Orthogonal Groups as symmetric spaces - Seminar

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February 13, 2009

The following makes extensive use of Jan Brunier's arcticle on Hilbert Modular forms contained in "The 1-2-3 of Modular Forms".

1 Quadratic Spaces and Orthogonal Groups

The following definitions are far more general than what is needed. To simplify things you may generally assume in the following that $R = k$ is a field of characteristic not 2 and that M is a vector space over k. The only more general setting we should need, is to consider discrete submodules of these.

Definition 1.1. Let R be a commutative ring with unity, R^* the group of units, V a finitely generated R-module. A quadratic form on V is a mapping $Q : V \to R$ s.t.

1. $Q(rx) = r^2 Q(x)$ for all $r \in R$ and $x \in V$
2. $B(x, y) := Q(x + y) - Q(x) - Q(y)$ is bilinear

The pair (V, Q) will be called a quadratic module (or quadratic space) over R.

In general the first condition follows from the second if $2 \in R^*$. (in this case $Q(x) = \frac{1}{2}B(x, x)$)

Definition 1.2. Let $x, y \in V$, they are said to be "orthogonal" if $B(x, y) = 0$

Using this definition one can define the notion of the orthogonal complement to a set.

Definition 1.3. Let $A \subset V$ then $A^\perp := \{x \in V | B(x, y) = 0 \forall y \in A\}$ is called the orthogonal complement.

We now wish to define the notions of morphisms between quadratic spaces, we use the obvious definition.

Definition 1.4. Let (V, Q) and (V', Q') be quadratic spaces, an R-linear map $\sigma : V \to V'$ is called an isometry if for all $x \in V$ we have $Q'(\sigma(x)) = Q(x)$.

Example. *Reflections: for an element $x \in M$ s.t. $Q(x) \in R^*$ define $\tau_x : M \to M$ via $\tau_x(y) = y - B(y, x)Q(x)^{-1}x$ This is an isometry and in the vector space case can be seen as the reflection in the hyperplane $x^\perp$.*

Eichler Elements: Let $u \in M$ be isotropic ($Q(u) = 0$) and $v \in M$ be s.t. $B(u, v) = 0$ define $E_{u,v}(y) = y + B(y, u)u - B(y, v)v - B(y, u)Q(v)u$

We now define the objects we actually wish to study, that is the orthogonal group for a quadratic space.

Definition 1.5. Let (V, Q) be a quadratic space, the orthogonal group of V is: $O_V := \{\sigma \in \mathrm{Aut}(V) | \sigma \text{is an isometry}\}$.

Definition 1.6. Suppose V is free and $v_1, ...v_n$ is a basis for V, let $S := (B(v_i, v_j))_{i,j}$ then the element $det(S) \in R/(R^*)^2$ is independent of the choice of basis and is called the discriminant $d(V)$. The space is non-degenerate if $d(V) \neq 0$.

Remark. Choosing a symmetric matrix S and a basis is equivalent to giving a quadratic form.

Gram Schmidt process allows construction of an orthogonal basis for non-degenerate spaces. Doing so allows us to always view our quadratic form as being given by $a_1 v_1^2 + ... + a_n v_n^2$ for some elements $a_i \in R$.

Choosing a basis for V also allows us to view the the elements of the orthogonal group as being contained in "$GL_n(R)$" which allows us to make the following definition.

Definition 1.7. The special orthogonal group is then $SO_V := \{\sigma \in O_V | \det(\sigma) = 1\}$

Example. *Having a basis, and viewing the elements of V as column vectors allows us express B as:*

$B(x, y) = {}^t x S y$

As such the statement, $M \in O_V$ amounts to saying ${}^t M S M = S$.

In the standard linear algebra case this is just ${}^t M M = id$ which gives us the usual notion of orthogonal matricies

Theorem 1.8. *Let $R = k$ be a field of characteristic not 2, M a regular quadratic space (That is, that for all $x \in M, x^\perp \neq M$). Then O_M is generated by reflections and SO_M is those that are products of an even number of them.*

Remark. The result over $\mathbb{R}$ is that every orthogonal matrix preserves a plane, this comes out of properties of the eigenvalues and diagnolization over $\mathbb{R}$. Once one has this one only needs to deal with the 2 dimensional case.

It is not clear to me if the same arguments can be carried out over general fields.

2 The Lie Algebra - Informal

For a Lie Group, the idea of the lie algebra is that it is supposed to be the elements of the group that are infinitesimally close to the identity of the group. That is the elements X s.t. $(1 + X) \in G$ and X is "close" to 1. That is to say, it is precisely the tangent space to the Manifold at the identity (T_e). One of the motivations for looking at this space is that in general one typically finds that neighborhood of the identity generates most of the group, and as such understanding this infinitesimal neighborhood may give much information.

For the case of the algebraic group GL_n, the map $GL_n \to \mathbb{A}^{n^2}$ makes it clear that the coordinate functions are just the components of the matrix (to be pedantic one should look at first an embedding into M_{n+1} we will not do this here). It is also easy to see, since the condition of being in GL_n is an open condition on $\mathbb{A}^{n^2}$ that you can travel in any direction (at least a small distance) from the identity and stay in GL_n and that these directions are all independent. Thus we have that $Lie(GL_n) = M_n$. We note that the exponential map $exp : A \mapsto 1 + A + A^2/2! + A^3/3!...$ maps $Lie(GL_n) \to GL_n$.

More rigourously one might want to use the dual of the cotangent space by computing $\mathfrak{m}_e/\mathfrak{m}_e^2$.

Actually computing the lie algebra for O_V can be done in several ways. The seemingly informal ways we present now, actually can be made rigorous, though doing so thoroughly could be quite painful in general. Essentially one needs to make the agrement, that the operations we perform on matricies, can be carried out componentwise to the same effect.

Example (Lie Algebra of an Orthogonal Group)**.** *Let V be a V.S. over k a field of characteristic not 2, fix a basis for V and S a symmetric matrix with non-zero determinant. Let Q be the quadratic form given by $Q(x) := xS^{tr}x$.*

Fixing a basis for V gave us an isomorphism $\mathrm{Aut}(V) \simeq GL_n(k)$ under which $O_V \simeq \{M \in GL_n(k) | MS^{tr}M = S\}$ We then wish to view the tangent space of O_V as a subspace of the tangent space of GL_n

The tangent space is then the elements $X \in M_n(k)$ s.t. $(1 + X) \in O_V$ "mod squares". that is: $(1 + X)S^{tr}(1 + X) = S$ which gives the condition $XS + S^{tr}X + XS^{tr}X = 0$ where we consider $XS^{tr}X$ a square since it has 2 Xs. Rigourously $XS^{tr}X$ is actually a square in the sense we mean, componentwise all the functions it contains will be generated by products of 2 coordinate functions each from the maximal ideal, and it is this that we are modding out by.

Alternatively "differentiating" the condition $MS({}^{tr}M) = S$ yields $(dM)S + S({}^{tr}dM) = 0$ and taking dM as $X \in M_n(k)$ gives the same relation.

We will now compute the dimension of the Lie Algebra, which from general theory will also be the dimension of the orthogonal group (as a manifold).

It suffices to consider things over the algebraic closure of k since dimensions won't change under extension(flatness), and here we may assume our quadratic form is the most trivial one given by $S = id_n$ the identity matrix. The condition $XS + S^{tr}X = 0$ then just says X is skew symmetric. The space of skew symmetric matricies in $M_n(k)$ is easily seen to have dimension $(n)(n - 1)/2$

3 The Clifford Algebra - For an Orthogonal Group

As before, Let R be a commutative ring with unity, and let (V, Q) be a finitely generated quadratic space over R. For an R algebra A we will denote $Z(A)$ its center.

Definition 3.1. Let T_V be the tensor algebra of V, that is $T_V := \bigoplus_{m=0}^{\infty} V^{\otimes m}$.

Let I_V be the two-sided ideal in T_V generated by the elements $v \oplus v - Q(v)$ for $v \in V$.

Then the Clifford algebra for V is $C_V := T_V/I_V$.

Notice that V injects naturally into C_V

The Clifford Algebra satisfies the following universal property:

Proposition 3.2. *Let A be any R-algebra, $f : V \to A$ an R-linear map s.t. $f(v)^2 = Q(v)id_A$ for all $v \in V$ then there exists unique $g : C_V \to A$ s.t. $f(v) = g(v)$.*

Proof. (sketch) The proof of this fact is to appeal to the universal property of tensor products and to note that the map descends to the quotient by I_V because of the condition $f(v)^2 = Q(v)id_A$. $\square$

In particular the assignment $V \to C_V$ is a functor from category of quadratic spacers over R to category of associative R algebras.

Example. $C_{0,0} = \mathbb{R}$, $C_{1,0} = \mathbb{R} \oplus \mathbb{R}$, $C_{0,1} = \mathbb{C}$, $C_{2,0} = M_2(\mathbb{R})$, $C_{1,1} = M_2(\mathbb{R})$, $C_{0,2} = \mathbb{H}$

Notice that because the relation defining I_V involves only tensors of even length, there is a natural $\mathbb{Z}/2\mathbb{Z}$ grading on C_V s.t. $C_V = C_V^0 \oplus C_V^1$ where C_V^0, C_V^1 are the even and odd length tensors respectively. Note that C_V^0 is a subalgebra, but C_V^1 is just a vector subspace.

Noting that multiplication by -1 on V is an isometry and induces the canonical automorphism J of the clifford algebra it is easy to see that (if 2 is a unit in R) $C_V^0 = \{x \in C_V | J(x) = x\}$

There is another involution (called the canonical involution) on C_V, ${}^t : x_1 \otimes ... \otimes x_n \mapsto x_n \otimes ... \otimes x_1$.

We can use this to define the clifford norm $N(x) = {}^t x x$. Note that this extends Q from V to C_V.

We now fully restrict our attention to the case $R = k$ a field of characteristic not 2. Let (V, Q) be a non-degenerate quadratic space and let $v_1, ..., v_n$ be an orthonormal basis of V. set $\delta = v_1 \otimes ... \otimes v_n$ in C_V.

Theorem 3.3. *The center of C_V is given by: $Z(C_V) = k$ if n is even, $k + \delta k$ if n is odd. The center of C_V^0 is given by: $Z(C_V^0) = k + \delta k$ if n is even, k if n is odd.*

Proof. (sketch)

It is easy to check that the centers contain the given elements.

To check that those elements are the center is a technical argument.

First observe that if a basis vector v_i is to commute with a linear combination of tensors it will need to commute with each elementary tensor.(provided things are expressed in terms of tensors of the orthogonal basis elements).

A basis vector v_i commutes with an elementary tensor iff it does not appear it in and the tensor is of odd length or it does appear and the tensor is of even length.

Thus, if an elementary tensor is to commute with every basis vector, it must either contain none and be of even length or all and be of odd length. This completes the argument □

Example. *if $n = 1$ then $C_V \simeq k[X]/(X^2 - d(V)/2)$*

if $n = 2$ then C_V is a quaternion algebra of type $(Q(v_1), Q(v_2))$ and $C_V^0 \simeq k[X]/(X^2 + d(V))$

if $n = 3$ then C_V^0 is a quaternion algebra of type $(-Q(v_1)Q(v_2), -Q(v_2)Q(v_3))$.

if $n = 4$ then C_V^0 is a quaternion algebra of type $(-Q(v_1)Q(v_2), -Q(v_2)Q(v_3))$ over $Z(C_V^0)$

Moreover the conjugation and norm on C_V^0 correspond to the main involution and clifford norm.

4 The Spin Group

The goal of this section is to define the Spin Group, which is meant to be the covering space of the orthogonal group.

Remark. We would like to remark 2 things, firstly, given any Semi-simple group G, there are two natural other groups to consider. These are These are the Adjoint group, $G/Z(G)$ and the universal covering space of the group. In general the former has finite index in the latter, and many of the properties of one are shared by the others, In particular the share Lie algebras. Both of these 2 groups are often easier to study than the original.

Secondly, I would like to remark that the term spin group, comes from physics. The notion is that physical laws should not depend on which coordinate system we are using to measure things and so isometries of the universe should be equally configurations. It turned out that under appropriate modelling of subatomic particles there was an extra degree of freedom for the configuration space that did not correspond to vector valued locations or directions but was rather a binary measure on particles. It was decided that this was the 'spin' of the particle. Once one actually applies mathematical language to this model of the universe, this extra spin parameter corresponds to the what is additionally captured by the covering space of an orthogonal group.

The first step towards the construction of this covering is to define the Clifford Group

Definition 4.1. The Clifford Group is defined to be:
$CG_V := \{x \in C_V | x \text{ invertable and } xVJ(x)^{-1} = V\}$

It is an easy check to see that this is a group.

Notice that for each $x \in CG_V$ have $\alpha_x(v) = xvJ(x)^{-1}$ an automorphism of V. We thus have a representation: $\alpha : CG_V \to \mathrm{Aut}_R(V)$ called the vector representation.

We observe that ${}^t : CG_V \to CG_V$ and thus so does the clifford norm.

Lemma 4.2. *If $R = k$ a field of characteristic not 2, then $ker(\alpha) = k^*$ and the clifford norm gives a homomorphism $N : CG_V \to k^*$.*

Proof. It is easy to see that k^* is in the $ker(\alpha)$, this is because the J map acts trivially on C_V^0.

Conversely let $x \in ker(\alpha)$. we can write $x = x_0 + x_1$ with $x_i \in C_V^i$. We have then that:
$xvJ(x)^{-1} = v$ for all $v \in V$

Thus $(x_0 + x_1)v(x_0 - x_1)^{-1} = v$ for all $v \in V$.

So then we get: $x_0v + x_1v = vx_0 - vx_1$ looking at the C_V^i components and noting that V generates C_V as an algebra we get that: $x_0 \in Z(C_V) \cap C_V^0 = k^*$.

To show that $x_1v = vx_1 \Rightarrow x_1 = 0$ is done similarly to computing the center of C_V.

This completes the first assertion that $ker(\alpha) = k^*$.

We next notice that, for $v \in V$ we have:
$\alpha_x(v) = -{}^tJ(\alpha_x(v))$ so that:
$xvJ(x)^{-1} = {}^tx^{-1}vJ({}^tx)$ so thus:
$N(x)vJ(N(x))^{-1} = v$ in particular $N(x) \in k^*$.

□

Lemma 4.3. *For each $x \in CG_V, \alpha_x$ is an isometry.*

Proof. for $v \in V$ we have:
$Q(a_x(v)) = N(a_x(v)) =^t J(x^{-1})^t v^t xxvJ(x^{-1}) = Q(v)$. □

In particular this gives us a map $\alpha : CG_V \to O_V$. Moreover, if $x \in CG_V \cap V$, then $Q(x) \in R^*$ and α_x corresponds to the reflection in the plane $x^\perp$. This can be seen by checking directly the action of the elements and seeing that it corresponds to the map we had previously defined as the reflection.

Definition 4.4. We define the groups $GSpin_V$ and $Spin_V$ as follows:
$GSpin_V := CG_V \cap C_V^0$
$Spin_V := \{x \in GSpin_V | N(x) = 1\}$

In the case where reflections generate the orthogonal group (which is the case when V is a regular quadratic space) we get exact sequences:

$$1 \to k^* \to CG_V \xrightarrow{\alpha} O_V \to 1$$

And since SO_V are generated by even numbers of reflections:

$$1 \to k^* \to CSpin_V \xrightarrow{\alpha} SO_V \to 1$$

We then have, that since $N : CG_V \to k^*$ we can construct an induced homomorphism:

$$\theta : O_V \to k^*/(k^*)^2$$

called the spinor norm. It is defined by taking a section of α and computing the norm in CG_V, one must check that different choices of sections give the same result up too $(k^*)^2$, this is an easy check. We observe that for τ_v the reflection at v we get $\theta(\tau_v) = Q(v)$

We then obtain the exact sequence:

$$1 \to \mu_2 \to Spin_V \xrightarrow{\alpha} SO_V \xrightarrow{\theta} k^*/(k^*)^2$$

We remark that over the algebraic closure of k this would tell us precisely that $Spin_V$ is the cover of SO_V.

For the purpose of doing computations in low dimensions, the following lemma is usefull.

Lemma 4.5. *If $dim(V) \leq 4$ then $GSpin_V = \{x \in C_V^0 | N(x) \in k^*\}$, $Spin_V := \{C_V^0 | N(x) = 1\}$*

Proof. The important observation in the proof, is that for $dim(V) \leq 4$ we have $V = \{g \in C_V^1 | g^t = g\}$. To prove that assertion just amounts to checking that a tensor of length 3 of orthogonal elements satisfies $x^t = -x$.

The other observations of note then are that if $N(x) \in k^*$ then $x^t N(x)^{-1}$ is the inverse of x and that $N(x)vN(x)^{-1} = v$ implies that $xvx^{-1} = (xvx^{-1})^t$ would imply $x \in CG_V$. $\square$

Example. *From the examples of the previous section, we get that the Spin groups correspond to the norm 1 elements of a quaternion algebra.*

5 The Symmetric Space

Let (V, Q) be a quadratic space over $\mathbb{Q}$. $V(\mathbb{R}) := V \otimes \mathbb{R}$ will be isomorphic to $\mathbb{R}^{p,q}$ for some choice of p, q.

If $K \subset O_V(\mathbb{R})$ is a maximal compact subgroup, then it will turn out that $O_V(\mathbb{R})/K$ is a symmetric space (every point has a symmetry where it is the unique local fixed point). It turns out that these only have complex structures (ie will be hermitian) if one of p or q is 2. Since interchanging p, q does not change the orthogonal group (it amounts to replacing Q by $-Q$) we suppose that $p = 2$.

We wish to construct these spaces along with their complex structure for this case.

Remark. For the next while, we will be discussing the structure of $\mathbb{R}$ points, and as such the only invariants of significance are these values p, q. However, when we mention 'Rational Boundry Points' all of a sudden the remaining details about the structure over $\mathbb{Q}$ become important. So although topologically, the spaces we define in what follows may be isomorphic, the rational structures on them may not be so simple.

5.1 The Grassmanian - Maximal Compacts

We consider (V, Q) to be of type $(2, n)$

We consider the Grassmannian of 2-dimensional subspaces of $V(\mathbb{R})$ on which the quadratic form Q restricts to one which is positive definite.

$$Gr(V) := \{v \subset V | dim(v) = 2, Q|_v > 0\}$$

By Witt's extension theorem, the group O_V will act transitively on $Gr(V)$. If we fix $v_0 \in Gr(V)$ then its stabilizer K_{v_0} will be a maximal compact subgroup (and $K \simeq O(2) \times O(n)$ over $\mathbb{R}$). Thus $Gr(V)/K_{v_0}$ realizes a hermitian symmetric space.

Remark. Though this is an easy realization of the space, It is not clear from this construction what the complex structure should be.

5.2 The Projective Model - Complex Structure

We consider the complexification $V(\mathbb{C})$ of the space V and the projectiviztion $P(V(\mathbb{C}))$.

We then consider the zero quadric:

$$N := \{[Z] \in P(V(\mathbb{C})) | (Z, Z) = 0\}$$

It is a closed algebraic subvariety of the projective space.

$$\kappa := \{[Z] \in P(V(\mathbb{C})) | (Z, Z) = 0, (Z, \overline{Z}) > 0\}$$

is a complex manifold of dimension n consisting of 2 connected components.

Remark. One must check that these spaces are in fact well defined, that is that the conditions do not depend on a representative Z. Indeed $(cZ, cZ) = c^2(Z, Z)$ and $(cZ, \overline{cZ}) = c\overline{c}(Z, \overline{Z})$.

The assertions about the dimension and connected components is easily seen once we consider the tube domain model later. However one can see that projectivization removes one dimension as does the $(Z, Z) = 0$ condition, the $(Z, \overline{Z}) > 0$ condition cuts the space into 2 components separating Z from $\overline{Z}$ and removing the real points.

The orthogonal group $O_V(\mathbb{R})$ acts transitively on κ

Remark. One must first check that it acts (conditions are preserved) this is easy once we rephrase conditions below.

To show transitivity observe:
for $Z \in V(\mathbb{C})$ the condition $[Z] \in \kappa$ is:
$(X + iY, X + iY) = 0$ and $(X + iY, X - iY) > 0$ but:
$(X + iY, X + iY) = (X, X) - (Y, Y) + 2i(X, Y)$ and $(X + iY, X - iY) = (X, X) + (Y, Y)$.
thus $[Z] \in \kappa$ iff $(X, X) = (Y, Y) > 0$ and $(X, Y) = 0$.
Thus, we can get an O in $O_V(\mathbb{R})$ that maps $X \mapsto X'$ $Y \mapsto Y'$ via witts extension theorem.
This O will then map $[Z] to [Z']$.

The subgroup $O^+_V(\mathbb{R})$ of elements whose spinor norm equals the determinant preserves the 2 components of κ whereas $O_V - O^+_V(\mathbb{R})$ interchanges them. This fact is most easily seen once you have the isomorphism to the $Gr(V)$ model and look at how the spinor norm and determinant check if the element preserves orientation of positive definite planes. Pick one component of κ denote it κ^+.

For $Z \in V(\mathbb{C})$ we will write $Z = X + iY$ where $X, Y \in V(\mathbb{R})$.

Lemma 5.1. *The assignment $[Z] \mapsto v(Z) := \mathbb{R}X + \mathbb{R}Y$ defines a real analytic isomorphism $\kappa^+ \to Gr(V)$*

Proof.

- Must check well definedness.

  The mapping gives us a pos-def plane by checking conditions.

  That it does not depend on representative amounts to checking that multiplying by $\mathbb{C}^*$ just rotates and rescales the plane. indeed $(a + ib)(X + iY) = (aX - bY) + i(aY + bX)$ so we have just changed the basis for the plane.

- Must check subjectivity.

  This amounts to the iff in the condition for inclusion in κ above plus noting that can pick $[X + iY]$ or $[Y + iX]$ at least one of which is in κ^+, that is the assertion that $[X + iY]$ and $[X - iY]$ are in different components of κ

- Must check analyticity.

  This amounts to saying that we can continuously analytically assign orientations to the elements of the grassmannian. This follows (either from what follows and considering homology) or by observing that $O^+_V(\mathbb{R})$ acts transitively and continuously analytically on $G(V)$ so it can be used to continuously analytically assign an orientation to each plane.

□

5.3 The "Tube Domain" Model

Remark. The term tube domain appears to refer to the imposition of positivity conditions on only the imaginary part of your vectors.

Pick e_1 a non-zero isotropic vector in V, pick e_2 s.t. $B(e_1, e_2) = 1$
define $V := W \cap e_2^\perp \cap e_1^\perp$, we then may express elements of $V(\mathbb{C})$ as (z, a, b)
Then we get $V = W \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}e_1$ and W is a quadratic space of type $(1, n-1)$ (a Lorentzian).

Definition 5.2. We define the tube domain $H := \{z \in W(\mathbb{C}) | Q(\Im(z)) > 0\}$.

Lemma 5.3. *The map $\psi : H \to \kappa$ given by $\psi(z) \mapsto [(z, 1, -Q(z) - Q(e2))]$ is biholomorphic.*

Proof. One first checks that this is well defined,
Indeed we have that:

$$Q(z + e_2 + (-Q(z) - Q(e2))e_1) = Q(z) + Q(e_2) + B(e_2, e_1)(-Q(z) - Q(e_2)) = 0$$

.

Additionally:

$$
\begin{aligned}
&B(z+e_2+(-Q(z)-Q(e_2))e_1, \overline{z+e_2+(-Q(z)-Q(e_2))e_1}) \\
&= B(z+e_2+(-Q(z)-Q(e_2))e_1, \overline{z}+e_2+(-Q(\overline{z})-Q(e_2))e_1) \\
&= B(z,\overline{z})+2Q(e_2)+(-Q(z)-Q(e_2))+(-Q(\overline{z})-Q(e_2)) \\
&= B(z,\overline{z})-(1/2)B(z,z)-(1/2)B(\overline{z},\overline{z}) \\
&= (1/2)(B(z,\overline{z}-z)+B(z-\overline{z},\overline{z})) \\
&= (1/2)(B(2\Im(z),z)-B(2\Im(z),\overline{z})) \\
&= 2B(\Im(z),\Im(z)) \\
&= Q(\Im(z)) \qquad > 0
\end{aligned}
$$

for $[Z] \in \kappa$ that X, Y span a positive definite plane tells us that $B(Z, e_2) \neq 0$, this is because W contains no positive definite plane, and so Z interacts with $e_1 \oplus e_2$, however e_1 is isotropic so there it is not orthogonal to e_2. We can thus write $[Z] = [(z, 1, b)]$. Reversing the above calculations give us that $b = -Q(z) - Q(e_2)$ and $Q(\Im(z)) > 0$.

The biholomorphicity is infered from the fact that in one directions we have a map that is essentially polynomial, in the reverse it is essentially a projection. ☐

H ends up having 2 components, this follows from the fact that $W(\mathbb{R})$ is a space of type $(1, n-1)$ and by inspecting the defining condition, one of these thus corresponds to κ^+ we shall label that one H^+.

It is this H^+ that is the analog of the usual upper half plane, we have an action of $O_V^+(\mathbb{R})$ acting on it through its action on κ. This action as before will be transitive.

6 Discrete Subgroups - Lattices

Let V be a non-degenerate quadratic space over $\mathbb{Q}$ of type $(2, n)$.

Definition 6.1. A Lattice in V is a $\mathbb{Z}$-module L s.t. $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$

We say a lattice L is integral if $B(x, y) \in \mathbb{Z} \forall x, y \in L$.

We say a lattice is even if $Q(x) \in \mathbb{Z} \forall x \in L$.

The dual lattice is $L^\vee := \{x \in V | B(x, y) \in \mathbb{Z} \forall y \in L\}$

Note that L is integral iff $L \subset L^\vee$ in which case $L^\vee/L$ is a finite abelian group called the discriminant group. $|L^\vee/L| = |det(S)|$ where S is the matrix for Q coming from a lattice basis for L. For the remainder L is an even lattice. Then $O_L \subset O_V$ is a discrete subgroup.

Let $\Gamma \subset O_L$ be a subgroup of finite index then Γ acts properly discontinuously (every element of the space has a nhd s.t. the group maps the nhd outside that nhd) on $Gr(V)$, κ^+, and H^+.

We consider the space: $Y(\Gamma) := \Gamma \setminus H^+$, it is a normal complex space and is compact if and only if V is anisotropic (does not take on the value 0 over $\mathbb{Q}$). If it is not compact, it can be compactified by adding rational boundary components, in the κ^+ model these are the non-trivial isotropic subspaces of $V(\mathbb{R})$.

If an isotropic boundry component is a line in $V(\mathbb{R})$ then we call it a special point, otherwise the component is called generic. A special point is considered to be a 0-dimensional boundary component.

If $F \subset V(\mathbb{R})$ is an isotropic subspace, the set of all generic boundary points corresponding to elements of $F(\mathbb{C})$ is called a one-dimensional boundary component.

Lemma 6.2. *There is a bijective correspondance between boundary components of κ^+ in N and non-zero isotropic subspaces of $V(\mathbb{R})$ of a corresponding dimension.*

We consider a boundary component to be rational if the corresponding space F is defined over $\mathbb{Q}$. we define $(\kappa^+)^*$ to κ^+ together with its rational boundary components.

$O_V(\mathbb{Q}) \cap O_V^+(\mathbb{R})$ acts on $(\kappa^+)^*$ and by the theory of baily-borel, $X(\Gamma) := (\kappa^+)^*/\Gamma$ together with baily-borel topology is a compact hausdorff space which can be given a complex structure. (one can construct an ample line bundles, hence it is projective algebraic)

Remark. does he mean quotient on left?

6.1 Heegner Divisors

These are special divisors on the space $X(\Gamma)$.

For $\lambda \in L^\vee$ with $Q(\lambda) < 0$ then $V_\lambda := \lambda^\vee \subset V$ is a rational quadratic space of type $(2, n-1)$. If we consider this in κ^+ we get: $H_\lambda = \{[Z] \in \kappa^+ | (Z, \lambda = 0)\}$ which is an analytic divisor.

In H^+ this would be $H_\lambda = \{z \in H^+ | aQ(z) - (z, \lambda_W) - aQ(e_2) - b = 0\}$ $(\lambda = \lambda_W + ae_2 + be_1)$.

Fix $\beta \in L^\vee/L$ and $m < 0$ then $H(\beta, m) = \sum_{\lambda \in \beta + L, Q(\lambda) = m} H_\lambda$ called a Heegner givisor of discriminant (β, m). (If Γ acts trivially on $L^\vee/L$ this descends to an algebraic divisor on $Y(\Gamma)$.

Can also consider $H(m) = (1/2) \sum_\beta H(\beta, m)$ which is Γ invariant and thus descends to $Y(\Gamma)$.

7 Modular Forms for O(2,n)

Let $\overline{\kappa}^+ = \{Z \in V(\mathbb{C}) | [Z] \in \kappa^+\}$ be the cone over κ^+.

Definition 7.1. Let $k \in \mathbb{Z}$, χ be a character of Γ. A meromorphic function on $\overline{\kappa}^+$ is a modular form of weight k and character χ for the group Γ if:

1. F is homogeneous of degree $-k$, ie $F(cZ) = c^{-k} F(Z)$ for $c \in \mathbb{C} - \{0\}$.
2. F is invariant under Γ, ie $F(gZ) = \chi(g) F(Z)$ for any $g \in \Gamma$.
3. F is meromorphic on the boundary.

If F is holomorphic on $\overline{\kappa}^+$ and on the boundary, we call it a holomorphic modular form.

The Koecher principle implies condition (3) is automatic if the witt rank of V (dimension of maximal isotropic subspace) is less than n. Note that for type $(2, n)$ the Witt rank is always less than 2, and will often be less.

7.1 Siegel Theta Function

Examples of modular forms on these spaces can be constructed either through eisenstien series or using lifts via the Siegal Theta Function, we shall not describe this here.

8 Isomorphism of Hilbert Modular Group and 0(2,2)

Let $F/\mathbb{Q}$ be a real quadratic field, so $F = \mathbb{Q}(\sqrt{d})$. Consider the 4-dimensional $\mathbb{Q}$ vector space $\mathbb{Q} \oplus \mathbb{Q} \oplus F$. With the quadratic form given by $Q(a, b, x + y\sqrt{d}) = (x + y\sqrt{d})(x - y\sqrt{d}) - ab$. Then V is a rational quadratic space of type $(2, 2)$ so all of the proceeding construction applies.

We consider the basis $v_1 = (1, 1, 0)$, $v_2 = (1, -1, 0)$, $v_3 = (0, 0, 1)$, $v_4 = (0, 0, \sqrt{d})$.

We then have (as in the notation of clifford algebras) that $\delta^2 = d$ and so $Z := Z(C_V^0) = \mathbb{Q} + \mathbb{Q}\delta$ and moreover $C_V^0 = Z + Zv_1v_2 + Zv_2v_3 + Zv_1v_3$ is isomorphic to the split quaternion algebra over $M_2(F)$. The map coming from linearly extending:

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $v_1v_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $v_2v_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $v_1v_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The canonical involution in C_V^0 is given by:

$$* : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The clifford norm is given by the determinant:

$$N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$$

We thus have that $Spin_V \cong SL_2(F) \cong Res_{F/\mathbb{Q}}(SL_2)$. Thus $\Gamma_F = SL_2(\mathcal{O}_F)$ are arithmetic subgroups of $Spin_V$. In fact, one can show that $\Gamma_F = Spin_L$ where L is the lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathcal{O}_F \subset V$.

We now explicitly describe the vector representation (That is, how does $Spin_V$ act on V). let $\sigma : x \mapsto v_1 x v_1^{-1}$ be $Ad(v_1)$. then $\delta^\sigma = -\delta$. Then σ agrees with conjugation on F when acting on the center of C_V^0. On $M_2(F)$ the action is expressed as:

$$\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} d' & -c' \\ -b' & a' \end{pmatrix}$$

Let $\overline{V} = \{X \in M_2(F) | X^* = X^\sigma\} = \{X \in M_2(F) | X^{tr} = X'\} = \{\begin{pmatrix} a & v' \\ v & b \end{pmatrix} | a, b \in \mathbb{Q}, v \in F\}$

We see in particular that V is isomorphic to $\overline{V}$

So from now on we work with $\overline{V}$. Where the quadratic and bilinear forms are given by:

$$\overline{Q}(X) = -det(X) \text{ and } \overline{B}(X, Y) = -tr(XY^*)$$

Moreover, $Spin_V \cong SL_2(F)$ acts via $g \circ X = gXg^{-\sigma} = gXg'^{tr}$

We next notice that we have: $\overline{V}(\mathbb{C}) = M_2(\mathbb{C})$ (really! it isn't a subspace) and so:

$$\kappa = \{[Z] \in P(M_2(\mathbb{C})) | det(Z) = 0, -tr(Z\overline{Z}^*) > 0\}$$

Take $e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Observe that $Q(e_1) = 0$ and $B(e_1, e_2) = 1$.

We set $W = \overline{V} \cap e_1^\perp \cap e_2^\perp$. Noting that $B(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e_1) = d$ and $B(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, e_2) = -a$. We conclude that $W = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in V\}$

Then $W(\mathbb{C}) \cong \mathbb{C}^2$ and $H \cong \{(z_1, z_2) \in \mathbb{C}^2 | \operatorname{im}(z_1 z_2) > 0\}$

For $z = (z_1, z_2) \in H$ put:

$$M(z) = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}$$

This corresponds to the map $H \to \kappa$ from before.

If we choose for H^+ the component where $\operatorname{im}(z_1) = \operatorname{im}(z_2) = 1$ then it is immediately clear we have an isomorphism $\mathbb{H}^2 \cong H^+ \cong \kappa^+$.

This map commutes with the action of $SL_2(F)$ where the action on κ^+ is given as before (that is through $Spin_V \cong SL_2(F)$).

In particular $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$ acts on $M(z) \in \kappa^+$ as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} = \begin{pmatrix} (az_1 z_2 + bz_2)a' + (az_1 + b)b' & (az_1 z_2 + bz_2)c' + (az_1 + b)d' \\ (cz_1 z_2 + dz_2)a' + (cz_1 + d)c' & (cz_1 z_2 + dz_2)c' + (cz_1 + d)d' \end{pmatrix}$$

And acting on H as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (z_1, z_2) = (\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'})$$

Applying M gives:

$$\begin{pmatrix} \frac{az_1+b}{cz_1+d} \frac{a'z_2+b'}{c'z_2+d'} & \frac{az_1+b}{cz_1+d} \\ \frac{a'z_2+b'}{c'z_2+d'} & 1 \end{pmatrix} = N(cz + d) \begin{pmatrix} (az_1 + b)(a'z_2 + b') & (az_1 + b)(c'z_2 + d') \\ (a'z_2 + b')(cz_1 + d) & (cz_1 + d)(c'z_2 + d') \end{pmatrix}$$

In particular this shows that $\gamma M(z) = N(cz + d)M(\gamma z)$.

In particular we see that for parallel weight modular forms, our definitions agree.