Questions in the Theory of Orthogonal Shimura Varieties

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Motivation

The motivation for all of this work is the study of orthogonal Shimura varieties. A lot of problems in the theory of Shimura varieties revolve around special points and modular forms. Two natural problems, and those that motivate us are:

- Characterizing special points on orthogonal Shimura varieties.
- Finding explicit dimension formulas for the spaces of modular forms on orthogonal Shimura varieties.

The problems we actually solve are all applicable to the above problems, but end up begin of interest to other people for entirely different reasons.

Throughout these slides k will be a finite extension of either $\mathbb{Q}$ of $\mathbb{Q}_p$.

First Question

Characterizing Special Points on Orthogonal Shimura Varieties

aka

Characterization of Algebraic Tori in Orthogonal Groups

Étale Algebras, Tori, and Orthogonal Groups

Given an étale algebra with involution $(E, \sigma)/k$, and $\lambda \in (E^{\sigma})^{\times}$. We define a quadratic form $q_{E,\sigma,\lambda} : E \to k$ by:

$$q_{E,\sigma,\lambda}(x) = \frac{1}{2} Tr_{E/k}(\lambda x \sigma(x)).$$

This makes $(E, q_{E,\sigma,\lambda})$ a quadratic space over k of dimension $\dim(E)$.

Moreover, given such an étale algebra, we define the algebraic torus $T_{E,\sigma}$ by specifying that its points over a field K/k are:

$$T_{E,\sigma}(K) := \{x \in (E \otimes_k K)^{\times} \mid x\sigma(x) = 1\}.$$

There is a natural inclusion:

$$T_{E,\sigma} \hookrightarrow O_{q_{E,\sigma,\lambda}}.$$

What makes this construction special is that all maximal tori in all orthogonal groups have this shape, and moreover, all quadratic forms containing such a torus come from this construction.

Theorem (BCKM)

Let (V, q) be a quadratic space over k of even dimension. If a torus:

$$T \hookrightarrow O_q$$

as a maximal torus then:

- *There exists (E, σ) over k with $T \simeq T_{E,\sigma}$.*
- *There exists $\lambda \in (E^\sigma)^\times$ with $q \simeq q_{E,\sigma,\lambda}$.*

The previous result describes all the tori, but some questions still remain:

For a fixed q, which (E, σ) actually appear?

Conversely, for a fixed (E, σ) which q can we obtain?

Given q and (E, σ) how to check if there exist λ such that $q \simeq q_{E,\sigma,\lambda}$?

We can explicitly compute the invariants.

Theorem (F)

The invariants of $q_{E,\sigma,\lambda}$ (which determine it) are:

- *Discriminant is:* $D(q_{E,\sigma,\lambda}) = (-1)^{n/2}\delta_{E/k}$.
- *Hasse invariant is:*

$$H(q_{E,\sigma,\lambda}) = Cor_{E^\sigma/k}(((-1)^{n/2}\lambda f_z'(z), z)_{E^\sigma})(-1,-1)_k^{n(n-2)/8}$$

  where $z \in E^\sigma$ *is such that* $\sqrt{z}$ *primitively generates* E *and* f_z *is its minimal polynomial.*
- *Signature is* $(2r+2s+t, n-(2r+2s+t))_\rho$ *where*
  - s *is the number of real places of* E^σ *over* ρ *that become complex in* E *having* $\lambda > 0$,
  - r *is the number of complex places of* E^σ *over* ρ, *and*
  - t *is the number of real places of* E^σ *over* ρ *that stay real in* E.

When Does a Given Torus Embed?

We have a concrete conditions on embedding tori.

Theorem (F)

If $T_{E,\sigma}$ is a maximal torus in O_q then for all σ-types ϕ of (E,σ) we have $E^\phi \hookrightarrow Cliff^+(V,q)$ with the canonical involution restricting to σ on E^ϕ.

Note: the above tells us about tori in spin groups.

Theorem (F)

Suppose k is a finite extension of $\mathbb{Q}_p$. Let (E,σ) be an etale algebra with involution of dimension n and (V,q) be a quadratic space of dimension n then the torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if:

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- E^ϕ *splits* $Cliff^+(V,q)$ *for all* σ*-types* ϕ.

Note: the corresponding theorem at Archimedian places is a simple condition on signatures.

Locally Everywhere Conditions

Now let k be a finite extension of $\mathbb{Q}$. We get locally everywhere embedding from global conditions.

Theorem (F)

Let (E,σ) be an étale algebra with involution of dimension n and (V,q) be a quadratic space of dimension n then the torus:

$$T_{E,\sigma} \otimes k_{\mathfrak{p}} \hookrightarrow O_q \otimes k_{\mathfrak{p}}$$

for all places (including infinite) $\mathfrak{p}$ of k if and only if

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- *E^{ϕ} splits Cliff$^{+}(V,q)$ for all ϕ.*
- *The signature condition at all real places.*

Note: the conditions are global, the conclusion is still only locally everywhere.

Global Conditions

Unfortunately, an example of Prasad and Rapinchuk shows that the local-global principle fails in general. However, there are cases where we can show it still holds, for example:

Theorem (F)

Let (E, σ) be a number field with involution with $\dim(E) = \dim(V)$. The torus $T_{E,\sigma} \hookrightarrow O_q$ if and only if

- $D(q) = (-1)^{n/2}\delta_{E/k}$.
- *E^ϕ splits Cliff$^+(V, q)$ for all ϕ.*
- *The signature condition at all real places.*

Note: The above theorem is also true in the specific case of E a CM-algebra (the local-global principle for this case is explained in my thesis, but also follows from the work of Bayer-Fluckiger).

Second Question

Lead Terms in Dimension Formulas for Spaces of Modular Forms

aka

Local Densities

What are Local Densities?

The local density:

$$\beta_{\mathfrak{p}}(L, M) = \operatorname*{Lim}_{r\to\infty} q^{\frac{rm(m+1-2n)}{2}} \left|\{\phi : M \to N/\pi^r \mid Q_N(\phi(x)) = Q_M(x)\ (2\pi^r)\}\right|$$

is a way to measure the volume of the set of isometries between lattices L and M.

We prove formulas for:

$$\beta_{\mathfrak{p}}(L, L)$$

for L a general lattice over an arbitrary finite extension of $\mathbb{Z}_p$ with an arbitrary p. Such formulas existed previously for very few extensions of $\mathbb{Z}_2$.

Every lattice over a finite extension of $\mathbb{Z}_p$ has at least one Jordan decomposition. These decompose the lattice into a direct sum of rescalings of unimodular lattices.

For $p \neq 2$ unimodular lattices are classified by discriminant.

For $p = 2$ the classification of unimodular lattices consists of:

1. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$,
2. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus \begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}(\alpha-4\rho)\pi^{r-t} \end{pmatrix}$,
3. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 0 \end{pmatrix} \oplus (-\delta_L)$,
4. $H^n \oplus \begin{pmatrix} \pi^s & 1 \\ 1 & 4\rho\pi^{-s} \end{pmatrix} \oplus (-(1-4\rho)\delta_L)$,
5. $\begin{pmatrix} a\pi^t & 1 \\ 1 & -a^{-1}\alpha\pi^{r-t} \end{pmatrix}$ or
6. $(-(1-\alpha\pi^r))$

with conditions on r, s, t, a, α, ρ.

This theorem reduces the computation of local densities for unimodular lattices to computations with unimodular lattices of rank at most 4.

Theorem (F)

Consider a unimodular lattice Λ. Then Λ has a decomposition $\Lambda = L(e) \oplus L$, where $L(e)$ is a maximal even-dimensional, even, unimodular sublattice of Λ and L has rank at most 4. Let $\ell = \mathrm{rank}(L)$ and $2n = \mathrm{rank}(L(e))$. Then:

$$\beta_R(\Lambda, \Lambda) = [L : L^{(2)}]^{-2n} \cdot \xi \cdot \beta_R(L, L) \cdot \prod_{e=1}^{n}(1 - q^{-2e}),$$

where:

$$\xi = \begin{cases} 2(1 + \chi(L(e))q^{-n})^{-1} & L(e) \text{ non-trivial, independent of choices} \\ 1 & \text{otherwise.} \end{cases}$$

$L^{(2)}$ is the sublattice of L consisting of elements with $Q(x) \in (2)$.

The Special Cases

We can compute the local density for the low rank cases in terms of the parameters r, s, t.

Theorem (F)

Consider a unimodular lattice L of rank at most 4 over a 2-adic ring R with no even unimodular factors. Then:

- *Case $n = 4$.*
$$\beta_R(L,L) = 4q^{-3\nu(2)+2t-2-(r-t-s)/2} \begin{cases} q^{(r-t-t-1)/2+1} & r-t \leq \nu(2) \\ q^{\lfloor(\nu(2)-t)/2\rfloor+1} & \nu(2) \leq r-t \end{cases}$$
- *Case $n = 3$.*
$$\beta_R(L,L) = 4q^{(1-t)/2}.$$
- *Case $n = 2$.*
$$\beta_R(L,L) = \begin{cases} 4q^{(r-1)/2-\nu(2)} & r-t \leq \nu(2) \\ 2q^{-\lceil(\nu(2)-t)/2\rceil} & \nu(2) < r-t. \end{cases}$$
- *Case $n = 1$.*
$$\beta_R(L,L) = 2.$$

Note: we can compute $[L : L^{(2)}]$ and if $L(e)$ is independent of choices.

We can compute the local density by knowing the set of all Jordan decompositions.

Let JD_L denote the set of all Jordan decompositions up to isomorphism for the lattice L. For $I \in JD_L$ let L_i^I denote the i^{th} Jordan block.

Theorem (F)

With the notation as above, for L an arbitrary lattice, we have:

$$\beta_R(L, L) = q^{\tilde{w}} \left(\sum_{I \in JD_L} \prod_i \beta_R(\tilde{L}_i^I, \tilde{L}_i^I)^{-1} \right)^{-1},$$

where $\tilde{L}_i^I$ is the unimodular rescaling of L_i^I and $\tilde{w}$ is given by:

$$\tilde{w} = \sum_i i \cdot n_i \left(\sum_{j>i} n_j \right) + \sum_i (n_i(n_i + 1)/2).$$

Note: effectively enumerating the set JD_L is non-trivial.

Third Question

Putting Together the First Two Questions

aka

Transfer of Lattices

Why Transfer?

The first topic told us we should care about the quadratic forms:

$$\mathrm{Tr}_{E/k}(\lambda x \sigma(x))$$

because these are related to CM-points, which connect to class field theory via modular forms.

The second topic was concerned about local densities, because Hirzebruch-Mumford proportionality and Riemann-Roch tell us this relates to the lead term in a dimension formula for spaces of modular forms.

So the question is, can we use the structure of E to understand the arithmetic volume of lattices contained in E with the quadratic forms:

$$\mathrm{Tr}_{E/k}(\lambda x \sigma(x))?$$

Example of Arithmetic Volumes for Hermitian forms over $\mathbb{Q}$

Consider the ring of integers of $\mathbb{Q}(\zeta_p)$ as a lattice over $\mathbb{Z}$ with quadratic form:

$$\tfrac{1}{2}\operatorname{Tr}_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\lambda x \sigma(x)).$$

Let $\lambda = 2\mathfrak{q}$, where $(\mathfrak{q})|q \neq p$ is prime and $(q,p)_{\mathfrak{p}} = -1$, set $n_q = \nu_q(N_{F/\mathbb{Q}}(\mathfrak{q}))$ and suppose $(\mathfrak{q}) \neq (q)$ and $\mathfrak{q}$ is totally positive, the arithmetic volume (ie. the product over all p of the local densities) is:

$$2^2 p^{(p-2)(p-1)/2} q^{n_q(n_q-1)/2} (1-q^{n_q}) \left(1 + q^{(p-1)/2-n_q}\right) \prod_{i=1}^{n_q} \left(1-q^{2i}\right)^{-1}$$
$$\prod_{i=1}^{p-1-n_q} \left(1-q^{2i}\right)^{-1} \prod_{\ell \neq q} \left(\left(1 + \left(\frac{\ell}{p}\right) \ell^{(p-1)/2}\right) \prod_{i=1}^{(p-1)/2} (1-\ell^{-2i})^{-1} \right).$$

One can work out explicitly many other cases. I do a number of examples with $\mathbb{Q}(\zeta_p)$ in my thesis.

What we are looking at is a special case of the following:
If L is an extension of K, and R_1 and R_2 are respectively the maximal orders of K and L then any R_2 lattice L is also an R_1 lattice with the associated quadratic form:

$$q_{R_1}(x) = \mathrm{Tr}_{L/K}(q_{R_2}(x)).$$

We call this construction transfer.

In order to compute the local densities for the Hermitian lattices in which we are interested, we need to understand the structure of lattices that arise this way.

The rational invariants (discriminant, signatures, Hasse invariants) are known in many cases; we will focus on the integral invariants.

What we can show

We are able to obtain sufficient detail about the structure of the Jordan decompositions of lattices under transfer in order to get formulas for the local densities of Hermitian lattices when the base ring is $\mathbb{Z}_p$.

Theorem (F)

We can compute the dimensions, the modularity, the discriminant mod π and whether or not the quadratic form represents an element of $R^\times$ for each of the unimodular components of a Jordan decomposition.

We do this explicitly for an arbitrary unary, binary or Hermitian form.

The End

Thank you.