Representation Densities

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Notation

- Let k be a number field.
- Let $\mathcal{O}$ be the ring of integers of k.
- Let $\mathfrak{p}$ denote a fixed prime of $\mathcal{O}$.
- We will denote by $q = p^n = |\mathcal{O}/\mathfrak{p}\mathcal{O}|$.

Definition

By a *lattice* L of rank ℓ over $\mathcal{O}$ we mean a projective $\mathcal{O}$-module of rank ℓ which is equipped with a (non-degenerate) k-valued bilinear form b_L.

We shall denote by $\mathcal{O}_\mathfrak{p}$, $L_\mathfrak{p}$ the completed localizations of the objects at the prime $\mathfrak{p}$. We shall denote by π a uniformizer of $\mathfrak{p}$. L and M shall always denote $\mathcal{O}$ lattices of ranks ℓ, and m.

The Definition

For L and M two $\mathcal{O}_{\mathfrak{p}}$-lattices the *representation density* $\beta_{\mathfrak{p}}(L, M)$ is:

$$\operatorname*{Lim}_{r\to\infty} q^{\frac{r\ell(\ell+1-2m)}{2}} \left|\{\phi : L \to M/\pi^r \mid Q_M(\phi(x)) = Q_L(x)\ (2\pi^r)\}\right|$$

This can be reinterpreted as the volume of the set of isometries from $L_{\mathfrak{p}}$ to $M_{\mathfrak{p}}$ with respect to a natural metric.
For L and M two $\mathcal{O}$-lattices the *arithmetic volume* (of the isometries between them) is defined (roughly) to be:

$$\mathrm{Vol}_{arith}(L, M) \doteq \prod_{\mathfrak{p}} \beta_{\mathfrak{p}}(L_{\mathfrak{p}}, M_{\mathfrak{p}}).$$

(Note that there are some normalization ambiguities that can arise, which I ignore for the purpose of this talk)

Siegel-Minkowski Theorem

Arithmetic volumes give information about the collection of lattices which are locally isomorphic to a given lattice.

Theorem (Siegel-Minkowski)

Let L be a totally definite $\mathcal{O}$-lattice, then:

$$\sum_{L_i \in \text{gen}(L)} \frac{1}{|\text{Aut}(L_i)|} = \frac{1}{\text{Vol}_{arith}(L, L)}.$$

Among other things, this theorem provides for an efficient way to check that you have finished enumerating the genus of a definite lattices.

Generalizations using $\text{Vol}_{arith}(L, M)$ (Siegel-Weil Formula) gives a relationship between Fourier coefficients of Siegel-Eisenstein and Siegel-theta series.

An alternative generalization, by way of computing the Tamagawa volumes is the following:

Theorem (Volumes of Locally Symmetric Spaces)

Let L be an $\mathcal{O}$-lattice, then:

$$\mathrm{Vol}(\mathrm{O}(L)\backslash \mathrm{O}(\mathbb{R})/K) \doteq \frac{2}{\mathrm{Vol}_{arith}(L,L)}\beta_\infty(L,L)^{-1}.$$

They have a role in computing volumes of locally symmetric spaces, which can in turn be used to compute the lead term in the formulas for the dimensions of spaces of modular forms on the locally symmetric space.

Known Results

Many results on representation densities are known:

- Smith (1867), Minkowski (1885), Siegel (1935) .
- Pall (1965), Watson (1976), Conway-Sloan (1988), Kitaoka (1993) $\beta_{\mathfrak{p}}(L, L)$ over $\mathbb{Z}_p$.
- Katsurada (1999) $\beta_{\mathfrak{p}}(L, M)$ over $\mathbb{Z}_2$.
- Shimura (1999) $\beta_{\mathfrak{p}}(L, L)$ when L is maximal, over any finite extension of $\mathbb{Z}_p$.
- Hironaka-Sato (2000) $\beta_{\mathfrak{p}}(L, M)$ over $\mathbb{Z}_p$ when $p \neq 2$.
- Gan-Yu (2000) $\beta_{\mathfrak{p}}(L, L)$ over finite extensions of $\mathbb{Z}_p$ for $p \neq 2$.
- Cho (2012) $\beta_{\mathfrak{p}}(L, L)$ over unramified extensions of $\mathbb{Z}_2$.

However, formulas for all cases are not yet known.
Our results give formulas for $\beta_{\mathfrak{p}}(L, L)$ when L is unimodular over any finite extension of $\mathbb{Z}_p$ (including $p = 2$). We also give reduction formulas to compute $\beta_{\mathfrak{p}}(L, L)$ for arbitrary L in terms of the collection of all of its Jordan decompositions.

Jordan Decompositions

Before we can give our results we need to recall a few facts about lattices over $\mathcal{O}_p$.

Every lattice over a finite extension of $\mathbb{Z}_p$ has at least one Jordan decomposition. These decompose the lattice into a direct sum of rescalings of unimodular lattices.

So every lattice can be decomposed as:

$$L = \bigoplus_i \pi^i L_i$$

where the L_i are unimodular lattices, and $\pi^i L_i$ means the rescaling of L_i by a factor of π^i.

It is important to note, that when $p = 2$, such a decomposition is not necissarily unique.

Unimodular Lattices

For $p \neq 2$ unimodular lattices are classified by discriminant. That is, there are exactly 2 options.

For $p = 2$ the classification of unimodular lattices consists of:

1. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$,
2. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}(\alpha-4\rho)\pi^{r-t} \end{pmatrix}$,
3. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus (-\delta_L)$,
4. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus (-(1-4\rho)\delta_L)$,
5. $\begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$ or
6. $(-(1-\alpha\pi^r))$

H is a rank 2 hyperbolic lattice, $\delta_L = 1 + \alpha\pi^r$ is the discriminant of L, so r odd or $r = 2v_\pi(2)$, the element $a\pi^t$ is of minimal valuation represented by L, we have innequalities $t < s \leq v(2), r - t$, the numbers s and t have different parities, the element ρ is such that $x^2 + x + \rho$ is irreducible mod π.

The main point is that the complexity in the $p = 2$ case is captured in 'rank 4' direct factors and there are 'many' invariants.

This theorem says we can reduce the problem for unimodular to those 'rank at most 4' factors.

Theorem (F)

Consider a unimodular lattice Λ. Then Λ has a decomposition $\Lambda = L(e) \oplus L$, where $L(e)$ is a maximal even-dimensional, even, unimodular sublattice of Λ and L has rank at most 4. Let $\ell = \mathrm{rank}(L)$ and $2n = \mathrm{rank}(L(e))$. Then:

$$\beta_R(\Lambda, \Lambda) = [L : L^{(2)}]^{-2n} \cdot \xi \cdot \beta_R(L, L) \cdot \prod_{e=1}^{n} (1 - q^{-2e}),$$

where:

$$\xi = \begin{cases} 2(1 + \chi(L(e))q^{-n})^{-1} & L(e) \textit{ non-trivial, independent of choices} \\ 1 & \textit{otherwise}. \end{cases}$$

$L^{(2)}$ is the sublattice of L consisting of elements with $Q(x) \in (2)$.

The Special Cases

This theorem says the rank at most 4 cases are handed using only the parameters r, s, t.

Theorem (F)

Consider a unimodular lattice L of rank at most 4 over a 2-adic ring R with no even unimodular factors. Then:

- *Case $n = 4$.*

$$\beta_R(L, L) = 4q^{-3\nu(2)+2t-2-(r-t-s)/2} \begin{cases} q^{(r-t-t-1)/2+1} & r - t \leq \nu(2) \\ q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \nu(2) \leq r - t \end{cases}$$

- *Case $n = 3$.*

$$\beta_R(L, L) = 4q^{(1-t)/2}.$$

- *Case $n = 2$.*

$$\beta_R(L, L) = \begin{cases} 4q^{(r-1)/2-\nu(2)} & r - t \leq \nu(2) \\ 2q^{-\lceil(\nu(2)-t)/2\rceil} & \nu(2) < r - t. \end{cases}$$

- *Case $n = 1$.*

$$\beta_R(L, L) = 2.$$

Note: we can compute $[L : L^{(2)}]$ and all other terms of previous theorem too.

Arbitrary (Not Necessarily Unimodular) Lattices

We can compute the local density by knowing the set of all Jordan decompositions.

Let JD_L denote the set of all Jordan decompositions up to isomorphism for the lattice L. For $I \in JD_L$ let L_i^I denote the i^{th} Jordan block.

Theorem (F)

With the notation as above, for L an arbitrary lattice, we have:

$$\beta_R(L, L) = q^{\tilde{w}} \left(\sum_{I \in JD_L} \prod_i \beta_R(\tilde{L}_i^I, \tilde{L}_i^I)^{-1} \right)^{-1},$$

where $\tilde{L}_i^I$ is the unimodular rescaling of L_i^I and $\tilde{w}$ is given by:

$$\tilde{w} = \sum_i i \cdot n_i \left(\sum_{j>i} n_j \right) + \sum_i (n_i(n_i + 1)/2).$$

Note: effectively enumerating the set JD_L is non-trivial.

- First theorem is a minor modification of a similar results for $\mathbb{Z}$.
- For second theorem, first we count change of basis that transform lattice to look like:

$$\begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \bigoplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}(\alpha - 4\rho)\pi^{r-t} \end{pmatrix}.$$

  (rank 4 case), we then count the number of ways to represent our lattice in this canonical way. Finally, we divide, obtaining the number of isometries. This turns out to be much easier than counting isometries directly.
- For the third theorem the main idea is the following, if g is a change of basis matrix for the lattice then $P(g \in \mathrm{O}_L)$ is:

$$P(g \text{ gives a JF})P(g \text{ gives specific JF} \mid \text{gives a JF})P(g \in \mathrm{O}_L \mid \text{ gives specific JF})$$

  We then take the rightmost term to the left hand side, sum over all specific Jordan forms that are possible, remember how probabilities relate to counting, easily compute a few of the terms.

Open Problems

1. We still lack formulas for $\beta_{\mathfrak{p}}(L, M)$ when $\mathfrak{p}|2$.
   Some of our methods generalize well, others require lots of work, and some pieces need entirely new ideas.
2. We don't have an explicit mehtod of how to enumerate all possible Jordan decompositions.
   The book of O'meara leads one to believe this can be implemented computationally, one can ask if there are clean descriptions that allow for good local density formulas. Noting that 'good' formulas exist for unramified extensions.

The End

Thank You.