SYMMETRIC SPACE EXAMPLES

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1. General Setup

We recall the basic definitions.

Definition 1.1. A symmetric space is a **riemannian manifold** M such that for each $x \in M$ there exists an isometric involution s_x of M for which x is locally a unique fixed point.

M is said to be **hermitian** if it has a complex structure making it hermitian.

It is a basic consequence of the definition that we shall have:

Proposition 1.2. *Fix $x \in M$, $G = \mathrm{Isom}(M)$, $K = \mathrm{Stab}_G(x)$ and let s_x act on G by conjugation then $M \simeq G/K$ and $(G^{s_x})^0 \subset K \subset G^{s_x}$.*

Moreover, given any real lie group G, inner automorphism s of order 2 and K such that $(G^s)^0 \subset K \subset G^s$ then $M = G/K$ is a symmetric space.

The requirement that M have complex structure is more subtle, Essentially the requirement is the existance of an isometry i of M such that $i^2 = s$. In the event that M is irreducible, this amounts to saying that $Z(K) = \mathrm{SO}_2(\mathbb{R})$.

There are three main types of symmetric spaces:

(1) Compact Type
 In general these come from compact lie groups G,
(2) Non-Compact Type
 In general these arise when K^0 is the maximal compact connected lie subgroup of G, or equivalently when s_x is what is known as a cartan involution
(3) Euclidean Type
 These generally arise as quotients of Euclidean space by discrete subgroups.

There are boundary cases to the above, for example $\mathrm{SO}_2(\mathbb{R})$ is $\mathbb{R}/\mathbb{Z}$ Euclidean type even though it is compact. The definitions are made precise by looking at the Lie algebra's.

Proposition 1.3. *Every Symmetric space decomposes into a product of the three types.*

We are interested most in the case where we are either compact or non-compact and $G = G(\mathbb{R})$ are the real points of a reductive algebraic group. (Maximal (closed) connected unipotent [upper triangularizable with 1's on diagonal] normal subgroup is trivial)

(Semi-simple is no connected normal abelian subgroup [implies trivial center])

(Borel is Maximal connected solvable closed)

(Parabolic is contains a Borel)

Proposition 1.4. *There exists a duality between the compact and non-compact types, under which, if M is of the compact type, there exists a dual symmetric space $\breve{M}$ such that $M \hookrightarrow \breve{M}$.*

For M a hermitian symmetric space of the non-compact type and $\breve{M}$ its compact dual we have the primary objects of interest are:

- The Lie algebra $\mathfrak{g}$ of G.

- The Lie sub-algebra $\mathfrak{k} \subset \mathfrak{g}$ of K.
- The Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{Ad}(X) \circ \mathrm{Ad}(Y))$ on $\mathfrak{g}$.
- The orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ under B.
- The center U of K and its lie algebra $\mathfrak{u}$.

Through these one can construct:

- A G invariant metric on M (via B).
- The Dual lie algebra $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$.
- The ideals $\mathfrak{p}^+, \mathfrak{p}^- \subset \mathfrak{p}_{\mathbb{C}}$ which are the eigenspaces of $\mathfrak{u}$.
- The parabolic subgroup $P^- = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$.
- The embedding $M = G/K \hookrightarrow G_{\mathbb{C}}/P^- \simeq \check{M}$.

To compute Lie algebra's we wish to view them as an infinitessimal neighbourhood of the identity. We shall view all Lie algebra's for matrix groups (or groups which are isogenous to matrix groups) as sub-Lie algebras of that of GL_n. In particular they are subalgebras of M_n with lie bracket $[X,Y] = XY - YX$.

So $\mathfrak{g} = \mathrm{Lie}(G) = \{X \in M_{2g} | 1 + \epsilon X \in G\}$ where we view $\epsilon \in \mathbb{R}[\epsilon]/(\epsilon^2)$.

We might then expect the killing form to be up to constant multiples that of $\mathfrak{gl}_n$ in particular a constant multiple of $\mathrm{Tr}(XY)$. One can can certainly produce subalgebras for which this would not hold. However, for semi-simple lie algebras one has that the Killing form must always be a constant multiple of that for $\mathfrak{gl}_n$.

2. Symplectic Group

We now consider the case $G = \mathrm{Sp}_{2g}(\mathbb{R})$.

Recall that $G = \{g \in \mathrm{GL}_{2g} | g^t J g = J\}$ where $J = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}$.

So writing $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this gives A^tC, D^tB symmetric and $A^tD - C^tB = \mathrm{Id}_g$. From this one can conclude that $g^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$, Indeed:

$$\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^tA - B^tC & D^tB - B^tD \\ A^tC - C^tA & A^tD - C^tB \end{pmatrix} = \mathrm{Id}_{2g}$$

We now see that $U_g = \mathrm{SO}_{2g} \cap \mathrm{Sp}_{2g}$, since SO_{2g} are the elements of GL_{2g} fixed by transpose inverse the above formula for the inverse gives the result. ($U_g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.)

From this we see that the involution $s : g \mapsto (g^t)^{-1}$ of G is such that $K = U_g = G^s$. One can recover this as an inner automorphism as conjugation by J.

$$J^{-1} M J M^t = \mathrm{Id}_2\, g$$

From which we see conjugation by J gives the inverse of the transpose.

2.1. **Why is U_g maximal compact?**

Noticing that when A is Id_g except -1 on the i diagonal then any quadratic form stabilized by K is of the form $\begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$ with all matricies diagonal. The matricies $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ allow us to show that W, X, Y, Z must be a constant multiple of hte identity. The matrix J then implies that $W = Z$ and $X = -Y$. This gives us a complete characterization of every bilinear form preserved by U_g under its usual representation.

To see that all the compacts in GL_n are in orthogonal groups consider:

$$Q(\vec{x}) = \int_G |g\vec{x}|^2 \, dg$$

which is a non-degenerate G invariant positive definite quadratic form whenever G is compact.

2.2. **Some remarks.** One should be warned that the isometry group of $\mathrm{Sp}_{2g}(\mathbb{R})/U_g(\mathbb{R})$ is actually $\mathrm{PSp}_{2g}(\mathbb{R})$ and not $\mathrm{Sp}_{2g}(\mathbb{R})$ as $-\mathrm{Id}_{2g}$ acts trivially.

It is perhaps also worth noting that even though $\mathrm{Sp}_{2g}(\mathbb{C})$ is simply connected, $\mathrm{Sp}_{2g}(\mathbb{R})$ has non-trivial fundamental group and hence admits non-trivial non-algebraic covers. The double cover is the metaplectic group.

2.3. **Lie Algebra's.** The Lie algebra $\mathfrak{sp}_{2g}$ is given by:

$$\mathfrak{sp}_{2g} = \{X \in M_{2g} | (\mathrm{Id}_{2g} + X^t\epsilon) J (\mathrm{Id}_{2g} + X\epsilon) = J\}$$

The condition is then $X^t J + JX = 0$. For a matrix $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ this becomes C, B symmetric and $A = -D^T$.

(Since the lie algebra of $\mathfrak{so}_{2g}$ is skew-symmetric matricies) we have that the lie subalgebra $\mathfrak{k}$ of $K = \mathfrak{so}_{2g} \cap \mathfrak{sp}_{2g}$ is given by $X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where A is skew symmetric and B is symmetric.

One can compute the killing form to be given by $B(X,Y) = (2g+2)\,\mathrm{Tr}(XY)$. (This is done by computing $B(X,X)$ for X coming from a cartan subalgebra and using conjugacy invariance and density of these elements.

We can then check that $\mathfrak{p} = \mathfrak{k}^\perp$ are the elements $X = \begin{pmatrix} D & C \\ C & -D \end{pmatrix}$ where C, D are symmetric. The space is naturally a complementary space, the calculation

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} D & C \\ C & -D \end{pmatrix} = \begin{pmatrix} AD + BC & * \\ * & -(AD + BC) \end{pmatrix}$$

shows that it is perpendicular.

The center of $K = U_g$ is $Z(U_g) = \{\begin{pmatrix} a\,\mathrm{Id}_g & b\,\mathrm{Id}_g \\ -b\,\mathrm{Id}_g & a\,\mathrm{Id}_g \end{pmatrix}\} \simeq SO_2(\mathbb{R})$ its lie algebra is $\mathfrak{u} = \{\begin{pmatrix} 0 & x\,\mathrm{Id}_g \\ -x\,\mathrm{Id}_g & 0 \end{pmatrix}\}$.

We observe that the adjoint action of $\mathfrak{u}$ on $\mathfrak{g}$ has the following eigenspaces:

The zero eigenspace is $\mathfrak{k}$.

The $+$ eigenspace $\mathfrak{p}^+ = \{\begin{pmatrix} C & iC \\ iC & -C \end{pmatrix} \mid C \text{ symmetric}\}$.

The $-$ eigenspace $\mathfrak{p}^- = \{\begin{pmatrix} C & -iC \\ -iC & -C \end{pmatrix} \mid C \text{ symmetric}\}$.

The corresponding parabolic subgroup is P^-

$$\{\begin{pmatrix} 1 + C & -iC \\ -iC & 1 - C \end{pmatrix} \mid C \text{ symmetric}\} \cdot U_g(\mathbb{C})$$

This is precisely the subgroup of matricies in $\mathrm{Sp}_{2g}(\mathbb{C})$ which perserve the subspace $F = \mathrm{span}(x, -ix)$ (where x are vector of dimension g).

2.4. **The compact form.** The compact form of the group corresponds to the lie subalgebra $\mathfrak{k} \oplus i\mathfrak{p}$.

The associated lie group is Sp_g which is the subgroup of invertible $g \times g$ quaternionic matricies which preserve the hermitian pairing on $\mathbb{H}^g$ given by $\sum_i \overline{x_i} y_i$.

One views this as a subgroup of GL_{2g} by the usual interpretation of $\mathbb{H}$ as a matrix group. $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$ Under this interpretation one sees that transpose conjugate is conjugation.

One can then view Sp_g as $\{M = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}\}$ which satisfy $M\overline{M}^t = \mathrm{Id}_{2g}$.

$$\begin{pmatrix} \overline{A}^t & -B^t \\ \overline{B}^t & A^t \end{pmatrix} \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} = \begin{pmatrix} \overline{A}^t A - B^t \overline{B} & \overline{A}^t B - B^t \overline{A} \\ * & * \end{pmatrix}$$

The condition is thus that $A^t B$ is hermitian symmetric and $\overline{A}^t A - B^t \overline{B} = \mathrm{Id}_g$. From this one concludes the inverse is $\begin{pmatrix} \overline{A}^t & -B^t \\ \overline{B}^t & A^t \end{pmatrix}$ and again we see $U_g = \mathrm{O}_{2g} \cap \mathrm{Sp}_g$.

3. Orthogonal Group

Now the case $G = \mathrm{SO}(2, n)(\mathbb{R})$

Recall that $G = \{g \in \mathrm{GL}_n \,|g^t J g = J\}$ where $J = \begin{pmatrix} -\mathrm{Id}_2 & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix}$.

A maximal compact here is $K = \mathrm{SO}(2, n)(\mathbb{R}) \cap \mathrm{SO}(2+n)(\mathbb{R}) = S(\mathrm{O}(2) \times \mathrm{O}(n))$.

The involution again is s given by transpose-inverse (which again corresponds to conjugation by J)

The lie algebra is those matricies such that $X^t J + JX = 0$, This condition implies that $X = \begin{pmatrix} A & C \\ C^t & D \end{pmatrix}$ where A, D are skew symmetric of dimension $2, n$ respectively (C is arbitrary).

The killing form is given by $n\,\mathrm{Tr}(XY)$.

The lie subalgebra of $\mathfrak{k}$ is $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ where A, D are skew symmetric, and thus $\mathfrak{p}$ are the matrices $\begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix}$.

For $n \neq 2$ the center of K is the S part of its O_2 factor, its lie algebra $\mathfrak{u}$ is given by $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. (For $n = 2$ the symmetric space is not irreducible, indeed it is given by $\mathbb{H} \times \mathbb{H}$.)

The eigenspaces for its adjoint action decompose C^t as (x, ix) and $(x, -ix)$ where x is a $1 \times n$ column vector.

The parabolic P^- is then precisely the stabilizer of $(1, i, 0, \ldots, 0)$.

The compact dual group is $\mathrm{SO}(2+n)$.