Spectral Sequence for Filtration - Secret Seminar

Andrew Fiori

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Recall that given a chain complex $C_n, d : C_{n+1} \to C_n$ we want to compute:

$$H_n = \ker(d_n)/\operatorname{im}(d_{n+1})$$

Recall that a spectral sequence is: $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q-r}$ with $E^{r+1}_{pq} \simeq \ker(d^r_{pq})/\operatorname{im}(d^r_{p+r,q-r+1})$. For a bounded spectral sequence we obtain E^∞_{pq} the eventual stabilization.

Let p_0 be the smallest p with $E_{pq} \neq 0, p+q=n$ then set

$$F_pH_n = 0 \quad \forall p < p_0$$

$$F_{p_0}H_n = E_{p_0,n-p_0}$$

inductively choose extension classes F_pH_n for $p > p_0$:

$$0 \to F_{p-1}H_n \to F_pH_n \to E_{p,n-p}$$

Then F_pH_n is a filtration of $H_n = \cap_p F_pH_n$.

The spectral sequence converges to any H_n as constructed.

It would be convenient if given a C_n we were able to construct an $E^r_{p,q}$ that would have $H_n(E^r_{p,q}) = H_n(C_n)$.

Definition 0.1. A **filtration** F_p on a chain complex C is as usual given by:

$$\cdots \subseteq F_{p-1}C \subseteq F_p \subseteq \cdots \subseteq C$$

We say it is **bounded below** if $F_pC = 0$ for some p and **bounded above** if $F_pC = C$ for some p. We call it **canonically bounded** if $F_{-1}C = 0$ and $F_nC_n = C_n$. We say it is **exhaustive** if $C = \cup F_pC$. We shall say it is **Hausdorff** if $0 = \cap F_pC$. We say it is **complete** if $C = \varprojlim C/F_pC$.

Given a filtration F of C we will get a spectral sequence:

$$E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}$$

The following is $E^0_{-1,-1}$ to $E^0_{2,2}$.

$$\begin{array}{cccc}
F_{-1}C_{-2}/F_{-2}C_{-2} & F_0C_{-1}/F_{-1}C_{-1} & F_1C_0/F_0C_0 & F_2C_1/F_1C_1 \\
F_{-1}C_{-1}/F_{-2}C_{-1} & F_0C_0/F_{-1}C_0 & F_1C_1/F_0C_1 & F_2C_2/F_1C_2 \\
F_{-1}C_0/F_{-2}C_0 & F_0C_1/F_{-1}C_1 & F_1C_2/F_0C_2 & F_2C_3/F_1C_3 \\
F_{-1}C_1/F_{-2}C_1 & F_0C_2/F_{-1}C_2 & F_1C_3/F_0C_3 & F_2C_4/F_1C_4
\end{array}$$

$$E^1_{p,q} = H_{p+q}(E^0_{p*})$$

Since we want to get our spectral sequence to basically act like the quotient of a filtration of the homology, we simply will force the spectral sequence to be an approximation living inside the filtration we are given for free.

We define what should be approximations of cycles, (things that are almost in the kernel)

$$A^r_{p,q} = \{c \in F_pC_{p+q} | d(c) \in F_{p-r}C_{p+q-1}\}$$

(This would be a cycle for the computation of $H_{p+q}(F_pC/F_{p-r}C)$) We then consider the image inside the filtration:

$$Z^r_{p,q} = A^r_{p,q}/F_{p-1}C_{p+q}$$

Notice that $A^r_{p,q} \cap F_{p-1}C_{p+q} = A^{r-1}_{p-1,q}$ and thus $Z^r_{p,q} = A^r_{p,q}/A^{r-1}_{p-1,q}$. We now consider the approximate coboundaries:

$$B^r_{p,q} = d(A^{r-1}_{p+r-1,q-r})/F_{p-1}C_{p+q})$$

(these would be coboundaries for $B_{p+q}(F_{p+r-1}C/F_{p-1}C) \cap F_pC_{p+q}/F_{p-1}C_{p+q})$

We now define the spectral sequence:

$$E^r_{p,q} = Z^r_{p,q}/B^r_{p,q} \simeq A^r_{p,q}/(d(A_{p+r-1,q}) + A^{r-1}_{p-1,q})$$

Lemma 0.2.

- *d naturally maps $A^r_{p,q}$ to $A^{r+1}_{p-r,q+r-1} \subset F_{p-r}C_{p+q-1}$. thus:*

$$d^r : E^r_{p,q} \to E^r_{p-r,q-r}$$

- *$d(A^r_{p,q}) \cap F_{p-r-1}C_{p+q} = d(A^{r+1}_p)$ so that $B^{r+1}_{p-r,p+r-1} = d(A^r_{p,q})/d(A^{r+1}_{p,q}$*
- *d naturally maps $Z^r_{p,q}$ to $B^{r+1}_{p-r,q+r-1}$.*

  The kernel of this map is trial.
- *d naturally maps $Z^r_{p,q}/Z^{r+1}_{p,q} \simeq A^r_{p,q}/(A^{r+1}_{p,q} + A^r_{p-1,q+1} - 1)$ to $B^{r+1}_{p-r,q+r-1}/B^r_{p-r,q+r-1}$*
- *One then has that:* $\ker(d^r) = Z^{r+1}_{p,q}/B^r_{p,q}$ *and* $\mathrm{im}(d^r) = B^{r+1}_{p-r,q+r}/B^r_{p-r,q+r}$
- *So we can see that $E^{r+1}_{p,q}$ is the homology of d^r.*

Now, we have a spectral sequence, the question is what if anything does it converge too?
Define: $A^\infty_{p,q} = \cap_r A^r_{p,q}$ and $Z^\infty_{p,q} = \cap_r Z^r_{p,q}$ and $B^\infty_{p,q} = \cup_r B^r_{p,q}$.
Notice that we always have that $A^\infty_{p,q}/F_{p-1}C_{p+q} \hookrightarrow Z^\infty_{p,q}$ but that we may not have equality.
Define:

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$$
$$e^\infty_{p,q} = A^\infty_{p,q}/(B^\infty_{p,q} + F_{p-1}C_{p+q})$$

- The filtration F_p being bounded below implies that the spectral sequence is bounded below.
- The filtration F_p being hausdorff will imply that $A^\infty_{p,q} = \ker(d : F_pC_{p+q})$.

- The filtration F_p being exhaustive will imply that $B_{p,q}^{\infty} = \mathrm{im}(d : C_{p+q-1})$
- We next notice that, given that we have a filtration on C, we automatically have a filtration on H (via the image of $H_n(F_pC)$. If F was exhaustive on C then it is too on H, likewise for bounded below.

  THIS DOES NOT HOLD FOR HAUSDORFF!! but bounded below still implies hausdorff.
- If the filtration is exhaustive and Hausdorff. We actually have $F_pH_n(C)/F_{p-1}H_n(C) \simeq e_{p,q}^{\infty}$.

So, wlog replace the filtration by its hausdorf/exhastification.

We will still have issues if the Homology is not Hausdorff...

Theorem 0.3. *Suppose C is complete We have the sequence*

$$0 \to \varprojlim{}^1 H_{n+1}(C/F_pC) \to H_n(C) \to \varprojlim H_n(C/F_pC) \to 0$$

Has $\varprojlim{}^1 H_{n+1}(C/F_pC) = \cap_p F_pH_n(C)$. *And* $H_*(C)/\cap F_pH_n(C) \simeq \varprojlim H_n(C)/F_pH_n(C) \simeq \varprojlim H_n(C/F_pC)$

Corollary 0.4. *If the spectral sequence weakly converges then $H_*(C)$ and $H_*(\hat{C})$ have the same completion.*

Theorem 0.5. *Let C be complete and exhaustive*

- *if bounded below it converges.*

  We have that the $A_{p,q}^r$ stabilize, this gives us $e_{p,q} = E_{p,q}$.

  We also have that $H(C)$ must be hausdorff.
- *if the spectral sequence is regular then it weekly converges to $H_n(C)$.*

  Regularity shows that $Z_{p,q}^r$ stabilizes, various technical results give us that $e_{p,q}^{\infty} = E_{p,q}^{\infty}$.
- *If the spectral sequence is regular and bounded above then it converge*

  We must show that Lim^1 vanishes.