan decompositions over 2-adic rings are not typically unique. For the special case of $\mathbb{Z}_2$, a complete description of the relations between different Jordan decompositions can be found in [Nik79]. We shall make some use of this in the sequel.

Theorem 12 (Uniqueness of Jordan decompositions) *Let $\Lambda = \bigoplus_{i=1}^{r_1} L_i = \bigoplus_{j=1}^{r_2} K_j$ be two Jordan decomposition of a lattice over a p-adic ring with L_i being $\mathfrak{a}_i$-modular and K_j being $\mathfrak{b}_j$-modular, $\mathfrak{a}_{i_1}|\mathfrak{a}_{i_2}$ for $i_1 < i_2$, and $\mathfrak{b}_{j_1}|\mathfrak{b}_{j_2}$ for $j_1 < j_2$. Then:*

1. $r_1 = r_2$,
2. $\mathfrak{a}_i = \mathfrak{b}_i$,
3. $\operatorname{rank} L_i = \operatorname{rank} K_i$,
4. $\mathfrak{N}_{L_i} = \mathfrak{a}_i$ *if and only if* $\mathfrak{N}_{K_i} = \mathfrak{a}_i$, *and*
5. *if* $p \neq 2$ *then* $L_i \simeq K_i$.

See [O'M00, 91:9].

Theorem 13 *For $p \neq 2$ and a p-adic ring R, the isomorphism classes of unimodular lattices Λ over R are classified by their rank and discriminant.*

See [O'M00, 92:1].

Theorem 14 *For $p = 2$ and a p-adic ring R, the isomorphism classes of unimodular lattices Λ over R are classified by their rank, discriminant, Hasse invariant and norm groups.*

See [O'M00, 93:16].

Remark 15 *Over $\mathbb{Z}_2$ specifying the norm group for a unimodular lattice is equivalent to specifying if the lattice is even or odd.*

4 Local Densities

In this section we focus our attention on what are called interchangeably representation densities, local densities or arithmetic volumes. Throughout this section we shall continue to assume that R is a p-adic ring, with maximal ideal $\mathfrak{p}$. We shall denote by π a uniformizer and $q = |R/\mathfrak{p}R|$ the size of the residue field, which is finite by assumption. We shall fix an additive Haar measure on R, normalized so that the volume of R is 1.

Local density gives a way of assigning volumes to the sets:

$$\mathrm{Isom}(\Lambda_1, \Lambda_2) = \{\phi \in \mathrm{Hom}_R(\Lambda_1, \Lambda_2) \mid b_{\Lambda_2}(\phi(x), \phi(y)) = b_{\Lambda_1}(x, y)\},$$

of isometric embeddings from Λ_1 to Λ_2. Such sets are typically infinite, so simply counting elements is insufficient.

There are many subtly different definitions whose values often differ only by constants. We shall use the following definition:

Definition 16 *Let L and M be lattices over a p-adic ring R, with bilinear forms b_L and b_M. Consider the map $\mathcal{F}_{b_L} : \mathrm{Hom}_R(M, L) \to \mathrm{Sym}^2(M^\vee)$ given by:*

$$(\mathcal{F}_{b_L}(\phi))(x, y) = b_L(\phi(x), \phi(y)).$$

Some references define the local density at R to be:

$$\alpha_R(b_M, b_L) = \alpha_R(M, L) = \frac{1}{2} \lim_{U \to b_M} \frac{\int_{\mathcal{F}_{b_L}^{-1}(U)} dX}{\int_U dT}.$$

Here $dX = \prod_{ij} dx_{ij}$ and $dT = \prod_{i \le j} dt_{ij}$ are the standard measures when viewing the spaces as matrix spaces with respect to some chosen basis. The limit is taken over the directed family of open subset U of $\mathrm{Sym}^2(M^\vee)$ containing b_M. By [Han05, Lemma 2.2] this does not depend on the choice of integral basis.

*We define the **local density** to be:*

$$\beta_R(M, L) = (q^{-\mathrm{rank}(M) v_\pi(2)}) \alpha_R(M, L).$$

When $R = \mathcal{O}_\mathfrak{p}$ one often denotes the local densities by $\beta_\mathfrak{p}$ rather than β_R.

Computing local densities is in general highly technical and the resulting formulas are quite complicated in the general case. The formulas we shall make use of are from Kitaoka.

Theorem 17 (Kitaoka) *Let L be a $\mathbb{Z}_p$-lattice. Let $L = \oplus_i L_i$, where the L_i are non-trivial p^{a_i}-modular lattices with distinct a_i. Let $L_i(e)$ be any maximal even dimensional unimodular even sublattice such that we may write $p^{-a_i} L_i = L_i(e) \oplus L_i(o)$. Define the following values:*

$$\begin{aligned} n_i &= \mathrm{rank}(L_i), \\ n_i(e) &= \mathrm{rank}(L_i(e)), \\ s &= \left|\{i \mid n_i \neq 0\}\right|, \\ w &= \sum_i a_i n_i \left((n_i + 1)/2 + \sum_{a_j > a_i} n_j \right), \\ \chi(L_i(e)) &= \begin{cases} 1 & L_i(e) \text{ is split} \\ -1 & \text{otherwise,} \end{cases} \end{aligned}$$

and set

$$\chi(i) = \begin{cases} 0 & n_i = 0 \\ 0 & p \neq 2 \text{ and } n_i \text{ odd} \\ 0 & p = 2 \text{ and one of } a_i - 1, a_i + 1 \text{ blocks is odd} \\ 0 & p = 2, L_i \text{ odd, } n_i \text{ even, and } D(L_i) \neq (-1)^{n_i/2} \pmod 4 \\ \chi(L_i(e)) & \text{otherwise.} \end{cases}$$

For $p \neq 2$ set $t = 0$ and $u = 0$, if $p = 2$ set:

$$t = \sum_i \begin{cases} 0 & L_i = 0 \text{ and } a_i - 1, a_i + 1 \text{ blocks are even} \\ -1 & L_i = 0, \text{ one of } a_i - 1, a_i + 1 \text{ blocks is odd} \\ 0 & L_i \neq 0 \text{ is even} \\ 0 & L_i \text{ is odd } a_i + 1 \text{ block is even} \\ 1 & L_i \text{ is odd } a_i + 1 \text{ block is odd,} \end{cases}$$

and

$$u = \sum_i \begin{cases} n_i & L_i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Finally set:

$$E_i = 1 + \chi(i) p^{-n_i(e)/2} \quad \text{and} \quad P(m) = \prod_{j=1}^{m} (1 - p^{-2j}).$$

Then we have the following formula for the local density:

$$\beta_p(L, L) = 2^{s-t} p^{w-u} \prod_i P\left(\left\lfloor \tfrac{n_i(e)}{2} \right\rfloor\right) E_i^{-1}.$$

Proof This is only a slight modification of [Kit93, Thm 5.6.3], we have adjusted the definition of E, introduced the value u and modified t accordingly. □

Remark 18 *The following corollaries are useful for explicitly computing local densities in special cases. They eliminate the need to explicitly find all the invariants of the Jordan blocks. The ideas used here shall be used extensively when we compute local densities in Section 6.*

The key observation that makes these corollaries possible is that the formula above for local density only depends on $L_i(e)$ if its isomorphism class does not depend on any choices. In particular the contribution that comes from the p^{a_i}-modular block cannot depend on the choice of Jordan decomposition, hence we only need to identify the isomorphism class of $L_i(e)$ for one choice of Jordan decomposition.

Corollary 19 *The local density of a unimodular lattice over $\mathbb{Z}_p$ with $p \neq 2$ is determined entirely by its rank and discriminant mod p.*

The local density of a unimodular lattice over $\mathbb{Z}_2$ is determined entirely by its rank, discriminant mod 4, Hasse invariant and norm group.

Proof Over $\mathbb{Z}_p$ in the non-dyadic case this information determines the lattice, hence the local density.

In the dyadic case, this follows by observing that we can compute χ as follows:

If $n - n(e) = 2$ and $D = (-1)^{n(e)/2} \pmod 4$ then by [Nik79] the isomorphism class of $L(e)$ is not well defined, and in Theorem 17 we have $\chi = 0$ in this case. Otherwise, by noting that the Hasse invariant of the odd part is trivial, we can easily compute the Hasse invariant of $L(e)$. Then, by observing that $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$ have different Hasse invariants we can distinguish the two cases using Hasse invariants. We conclude that if χ is not zero as above then χ is given by:

$$\chi = \begin{cases} (-1,-1)^{n(e)(n(e)-2)/8} H & n = n(e) \\ ((-1)^{n(e)/2}, (-1)^{n(e)/2} D)(-1,-1)^{n(e)(n(e)-2)/8} H & \text{otherwise.} \end{cases}$$

□

Corollary 20 *Suppose $p \neq 2$ and L_p is a $\mathbb{Z}_p$-lattice with exactly 2 Jordan blocks which are p^j, p^{j+1} modular and of dimension n_j, n_{j+1}, respectively. Then the Local density of L_p is determined entirely by the ranks of the blocks, and the discriminant D and Hasse invariant H of L_p.*

In particular the local density is:

$$4q^{j(n_j+n_{j+1})(n_j+n_{j+1}+1)/2+n_{j+1}(n_{j+1}+1)/2} \prod_{i=1}^{\lfloor n_j/2 \rfloor} (1-q^{-2i}) \prod_{i=1}^{\lfloor n_{j+1}/2 \rfloor} (1-q^{-2i})\xi,$$

where:

$$\xi = \begin{cases} (1+\chi(j)q^{n_j/2})^{-1}(1+\chi(j+1)q^{n_{j+1}/2})^{-1} & n_j, n_{j+1} \text{ even} \\ (1+\chi(j)q^{n_j/2})^{-1} & n_j \text{ even and } n_{j+1} \text{ odd} \\ (1+\chi(j+1)q^{n_{j+1}/2})^{-1} & n_j \text{ odd and } n_{j+1} \text{ even} \\ 1 & \text{otherwise.} \end{cases}$$

One can compute $\chi(i)$ as:

$$\chi(i) = \begin{cases} 0 & n_i \text{ odd} \\ (p,-1)_p^{(i+1)(n_j+n_{j+1})/2}(p,D)_p^{i+1} H & \text{both blocks even} \\ (p,-1)_p^{(i+1)(n_j+n_{j+1}-1)/2} H & \text{otherwise.} \end{cases}$$

Proof One only needs to check that the computations for $\chi(i)$ are accurate, otherwise this is simply evaluating the Theorem 17 in this case. Checking χ is simply a matter of computing the Hasse invariant for a diagonal form and its rescaling by p. Then by observing the dependence on the discriminant of each block in the various cases we may conclude the result. □

Corollary 21 *Suppose $p = 2$ and L_p is a $\mathbb{Z}_p$-lattice with exactly 2 Jordan blocks which are p^j, p^{j+1} modular and of dimension n_j, n_{j+1}, respectively. Then the Local density of L_p is determined entirely by the ranks and parities of the blocks and the discriminant and Hasse invariants of L_p. Note that a method for computing the local densities is made explicit in the proof.*

Proof We shall denote by D and H the discriminant and Hasse invariant of L_p and by D_i and H_i the discriminant and Hasse invariants of the ith modular block. Note that for a unimodular lattice over $\mathbb{Z}_2$ one can compute that $\chi = 0$ when $n - n(e) = 2$ and $D = (-1)^{n(e)/2} \pmod 4$ otherwise χ is given by:

$$\chi = \begin{cases} (-1,-1)^{n(e)(n(e)-2)/8} H & n = n(e) \\ ((-1)^{n(e)/2}, (-1)^{n(e)/2} D)_2 (-1,-1)_2^{n(e)(n(e)-2)/8} H & \text{otherwise.} \end{cases}$$

This is based on the observation that in the first case the isomorphism class is not well defined, and in the latter two cases the Hasse invariant of the odd part is trivial, hence we can easily compute the Hasse invariant of $L(e)$. Noting that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ have different Hasse invariants allows us to distinguish them in this way. We shall make implicit use of this in the following. Set:

$$w = j(n_j + n_{j+1})(n_j + n_j + 1)/2 + n_{j+1}(n_{j+1} + 1)/2.$$

There are 4 cases to consider depending on the parities of the blocks.

1. Both the p^j and p^{j+1} blocks are odd.
   By checking the cases as listed in [Nik79] one can confirm that in this case the lattice L_p has at least 4 (and potentially more) Jordan decompositions.
   Importantly, one can check that Kitaoka's formula (Theorem 17) depends only on the ranks of the Jordan blocks and not otherwise on the isomorphism class.
   In particular the local density is:
   $$2^{w+n+5} \prod_{i=1}^{n_j(e)/2} (1 - p^{-2i}) \prod_{i=1}^{n_{j+1}(e)/2} (1 - p^{-2i}).$$
2. The p^j block is odd and the p^{j+1} block is even.
   Again by checking the cases as listed in [Nik79] one can confirm that in this case L_p has 2 Jordan decompositions. One can check that Kitaoka's formula depends on the isomorphism class of the p^j, but importantly the result of the formula is independent of which Jordan decomposition we use.
   Without loss of generality we may use the Jordan decomposition where the p^{j+1} block is hyperbolic (which exists by [Nik79]). With this assumption the p^{j+1} block has determinant $(-1)^{n_{j+1}/2}$ and Hasse invariant $(-1,-1)^{\ell(\ell+2)/8}$. We can thus determine both the determinant and Hasse invariant of the p^j block. The determinant is $(-1)^{n_{j+1}/2} D$ and the Hasse invariant is:
   $$(-1,-1)_2^{n_{j+1}(n_{j+1}+2)/8 + n_{j+1}/2} (-1, D)_2^{n_{j+1}/2}.$$
   Consequently, computations as in Corollary 19 allow us to conclude that if $n_j - n_j(e) = 2$ and $D = (-1)^{(n_j(e)+n_{j+1})/2} \pmod 4$ then $\chi(j) = 0$ and that otherwise $\chi(j)$ is give by:
   $$(2, D)_2^{j(n_j+n_{j+1}-1)} (-1,-1)_2^{(n_{j+1}+n_j(e))(n_{j+1}+n_j(e)+2)/8} (D, -1)_2^{(n_{j+1}+n_j(e))/2} H.$$
   The local density can then be explicitly computed as:
   $$2^{w+n_j+3} (1 + \chi(j) p^{n_j(e)/2})^{-1} \prod_{i=1}^{n_j(e)/2} (1 - p^{-2i}) \prod_{i=1}^{n_{j+1}(e)/2} (1 - p^{-2i}).$$

3. The p^j block is even and the p^{j+1} block is odd.
   This case is symmetric to the above case.
   Without loss of generality we may use the Jordan decomposition where the p^{j+1} block is hyperbolic. Thus this block has determinant $(-1)^\ell$ and Hasse invariant $(-1)^{\ell/2}$. We can thus determine both the determinant and Hasse invariant of the p^{j+1} block. Consequently, computations as in Corollary 19 allow us to conclude that if $n_{j+1} - n_{j+1}(e) = 2$ and $D = (-1)^{(n_{j+1}(e)+n_j)/2}$ (mod 4) then $\chi(j+1) = 0$ and otherwise $\chi(j+1)$ is:

$$(2,D)_2^{(j+1)(n_j+n_{j+1}-1)}(-1,-1)_2^{(n_j+n_{j+1}(e))(n_j+n_{j+1}(e)+2)/8}(D,-1)_2^{(n_j+n_{j+1}(e))/2}H.$$

   The local density can then be explicitly computed as:

$$2^{w+n_{j+1}+3}(1+\chi(j+1)p^{n_{j+1}(e)/2})^{-1}\prod_{i=1}^{n_j(e)/2}(1-p^{-2i})\prod_{i=1}^{n_{j+1}(e)/2}(1-p^{-2i}).$$

4. Both the p^j and p^{j+1} blocks are even.
   In this case there is a unique Jordan decomposition. We also note that the discriminants of the unimodular blocks are $(-1)^{n_j/2}$ mod 4. As $\chi(i) = (2, D_i)_2$, the goal is to solve for $(2, D_i)_2$. We have that:

$$1 = (D_j, D_{j+1})_2, \text{ and } H_i = (2, D_i)_2(-1,-1)_2^{n_i(n_i+2)/8}.$$

   It follows that:

$$\begin{aligned} H &= H_j H_{j+1}(D_j, D_{j+1})_2(2, D_j)_2^{j+1}(2, D_{j+1})_2^j \\ &= (-1,-1)_2^{n(n+2)/8}(2, D_j)_2^{j+1}(2, D_{j+1})_2^j. \end{aligned}$$

   Thus we may solve:

$$\chi(i) = (-1,-1)_2^{n(n+2)/8}(2, D)_2^i H.$$

   Therefore the local density can be explicitly computed as:

$$2^{w+2}(1+\chi(j)p^{n_j(e)/2})^{-1}(1+\chi(j+1)p^{n_{j+1}(e)/2})^{-1}\prod_{i=1}^{n_j(e)/2}(1-p^{-2i})\prod_{i=1}^{n_{j+1}(e)/2}(1-p^{-2i}).$$

□

5 Transfer of Lattices

Suppose $R_1 \hookrightarrow R_2$ is an inclusion of commutative rings which gives R_2 the structure of a finitely presented projective R_1-module. The natural maps:

$$R_2 \longrightarrow \mathrm{Hom}_{R_1}(R_2, R_2) \xrightarrow{\sim} \mathrm{Hom}_{R_1}(R_2, R_1) \otimes_{R_1} R_2 \longrightarrow R_1$$

in the category of R_1-modules induce a map:

$$\mathrm{Tr}_{R_2/R_1} : R_2 \to R_1.$$

In this setting, given any quadratic module (L_{R_2}, q_{R_2}) over R_2, one can construct a quadratic module (L_{R_1}, q_{R_1}) over R_1 by viewing L_{R_2} as a module over R_1 and taking $q_{R_1}(x) = \mathrm{Tr}_{R_2/R_1}(q_{R_2}(x))$. We shall refer to this as **transfer**.

The purpose of this section is to study properties of this process over p-adic rings. We are particularly interested in the transfer of Hermitian lattices, that is, quadratic forms of the form:

$$q_{R_2}(x) = \tfrac{1}{2}\,\mathrm{Tr}_{R_3/R_2}(\lambda x\sigma(x)) = \lambda x\sigma(x),$$

where $x \in R_3$ a quadratic extension of R_2, σ the nontrivial automorphism of R_3/R_2, and λ is a unit in the fraction field of R_2. The subsections of this section are organized as follows:

(5.1) We give some basic results about trace forms for local fields.
(5.2) We compute invariants for the forms q_{R_1}.
(5.3) We describe Jordan decompositions when $p \neq 2$ for both unary and binary forms.
(5.4) We describe Jordan decompositions when $p = 2$ for both unary and binary forms.

In the Section 6 we shall use these results to compute local densities for Hermitian lattices over $\mathbb{Q}$.

5.1 Trace Forms for Local Fields

The next few lemmas are important for various computations.

Lemma 22 (Euler) *Let $L = F(z)$ be a finite separable extension of F of degree m with $f_z(x) \in \mathcal{O}_F[x]$ the minimal (monic) polynomial of z. We then have:*

$$\mathrm{Tr}_{L/F}\left(\frac{z^\ell}{f_z'(z)}\right) = \begin{cases} 1 & \ell = m-1 \\ 0 & 0 \le \ell < m-1. \end{cases}$$

See [Ser79, III.6 Lemma 2].

Lemma 23 *Let L/F be a totally ramified extension of local fields of degree m. Let $z = \pi_L$ be a uniformizer of $\mathcal{O}_L$ and $f_z(x)$ be the minimal (monic) polynomial of z. Then f_z is an Eisenstein polynomial and the collection $1, z, z^2, \ldots, z^{m-1}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_L$ and $N_{L/F}(z)$ is a uniformizer of F.*

See [Ser79, Prop I.6.18].

Lemma 24 *Let L/F be a totally ramified extension of local fields of degree m. Let $z = \pi_L$ be a uniformizer of $\mathcal{O}_L$ and $f_z(x)$ be the minimal (monic) polynomial of z. Then for $0 \le \ell \le m-1$ and k any integer, we have:*

$$\nu_F\left(\mathrm{Tr}_{L/F}\left(\frac{z^{km+\ell}}{f_z'(z)}\right)\right) \ge k.$$

Moreover, this is an equality if $\ell = m-1$.

Proof As $\pi_F = N_{L/F}(z)$ is a uniformizer of F we write $z^m = u\pi_F$. We see that:

$$\mathrm{Tr}_{L/F}\left(\frac{z^{km+\ell}}{f_z'(z)}\right) = \pi_F^k \,\mathrm{Tr}_{L/F}\left(\frac{u^k z^\ell}{f_z'(z)}\right).$$

As $u^k z^\ell \in \mathcal{O}_L$ write:

$$u^k z^\ell = \sum_{i=0}^{m-1} a_i z^i,$$

with $a_i \in \mathcal{O}_F$. Then:

$$\mathrm{Tr}_{L/F}\left(\frac{u^k z^\ell}{f_z'(z)}\right) = a_{m-1} \in \mathcal{O}_F.$$

The result follows immediately.

To show we have an equality if $\ell = m-1$ write:

$$u^k = \sum_{i=0}^{m-1} a_i z^i.$$

Then we compute that:

$$\mathrm{Tr}_{L/F}\left(\frac{u^k z^\ell}{f_z'(z)}\right) = \sum_{i=0}^{m-1} a_i \,\mathrm{Tr}_{L/F}\left(\frac{z^{m-1+i}}{f_z'(z)}\right) = a_0 \pmod{\pi_F}.$$

As $v_L(u) = 0$ it follows that $v_F(a_0) = 0$, which concludes the result. □

Example 25 *We have the following special cases of the above. Write the minimal monic polynomial f_z of z as*

$$f_z(X) = \sum_{i=0}^{m} a_i X^i.$$

Then:

$$\mathrm{Tr}_{L/F}\left(\frac{z^\ell}{f_z'(z)}\right) = \begin{cases} -a_{m-1} & \ell = m \\ a_{m-1}^2 - a_{m-2} & \ell = m+1 \\ 1/a_0 & \ell = -1 \\ a_1/a_0^2 & \ell = -2. \end{cases}$$

The results for other powers can also be computed directly from the coefficients.

The following Lemma is immediate.

Lemma 26 *Transfer commutes with orthogonal direct sums.*

5.2 Invariants of q_{R_1}

The most basic of questions is to understand the standard invariants of the quadratic modules which result from transfer.

Theorem 27 (Discriminants) *Let R_2/R_1 be an extension of p-adic rings or orders in number fields. Suppose L is an R_2-lattice (and hence also an R_1-lattice) which is free over R_2 with quadratic form q_{R_2}. Suppose that R_2 is free over R_1. Consider the form $q_{R_1}(y) = \mathrm{Tr}_{R_2/R_1}(q_{R_2}(y))$ as a quadratic form on L viewed as an R_1-lattice. Then:*

$$\delta_{q_{R_1}} = N_{R_2/R_1}(\delta_{q_{R_2}})\delta^n_{R_2/R_1},$$

where δ_{R_2/R_1} is the usual discriminant relative to the trace form.

Proof If q_{R_1} is diagonalizable then by multiplicitivity of determinants and norms we may reduce the problem to studying the unary case. In this setting we have the usual argument (see [Fio12, Lemma 3.1]). The argument works integrally. Note that in the argument cited one can use $\{z_\ell\}$, any basis for the ring of integers, and this basis need not be a power basis $\{z^\ell\}$.

More generally we need to work with lattices which may not be diagonalizable. Consider $L' \subset L$ a free diagonalizable lattice in the same quadratic module. There exists a basis for L and a matrix $M = \mathrm{diag}(a_1, \ldots, a_n)U$, where $a_i \in R_2^\times$ and U is an upper triangular unipotent matrix with respect to which $L' = ML$. The discriminant of L' differs from that of L by $\prod_i^n a_i^2$.

Fix a basis for R_2 over R_1. For $x \in R_2$ let (x) denote the matrix for x acting on R_2 as an R_1-module in this basis.

Passing to R_1 the matrix which realizes L' as a submodule can be taken to have a block decomposition $M' = \mathrm{diag}((a_1), \ldots, (a_n))U'$, where U' is the matrix whose blocks are (U_{ij}). The determinant of $(a_i) = N_{R_2/R_1}(a_i)$, and hence the determinant of this change of basis becomes the norm of the original change of basis. We thus relate $\delta_{L,q_{R_1}}$, $\delta_{L,q_{R_2}}$, $\delta_{L',q_{R_2}}$ and $\delta_{L',q_{R_1}}$ by

$$\begin{aligned}
\delta_{L,q_{R_1}} &= N_{R_2/R_1}\left(\prod_i a_i\right)\delta_{L',q_{R_1}} \\
&= N_{R_2/R_1}\left(\prod_i a_i\right) N_{R_2/R_1}\left(\delta_{L',q_{R_2}}\right)\delta^n_{R_2/R_1} \\
&= N_{R_2/R_1}\left(\delta_{L,q_{R_2}}\right)\delta^n_{R_2/R_1}.
\end{aligned}$$

The formula thus holds for L. □

Theorem 28 (Hasse Invariants) *Let R_2/R_1 be an extension of p-adic rings. Let L be an R_2-lattice of rank n with quadratic form q_{R_2}. Denote by*

$$Q_{R_2/R_1,\lambda}(x) = \mathrm{Tr}_{R_2/R_1}(\lambda x^2) \quad \textit{and by} \quad d = N_{R_2/R_1}(D(q_{R_2})).$$

We will consider the form:

$$q_{R_1} = \mathrm{Tr}_{R_2/R_1}(q_{R_2}).$$

Continue to denote $(\cdot,\cdot)_{R_1}$ the Hilbert symbol. We have the following results:

1. *The form* q_{R_1} *has Hasse invariant:*

$$H_{R_1}(q_{R_1}) = H_{R_1}(Q_{R_2/R_1,1})^{n+1} H_{R_1}(Q_{R_2/R_1,D(q_{R_2})})(\delta_{R_2/R_1}, d)_{R_1}^{n+1} H_{R_2}(q_{R_2}).$$

We view these all as being in the same cohomology group $H^2(K_1, \{\pm 1\})$ *by identifying the different groups with* $\{\pm 1\}$ *or equivalently via corestriction, which is injective for local fields.*

2. *If* $p \neq 2$ *and the extension* R_2/R_1 *is unramified, then:*

$$H_{R_1}(q_{R_1}) = H_{R_2}(q_{R_2})(\pi_{R_1}, (-1)^{n(n-1)/2}\delta_{R_2/R_1} d)_{R_1}^{v_{R_2}(D(q_{R_2}))}.$$

3. *Consider the case* $p \neq 2$, $u \in R_1^\times$ *and* R_2/R_1 *is totally ramified. Write* λ *as*

$$\lambda = \frac{\pi_{R_2}^k}{u f'(\pi_{R_2}) \pi_{R_2}^\ell},$$

where f *is the minimal polynomial of* π_{R_2}. *The form* $Q_{R_2/R_1,\lambda}$ *has Hasse invariant:*

$$H_{R_1}(Q_{R_2/R_1,\lambda}) = (\pi_{R_1}, u)_{R_1}^{n(n-\ell)} (\pi_{R_1}, -1)_{R_1}^{k(n^2(n-1)/2+\ell^2(1-n)) - \ell(n-\ell)(n-\ell-1)/2}.$$

4. *Suppose* $p = 2$ *and the extension is Galois. The form* $Q(x) = \mathrm{Tr}_{R_2/R_1}(x^2)$ *has Hasse invariant:*

$$H_{R_1}(Q) = \begin{cases} (-1,-1)^{(n^2-1)/8} & n = 1 \pmod 2 \\ (\delta_{R_2/R_1}, (-1)^{(n+2)/4})_{R_1} & n = 2 \pmod 4 \\ 1 & n = 0 \pmod 4 \text{ and } -1 \in R_2^2 \\ (-1,-1)_{R_1}(2, \delta_{R_2/R_1})_{R_1} & n = 4 \pmod 8), -1 \in N_{R_2/R_1}(R_2) \\ -(-1,-1)_{R_1}(2, \delta_{R_2/R_1})_{R_1} & n = 4 \pmod 8), -1 \notin N_{R_2/R_1}(R_2) \\ (2, \delta_{R_2/R_1})_{R_1} & \text{otherwise.} \end{cases}$$

The first and fourth statements are [Epk89, Lemma 1 and Theorem 1], respectively; the second and third are [Fio12, Lemma 4.1 and 4.3] , respectively.

Remark 29 *The above theorem fails to provide a complete description of how to compute Hasse invariants for certain dyadic fields. This is remedied for binary forms of the following special type.*

Theorem 30 (Invariants for Hermitian Lattices) *Suppose* R_3 *is a* p*-adic ring with an involution* σ. *Let* $z \in R_2 = R_3^\sigma$ *be such that* $\sqrt{z}$ *generates* $R_3[\frac{1}{p}]$ *as a* $R_1[\frac{1}{p}]$*-algebra (note that by [Fio12, Prop. 3.5] such a* z *exists). View* R_3 *as a binary* R_2*-lattice with quadratic form:*

$$q_{R_2}(x + y\sqrt{z}) = \lambda((x + y\sqrt{z})\sigma(x + y\sqrt{z})) = \lambda x^2 - z\lambda y^2$$

so that $D(q_{R_2}) = -z$ *and* $H(q_{R_2}) = (\lambda, z)$. *Let* f *be the minimal monic polynomial for* z *over* R_1 *and* $m = [R_2 : R_1]$. *Then:*

$$H(q_{R_1}) = \mathrm{Cor}_{R_2/R_1}((z, -\lambda f_z'(z))_{R_2}) \cdot (N_{R_2/R_1}(z), -1)_{R_1}^{m-1} \cdot (-1,-1)_{R_1}^{m(m-1)/2}.$$

See [Fio12, Theorem 3.8].

Theorem 31 (Modularity for Unramified Transfer) *Suppose that R_2/R_1 is an unramified extension of p-adic rings and that L is a π^r-modular R_2-lattice with quadratic form q_{R_2}. Then L is also π^r-modular as an R_1-lattice. Moreover, the valuation of the norm ideal $\mathfrak{N}_L$ and scale ideal $\mathfrak{S}_L$ are unchanged. In particular, Jordan decompositions are taken to Jordan decompositions.*

Proof It is clear that we have:

$$\mathfrak{N}_{L/R_1} = \mathrm{Tr}_{R_2/R_1}(\mathfrak{N}_{L/R_2}) \text{ and } \mathfrak{S}_{L/R_1} = \mathrm{Tr}_{R_2/R_1}(\mathfrak{S}_{L/R_2}).$$

Indeed, picking an element $x \in L$, where $\nu(q_{R_2}(x))$ is minimal write $q_{R_2}(x) = u\pi^t$ with π a uniformizer of R_1 and u a unit. Then $q_{R_1}(ax) = \pi^t \,\mathrm{Tr}_{R_2/R_1}(ua^2)$. For $p \neq 2$ the unimodularity $\mathrm{Tr}_{R_2/R_1}(ua^2)$ implies that there exists $a \in R_2$ for which this is a unit. For $p = 2$ notice that $a \mapsto \mathrm{Tr}_{R_2/R_1}(ua^2)$ is surjective on the residue field. The claim for $\mathfrak{N}_{L/R_1}$ follows immediately, the proof for $\mathfrak{S}_{L/R_1}$ is similar.

The question of π^r modularity now follows from the observation that L is π^r-modular if and only if $\mathfrak{S}_L = (\pi^r)$ and $\mathfrak{S}_{L^\#} = (\pi^{-r})$. □

With the above result in hand, we shall for the time being restrict to the case of totally ramified extensions.

5.3 Ramified Transfer Over Non-Dyadic p-adic Rings

The case of $p \neq 2$ is simpler for both unary and Hermitian forms. We thus present the results for this case separately. The important feature we will show is that in both the unary and binary cases we know that there are at most two Jordan blocks and that their modularity differs by a power of π_{R_1}. We may thus completely recover the invariants of the blocks as in Corollary 20.

Let R_2/R_1 be a totally ramified extension of p-adic rings of degree m. Let π_{R_2} be a uniformizer of R_2 and set $\pi_{R_1} = N_{R_2/R_1}(\pi_{R_2})$ to be a uniformizer of R_1. Let $f(X) = f_{\pi_{R_2}}(X)$ be the minimal monic polynomial of π_{R_2} over R_1. Suppose $u_1 \in R_1^\times$, $u_2 \in R_2^\times$, $v \in R_2^\times$, $0 \leq \ell \leq m$, $k \in \mathbb{Z}$, set $u = u_1u_2$ and set:

$$\lambda = \frac{\pi_{R_1}^k}{u_1u_2v^2\pi_{R_2}^\ell f'(\pi_{R_2})}.$$

We note that if the residue characteristic is not 2, then for any given λ in the fraction field of R_2 there exists (non-unique) corresponding values for u_1, v, ℓ, k with $u_2 = 1$. We shall thus assume in this section that the constant u_2, as introduced above, is 1.

Now denote by $q_{R_2}(x)$ the R_2-quadratic form on R_2 given by λx^2, and by $q_{R_1}(x)$ the R_1-quadratic form on R_2 given by $q_{R_1}(x) = \mathrm{Tr}_{R_2/R_1}(\lambda x^2)$. Consider:

$$M_1 = \mathrm{span}\{v, \ldots, v\pi_{R_2}^{\ell-1}\} \text{ and } M_2 = \mathrm{span}\{uv\pi_{R_2}^\ell, \ldots, uv\pi_{R_2}^{m-1}\}$$

as quadratic submodules of R_2. These submodules will play important roles in the construction of Jordan decompositions.

Theorem 32 (Non-Dyadic Ramified Transfer for Unary Forms) *With all the notation as above the orthogonal decomposition $R_2 = M_1 \oplus M_2$ is a Jordan decomposition with M_1 and M_2 being, respectively, $\pi_{R_1}^{k-1}$ and $\pi_{R_1}^{k}$ modular. Moreover, the discriminants of $\frac{1}{\pi_{R_1}^{k-1}} q_{R_1}|_{M_1}$ and $\frac{1}{\pi_{R_1}^{k}} q_{R_1}|_{M_2}$ are, respectively:*

$$D\left(\frac{1}{\pi_{R_1}^{k-1}} q_{R_1}|_{M_1}\right) = (-1)^{\ell(\ell+1)/2 - m\ell} u^{-\ell} \text{ and}$$
$$D\left(\frac{1}{\pi_{R_1}^{k}} q_{R_1}|_{M_2}\right) = (-1)^{(m-\ell)(m-\ell-1)/2} u^{m-\ell}.$$

See [Fio12, Lemma 4.3].

In addition to the above notation, suppose that R_3/R_2 is a quadratic extension with involution σ. Fix w a non-square element of $R_1^{\times}$. Writing $x = x_1 + x_2\sqrt{\delta_{R_3/R_2}}$ consider the quadratic form on R_3 given by:

$$q_{R_3/R_1}(x) = \tfrac{1}{2}\operatorname{Tr}_{R_3/R_1}(\lambda x \sigma(x)) \simeq \operatorname{Tr}_{R_2/R_1}(\lambda x_1^2) - \operatorname{Tr}_{R_2/R_1}(\lambda \delta_{R_3/R_2} x_2^2).$$

Then set $\lambda' = \lambda\delta_{R_3/R_2}$, $k' = k$, $u_2' = 1$ and choose u_1', v', ℓ' so that

$$\lambda' = \frac{\pi_{R_1}^{k}}{u' v'^2 \pi_{R_2}^{\ell'} f'(\pi_{R_2})}.$$

Let q_{R_1}', M_i' be defined similarly to q_{R_1}, M_i using λ' instead of λ so that

$$q_{R_3/R_1}(x) = q_{R_1}(x_1) - q_{R_1}'(x_2).$$

Now define $N_i = M_i \oplus -M_i'$ and $\widetilde{N_1} = \frac{1}{\pi_{R_1}^{k-1}} N_1$ and $\widetilde{N_2} = \frac{1}{\pi_{R_1}^{k}} N_2$ their unimodular rescalings.

Theorem 33 (Non-Dyadic Ramified Transfer for Hermitian Forms) *With all the notation as above the orthogonal decomposition $R_3 = N_1 \oplus N_2$ is a Jordan decomposition for R_3 with the form q_{R_3/R_1}. The sublattices N_1 and N_2 are, respectively, $\pi_{R_1}^{k-1}$ and $\pi_{R_1}^{k}$-modular. Moreover:*

- *If $\delta_{E/R_2} = w$ then $D(\widetilde{N_1}) = (-1)^{-\ell} w^{-\ell}$ and $D(\widetilde{N_2}) = (-1)^{\ell-m} w^{\ell-m}$.*
- *If $\delta_{E/R_2} = \pi_{R_2}$ then $D(\widetilde{N_1}) = (-1)^{m+1} u$ and $D(\widetilde{N_2}) = -u$.*
- *If $\delta_{E/R_2} = w\pi_{R_2}$ then $D(\widetilde{N_1}) = (-1)^{m-1} u w^{1-\ell}$ and $D(\widetilde{N_2}) = -uw^{\ell-m+1}$.*
- *If $\delta_{E/R_2} = 1$, then $D(\widetilde{N_1}) = (-1)^{-\ell}$ and $D(\widetilde{N_2}) = (-1)^{\ell-m}$.*

See [Fio12, Lemma 4.4].

5.4 Ramified Transfer Over Dyadic p-adic Rings

The case of $p = 2$ is more complex for a variety of reasons, the failure of diagonalizability being the most prominent. In this section we obtain results on Jordan decompositions similar to those of the previous section keeping track of the additional information about norm ideals. In order to account for non-diagonalizability, we must consider both unary and binary lattices separately.

We begin exactly as in the non-dyadic case. Let R_2/R_1 be a totally ramified extension of p-adic rings of degree m. Let π_{R_2} be a uniformizer of R_2 and set $\pi_{R_1} = N_{R_2/R_1}(\pi_{R_2})$ to be a uniformizer of R_1. Let $f(X) = f_{\pi_{R_2}}(X)$ be the minimal monic polynomial of π_{R_2} over R_1. Suppose $u_1 \in R_1^\times$, $u_2 \in R_2^\times$, $v \in R_2^\times$, $0 \le \ell \le m$, $k \in \mathbb{Z}$, set $u = u_1 u_2$ and set:

$$\lambda = \frac{\pi_{R_1}^k}{u_1 u_2 v^2 \pi_{R_2}^\ell f'(\pi_{R_2})}.$$

Now denote by $q_{R_2}(x)$ the R_2-quadratic form on R_2 given by λx^2, and by $q_{R_1}(x)$ the R_1-quadratic form on R_2 given by $q_{R_1}(x) = \mathrm{Tr}_{R_2/R_1}(\lambda x^2)$. Consider:

$$M_1 = \mathrm{span}\{v, \ldots, v\pi_{R_2}^{\ell-1}\} \text{ and } M_2 = \mathrm{span}\{uv\pi_{R_2}^{\ell}, \ldots, uv\pi_{R_2}^{m-1}\}$$

as quadratic submodules of R_2. Note, one key difference between the dyadic and non-dyadic cases is that we may no longer make the assumption that $u_2 = 1$. This is relevant in the following theorems.

Theorem 34 (Dyadic Ramified Transfer for Unary Forms) *With all the notation as above the orthogonal decomposition $R_2 = M_1 \oplus M_2$ is a Jordan decomposition with M_1 and M_2 being, respectively, $\pi_{R_1}^{k-1}$ and $\pi_{R_1}^k$-modular. They differ in modularity by a multiple of π_{R_1}, hence their discriminants may depend on the choice of Jordan decomposition. Set:*

$$\widetilde{M_1} = \frac{1}{\pi_{R_1}^{k-1}} M_1 \qquad \text{and} \qquad \widetilde{M_2} = \frac{1}{\pi_{R_1}^{k}} M_2.$$

We can in general only determine if $\mathfrak{N}_{\widetilde{M_i}}$ is R_1. We have the following cases:

- $\mathfrak{N}_{\widetilde{M_1}} \subset (\pi_{R_1})$ *if ℓ is even and* $u_2 \cong \frac{\pi_{R_1}}{\pi_{R_2}^m} \pmod{R_2^2 \pi_{R_2}^\ell}$. *Otherwise* $\mathfrak{N}_{\widetilde{M_1}} = R_1$.
- $\mathfrak{N}_{\widetilde{M_2}} \subset (\pi_{R_1})$ *if $m - \ell$ is even and* $u_2 \cong 1 \pmod{R^2 \pi_{R_2}^{m-\ell}}$. *Otherwise* $\mathfrak{N}_{\widetilde{M_2}} = R_1$.

Proof One easily checks by Lemma 22 that $M_1 \perp M_2$.

Next we consider the matrix for M_1, it is of the form $(a_{ij})_{i,j}$, where the a_{ij} satisfy:

1. $a_{i_1 j_1} = a_{i_2 j_2}$ whenever $i_1 + j_1 = i_2 + j_2$.
2. $v_{R_1}(a_{i,\ell-i}) = k - 1$.
3. $v_{R_1}(a_{i,j}) > k - 1$ whenever $i + j > \ell$.
4. If ℓ is even and $u_2 \cong \frac{\pi_{R_2}^m}{\pi_{R_1}} \pmod{R_2^2 \pi_{R_2}^\ell}$, then $v_{R_1}(a_{ii}) > k-1$ for all i. Otherwise there exists i with $\nu(a_{ii}) = k - 1$.

The first statement is immediate, the second and third follow from Lemma 24. The last statement is seen as follows. Firstly, the statement depends only on the square class of u_2. This is true even though modifying u_2 changes the basis as the conclusion about the norm groups we are making is independent of choice of Jordan decomposition (see Theorem 12). We may thus choose to write:

$$u_2 = 1 + c_1\pi_{R_2} + c_3\pi_{R_2}^3 + \cdots \pmod{\pi_{R_2}^\ell}$$

with $c_i \in R_1$. Now by taking $x = \pi_{R_2}^{(\ell-i)/2}$ and setting $\mathrm{Tr}_{R_2/R_1}(\lambda x^2) = 0 \pmod{\pi_{R_1}^k}$ we can solve for $c_i \mod \pi_{R_1}$ in terms of c_j with $j < i$ (the equations involve the coefficients of f but these are constant). Explicitly we are solving:

$$c_i = \pi_{R_1}(\mathrm{Tr}_{R_2/R_1}(\pi^{-1-i}) + \sum_{j<i} c_j \,\mathrm{Tr}_{R_2/R_1}(\pi^{j-i-1})) \pmod{\pi_{R_1}}.$$

Lemma 24 tells us that the right hand side makes sense. As this is solvable we conclude that up to squares there is a unique value of u_2 modulo $\pi_{R_2}^\ell$ which makes all values of the quadratic form be contained in $\pi_{R_1}R_1$. Observing that $u_2 = \pi_{R_2}/\pi_{R_1}^m$ does this allows us to conclude the result.

We now consider the matrix for M_2, it is of the form $(b_{ij})_{i,j}$, where the b_{ij} satisfy:

1. $b_{i_1j_1} = b_{i_2j_2}$ whenever $i_1 + j_1 = i_2 + j_2$.
2. $v_{R_1}(b_{i,m-\ell-i}) = k$.
3. $v_{R_1}(b_{i,j}) > k$ whenever $i + j > m - \ell$.
4. If $m - \ell$ is even and $u_2 \cong \frac{\pi_{R_2}^m}{\pi_{R_1}} \pmod{R^2\pi_{R_2}^{m-\ell}}$, then $v_{R_1}(b_{ii}) > k$ for all i. Otherwise there exists i with $v_{R_1}(b_{ii}) = k$.

The arguments are identical to those for M_1 except that 1 is the necessary congruence. □

Another difference in the dyadic case is the need to consider binary forms, in particular those that do not decompose as direct sums of unary forms. Taking λ and all the associated constants as above we consider L over R_2 of the form:

$$\lambda \begin{pmatrix} u_3\pi^a & 1 \\ 1 & u_4\pi^{a+b} \end{pmatrix} = \frac{\pi_{R_1}^k}{u_1u_2v^2} \begin{pmatrix} \frac{u_3}{f'(\pi_{R_2})\pi_{R_2}^{\ell-a}} & \frac{1}{f'(\pi_{R_2})\pi_{R_2}^{\ell}} \\ \frac{1}{f'(\pi_{R_2})\pi_{R_2}^{\ell}} & \frac{u_4}{f'(\pi_{R_2})\pi_{R_2}^{\ell-a-b}} \end{pmatrix}$$

with $a > 0$ and $b \geq 0$.

We use the basis:

$$\{v, \dots v\pi_{R_2}^{\ell-1}\}e_1 \cup \{vu\pi_{R_2}^{\ell}, \dots, vu\pi_{R_2}^{m-1}\}e_1,$$
$$\{v, \dots v\pi_{R_2}^{\ell-1}\}e_2 \cup \{vu\pi_{R_2}^{\ell}, \dots, vu\pi_{R_2}^{m-1}\}e_2,$$

where e_1, e_2 denote respectively the first and second coordinates of L.

Define the following quadratic submodules with the given basis:

$$M_1 = \{v, \dots v\pi_{R_2}^{\ell-1}\}e_1, \quad M_1' = \{v, \dots, v\pi_{R_2}^{\ell-1}\}e_2,$$
$$M_2 = \{vu\pi_{R_2}^{\ell}, \dots, vu\pi_{R_2}^{m-1}\}e_1, \quad M_2' = \{vu\pi_{R_2}^{\ell}, \dots, vu\pi_{R_2}^{m-1}\}e_1.$$

Also define $N_1 = M_1 + M_1'$ and $N_2 = M_2 + M_2'$. Note these are not orthogonal decompositions. We are considering the span of both in the ambient space. Note also that N_1 and N_2 need not be orthogonal complements.

Theorem 35 (Dyadic Ramified Transfer for Binary Forms) *With all the notation as above. The lattice R_1 with q_{R_1} has 2 Jordan blocks, $\tilde{N}_1$ and $\tilde{N}_2$ of modularities $\pi_{R_1}^{k-1}$ and $\pi_{R_1}^{k}$, respectively. They differ in modularity by a multiple of π_{R_1}. We can only in general determine if the norm ideals are R_1.*

- $\mathfrak{N}_{\widetilde{N_1}} \subset (\pi_{R_1})$ *if and only if* $\max(\ell - a, 0)$ *and* $\max(\ell - a - b, 0)$ *are even, and* $u_2u_3 \cong \pi_{R_1}/\pi_{R_2}^m \pmod{\pi_{R_2}^{\ell-a}}$ *and* $u_2u_4 \cong \pi_{R_1}/\pi_{R_2}^m \pmod{R_1^2\pi_{R_2}^{\ell-a-b}}$
- $\mathfrak{N}_{\widetilde{N_2}} \subset (\pi_{R_1})$ *if and only if* $\max(m-\ell-a, 0)$ *and* $\max(m-\ell-a-b, 0)$ *are even, and* $u_2u_3 \cong 1 \pmod{\pi_{R_2}^{m-\ell-a}}$ *and* $u_2u_4 \cong 1 \pmod{R_1^2\pi_{R_2}^{m-\ell-a-b}}$.

Note that $\tilde{N}_1$ and $\tilde{N}_2$ may or may not be simultaneously N_1 and N_2, see Remark 36.

Proof Viewing the underlying space under the basis M_1, M_1', M_2, M_2' as above the matrix for q_{R_1} is of the form:

$$\begin{pmatrix} A & B^t & D^t & 0 \\ B & C & 0 & E^t \\ D & 0 & F & G^t \\ 0 & E & G & H \end{pmatrix}.$$

The blocks (that is the submatrices $A, \ldots, H$) have the following properties:

1. A, B, C are ℓ by ℓ matrices and, F, G, H are $m-\ell$ by $m-\ell$ matrices.
2. For all the blocks we have $*_{i_1j_1} = *_{i_2j_2}$ whenever $i_1+j_1 = i_2+j_2$. In particular, the square blocks are symmetric.
3. $\nu(*_{ij}) \geq k-1$ for all blocks and all i, j. Furthermore,
$$\begin{aligned} &\nu(A_{ij}) > k-1 \text{ for } i+j > \ell - a, \\ &\nu(B_{ij}) > k-1 \text{ for } i+j > \ell, \\ &\nu(B_{ij}) = k-1 \text{ for } i+j = \ell, \\ &\nu(C_{ij}) > k-1 \text{ for } i+j > \ell - a - b, \\ &\nu(D_{ij}), \nu(E_{ij}) > k-1 \text{ for all } i, j, \\ &\nu(F_{ij}) > k \text{ for } i+j > m-\ell-a, \\ &\nu(G_{ij}) > k \text{ for } i+j > m-\ell, \\ &\nu(G_{ij}) = k \text{ for } i+j = m-\ell, \text{ and} \\ &\nu(H_{ij}) > k \text{ for } i+j > m-\ell-a-b. \end{aligned}$$
4. The discriminant of $\frac{1}{\pi_{R_1}^{k-1}}\left(\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right)$ and the discriminant of $\frac{1}{\pi_{R_1}^{k-1}}\left(\begin{smallmatrix} F & G^t \\ G & H \end{smallmatrix}\right)$ are units mod π_{R_1}.
5. There are changes of basis which realize both N_1 and N_2 as Jordan blocks (though not simultaneously).
Hence the questions of whether the norm ideals of the rescaled Jordan blocks are contained in R_1 are determined by $\frac{1}{\pi_{R_1}^{k-1}}\left(\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right)$ and $\frac{1}{\pi_{R_1}^{k}}\left(\begin{smallmatrix} F & G^t \\ G & H \end{smallmatrix}\right)$.
6. The lattice N_1 is odd unless $\max(\ell-a, 0)$ and $\max(\ell-a-b, 0)$ are even, and $u_2u_3 \cong \pi_{R_1}/\pi_{R_2}^m \pmod{\pi^{\ell-a}}$ and $u_2u_4 \cong \pi_{R_1}/\pi_{R_2}^m \pmod{R_1^2\pi^{\ell-a-b}}$
7. The lattice N_2 is odd unless $\max(m-\ell-a, 0)$ and $\max(m-\ell-a-b, 0)$ are even, and $u_2u_3 \cong 1 \pmod{\pi^{m-\ell-a}}$ and $u_2u_4 \cong 1 \pmod{R_1^2\pi^{m-\ell-a-b}}$.

Points (1) and (2) are direct checks. Point (3) uses Lemma 24. Point (4) is elementary yet tedious to check. First observe that since modulo π_{R_1} the matrix $\frac{1}{\pi_{R_1}^{k-1}}\left(\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right)$ is of the form:

$$\left(\begin{smallmatrix} * & * & u \\ * & X & 0 \\ u & 0 & 0 \end{smallmatrix}\right),$$

where X is a $2\ell-2$ by $2\ell-2$ block, it has determinant $-u^2\det(X)$. We may iterate this procedure on X until X is of the form:

$$\begin{pmatrix} \tilde{A} & \tilde{B}^t \\ \tilde{B} & \tilde{C} \end{pmatrix}$$

with $\tilde{A}, \tilde{B}, \tilde{C}$ being $\ell-a-b$ by $\ell-a-b$ blocks. We may iterate until X has additional non-zero entries on the bottom row and rightmost column. Now use the fact that:

$$\det\left(\begin{smallmatrix} \tilde{A} & \tilde{B}^t \\ \tilde{B} & \tilde{C} \end{smallmatrix}\right) = \det(\tilde{C})\det(\tilde{A} - \tilde{B}^t\tilde{C}^{-1}\tilde{B}),$$

combined with the observation that:

$$\tilde{A} - (\tilde{B}^t\tilde{C}^{-1}\tilde{B})_{ij} \in \begin{cases} \pi_{R_1}R_1 & i+j > \ell-a-b \\ R_1^* & i+j = \ell-a-b \end{cases}$$

to conclude the result. We may perform an analogous argument for $\left(\begin{smallmatrix} F & G^t \\ G & H \end{smallmatrix}\right)$.

For point (5) notice that the change of bases needed are, respectively:

$$\begin{pmatrix} \mathrm{Id} - \left(\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} D^t & 0 \\ 0 & E^t \end{smallmatrix}\right) \\ 0 \qquad\qquad \mathrm{Id} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathrm{Id} & 0 \\ -\left(\begin{smallmatrix} F & G^t \\ G & H \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}\right) & \mathrm{Id} \end{pmatrix}.$$

The matrices $\left(\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right)^{-1}$ and $\left(\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}\right)\left(\begin{smallmatrix} F & G^t \\ G & H \end{smallmatrix}\right)^{-1}$ are integral by points (3) and (4). One sees that orthogonal complements of N_2 and N_1 are preserved, respectively, modulo $\pi_{R_1}^{k-1}$ and $\pi_{R_1}^{k}$. Hence they are modular and we indeed have a Jordan decomposition.

The arguments for (6) and (7) are analogous to that of the previous lemma. Indeed, one has norm ideal R_1 if and only if the diagonal contains a unit. Hence the problem reduces to considering the blocks on the diagonal, and we are reduced to the situation of the previous lemma, (except that we have now two different subblocks to check for each Jordan decomposition). □

Remark 36 *Note that though N_1 and N_2 are Jordan blocks for some Jordan decompositions, it is not necessarily true that the space for q_{R_1} is isomorphic to $N_1 \oplus N_2$ as N_1 and N_2 may not be Jordan blocks in the same decomposition. In the above theorem one can take either $\tilde{N}_1 = N_1$ or $\tilde{N}_2 = N_2$, though not necessarily both at the same time.*

We now move to the special case of forms which arise from Hermitian forms. We quickly review the possible quadratic extensions R_3/R_2 of a 2-adic ring. On the level of their fields of fractions they are of the form $K(\sqrt{z})$. We therefore look at the various cases for z before describing the resulting lattices in each case.

- $z = a\pi_{R_2}$ for $a \in R_2^\times$.
  Then the extension is ramified, has uniformizer $\sqrt{a\pi_{R_2}}$, the discriminant is $\delta_{R_3/R_2} = 4a\pi_{R_2}$, and the ring of integers has integral basis:
  $$1, \sqrt{a\pi_{R_2}}.$$
  In this basis the Hermitian form $q_{R_2} = \frac{1}{2}\operatorname{Tr}_{R_3/R_2}(\lambda x\sigma(x))$ has matrix:
  $$\lambda \begin{pmatrix} 1 & 0 \\ 0 & -a\pi_{R_2} \end{pmatrix}.$$
  In this case $k = \left\lceil \frac{v_{R_2}(2\lambda f'(\pi_{R_2}))+1}{2m} \right\rceil$ and $\ell = -(v_{R_2}(\lambda f'(\pi_{R_2})) - mk)$.
- $z = 1 + a\pi_{R_2}^{2r+1}$ for $0 \le r < v_{\pi_{R_2}}(2)$ and $a \in R^\times$.
  Then the extension is ramified, has uniformizer $\frac{1+\sqrt{1+a\pi_{R_2}^{2r+1}}}{\pi_{R_2}^r}$, the discriminant is $\delta_{R_3/R_2} = \frac{4}{\pi_{R_2}^{2r}}(1 + a\pi_{R_2}^{2r+1})$, and the ring of integers has integral basis:
  $$1, \frac{1+\sqrt{1+a\pi_{R_2}^{2r+1}}}{\pi_{R_2}^r}.$$
  In this basis the Hermitian form $q_{R_2} = \frac{1}{2}\operatorname{Tr}_{R_2(R_3/R_2}(\lambda x\sigma(x))$ has matrix:
  $$\lambda \frac{1}{\pi_{R_2}^r} \begin{pmatrix} \pi_{R_2}^r & 1 \\ 1 & -a\pi_{R_2}^{r+1} \end{pmatrix}.$$
  In this case $k = \left\lceil \frac{v_{R_2}(\lambda f'(\pi_{R_2}))-r}{m} \right\rceil$ and $\ell = -(v_{R_2}(\lambda f'(\pi_{R_2})) - r - mk)$.
- $z = 1 + b\pi^{2r}$ for $r = v_{\pi_{R_2}}(2)$ and $x^2 + \frac{2}{\pi_{R_2}^r}x - b$ irreducible mod π_{R_2} .
  Then the extension is unramified, has uniformizer π_{R_2}, the discriminant is $\delta_{R_3/R_2} = (1 + b\pi^{2r})$, and the ring of integers has integral basis
  $$1, \frac{1+\sqrt{1+b\pi_{R_2}^{2r}}}{\pi_{R_2}^r}.$$
  In this basis the Hermitian form $q_{R_2} = \frac{1}{2}\operatorname{Tr}_{R_3/R_2}(\lambda x\sigma(x))$ has matrix:
  $$\lambda \frac{1}{\pi_{R_2}^r} \begin{pmatrix} \pi_{R_2}^r & 1 \\ 1 & -b\pi_{R_2}^{r} \end{pmatrix}.$$
  In this case $k = \left\lceil \frac{v_{R_2}(\lambda f'(\pi_{R_2}))-r}{m} \right\rceil$ and $\ell = -(v_{R_2}(\lambda f'(\pi_{R_2})) - r - mk)$.

Corollary 37 (Dyadic Ramified Transfer for Hermitian Lattices) *Let R_3 be the maximal order of $R_2[\frac{1}{2}](\sqrt{z})$, R_2 and R_1 being as above. Let*

$$\lambda = \frac{\pi_{R_1}^k}{u_1 u_2 v^2 f'(\pi_{R_2})\pi_{R_2}^\ell}$$

with $u_1 \in R_1^\times$, $u_2, v \in R_2^\times$, $0 \le \ell \le m$, $k \in \mathbb{Z}$ *and where* $f(X)$ *is the minimal monic polynomial of* π_{R_2}*. Consider the Hermitian forms*

$$q_{R_2}(x) = \tfrac{1}{2}\operatorname{Tr}_{R_3/R_2}(\lambda x \sigma(x)) \quad \textit{and} \quad q_{R_1}(x) = \operatorname{Tr}_{R_2/R_1}(q_{R_2}(x)).$$

The form q_{R_1} *has two Jordan blocks* $\tilde{N}_1$ *and* $\tilde{N}_2$*, they are* $\pi_{R_1}^{k-1}$ *and* $\pi_{R_2}^{k}$*-modular, respectively. Moreover, we have:*

- *If* $z = a\pi_{R_2}$ *then the blocks are of dimension* $2\ell - 1$ *and* $2(m-\ell)+1$*, respectively. Both blocks have* $\mathfrak{N} = R_1$*.*
- *If* $z = (1 + a\pi_{R_2}^{2r+1})$ *then the blocks are of dimension* 2ℓ *and* $2(m-\ell)$*, respectively. The block* $\tilde{N}_1$ *has* $\mathfrak{N}_{\tilde{N}_1} = R_1$ *if and only if* $r < \ell$ *whereas* $\tilde{N}_2$ *has* $\mathfrak{N}_{\tilde{N}_2} = R_1$ *if and only if* $r < m - \ell$*.*
- *If* $z = (1 + b\pi^{2r})$ *then the blocks are of dimension* 2ℓ *and* $2(m-\ell)$*, respectively. Both blocks always satisfy* $\mathfrak{N} \subset (\pi_{R_1})$*.*

Proof The result is immediate by the proceeding discussion and Theorem 35. □

Remark 38 *As in Theorem 35 we do not give an explicit Jordan decomposition, we only prove one exists with the given properties. The blocks* $\tilde{N}_1$ *and* $\tilde{N}_2$ *are again both Jordan blocks in some decomposition, but not necessarily in the same decomposition.*

6 Computing Local Densities For Hermitian Forms over $\mathbb{Q}$

We now have all the tools in hand to carry out the task of computing the local densities for Hermitian lattices over $\mathbb{Q}$. This is what we shall do in this section.

The idea is as follows: given the ring of integers $\mathcal{O}$ of some étale algebra E over $\mathbb{Q}$, we wish to understand the local densities for the form

$$q(x) = \tfrac{1}{2}\operatorname{Tr}_{E/\mathbb{Q}}(\lambda x \sigma(x)),$$

where $\lambda \in E^\times$. For each prime p of $\mathbb{Q}$ we may write $E_p = \oplus_{\mathfrak{p}|p} E_{\mathfrak{p}}$, where the sum is over maximal ideals $\mathfrak{p}$ for the maximal order of E^σ. The first step is thus to understand the Jordan decompositions of the forms

$$q_{\mathfrak{p}}(x) = \tfrac{1}{2}\operatorname{Tr}_{E_{\mathfrak{p}}/\mathbb{Q}_p}(\lambda_{\mathfrak{p}} x \sigma_{\mathfrak{p}}(x)).$$

We may then combine these to understand the Jordan decompositions of the orthogonal direct sum $q_p = \oplus_{\mathfrak{p}} q_{\mathfrak{p}}$ with sufficient precision to compute the local density. We remind the reader of the ideas in the proof of Corollary 21. One of the keys of computing local densities is that whenever the invariants of a Jordan block are needed, they do not depend on the choice of Jordan decomposition. As such, we can allow ourselves to compute the invariants of a Jordan block for any representative Jordan decomposition. In the following we will be computing what we call **valid** invariants for each block. These represent an invariant that occurs for some choice of Jordan decomposition, not necessarily all choices.

6.1 Explicit Computations of Invariants For Jordan Blocks

Fix $\mathfrak{p}|p$ a maximal ideal dividing p in the maximal order of E^σ. Set R_3 be the maximal order of $E_\mathfrak{p}$, R_2 the maximal order of $E_\mathfrak{p}^\sigma$ and $R_1 = \mathbb{Z}_p$. Let $e_\mathfrak{p}$ and $f_\mathfrak{p}$ be, respectively, the ramification and inertial degrees of R_2 over R_1. Let $n_\mathfrak{p} = 2m_\mathfrak{p} = [R_3 : R_1]$. We shall denote by $\mathcal{D}_{R_i/R_j}$ the different ideal of R_i over R_j. Now set:

$$\delta_\mathfrak{p} = (-1)^{[R_2:R_1]} N_{R_2/R_1}(\tfrac{1}{4}\lambda^2 \mathcal{D}^2_{R_2/R_1}\delta_{R_3/R_2}).$$

This is the discriminant of the quadratic form q (see Theorem 27). Next, set:

$$H_\mathfrak{p} = \mathrm{Cor}_{R_2/R_1}((z, -\lambda f_z'(z))_{R_2})(N_{R_2/R_1}(z), -1)_{R_1}^{m_\mathfrak{p}-1}(-1,-1)_{R_1}^{m_\mathfrak{p}(m_\mathfrak{p}-1)/2},$$

where $\sqrt{z}$ primitively generates the fraction field of R_3 over that of R_1. This is the Hasse invariant (see Theorem 30).

Set $k_\mathfrak{p} = \left\lceil \frac{v_{R_1}(\delta_\mathfrak{p})}{n_\mathfrak{p}} \right\rceil$. The $k_\mathfrak{p}$ and $k_\mathfrak{p}-1$-modular blocks are those which may be non-trivial. Set:

$$n_{\mathfrak{p},i} = \begin{cases} n_\mathfrak{p} - v_{R_1}(\delta_\mathfrak{p}) \pmod{n_\mathfrak{p}}^* & i = k_\mathfrak{p} - 1 \\ v_{R_1}(\delta_\mathfrak{p}) \pmod{n_\mathfrak{p}}^* & i = k_\mathfrak{p} \\ 0 & \text{otherwise.} \end{cases}$$

These are the dimensions of the $k_\mathfrak{p}$ and $k_\mathfrak{p}-1$-modular blocks. The value of $k_\mathfrak{p}$ and the dimensions are clear by considering the discriminant and the fact that there are at most two non-trivial Jordan blocks. Note that for the $i = k_\mathfrak{p}$ case use $n_\mathfrak{p}$ as the representative for 0, for the $i = k_\mathfrak{p} - 1$ case use 0 (so that if there is only one non-trivial block it is the $k_\mathfrak{p}$-modular block).

Set $\ell_\mathfrak{p} = v_{R_2}(\lambda) + v_{R_2}(\mathcal{D}_{R_2/R_1}) + v_{R_2}(\delta_{R_3/R_2})/2 \pmod{e_\mathfrak{p}}$ (a representative between 0 and $e_\mathfrak{p}$). Then define:

$$\chi_{\mathfrak{p},i}(o) = \begin{cases} 0 & p = 2,\ i = k_\mathfrak{p}-1, k_\mathfrak{p} \text{ and } v_{R_2}(\delta_{R_3/R_2}) \text{ is odd} \\ 0 & p = 2,\ i = k_\mathfrak{p}-1 \text{ and } \ell_\mathfrak{p} < v_{R_2}(\delta_{R_3/R_2})/2 \\ 0 & p = 2,\ i = k_\mathfrak{p} \text{ and } e_\mathfrak{p} - \ell_\mathfrak{p} < v_{R_2}(\delta_{R_3/R_2})/2 \\ 1 & \text{otherwise.} \end{cases}$$

This value is 1 if $\mathfrak{N}_i \subset 2\mathfrak{S}_i$, and 0 otherwise. This follows immediately from the criterion for evenness of Corollary 37.

Set $n_{\mathfrak{p},i}(e) = 2\left\lfloor \frac{n_{\mathfrak{p},i}-1+\chi_{\mathfrak{p},i}(o)}{2} \right\rfloor$. This represents the dimension of the maximal even dimensional unimodular sublattice with $\mathfrak{N} \subset (2)$. Then:

$$\chi_{\mathfrak{p},i}(e) = \begin{cases} (p,-1)_p^{(i+1)m_\mathfrak{p}}(\delta_\mathfrak{p},p)_p^{i+1}(p,-1)_p^{n_{\mathfrak{p},i}/2} H_\mathfrak{p} & n_{\mathfrak{p},i} \neq 0 \text{ even}, p \neq 2 \\ (p,-1)_p^{im_\mathfrak{p}}(\delta_\mathfrak{p},p)_p^{i}(p,-1)_p^{(n_\mathfrak{p}+2)/2} H_\mathfrak{p} & n_{\mathfrak{p},i} \neq 0 \text{ odd}, p \neq 2 \\ (\delta_\mathfrak{p},2)_2^i(-1,-1)_2^{(n_\mathfrak{p}^2-2n_\mathfrak{p})/8} H_\mathfrak{p} & n_{\mathfrak{p},i}(e) = n_{\mathfrak{p},i} \neq 0, p = 2 \\ 1 & \text{otherwise.} \end{cases}.$$

The above calculations combine those of Corollaries 20 and 21. We note that for $p = 2$, it computes this accurately whenever it is well defined. If it is not well defined then the result is still valid for some Jordan decomposition.

Let u be a non-square in $R_1^\times$. For $p=2$ set $u=3$. Define:

$$\delta_{\mathfrak{p},i} = \begin{cases} 1 & (\chi_{\mathfrak{p},i}(o)=0 \text{ and } n_{\mathfrak{p},i} \text{ odd) or } n_{\mathfrak{p},i}=0 \\ (-1)^{n_\mathfrak{p}-n_{\mathfrak{p},i}/2}\delta_\mathfrak{p} & \chi_{\mathfrak{p},i}(o)=0, n_{\mathfrak{p},i} \text{ even} \\ (-1)^{\lfloor n_{\mathfrak{p},i}/2\rfloor} u^{(\chi_{\mathfrak{p},i}(e)-1)/2} & \text{otherwise.} \end{cases}$$

This represents a valid discriminant for the ith modular Jordan block. For $p=2$ the value is typically accurate mod 8. If $p=2, n_{\mathfrak{p},i}=1, m_\mathfrak{p}=1$ it is only accurate mod 4 but this case does not impact the following computations. The first two cases compute the discriminant when this block is odd. It does so assuming the complementary block is hyperbolic, since if this block were odd, there exists a Jordan decomposition with hyperbolic complementary block. In the final case we reverse engineer the discriminant based on whether or not the block is split using the computation of $\chi_{\mathfrak{p},i}(e)$ above.

We now set:

$$H_{\mathfrak{p},i} = \begin{cases} 1 & p \neq 2 \\ 1 & n_{\mathfrak{p},i}=1 \\ (-1,-1)_2^{(n_p-n_{p,i})(n_p-n_{p,i}-2)/8}(\delta_{\mathfrak{p},i},-1)_2^{m_\mathfrak{p}-n_{\mathfrak{p},i}/2}(\delta_\mathfrak{p},2)_2^i H_\mathfrak{p} & \text{otherwise.} \end{cases}$$

This represents a valid Hasse invariant for the ith modular block. The computations here are the same as those in Corollary 21. We compute it assuming the complementary block is even. If it is not, then the Hasse invariant of the ith block depends on a choice. Hence the result above is still valid.

Now we set:

$$\chi_{\mathfrak{p},i} = \begin{cases} 0 & n_{\mathfrak{p},i} \text{ is odd} \\ 0 & \chi_{\mathfrak{p},i-1}(o)\chi_{\mathfrak{p},i+1}(o)=0 \\ 0 & \chi_{\mathfrak{p},i}(o)=0 \text{ and } \delta_{\mathfrak{p},i}=(-1)^{(n_{\mathfrak{p},i}-1)/2} \pmod 4 \\ ((-1)^{n_{\mathfrak{p},i}/2}\delta_{\mathfrak{p},i},p)_p & p\neq 2, \\ (-1,-1)_2^{n_{\mathfrak{p},i}(n_{\mathfrak{p},i}-2)/8} H_{\mathfrak{p},i} & p=2. \end{cases}$$

This value is 0 precisely when the isomorphism class of $L_i(e)$ is not well-defined. The computation is based on those in the proof of Corollary 19.

6.2 Explicit Formulas for Local Densities

We now exploit the explicit computations of the previous section to give explicit formulas for the local densities in the various cases of interest.

The key terms which appear in the formulas are:

$$\begin{aligned}
t_{\mathfrak{p}} &= \sum_i (1-\chi_{\mathfrak{p},i}(o))n_{\mathfrak{p},i} + (1-\chi_{\mathfrak{p},i}(o))(1-\chi_{\mathfrak{p},i+1}(o)) - \\
&\qquad \sum_i \delta_{n_{\mathfrak{p},i},0}(1-\chi_{\mathfrak{p},i-1}(o)\chi_{\mathfrak{p},i+1}(o)), \\
s_{\mathfrak{p}} &= \left|\{i \mid n_{\mathfrak{p},i} \neq 0\}\right|, \\
w_{\mathfrak{p}} &= (k-1)[R_3:R_1]([R_3:R_1]+1)/2 + n_k(n_k+1)/2, \\
P_{\mathfrak{p},i} &= \prod_{j=1}^{\frac{n_{\mathfrak{p},i}(e)}{2}} (1-q^{-2j}), \\
E_{\mathfrak{p},i} &= (1+\chi_{\mathfrak{p},i}q^{-n_{\mathfrak{p},i}(e)/2})^{-1}, \\
P_{\mathfrak{p}} &= \prod_i P_{\mathfrak{p},i}, \\
E_{\mathfrak{p}} &= \prod_i E_{\mathfrak{p},i}^{-1}.
\end{aligned}$$

Theorem 39 *Let $R_1 = \mathbb{Z}_p$ and R_3 be the ring of integers of a p-adic field with involution σ and maximal ideal $\mathfrak{p}$. Suppose $\lambda \in (R_3^\sigma)^\times$. Consider the lattice $L = R_3$ with the form: $\frac{1}{2}\operatorname{Tr}_{R_3/R_1}(\lambda x \sigma(x))$. Using all the notation as above, we have:*

$$\beta_p(L,L) = 2^{s_{\mathfrak{p}}-t_{\mathfrak{p}}} q^{w_{\mathfrak{p}}} P_{\mathfrak{p}} E_{\mathfrak{p}}.$$

Proof The result follows immediately from Theorem 17 and all the computations of the relevant terms. □

We now combine what we know about the quadratic forms $q_{\mathfrak{p}}$ to get sufficient information about the form q_p to compute its local densities. We define the relevant constants in terms of the decomposed ones:

$$\begin{aligned}
n_{p,i} &= \sum_{\mathfrak{p}|p} n_{\mathfrak{p},i}, \\
\delta_{p,i} &= \prod_{\mathfrak{p}|p} \delta_{\mathfrak{p},i}, \\
\chi_{p,i}(o) &= \prod_{\mathfrak{p}|p} \chi_{\mathfrak{p},i}(o), \\
n_{p,i}(e) &= 2\left\lfloor \frac{n_i+1-\chi_{p,i}(o)}{2} \right\rfloor, \text{ and} \\
H_{p,i} &= \prod_{\mathfrak{p}|p} H_{\mathfrak{p},i} \prod_{\mathfrak{p}<\mathfrak{q}} (\delta_{\mathfrak{p},i},\delta_{\mathfrak{q},i}).
\end{aligned}$$

The above formulas are all direct computations. Now we set:

$$\chi_{\mathfrak{p},i} = \begin{cases} 0 & n_{\mathfrak{p},i} \text{ is odd} \\ 0 & \chi_{\mathfrak{p},i-1}(o)\chi_{\mathfrak{p},i+1}(o) = 0 \\ 0 & \chi_{\mathfrak{p},i}(o) = 0 \text{ and } \delta_{\mathfrak{p},i} = (-1)^{(n_{\mathfrak{p},i}-1)/2} \pmod 4 \\ ((-1)^{n_{\mathfrak{p},i}/2}\delta_{\mathfrak{p},i}, p)_p & p \neq 2, \\ (-1,-1)_2^{n_{\mathfrak{p},i}(n_{\mathfrak{p},i}-2)/8} H_{\mathfrak{p},i} & p = 2. \end{cases}$$

As before, this formula is based on the computations of Corollary 19. We may now introduce the terms which will appear in the formulas:

$$t_p = \sum_i (1-\chi_{p,i}(o))n_{p,i} + (1-\chi_{p,i}(o))(1-\chi_{p,i+1}(o)) - \sum_i \delta_{n_{p,i},0}(1-\chi_{p,i-1}(o)\chi_{p,i+1}(o)),$$

$$s_p = \left|\{i \mid n_{p,i} \neq 0\}\right|,$$

$$w_p = \sum_i i n_{p,i}((n_{p,i}+1)/2 + \sum_{j>i} n_{p,j}),$$

$$P_{p,i} = \prod_{j=1}^{\frac{n_{p,i}(e)}{2}} (1-q^{-2j}),$$

$$E_{p,i} = (1+\chi_{p,i}q^{-n_{p,i}(e)/2}).$$

Finally, define:

$$P_p = \prod_i P_{p,i} \quad \text{and} \quad E_p = \prod_i E_{p,i}^{-1}.$$

Theorem 40 *Let $\mathcal{O}_E$ be the ring of integers of a number field with involution. Using all the notation above the p-adic local density of the form $\frac{1}{2}\operatorname{Tr}_{E/\mathbb{Q}}(\lambda x\sigma(x))$ on $\mathcal{O}_E$ is:*

$$\beta_p(L,L) = 2^{s_p-t_p}q^{w_p}P_pE_p.$$

Proof Again, the result follows immediately from Theorem 17 and the above computations of the relevant terms. □

The above formula is complicated. This is largely by virtue of the fact that each $\mathfrak{p}|p$ could contribute to different Jordan blocks, and hence we must independently compute the invariants for each. One can thus in general expect no reasonable cancellation in the above formulas as there are cases where none occurs. The advantage of this formula over those of the Section 4 is that the formula is expressed entirely in terms of the invariants of the rings involved (and λ) and thus given a ring which one understands, one can compute this formula.

We now present a restricted case, that is, we shall suppose that $\lambda_{\mathfrak{p}}$ has small valuation for all $\mathfrak{p}$ so that $k=1$ and the final lattice has at most 2 Jordan blocks at each p. In particular assume that $0 \leq v_{\mathfrak{p}}(\lambda/2) + v_{\mathfrak{p}}(\delta_{\mathcal{O}_E/\mathcal{O}_{E^\sigma}})/2 + v_{\mathfrak{p}}(\mathcal{D}_{\mathcal{O}_{E^\sigma}/\mathbb{Z}}) \leq e_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of E^σ.

Under these assumptions we have:

- The dimension of the space is $n = 2m = [E:\mathbb{Q}]$.
- The dimensions of the Jordan blocks are:

$$n_{p,0} = n - v_p(N(\lambda/2)^2\delta_{E/\mathbb{Q}_p}) \quad \text{and} \quad n_{p,1} = v_p(N(\lambda/2)^2\delta_{E/\mathbb{Q}_p}).$$

- The conditions for the blocks to be odd are:
  - $\chi_{p,0}(o) = 0$ if and only if either $v_{\mathfrak{p}}(\delta_{E/E^\sigma})$ is odd or $e_{\mathfrak{p}} > v_{\mathfrak{p}}(\lambda) + v_{\mathfrak{p}}(\mathcal{D}_{E^\sigma/\mathbb{Q}})$ for some $\mathfrak{p}$.
  - $\chi_{p,1}(o) = 0$ if and only if either $v_{\mathfrak{p}}(\delta_{E/E^\sigma})$ is odd or $v_{\mathfrak{p}}(\delta_{E/E^\sigma}) > 2e_{\mathfrak{p}} - v_{\mathfrak{p}}(\lambda) - v_{\mathfrak{p}}(\mathcal{D}_{E^\sigma/\mathbb{Q}})$ for some $\mathfrak{p}$.

- As before one computes $n_{p,i}(e) = 2\left\lfloor \frac{n_i+1-\chi_{p,i}(o)}{2} \right\rfloor$.
- We have the following formula for $\chi_{p,i}$:

$$\chi_{p,i} = \begin{cases} 0 & n_i = 0 \text{ or } n_i \text{ odd} \\ 0 & p=2, \chi_{\mathfrak{p},i-1}\chi_{\mathfrak{p},i+1} = 0 \\ 0 & p=2, \delta_{\mathfrak{p}} = (-1)^{m-1} \pmod 4 \\ \mathrm{Cor}_{E_p^\sigma/\mathbb{Q}_p}((z,(-1)^m p^{i+1}\lambda f_z'(z))_{E_p^\sigma}) & p \neq 2, n_i \text{ even} \\ \mathrm{Cor}_{E_p^\sigma/\mathbb{Q}_p}((z,(-1)^m 2^i \lambda f_z'(z))_{E_p^\sigma}) & \text{otherwise,} \end{cases}$$

where $\sqrt{z}$ primitively generates the E over $\mathbb{Q}_p$.

Remark 41 *Notice that for all primes which are unramified in E and for which $v_p(N(\lambda)) = 0$ (or for $p = 2$ take $\lambda = 2$) the above formula for $\chi_{p,i}$ reduces to $((-1)^m D, p)_p$. The lack of symmetry at 2 is a consequence of our normalization of the form. The normalization we have chosen makes the Hasse invariant formula cleaner, but breaks the symmetry in this formula.*

Now set:

$$t_p = \begin{cases} (1-\chi_{p,0}(o))(n_{p,0}-1) + (1-\chi_{p,1}(o))(n_{p,1}-1) + \\ \qquad (1-\chi_{p,0}(o))(1-\chi_{p,1}(o)) & n_{p,0}n_{p,1} \neq 0 \\ (1-\chi_{p,0}(o))(n_{p,0}-2) + (1-\chi_{p,1}(o))(n_{p,1}-2) & \text{otherwise,} \end{cases}$$

$$s_p = \left|\{i \mid n_{p,i} \neq 0\}\right|, \text{ and}$$

$$w_p = n_{p,1}(n_{p,1}+1)/2.$$

Corollary 42 *Let $E/\mathbb{Q}$ be a finite extension with involution σ, supposing E is primitively generated by $\sqrt{z}$ over $\mathbb{Q}$ with $z \in E^\sigma$. Let $\lambda \in (E^\sigma)^\times$ with:*

$$0 \leq v_{\mathfrak{p}}(\lambda/2) + v_{\mathfrak{p}}(\delta_{\mathcal{O}_E/\mathcal{O}_{E^\sigma}})/2 + v_{\mathfrak{p}}(\delta_{\mathcal{O}_{E^\sigma}/\mathbb{Z}}) \leq e_{\mathfrak{p}},$$

for all primes $\mathfrak{p}$ of E^σ. Then with notation as above the local density of the form $\frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \sigma(x))$ is:

$$2^{s_p - t_p} q^w \prod_{j=1}^{\frac{n_{p,0}(e)}{2}} (1-q^{-2j}) \prod_{j=1}^{\frac{n_{p,1}(e)}{2}} (1-q^{-2j})(1+\chi_{p,0} q^{-n_{p,0}(e)/2})^{-1}(1+\chi_{p,1} q^{-n_{p,1}(e)/2})^{-1}.$$

Proof Once again this is an immediate application of Theorem 17 together with the above computations of the relevant terms. □

7 Example of $\mathbb{Q}(\mu_p)$

Fix a prime p of $\mathbb{Z}$. In this section we shall compute the local densities for the form

$$q_{E,\lambda} = \tfrac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(\lambda x \sigma(x)),$$

where $E = \mathbb{Q}(\mu_p)$ is the cyclotomic field of pth roots of unity, σ is complex conjugation, and λ is restricted in valuation so that:

$$0 \leq v_{\mathfrak{q}}(\lambda/2) + v_{\mathfrak{q}}(\delta_{\mathcal{O}_E/\mathcal{O}_{E^\sigma}})/2 + v_{\mathfrak{q}}(\mathcal{D}_{\mathcal{O}_{E^\sigma}/\mathbb{Z}}) \leq e_{\mathfrak{q}}$$

for all $\mathfrak{q}$.

We shall use the following 'elementary' facts.

- The ring of integers of E is $\mathcal{O}_E = \mathbb{Z}[\zeta_p^a]$ for each $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.
- The ring of integers of $F := E^\sigma$ is:
$$\mathcal{O}_F = \mathbb{Z}[\zeta_p + \zeta_p^{-1}] = \mathbb{Z}[(\zeta_p - \zeta_p^{-1})^2] = \mathbb{Z}[(\zeta_p^a - \zeta_p^{-a})^2]$$
for each $a \in (\mathbb{Z}/p\mathbb{Z})^\times$.
Denote by $z_a = (\zeta_p^a - \zeta_p^{-a})^2$ then z_a is totally negative and $E = \mathbb{Q}(\sqrt{z_a})$. Denote by f_z the minimal polynomial of z_a (this does not depend on a).
- There is a unique prime in each of $\mathcal{O}_E$ and $\mathcal{O}_F$ over p. Denote by $\mathfrak{p}$ the prime over p in $\mathcal{O}_F$.
- The discriminant of $E/\mathbb{Q}$ is $\delta_{E/\mathbb{Q}} = (-1)^{(p-1)/2}p^{p-2}$.
- Since $\zeta_p^2 \not\equiv 1 \pmod{\mathfrak{q}}$ for all $\mathfrak{q} \nmid p$ it follows that $\zeta_p^a - \zeta_p^{-a}$ and hence $(\zeta_p^a - \zeta_p^{-a})^2$ is a unit away from p.
- Since the different ideal is $\mathcal{D}_{F/\mathbb{Q}} = (f_z'(z_a))$ it follows that $f_z'(z_a)$ is a unit at all places away from p.
- The elements $\zeta_p^a - \zeta_p^{-a}$ and $(\zeta_p^a - \zeta_p^{-a})^2$ are uniformizing elements at p for the fields they generate.
This follows from the observation that the order $\mathbb{Z}[\sqrt{z_a}] = \mathcal{O}_F[\sqrt{z_a}]$ is maximal away from 2.
- The ramification degrees are $e_\ell = \begin{cases} p-1 & \ell = p \\ 1 & \text{otherwise} \end{cases}$.

In the formulas of the previous section we have the following:

- The dimension of the space is $[E : \mathbb{Q}] = p - 1$.
- The dimensions of the Jordan blocks are for $\ell \neq p$ are:
$$n_{\ell,0} = p - 1 - 2\nu_\ell(N_{F/\mathbb{Q}}(\lambda/2)) \text{ and } n_{\ell,1} = 2\nu_\ell(N_{F/\mathbb{Q}}(\lambda/2))$$
and for $\ell = p$ they are:
$$n_{\ell,0} = 1 - 2\nu_p(N_{F/\mathbb{Q}}(\lambda)) \text{ and } n_{\ell,1} = p - 2 + 2\nu_p(N_{F/\mathbb{Q}}(\lambda)).$$
Thus we set:
$$w_\ell = n_{\ell,1}(n_{\ell,1}+1)/2 \text{ and}$$
$$s_\ell = \begin{cases} 1 & \ell \neq p, \nu_\ell(N_{F/\mathbb{Q}}(\lambda)) = 0, \pm(p-1)/2 \\ 2 & \text{otherwise.} \end{cases}$$
- The parity of the Jordan blocks at 2 are:
$$\chi_{2,i}(o) = 1$$
so long as the blocks are non-trivial. This is true because the extension is unramified at 2. Consequently, $t_\ell = 0$ for all ℓ.
- The character for the blocks are computed as follows:
$$\chi_{\ell,i} = \begin{cases} 0 & \ell = p \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a, (-1)^{(p-1)/2}2^i\lambda f_z'(z_a))\right)_\ell & \ell = 2 \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a, (-1)^{(p-1)/2}\ell^{i+1}\lambda f_z'(z_a))\right)_\ell & \ell \neq 2, p. \end{cases}$$

We are thus interested in computing:

$$\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,\lambda)\right)\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a))\right).$$

For all $\ell\neq 2,p$ we have that z_a and $f_z'(z_a)$ are units and thus:

$$\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a))\right)_\ell=1.$$

For $\ell=2$ we have that:

$$\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a))\right)_2\cdot(-1)^{(p-1)(p-3)/8}$$

computes the Hasse invariant of the form (for $\lambda=1$). Since this Hasse invariant is 1 for all places (including infinite) other than p we can conclude that:

$$\begin{aligned}\mathrm{Cor}_{F/\mathbb{Q}}&((z_a,(-1)^{(p-1)/2}f_z'(z_a)))_2\\&=(-1)^{(p-1)(p-3)/8}\,\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a))\right)_p\end{aligned}$$

We are thus reduced to computing $\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a)\right)_p$. Observe that:

$$\begin{aligned}\left(z_a,(-1)^{(p-1)/2}f_z'(z_a)\right)_\mathfrak{p}&=\left(z_a,-z_a^{-1}\right)_\mathfrak{p}^{(p-3)/2}\left(z_a,(-1)^{(p-1)/2}f_z'(z_a)\right)_\mathfrak{p}\\&=(z_a,-1)_\mathfrak{p}\left(z_a,z^{-(p-3)/2}f_z'(z_a)\right)_\mathfrak{p}\\&=(z_a,-1)_\mathfrak{p}\prod_{a\neq b\in(\mathbb{Z}/p\mathbb{Z})^\times/\pm1}\left(z_a,1-\tfrac{z_b}{z_a}\right)_\mathfrak{p}.\end{aligned}$$

Now, we may use that z_a is a uniformizer and that:

$$\frac{z_b}{z_a}\cong\frac{a^2}{b^2}\pmod{z_a}.$$

It follows that the terms we wish to evaluate are actually:

$$\left(z_a,1-\tfrac{z_b}{z_a}\right)_\mathfrak{p}=\left(z_a,1-\tfrac{b^2}{a^2}\right)_\mathfrak{p}=\left(z_a,1-\tfrac{b}{a}\right)_\mathfrak{p}\left(z_a,1+\tfrac{b}{a}\right)_\mathfrak{p}.$$

The resulting expression now becomes:

$$\begin{aligned}\left(z_a,(-1)^{(p-1)/2}f_z'(z_a)\right)_\mathfrak{p}&=(z_a,-1)_\mathfrak{p}\prod_{\pm a\neq b\in(\mathbb{Z}/p\mathbb{Z})^\times}\left(z_a,1-\tfrac{b}{a}\right)_\mathfrak{p}\\&=(z_a,-2)_\mathfrak{p}.\end{aligned}$$

Applying the Corestriction map we have:

$$\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,-2)_\mathfrak{p}\right)=\left(N_{F/\mathbb{Q}}(z_a),-2\right)_p=\left((-1)^{(p-1)/2}p,-2\right)_p=(p,-2)_p.$$

From this we can conclude that:

$$\mathrm{Cor}_{F/\mathbb{Q}}\left((z_a,(-1)^{(p-1)/2}f_z'(z_a))\right)_2=(-1)^{(p-1)(p-3)/8}(p,-2)_p=1.$$

Now, for all $\ell \neq p$ we find:

$$\mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a,\ell)\right)_\ell = \left(\frac{(-1)^{(p-1)/2}p}{\ell}\right) = \left(\frac{\ell}{p}\right).$$

Thus we can conclude that:

$$\chi_{\ell,i} = \begin{cases} 0 & \ell = p \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a,\lambda)\right)_\ell & \ell = 2, i = 0 \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a,\lambda)\right)_\ell \left(\frac{\ell}{p}\right) & \ell = 2, i = 1 \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a,\lambda)\right)_\ell & \ell \neq p, i = 1 \\ \mathrm{Cor}_{E^\sigma/\mathbb{Q}}\left((z_a,\lambda)\right)_\ell \left(\frac{\ell}{p}\right) & \ell \neq p, i = 0. \end{cases}$$

7.1 Explicit Formulas for the Examples

Combining all of the above we can easily compute the product over all local densities for the following cases:

- Case $\lambda = 2$, the local density is:

$$2p^{(p-2)(p-1)/2}(1-p^{p-1})\prod_\ell\left(\left(1+\left(\frac{\ell}{p}\right)\ell^{(p-1)/2}\right)\prod_{i=1}^{(p-1)/2}(1-\ell^{-2i})^{-1}\right).$$

- Case $\lambda = 2\mu$, where $\mu \in \mathcal{O}_F^\times$ has a unique negative embedding and $(z_a,\mu)_\mathfrak{p} = -1$, the local density is:

$$2p^{(p-2)(p-1)/2}(1-p^{p-1})\prod_\ell\left(\left(1+\left(\frac{\ell}{p}\right)\ell^{(p-1)/2}\right)\prod_{i=1}^{(p-1)/2}(1-\ell^{-2i})^{-1}\right).$$

- Case $\lambda = 2\mu$, where $\mu \in \mathcal{O}_F^\times$ has a unique negative embedding and $(z_a,\mu)_\mathfrak{p} = 1$, the local density is:

$$2p^{(p-2)(p-1)/2}(1-p^{p-1})\frac{1-\left(\frac{2}{p}\right)2^{(p-1)/2}}{1+\left(\frac{2}{p}\right)2^{(p-1)/2}}$$
$$\prod_\ell\left(\left(1+\left(\frac{\ell}{p}\right)\ell^{(p-1)/2}\right)\prod_{i=1}^{(p-1)/2}(1-\ell^{-2i})^{-1}\right).$$

- Case $\lambda = 2\mathfrak{q}$, where $(\mathfrak{q})|q \neq p$ is prime and $(q,p)_\mathfrak{p} = -1$, set $n_q = \nu_q(N_{F/\mathbb{Q}}(\mathfrak{q}))$ and suppose $(\mathfrak{q}) \neq (q)$ and $\mathfrak{q}$ is totally positive, the local density is:

$$2^2p^{(p-2)(p-1)/2}q^{n_q(n_q-1)/2}\left(1-q^{n_q}\right)\left(1+q^{(p-1)/2-n_q}\right)\prod_{i=1}^{n_q}\left(1-q^{2i}\right)^{-1}$$
$$\prod_{i=1}^{p-1-n_q}\left(1-q^{2i}\right)^{-1}\prod_{\ell\neq q}\left(\left(1+\left(\frac{\ell}{p}\right)\ell^{(p-1)/2}\right)\prod_{i=1}^{(p-1)/2}(1-\ell^{-2i})^{-1}\right).$$

- Case $\lambda = 2\mathfrak{q}$, where $\mathfrak{q}|q \neq p, 2$ is prime and $(q,p)_\mathfrak{p} = 1$, set $n_q = \nu_q(N_{F/\mathbb{Q}}(\mathfrak{q}))$ and suppose $(\mathfrak{q}) \neq (q)$ and $\mathfrak{q}$ is totally positive, the local density is:

$$2^2 p^{(p-2)(p-1)/2} q^{n_q(n_q-1)/2} (1+q^{n_q}) \left(1+q^{(p-1)/2-n_q}\right) \prod_{i=1}^{n_q} \left(1-q^{2i}\right)^{-1} \prod_{i=1}^{p-1-n_q} \left(1-q^{2i}\right)^{-1} \frac{1-\left(\frac{2}{p}\right)2^{(p-1)/2}}{1+\left(\frac{2}{p}\right)2^{(p-1)/2}} \prod_{\ell \neq q} \left(\left(1+\left(\frac{\ell}{p}\right)\ell^{(p-1)/2}\right) \prod_{i=1}^{(p-1)/2} (1-\ell^{-2i})^{-1} \right).$$

Other more complicated combinations can be handled similarly.

8 Concluding Remarks

We have been able to accomplish two important goals:

1. Describe the structure of lattices that arise from transfer, in particular for Hermitian lattices.
2. Give a method for computing the local densities for Hermitian lattices over $\mathbb{Z}$ in terms of invariants of the rings.

In spite of this many interesting problems remain open:

- More refined computation of invariants for transfer of lattices over 2-adic rings. In particular, a complete description of the resulting norm group.
- Obtaining more general formulas for $\beta_p(L, M)$ where the lattices L and or M are Hermitian lattices.
- Attempting to expand the results to finite extensions of $\mathbb{Z}$, this requires more work on several fronts, especially the problem of more general explicit formulas for local densities.

Acknowledgments

I would like to thank Victoria de Quehen, Juan Restrepo and my Ph.D. Supervisor Prof. Eyal Goren for their help in editing various drafts of this paper. I would like to thank NSERC and the ISM for their financial support while doing this research. Finally, I would like to thank the various reviewers for their many helpful comments.

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