

EMBEDDINGS OF MAXIMAL TORI IN GROUPS OF TYPE F_4

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ABSTRACT. We classify maximal tori in groups of type F_4 over a local or global field of characteristic different from 2 and 3. We prove a local-global principle for embeddings of maximal tori in groups of type F_4 .

1. INTRODUCTION

Let k be a field, and let G be a semisimple group over k (not necessarily k -split). It is a natural problem to classify the isomorphism classes of maximal k -tori of G .

Assume that G is a classical group, and let (A, τ) be the associated central simple algebra with involution. Then it is well known that maximal tori in G are associated to certain étale algebras with involution (E, σ) inside (A, τ) . However, to find necessary and sufficient conditions for the existence of an embedding of (E, σ) in (A, τ) can be quite difficult, especially without restrictive assumptions on k .

The prototypical situation is that of special orthogonal groups. This case has been studied by several authors, and has found applications to many arithmetic problems, for example to special points in symmetric spaces; see e.g. [BCKM03], [Fio12], [Gil04], [PR10], [BF14], [Fio17]. In the majority of these works, k is assumed to be a local or global field: on the one hand, this allows to obtain clear and explicit results, and on the other, such fields are the most important in many applications.

Assume that k is a global field such that $\text{char } k \neq 2$, let (V, q) be a quadratic k -space of even dimension $2n$, and let $G = \text{SO}(V, q)$. Let (E, σ) be an étale algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$, and let $U(E, \sigma)$ be the k -torus of rank n defined by

$$U(E, \sigma)(R) := \{x \in (E \otimes_k R)^\times : x\sigma(x) = 1\}$$

for every k -algebra R .

By [BF14, Proposition 1.2.1], if T is maximal k -torus in $\text{SO}(V, q)$, there exists a unique étale subalgebra E' of $\text{End}(V)$ of rank $2n$ that is stable under the adjoint involution $\tau_q : \text{End}(V) \rightarrow \text{End}(V)$ such that $T = U(E', \tau_q|_{E'})$; moreover $\text{rank}_k E' = 2 \text{rank}_k (E')^{\tau_q} = 2n$; see Section 6 for more details. If (E, σ) is an étale algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$, we say that (E, σ) is *realizable* if there exists a maximal torus T of $\text{SO}(V, q)$ such that the associated (E', τ_q) is isomorphic to (E, σ) , and we say that T is of *type* (E, σ) ; see [BF14, §1.2].

The problem is then to give necessary and sufficient conditions on (E, σ) and (V, q) for the realizability of (E, σ) . The case of local fields is easier, so it seems

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natural to study the local case first, and then to attack the global case by proving a local-global principle. The local-global question that must be answered is the following: if (E_v, q_v) is realizable over k_v for every place v of k , is it true that (E, q) is realizable over k ?

It turns out that the answer to this question is negative, as shown by a counterexample of G. Prasad and A. Rapinchuk [PR10, Example 7.5]. In [Fio12], the first author gave necessary and sufficient conditions for the everywhere local realizability of (E, q) , that is, he answered the question of when (E_v, q_v) is realizable for every place v . Moreover, he proved the local-global principle in the case when E is a field. The failure of the local-global principle was later completely clarified by E. Bayer-Fluckiger [BF14]. In subsequent papers [BFLP15], [BFLP16], [BFLP18], E. Bayer-Fluckiger, T.-Y. Lee and R. Parimala more generally studied embeddings of maximal tori in classical groups.

We turn to the situation for exceptional groups. If G is of type G_2 , embeddings of maximal tori in G have been studied in [BGL15] and [Hoo18]. For outer forms of type D_4 (in particular, triality forms), classifications and local-global principles in many cases have been established in [Fio17].

In the present work, we analyze the case when G is of type F_4 over a global field k such that $\text{char } k \neq 2, 3$. In analogy with the case of orthogonal groups, we parametrize the tori which embed in some group of type F_4 using étale algebras with extra structure. More precisely, in Proposition 6.4 we show that maximal tori in groups of type F_4 can be constructed from a datum $\alpha = (L, E, \sigma, \beta)$, where L is a cubic étale k -algebra, (E, σ) is an étale L -algebra with involution such that $\text{rank}_L E = 2 \text{rank}_L E^\sigma = 8$ and β is an isomorphism of étale L -algebras defined in (6.1). We say that the pair (G, α) is *realizable* if there exists a maximal torus T of G with associated datum α . We give a functorial description of these tori in terms of α which is entirely analogous to that of tori $U(E, \sigma)$; see (6.2).

Our main result is a proof of the local-global principle for maximal tori inside G .

Theorem 1.1. *Let k be a global field of characteristic different from 2 and 3, let G be a k -group of type F_4 , and let α be a datum. Then (G, α) is realizable if and only if (G_v, α_v) is realizable for every place v of k .*

The validity of the local-global principle is quite surprising, especially because our proof relies on the results for orthogonal groups of type D_4 , for which the local-global principle does not hold, in general. More precisely, our proof of Theorem 1.1 uses the following new result on the local-global principle for embeddings of tori in orthogonal groups. Let (V, q) be a quadratic space of even dimension over a global field k , let (E, σ) be an étale algebra with involution over k , and assume that (E_v, q_v) is realizable for every place v of k . As we have said, this does not imply that (E, q) is realizable. However, in Proposition 10.1 we show that if q has trivial Clifford invariant, then (E, q) is realizable. In other words, the local-global principle for embeddings of maximal tori in orthogonal groups with trivial Clifford invariant holds.

Using Theorem 1.1 and a local analysis, we are able to classify maximal tori for all groups of type F_4 over global fields. Recall that there are exactly three real forms of type F_4 : the split form, the anisotropic form, and the form of real rank 1.

If (E, σ) is an étale algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$, we say that a real place w of E^σ is ramified if it extends to a complex place of E .

We denote by ρ_v the number of real places of E^σ above v which are not ramified in E . We have $0 \leq \rho_v \leq n$.

Theorem 1.2. *Let k be a global field of characteristic different from 2 and 3, let G be a group of type F_4 , and let $\alpha = (L, E, \sigma, \beta)$ be a datum. Then (G, α) is realizable if and only if for every real place v of k one of the following holds:*

- (i) G_v is split and either $L_v = \mathbb{R}^3$ and ρ_v is even, or $L_v = \mathbb{R} \times \mathbb{C}$ and ρ_v is odd;
- (ii) G_v is anisotropic, $L_v = \mathbb{R}^3$ and $\rho_v = 0$;
- (iii) G_v has real rank 1 and either $L_v = \mathbb{R}^3$ and ρ_v is even, or $L_v = \mathbb{R} \times \mathbb{C}$ and $\rho_v = 1$.

It is not difficult to describe the possible isomorphism classes of maximal tori in G_v , for every real place v ; see Proposition 8.4.

We conclude the paper with a brief discussion of the topic of rational conjugacy classes of tori in groups of type F_4 ; this can be found in Section 12.

2. CONSTRUCTION OF H - THE SPLIT CASE

Our main references for octonion algebras, twisted compositions and Albert algebras are [SV00] and [KMRT98].

Let k be a field. We assume that $\text{char } k \neq 2, 3$. We denote by \mathbb{O} the split octonion algebra over k . As a k -vector space, $\mathbb{O} = M_2(k)^{\oplus 2}$. For every

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(k)$$

we define

$$\bar{A} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$

so that $A\bar{A} = \bar{A}A = \det(A) \cdot I$. The product on \mathbb{O} is defined as

$$(A, B)(C, D) := (AC + \bar{D}B, DA + B\bar{C}).$$

The norm on \mathbb{O} is defined as $n(A, B) = \det(A) - \det(B)$. We denote by $\langle \cdot, \cdot \rangle$ the associated inner product on \mathbb{O} .

Let G_s be the split group of type F_4 . We have $G_s = \text{Aut}(J_s)$, where $J_s \subseteq M_3(\mathbb{O})$ is the k -split Albert algebra:

$$J_s := \left\{ \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix} : \xi_i \in k, c_i \in \mathbb{O} \right\}.$$

We denote by \cdot the matrix product in $M_3(\mathbb{O})$, and by juxtaposition the product in J_s , that is

$$xy = \frac{1}{2}(x \cdot y + y \cdot x)$$

for every $x, y \in J_s$.

Denote by Q the norm of J_s ; see [SV00, (5.3)]. It is a non-degenerate quadratic form on J_s . Let $\langle \cdot, \cdot \rangle$ be the bilinear form associated to Q , called the inner product of J_s . We denote by e the identity element of J_s . Let \det be the determinant form of J_s , as defined in [SV00, (5.7)]: it is a cubic form on J_s . Denote by $\langle \cdot, \cdot, \cdot \rangle$ the

trilinear form associated to \det . For every $x, y \in J_s$, define the cross product $x \times y$ as the unique element of J_s such that

$$\langle x \times y, z \rangle = 3 \langle x, y, z \rangle$$

for every $z \in J_s$; see [SV00, §5.2].

We set

$$L_s := \left\{ \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} : \xi_i \in k \right\}$$

and

$$M_s := \left\{ \begin{pmatrix} 0 & c_3 & \bar{c}_2 \\ \bar{c}_3 & 0 & c_1 \\ c_2 & \bar{c}_1 & 0 \end{pmatrix} : c_i \in \mathbb{O} \right\}.$$

It is clear that L_s is a subalgebra of J_s . We will give M_s the structure of an L_s -module; see Lemma 2.1. We have $M_s = L_s^\perp$. An element of M_s may be written as a triple (c_1, c_2, c_3) , where $c_i \in \mathbb{O}$. This yields a direct sum decomposition $M_s = J_1 \oplus J_2 \oplus J_3$, where $J_i \cong \mathbb{O}$ and $c_i \in J_i$. Thus

$$J_s = L_s \oplus M_s = L_s \oplus J_1 \oplus J_2 \oplus J_3.$$

It follows immediately from the definition of Q that the restriction of Q to the J_i is the octonion norm.

The subgroup of G_s fixing every element of L_s is isomorphic to Spin_8 , and the subgroup of automorphisms preserving L_s is $\text{Spin}_8 \rtimes S_3$; see [KMRT98, §39.19]. We denote by $H_s \cong \text{Spin}_8$ its connected component. By [KMRT98, §35.8], for every k -algebra R , we have

$$H_s(R) = \left\{ (g_1, g_2, g_3) \in \text{SO}_8^3(R) : g_1(x \cdot y) = \overline{g_3(\bar{x})} \cdot \overline{g_2(\bar{x})} \text{ for all } x, y \in \mathbb{O} \otimes_k R \right\}.$$

We now construct a k -split maximal torus T_s of G_s , following the exposition of [Mac14].

For every $\lambda \in k^\times$, let R_λ and L_λ be the linear automorphisms of \mathbb{O} given by right and left multiplication by

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

respectively. Consider the following elements of $\text{SO}_8^3(k)$:

$$s_\lambda := (R_\lambda, L_{\lambda^{-1}} \circ R_{\lambda^{-1}}, L_\lambda), \quad r_\lambda := (L_\lambda, R_\lambda, L_{\lambda^{-1}} \circ R_{\lambda^{-1}}).$$

One can check that $r_\lambda, s_\lambda \in \text{Spin}_8(k)$, and that the collection $\{r_\lambda, s_\lambda\}_{\lambda \in k^\times}$ generates a 2-dimensional split torus T_1 ; see [Mac14, 1.6].¹

The group Spin_8 has a k -split subgroup of type G_2 . Its k -points may be described as the triples $(g_1, g_2, g_3) \in \text{SO}_8^3(k)$ such that $g_i(1) = 1$; see [KMRT98, §35.16]. It is clear from this description that G_2 and T_1 intersect trivially. The group G_2 contains a subgroup isomorphic to SL_3 ; it is the stabilizer of

$$i := \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right),$$

¹More precisely, one should define r_λ and s_λ for every $\lambda \in R^\times$ and every k -algebra R , and then check that this defines a k -group homomorphism $\varphi : \mathbb{G}_m^2 \rightarrow G$. We let T_1 be the image of φ .

in the 7-dimensional representation of G_2 (trace-zero octonions); see [KMRT98, §36, Exercise 6] or [SV00, §2.2]. Denote by T_2 the maximal torus inside SL_3 constructed in [SV00, §2.3]. The tori T_1 and G_2 intersect trivially and commute; see [Mac14, 1.7]. We define T_s as the torus generated by T_1 and T_2 : it is a k -split maximal torus of G_s . The torus T_s acts trivially on L_s , and its action on $M_s = L_s^\perp$ has no non-zero fixed points.

We now want to endow (L_s, M_s) with the structure of a split twisted composition over k ; see [KMRT98, §36] for the definition. In the next paragraph we will carry out a similar construction for an arbitrary Albert algebra J . For this reason, it is useful to give an intrinsic definition first, and only later write it in coordinates.

Using multiplication in J_s and projections, one may construct k -bilinear maps

$$(2.1) \quad \mu_{L_s} : L_s \times L_s \rightarrow L_s, \quad \mu_{M_s} : M_s \times M_s \rightarrow M_s.$$

For example, μ_{L_s} is the composition

$$L_s \times L_s \hookrightarrow J_s \times J_s \rightarrow J_s \xrightarrow{\pi_s} L_s$$

where the second map is multiplication in J_s and π_s is projection onto L_s ; a similar construction gives μ_{M_s} . There is also a k -bilinear map

$$(2.2) \quad \begin{aligned} \eta_s : L_s \times M_s &\rightarrow M_s \\ (\xi, c) &\mapsto -2\xi c + \langle \xi, e \rangle c. \end{aligned}$$

One could also define η_s using the cross product on J_s ; see [SV00, p. 162]. That η_s really takes values in M_s will be shown in Lemma 2.1.

Finally, we have a k -bilinear map

$$\varphi_s : M_s \times M_s \rightarrow L_s,$$

given by the composition

$$M_s \times M_s \hookrightarrow J_s \times J_s \xrightarrow{\times} J_s \xrightarrow{\pi_s} L_s.$$

We define $q_s := -2\varphi_s$. If $\xi \in L_s$ and $c \in M_s$, we express them in coordinates as $\xi = (\xi_1, \xi_2, \xi_3)$ and $c = (c_1, c_2, c_3)$.

Lemma 2.1. *Let $\xi, \xi' \in L_s$ and $c, c' \in M_s$. The following identities hold:*

$$(2.3) \quad \mu_{L_s}(\xi, \xi') = (\xi_1, \xi_2, \xi_3)(\xi'_1, \xi'_2, \xi'_3) = (\xi_1 \xi'_1, \xi_2 \xi'_2, \xi_3 \xi'_3) \in L_s.$$

$$(2.4) \quad \eta_s(\xi, c) = -2\xi c + \langle \xi, e \rangle c = (\xi_1 c_1, \xi_2 c_2, \xi_3 c_3) \in M_s.$$

$$(2.5) \quad q_s(c, c') = (\langle c_1, c'_1 \rangle, \langle c_2, c'_2 \rangle, \langle c_3, c'_3 \rangle) \in L_s.$$

$$(2.6) \quad \mu_{M_s}(c, c) = (\overline{c_2 c_3}, \overline{c_3 c_1}, \overline{c_1 c_2}) \in M_s.$$

Proof. The verification of (2.3) is immediate.

We have

$$\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & c_3 & \overline{c_2} \\ \overline{c_3} & 0 & c_1 \\ c_2 & \overline{c_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi_1 c_3 & \xi_1 \overline{c_2} \\ \xi_2 \overline{c_3} & 0 & \xi_2 c_1 \\ \xi_3 c_2 & \xi_3 \overline{c_1} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & c_3 & \overline{c_2} \\ \overline{c_3} & 0 & c_1 \\ c_2 & \overline{c_1} & 0 \end{pmatrix} \cdot \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & \xi_2 c_3 & \xi_3 \overline{c_2} \\ \xi_1 \overline{c_3} & 0 & \xi_3 c_1 \\ \xi_1 c_2 & \xi_2 \overline{c_1} & 0 \end{pmatrix}.$$

Hence

$$(2.7) \quad \xi c = \frac{1}{2}((\xi_2 + \xi_3)c_1, (\xi_1 + \xi_3)c_2, (\xi_1 + \xi_2)c_3) \in M_s.$$

Using that $\langle \xi, e \rangle = \xi_1 + \xi_2 + \xi_3$, one concludes that

$$\eta_s(\xi, c) = -2\xi c + \langle \xi, e \rangle c = (\xi_1 c_1, \xi_2 c_2, \xi_3 c_3) \in M_s,$$

proving (2.4).

By [SV00, Lemma 5.2.1(i)] we have

$$x \times y = xy - \frac{1}{2} \langle x, e \rangle y - \frac{1}{2} \langle y, e \rangle x - \frac{1}{2} \langle x, y \rangle e + \frac{1}{2} \langle x, e \rangle \langle y, e \rangle e$$

for every $x, y \in J$. Using this formula together with

$$(2.8) \quad \begin{pmatrix} 0 & c_3 & \overline{c_2} \\ \overline{c_3} & 0 & c_1 \\ c_2 & \overline{c_1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & c'_3 & \overline{c'_2} \\ \overline{c'_3} & 0 & c'_1 \\ c'_2 & \overline{c'_1} & 0 \end{pmatrix} = \begin{pmatrix} c_3 \overline{c'_3} + \overline{c_2} c'_2 & \overline{c_2} c'_1 & c_3 c'_1 \\ c_1 c'_2 & \overline{c_3} c'_3 + c_1 \overline{c'_1} & \overline{c_3} c'_2 \\ \overline{c_1} c'_3 & c_2 c'_3 & c_2 \overline{c'_2} + \overline{c_1} c'_1 \end{pmatrix}.$$

one can verify without difficulty that

$$\varphi(c, c') = -\frac{1}{2}(\langle c_1, c'_1 \rangle, \langle c_2, c'_2 \rangle, \langle c_3, c'_3 \rangle);$$

see [SV00, p. 163]. This proves (2.5).

From (2.8), we have that

$$\mu_{M_s}((c_1, c_2, c_3), (c'_1, c'_2, c'_3)) = \frac{1}{2}(\overline{c_3} c'_2 + \overline{c'_3} c_2, \overline{c_1} c'_3 + \overline{c'_1} c_3, \overline{c_2} c'_1 + \overline{c'_2} c_1).$$

It follows that

$$\mu_{M_s}(c, c) = (\overline{c_2} c_3, \overline{c_3} c_1, \overline{c_1} c_2). \quad \square$$

Proposition 2.2. *Let $x^{*2} := \mu_{M_s}(x, x)$. Together with the maps previously defined, the quadruple $\Gamma_s = (L_s, M_s, q_s, \cdot^{*2})$ is a split twisted composition.*

Proof. This is a consequence of Lemma 2.1. By (2.3), we have an isomorphism $L_s \cong k^3$ of k -algebras. By (2.4), the map η_s makes M_s into an L_s -module, and we have $\dim_{L_s} M_s = 8$. By (2.5), q_s is a non-degenerate symmetric L_s -bilinear form. The associated quadratic form is

$$(2.9) \quad q_s(c) = (N(c_1), N(c_2), N(c_3)),$$

where $N : \mathbb{O} \rightarrow k$ is the octonion norm. To prove that $(L_s, M_s, q, \cdot^{*2})$ is a split composition, it remains to show that

$$(2.10) \quad q_s(c)q_s(c^{*2}) = N_{L_s/k}(q_s(c))$$

and

$$(2.11) \quad \xi(\xi c)^{*2} = N_{L_s/k}(\xi)c^{*2}$$

for every $\xi \in L_s, c \in M_s$. Here $N_{L_s/k}(\xi) = \xi_1 \xi_2 \xi_3 \in k$.

We have $N(\overline{x}) = N(x)$ and $N(xy) = N(x)N(y)$ for every $x, y \in \mathbb{O}$, hence by (2.9) we get

$$q_s(c^{*2}) = (N(c_2)N(c_3), N(c_1)N(c_3), N(c_1)N(c_2)).$$

We deduce that

$$q_s(c)q_s(c^{*2}) = N(c_1)N(c_2)N(c_3) = N_{L_s/k}(q_s(c)),$$

which proves (2.10). The proof of (2.11) is similar. \square

Lemma 2.3. *The groups H_s and T_s act on J_s via automorphisms of the twisted composition $\Gamma_s = (L_s, M_s, q_s, \cdot^{*2})$.*

Proof. Since T_s is contained in H_s , it is enough to prove the claim for H_s . By definition, H_s fixes L_s pointwise, the subspaces J_1, J_2, J_3 are H_s -stable, and the H_s -action on each $J_i \cong \mathbb{O}$ respects the octonion norm, so we only need to show that for every $h \in H_s(\bar{k})$ and every $m \in M_s(\bar{k})$ we have $h(m^{*2}) = h(m)^{*2}$. Let $m \in M_s(\bar{k})$ and $h \in H_s(\bar{k})$, and write

$$m^2 = l + n, \quad (hm)^2 = l' + n',$$

where $l, l' \in L_s(\bar{k})$ and $n, n' \in M_s(\bar{k})$. By definition, we have

$$m^{*2} = n, \quad h(m^{*2}) = h(n), \quad (hm)^{*2} = n'.$$

Since H_s acts on J_s via Albert algebra automorphisms, we have $h(m^2) = (hm)^2$. Since $J_s = L_s \oplus M_s$, we deduce that $h(n) = n'$. This means that $h(m^{*2}) = (hm)^{*2}$, as desired. \square

We have constructed a diagram $T_s \hookrightarrow H_s \hookrightarrow G_s$. In the next section, we will generalize this to maximal tori in arbitrary groups of type F_4 .

3. CONSTRUCTION OF H - THE GENERAL CASE

Let J be an Albert algebra over k . Similarly to the split case, we denote by Q the norm of J ; it is a non-degenerate quadratic form on J . We let $\langle \cdot, \cdot \rangle$ be the bilinear form associated to Q , called the inner product of J , and we denote by e the identity element of J .

We let \det be the determinant form of J , and we denote by $\langle \cdot, \cdot, \cdot \rangle$ the trilinear form associated to \det . For every $x, y \in J$, we define the cross product $x \times y$ as the unique element of J such that

$$\langle x \times y, z \rangle = 3 \langle x, y, z \rangle$$

for every $z \in J$.

Let G be a group of type F_4 over k , and let J be the Albert algebra associated to G , so that $G = \text{Aut}(J)$. By a Galois descent argument, every k -group of type F_4 arises in this way; see [KMRT98, Proposition 37.11].

Let T be a maximal k -torus of G (not necessarily k -split). Since $G_{\bar{k}}$ is split, we have a constructed in Section 2 a maximal torus T_s and a group $H_s \cong \text{Spin}_8$ such that $T_s \hookrightarrow H_s \hookrightarrow G_{\bar{k}}$. There exists $g \in G(\bar{k})$ such that $gT_{\bar{k}}g^{-1} = T_s$. The torus T acts on J via G . If we let L be the T -fixed subspace of J , then $gL_{\bar{k}}g^{-1} = L_s$, hence L is an étale algebra of degree 3 over k . If we set $M := L^\perp$, we have $J = L \perp M$ and $gM_{\bar{k}}g^{-1} = M_s$, hence the T -action on M has no non-zero fixed points.

We now define k -bilinear maps μ_L, μ_M, η, q , by mimicking the construction of $\mu_{L_s}, \mu_{M_s}, \eta_s$ and q_s . Using multiplication in J and projections, we construct

$$(3.1) \quad \mu_L : L \times L \rightarrow L, \quad \mu_M : M \times M \rightarrow M.$$

We define

$$(3.2) \quad \eta : L \times M \rightarrow M$$

$$(l, m) \mapsto -2lm + \langle l, e \rangle m.$$

As in the split case, one could also define η using the cross product \times on J ; see [SV00, p. 162].

Finally, we have a k -bilinear map

$$\varphi : M \times M \rightarrow L,$$

given by the composition

$$M \times M \hookrightarrow J \times J \xrightarrow{\times} J \xrightarrow{\pi} L,$$

where π denotes the projection onto L . We define $q := -2\varphi$.

The decomposition $J = L \perp M$ is known as the *Springer decomposition*; see [KMRT98, 38.A].

Proposition 3.1. *Let G be a group of type F_4 , not necessarily k -split, and let T be a maximal torus of G . Define $x^{*2} := \mu_M(x, x)$.*

- (a) *Together with the previously defined maps, $\Gamma = (L, M, q, \cdot^{*2})$ is a twisted composition algebra over k , and $J = L \perp M$ is a Springer decomposition.*
- (b) *The T -action on J induces an action on M via automorphisms of the twisted composition.*
- (c) *The group $\text{Aut}(\Gamma)^\circ$ is simply connected of type D_4 , it canonically embeds in G , and its image H is the unique subgroup of G of type D_4 containing T .*
- (d) *Every subgroup of G of type D_4 is uniquely of the form $\text{Aut}(\Gamma')^\circ$, for some twisted composition $\Gamma' = (L', M', q', \cdot^{*2})$ such that $J = L' \perp M'$ is a Springer decomposition.*

The quadratic form $M \rightarrow L$ associated to q will also be denoted by q .

Proof. (a) The maps are defined over k , and the properties that we want to prove can be checked on an algebraic closure of k , so we may assume that k is algebraically closed and $J = J_s$. Up to conjugation, we may assume that $T = T_s$. Then $L = L_s$ and $M = M_s$. We conclude using Proposition 2.2.

(b) After passing to an algebraic closure of k , the statement follows immediately from Lemma 2.3.

(c) We have $J = L \perp M$ by construction. By [KMRT98, Proof of Corollary 38.7], an automorphism of Γ extends uniquely to an automorphism of J as an Albert algebra. It follows that $\text{Aut}(J)$ is canonically a subgroup of G , and in particular this is true for $\text{Aut}(J)^\circ$; let H be its image in G . The fact that T is contained in H is an immediate consequence of (b). On an algebraic closure of k , the torus T is conjugate to T_s , say $gT_s g^{-1} = T$. Since L is the fixed locus of T and L_s is the fixed locus of T_s , we have that $gL_s = L$. The construction of Γ from L is the same as the construction of Γ_s from L_s , hence for every $x \in G$, x is an automorphism of Γ_s if and only if gxg^{-1} is an automorphism of Γ . We deduce that $gH_s g^{-1} = H$. Since $H_s \cong \text{Spin}_8$, we conclude that H and $\text{Aut}(\Gamma)^\circ$ are simply connected of type D_4 .

To show the uniqueness of H , we may assume that k is algebraically closed. The result is now a consequence of Borel-de Siebenthal theory, which classifies subgroups of maximal rank of semisimple groups; see [BDS49, p. 219]. In type F_4 , there are three maximal subgroups of maximal rank containing T , of type B_4 , $A_2 \times A_2$ and $A_1 \times C_3$. Using [BDS49, p. 219] again, we see that the last two do not contain subgroups of type D_4 , and the first contains exactly one.

(d) Let H be a subgroup of G of type D_4 , and let T is a maximal torus of H . Since H and G have the same rank, T is a maximal torus of H . By (c), it

follows that H is the unique simply connected subgroup of G of type D_4 containing T , and it is the connected component of the automorphism group of the twisted composition associated to T . \square

Remark 3.2. Let G be a k -group of type F_4 , let T be a maximal k -torus of G , and let $\Phi(G, T)$ be the corresponding root system. Since the $\text{Gal}(k)$ -action respects the length of a root, it is easy to see that the set of long roots of $\Phi(G, T)$ spans a k -group H of type D_4 and such that $T \hookrightarrow H \hookrightarrow G$. One can check that if $G = G_s$ and $T = T_s$, then $H = H_s$, thus showing that H is simply connected. We conclude that H is the spin group associated to some trialitarian algebra. However, Proposition 3.1 is a much more precise statement: for example, it shows that only spin groups associated to twisted compositions occur.

Starting from a twisted composition $\Gamma = (L, M, q, \cdot^{*2})$, there is a canonical way to construct an Albert algebra $J(L, M)$, explained in [SV00, §6.3]. Let $J := L \oplus M$ as a k -vector space. One defines a quadratic form $Q : J \rightarrow k$ as in [SV00, (6.17)], a crossed product $\times : J \times J \rightarrow J$ as in [SV00, (6.19)], and product on J as in [SV00, (6.21)]. By [SV00, Theorem 6.3.2], this makes J into an Albert algebra, which we denote by $J(L, M)$.

The algebra $J(L, M)$ is called the *Springer construction* associated with Γ ; see [KMRT98, §38.A] and in particular [KMRT98, Theorem 38.6], where the construction is carried out in an equivalent way. The orthogonal decomposition $J(L, M) = L \perp M$ is a Springer decomposition of the Springer construction.

Remark 3.3. Let J be a Jordan algebra over k , and let $\Gamma = (L, M, q, \cdot^{*2})$ be a twisted composition over k . Let $G := \text{Aut}(J)$ and let $H := \text{Aut}(J)^\circ$. If $J \cong J(L, M)$, by [KMRT98, Proof of Corollary 38.7], an automorphism of Γ uniquely extends to an automorphism of J as an Albert algebra, hence H is canonically a subgroup of G .

4. PRELIMINARIES ON MAXIMAL TORI IN ORTHOGONAL GROUPS

We set up the notation for quadratic spaces and étale algebras with involutions in accordance to [BF14, §1]. In this section, k is a field of characteristic different from 2.

4.1. Quadratic spaces. We say that (V, q) is a quadratic space over k if V is a finite-dimensional k -vector space and $q : V \times V \rightarrow k$ is a non-degenerate symmetric bilinear form. We use the same symbol q to denote the quadratic form associated with q . We denote by $\text{O}(V, q)$ or by $\text{O}(q)$ its orthogonal group. We let $\det(q) \in k^\times / k^{\times 2}$ be the determinant of q , and we define the discriminant of q as $\text{disc}(q) = (-1)^{\frac{m(m-1)}{2}} \det(q)$, where $m := \dim(q)$. The quadratic form q can be diagonalized, i.e. there exist $a_1, \dots, a_m \in k^\times$ such that $q \cong \langle a_1, \dots, a_m \rangle$.

We denote the Brauer group $\text{Br}(k)$ of k , viewed as an abelian group, and we let $\text{Br}_2(k)$ be its 2-torsion subgroup. If $a, b \in k^\times$, we denote by $(a, b) \in \text{Br}_2(k)$ the Brauer class of the quaternion algebra determined by a, b . The Hasse invariant of q is by definition $w(q) := \sum_{i < j} (a_i, a_j) \in \text{Br}_2(k)$.

We denote by $\tau_q : \text{End}(V) \rightarrow \text{End}(V)$ the adjoint involution of q , i.e. $q(f(x), y) = q(x, \tau_q(f)(y))$ for all $f \in \text{End}(V)$ and $x, y \in V$. If q and q' are quadratic spaces, we denote by $q \perp q'$ their orthogonal sum.

If $k = \mathbb{R}$, we say that q has signature (r, s) if q is isometric to $x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2$.

4.2. Maximal tori and étale algebras with involution. Let E be an étale algebra over k , let $\sigma : E \rightarrow E$ be a k -linear involution, and denote by E^σ the subalgebra of E fixed by σ . By definition, the pair (E, σ) is an étale algebra with involution. We will always assume that $\text{rank}_k E = 2\text{rank}_k E^\sigma$. We define the k -torus $U(E, \sigma)$ by setting

$$U(E, \sigma)(R) := \{x \in E \otimes_k R : x\sigma(x) = 1\}$$

for every commutative k -algebra R .

It is possible to give an alternative description of $U(E, \sigma)$, using Weil restrictions. If $E = E_1 \times E_2$, where E_1 and E_2 are σ -stable, it is clear that $U(E, \sigma) = U(E_1, \sigma|_{E_1}) \times U(E_2, \sigma|_{E_2})$. It is thus sufficient to describe $U(E, \sigma)$ in the cases when (i) E is a field and (ii) $E = F \times F$ and σ switches the two field factors F . In case (i) $U(E, \sigma) = R_{E^\sigma/k}(R_{E/E^\sigma}^{(1)}(\mathbb{G}_m))$, and in case (ii) $U(E, \sigma) = R_{F/k}(\mathbb{G}_m)$.

Let (V, q) be a quadratic space of dimension $2n$ over k . By [BF14, Proposition 1.2.1(i)], every maximal torus T of $\text{O}(V, q)$ is of the form $U(E, \tau_q)$, where $E \subseteq \text{End}(V)$ is an étale algebra τ_q -stable and $\text{rank}_k E = 2\text{rank}_k E^\sigma = 2n$; the étale subalgebra E is unique. Conversely, by [BF14, Proposition 1.2.1(ii)], for every étale algebra $E \subseteq \text{End}(V)$ stable under τ_q and such that $\text{rank}_k E = 2\text{rank}_k E^{\tau_q} = 2n$, the torus $U(E, \tau_q)$ is a maximal k -torus of $\text{O}(V, q)$.

Let (E, σ) be an étale algebra with involution. For every $\alpha \in E^\sigma$, let $q_\alpha : E \times E \rightarrow k$ be the non-degenerate symmetric bilinear form defined by $q_\alpha(x, y) := \text{Tr}_{E/k}(\alpha x \sigma(y))$. By [BF14, Proposition 1.3.1], $U(E, \sigma)$ is a maximal subtorus of the orthogonal group $\text{O}(V, q)$ if and only if there exists $\alpha \in E^\sigma$ such that (V, q) is isometric to (E, q_α) . When this is the case, we say that (E, q) is *realizable*, and that T is of *type* (E, σ) ; see [BF14, §1.2].

5. DESCRIPTION OF MAXIMAL TORI OF GROUPS OF TYPE F_4

By Proposition 3.1(a), every maximal torus in a group of type F_4 is contained inside the automorphism group of a twisted composition. Moreover, Proposition 3.1(c) and (d) give a complete understanding of groups of type D_4 inside a group of type F_4 . To be able to parametrize tori in groups of type F_4 , all that is left is the study of maximal tori in automorphism groups of twisted compositions.

Lemma 5.1. *Let H_1 and H_2 be reductive groups, and let $\pi : H_1 \rightarrow H_2$ be an isogeny. The map π sets up a correspondence between the maximal tori in H_1 and the maximal tori in H_2 .*

Proof. Let T_1 be a maximal k -torus of H_1 , and let $T_2 := \pi(T_1)$; it is a maximal k -torus of H_2 . Denote by N_i the normalizer of T_i in H_i , for $i = 1, 2$. The isogeny π induces a surjection $N_1 \rightarrow N_2$, because H_1 and H_2 have the same Weyl group scheme. It follows that the induced map $H_1/N_1 \rightarrow H_2/N_2$ between the varieties of maximal tori of H_1 and H_2 (see [Vos98, §4.1]) is an isomorphism. This concludes the proof. \square

Let $\Gamma = (L, M, q, \cdot^{*2})$ be a twisted composition, and let $H := \text{Aut}(\Gamma)^\circ$. By Proposition 3.1(c), H is a k -form of Spin_8 , split by L . We have an embedding

$$(5.1) \quad \iota : H \hookrightarrow R_{L/k}(\text{O}(M, q)),$$

that is constructed as follows. We have an embedding $j : H \hookrightarrow R_{L/k}(\text{Spin}(M, q))$; see [KMRT98, §44.B p. 561] or [Fio17, §3.4 p. 12]. Then ι is the composition of

j with the natural projection $R_{L/k}(\mathrm{Spin}(M, q)) \rightarrow R_{L/k}(\mathrm{O}(M, q))$. We call ι the trialitarian embedding associated to the twisted composition (L, M, q, \cdot^{*2}) . The base change ι_L is the usual trialitarian embedding of $\mathrm{Spin}(M, q)$ in a product of three orthogonal groups. In particular, ι is injective.

Let T be a maximal k -torus of H , and let $T_1 = \iota(T)$. The center of the centralizer of T_1 in $R_{L/k}(\mathrm{O}(M, q))$ is a k -torus T_2 containing T_1 .

Proposition 5.2. *The torus T_2 is the unique maximal torus of $R_{L/k}(\mathrm{O}(M, q))$ containing T_1 .*

Proof. This can be checked over an algebraic closure of k , where $R_{L/k}(\mathrm{O}(M, q))$ becomes a product of three orthogonal groups, T_2 becomes a product of three tori (one for each orthogonal group factor), and ι is the usual trialitarian embedding. Since T projects isomorphically under each of the three projections, every factor of T_2 has rank at least $\mathrm{rank} T = 4$. We conclude that T_2 is a maximal torus in $R_{L/k}(\mathrm{O}(M, q))$. Every maximal torus containing T_1 must contain the center of the centralizer of T_1 in $R_{L/k}(\mathrm{O}(M, q))$, which by definition is T_2 . This completes the proof. \square

Definition 5.3. Let T be a maximal k -torus of H . We let T' be the unique L -torus U of $\mathrm{O}(M, q)$ such that $T_2 = R_{L/k}(U)$.

For the uniqueness statement implicit in the definition, see [Fio17, Lemma 4.2] when k is infinite and [Fio17, Remark 4.3] when k is finite.² When k is infinite, the torus T' is the center of the centralizer of $T_2(k) \subseteq \mathrm{O}(M, q)(L)$.

The function $T \mapsto T'$ from the set of maximal k -tori in H to the set of maximal L -tori in $\mathrm{O}(M, q)$ is injective, but it is not necessarily surjective.

6. MAXIMAL TORI AND ÉTALE ALGEBRAS WITH INVOLUTION

Let (E, σ) be an étale algebra with involution over k such that $\mathrm{rank}_k E = 2 \mathrm{rank}_k E^\sigma = 2n$, and set

$$\Phi := \{ \varphi \subseteq \mathrm{Hom}_k(E, \bar{k}) : \varphi \cap \varphi\sigma = \emptyset, \varphi \cup \varphi\sigma = \mathrm{Hom}_k(E, \bar{k}) \}.$$

The action of $\mathrm{Gal}(k)$ on $\mathrm{Hom}(E, \bar{k})$ induces an action on Φ . We let E^Φ be the étale algebra corresponding to Φ . The action of σ on E induces an involution on $\mathrm{Hom}_k(E, \bar{k})$ which commutes with the $\mathrm{Gal}(k)$ -action, which in turn induces an involution on Φ . We denote by the same letter σ the corresponding involution on E^Φ , so that (E^Φ, σ) is an étale algebra with involution. It is not difficult to show that $\mathrm{rank}_k E^\Phi = 2 \mathrm{rank}_k (E^\Phi)^\sigma = 2^n$.

Let L be a cubic étale k -algebra, and let (E, σ) be an étale algebra with involution over L , such that $\mathrm{rank}_L E = 2 \mathrm{rank}_L E^\sigma = 8$. By [KMRT98, Corollary 18.28] we have an isomorphism $L \otimes_k L \cong L \times (L \otimes_k \Delta)$, where $\Delta = \Delta(L)$ is the discriminant algebra of L ; see [KMRT98, p. 291]. It follows that we have a k -isomorphism

$$(E, \sigma) \otimes_k L \cong (E, \sigma) \times (E_2, \sigma_2)$$

where (E_2, σ_2) is an étale algebra over $L \otimes_k \Delta$.

²For the proof of the main results of this paper, only the case when k is infinite is relevant.

Definition 6.1. A *datum* over k is a quadruple (L, E, σ, β) , where L is a cubic étale k -algebra, (E, σ) is an étale L -algebra with involution such that $\text{rank}_L E = 2 \text{rank}_L E^\sigma = 8$ and β is an isomorphism

$$(6.1) \quad \beta : (E, \sigma) \otimes_k L \xrightarrow{\sim} (E \times E^\Phi, \sigma \times \sigma).$$

There is an obvious notion of k -isomorphism of data over k , which we do not spell out explicitly.

Let α be a datum. We define the torus $T(\alpha)$ by setting

$$(6.2) \quad T(\alpha)(R) := \{x \in (E \otimes_k R)^\times : x\sigma(x) = 1, \Psi(x_2) = x_1 \otimes 1\}$$

for every k -algebra R . Here we denote $\beta(x \otimes 1) = (x_1, x_2)$ for every $x \in (E \otimes_k R)^\times$. Furthermore, the homomorphism of k -algebras $\Psi : E^\Phi \rightarrow E \otimes_k E^\Phi$ is defined by considering the $\text{Gal}(k)$ -equivariant homomorphism

$$\Psi\left(\sum_{\varphi \in \Phi} a_\varphi e_\varphi\right) := \sum_{\varphi_1 \cap \varphi_2 = \{\rho\}} a_{\varphi_1} a_{\varphi_2} e_\rho \otimes (e_{\varphi_1} + e_{\varphi_2} + e_{\varphi_1 \sigma} + e_{\varphi_2 \sigma})$$

over a separable closure k_{sep} of k , and then taking $\text{Gal}(k)$ -invariants; see [Fio17, Definition 4.25]. Here $(e_\varphi)_{\varphi \in \Phi}$ is the basis of orthogonal idempotents of $E^\Phi \otimes_k k_{sep}$. Using β , we may equivalently describe the R -points of $T(\alpha)$ as

$$T(\alpha)(R) = \{(x_1, x_2) \in ((E \times E^\Phi) \otimes_k R)^\times : x_i \sigma(x_i) = 1, \Psi(x_2) = x_1\}.$$

The next lemma makes precise the correspondence between data and maximal tori in groups of type F_4 . It is the analog of [BF14, Proposition 1.2.1] in type F_4 .

Lemma 6.2. (i) *Let $\Gamma = (L, M, q, \cdot^{*2})$ be a twisted composition over k , let $\iota : \text{Aut}(\Gamma)^\circ \hookrightarrow R_{L/k}(\text{O}(M, q))$ be the trialitarian embedding, and let T be a maximal torus in $\text{Aut}(\Gamma)^\circ$. There exists a unique τ_q -stable étale L -subalgebra E of $\text{End}_L(M)$ such that $\text{rank}_L E = 2 \text{rank}_L E^{\tau_q} = 8$ and such that $T = \iota^{-1}(R_{L/k}(U(E, \tau_q)))$. Moreover, there is a canonical isomorphism*

$$\beta_T : (E, \sigma) \otimes_k L \xrightarrow{\sim} (E \times E^\Phi, \sigma \times \sigma)$$

which makes $\alpha = (L, E, \tau_q, \beta_T)$ into a datum.

(ii) *Let $\alpha = (L, E, \sigma, \beta)$ be a datum. Then there exists a twisted composition $\Gamma = (L, M, q, \cdot^{*2})$ and an étale L -subalgebra $E' \subseteq \text{End}_L(M)$ such that $(E, \sigma) \cong (E', \tau_q)$, $T := \iota^{-1}(U(E', \tau_q|_{E'}))$ is a maximal torus in $\text{Aut}(\Gamma)^\circ$, and $\beta_T = \beta$.*

We will prove the lemma by invoking results of [Fio17, §4]. Analogous properties are established there for trialitarian algebras. We first need to address the compatibility of our constructions with those of [Fio17].

Remark 6.3. Let $\Gamma = (L, M, q, \cdot^{*2})$ be a twisted composition, and let $\Lambda = \text{End}(\Gamma) = (\text{End}_L(M), L, \tau_q, \eta)$ be the associated trialitarian algebra; see [KMRT98, Example 43.7]. Then $\text{Aut}(\Gamma)^\circ$ is canonically isomorphic to $\text{Spin}(\Lambda)$, where the latter group is the simply connected group associated to $\text{Aut}(\Gamma)$ (which is a twisted form of PGO_8^+). We also have a canonical L -isomorphism $\text{O}(M, q) \cong \text{O}(\text{End}_L(M), \tau_q)$. We obtain a commutative diagram

$$\begin{array}{ccc} \text{Aut}(\Gamma)^\circ & \xleftarrow{\iota} & R_{L/k}(\text{O}(M, q)) \\ \parallel & & \parallel \\ \text{Spin}(\Lambda) & \xleftarrow{\iota'} & R_{L/k}(\text{O}(\text{End}_L(M), \tau_q)), \end{array}$$

where ι and ι' are the trialitarian embeddings. Let T be a maximal torus of $\text{Aut}(\Gamma)^\circ$. By Proposition 5.2, $\iota(T)$ is contained in a unique maximal L -torus of $R_{L/k}(\text{O}(M, q))$; let (E, σ) be its type. If we view T as a torus of $\text{Spin}(\Lambda)$, by [Fio17, Lemma 4.31] there exists a unique maximal L -torus U of $R_{L/k}(\text{O}(\text{End}_L(M), \tau_q))$ containing $\iota'(T)$. By transport of structure along the isomorphism $\Lambda \cong \text{End}(\Gamma)$, it is immediate to see that U also has type (E, σ) .

Proof of Lemma 6.2. (i) Apply [Fio17, Lemma 4.31] to $\text{End}(\Gamma)$ to construct E , then apply [Fio17, Lemma 4.32] to construct β_T .

(ii) By [Fio17, Lemma 4.35] applied to $(\text{End}_L(M), L, \tau_q)$, there exist a trialitarian algebra $\Lambda' = (\text{End}_L(M), L, \tau_q, \eta')$ and a maximal torus T of $\text{Spin}(\Lambda')$ such that the procedure of [Fio17, Lemma 4.32, Lemma 4.35] associates to T the quadruple (L, E, σ, β) . By [KMRT98, Proposition 44.16(1)], $\Lambda' \cong \text{End}(\Gamma')$ for some twisted composition Γ' . By Remark 6.3, there exists a maximal torus of $\text{Aut}(\Gamma')$ with datum α . \square

The following proposition gives the precise relation between maximal tori in groups of type F_4 and data (L, E, σ, β) . In particular, it shows that such data allow a uniform description of all maximal tori in groups of type F_4 .

Proposition 6.4. (i) *Let J be an Albert k -algebra, let $G := \text{Aut}(J)$, and let T be a maximal torus of G . Let $\Gamma = (L, M, q, \cdot^{*2})$ be the twisted composition associated to T in Proposition 3.1, and let $\iota : \text{Aut}(\Gamma)^\circ \hookrightarrow R_{L/k}(\text{O}(M, q))$ be the trialitarian embedding. There exists a unique τ_q -stable étale L -subalgebra E of $\text{End}_L(M)$ such that $\text{rank}_L E = 2 \text{rank}_L E^{\tau_q} = 8$ and such that $T = \iota^{-1}(R_{L/k}(U(E, \tau_q)))$. Moreover, there is a canonical isomorphism*

$$\beta_T : (E, \sigma) \otimes_k L \xrightarrow{\sim} (E \times E^\Phi, \sigma \times \sigma)$$

which makes $\alpha = (L, E, \tau_q, \beta_T)$ into a datum.

(ii) *Let $\alpha = (L, E, \sigma, \beta)$ be a datum. There exist a group G of type F_4 and a maximal torus T of G such that the construction of (i) applied to T and G gives a datum isomorphic to α .*

Proof. Combine Proposition 3.1 with Lemma 6.2. \square

To complete the parallel with [BF14], we define the type of a maximal torus of a group of type F_4 , and a notion of realizability.

Definition 6.5. Let J be an Albert algebra over k , let $G := \text{Aut}(J)$, let T be a maximal k -torus of G , and let $\alpha = (L, E, \sigma, \beta)$ be a datum. We say that T is of *type α* if the datum associated to T in Proposition 6.4(i) is isomorphic to α . We say that (G, α) is *realizable* if there exists a torus of type α .

Remark 6.6. Let G be a group of type F_4 , let T be a maximal torus of G of type $\alpha = (L, E, \sigma, \beta)$, and let $\Gamma = (L, M, q, \cdot^{*2})$ be the twisted composition associated to T by Proposition 3.1. Then, by definition of type, (E, q) is realizable in the sense of orthogonal groups.

Lemma 6.7. *Let $\alpha = (L, E, \sigma, \beta)$ be a datum. Then:*

- (i) $\text{disc}_L(E) = \text{disc}_k(L)$ in $L^\times/L^{\times 2}$;
- (ii) *The Clifford L -algebra $C(M, q)$ is split, that is, (M, q) has trivial Clifford invariant.*

Here, we are identifying $\text{disc}_k(L) \in k^\times/k^{\times 2}$ with its image under the natural map $k^\times/k^{\times 2} \rightarrow L^\times/L^{\times 2}$. This map is injective; see [KMRT98, Proposition 18.34].

Proof. By Proposition 6.4(ii), there exist a k -group G of type F_4 and a maximal k -torus T of G of type α . Let $\Gamma = (L, M, q, \cdot^{*2})$ be the twisted composition associated to T by Proposition 3.1.

(i) By Remark 6.6 (E, q) is realizable. By [Fio17, Corollary 3.19], $\text{disc}(q) = \text{disc}_k(L) \in L^\times/L^{\times 2}$. By [BF14, Lemma 1.3.2] we have $\text{disc}(q) = \text{disc}_L(E) \in L^\times/L^{\times 2}$, hence $\text{disc}_L(E) = \text{disc}_k(L) \in L^\times/L^{\times 2}$.

(ii) If $L = k \times F$, the claim follows from [KMRT98, Proposition 44.13]. If L is a field, the map $\text{Res} : \text{Br}_2(k) \rightarrow \text{Br}_2(L)$ given by tensoring by L is injective, because the composition

$$\text{Br}_2(k) \xrightarrow{\text{Res}} \text{Br}_2(L) \xrightarrow{\text{Cores}} \text{Br}_2(k)$$

is given by $[A] \mapsto [A]^3 = [A]$. Here Cores is the corestriction map. Since $L \otimes_k L \cong L \times (L \otimes_k \Delta)$, the conclusion follows from the case when L is not a field. \square

One could describe E^Φ more explicitly. We refrain from doing so here, and instead refer the interested reader to [Fio17, §4.5.1].

7. LOCAL-GLOBAL PRINCIPLE FOR TORI IN ORTHOGONAL GROUPS

For the proof of Theorem 1.1, we will need to make use of the main results in [BF14]. In this section, we give an account of what we need.

The following observation will be useful in the sequel.

Lemma 7.1. *Let (E, σ) be an étale k -algebra with involution of rank $2n$. Assume that $E = R \times R$ for some étale algebra R , and that σ acts by switching the two factors. Then (E, q) is realizable if and only if $q \cong h_{2n}$ is hyperbolic.*

Proof. One shows that the quadratic form $q_\alpha : x \mapsto \text{Tr}_{R/k}(\alpha x^2)$ is hyperbolic for every $\alpha \in E^\sigma = R$, and then applies [BF14, Proposition 1.3.1]. See the proof of [BF14, Lemma 2.1.1]. \square

The next result characterizes maximal tori of an orthogonal group of type D_n over a local field in terms of the cohomological invariants of (E, σ) . Recall that every such torus is of the form $U(E, \sigma)$ for some étale algebra with involution (E, σ) such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$. Following [BF14, §2], we write Σ_k for the set of places of v , and we let $\Sigma_k^{\text{split}}(E)$ be the set of $v \in \Sigma_k$ such that all places of E^σ above v split in E . We denote by ρ_v the number of places of E^σ above v that ramify in E .

Proposition 7.2. *Let k be a global field, let v be a place of k , let (V, q) be a quadratic space of dimension $2n$ over k , and let (E, σ) be an étale k -algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$.*

- (a) *If $v \in \Sigma_k^{\text{split}}(E)$, then (E_v, q_v) is realizable if and only if q_v is hyperbolic.*
- (b) *If $v \notin \Sigma_k^{\text{split}}(E)$ is a finite place, then (E_v, q_v) is realizable if and only if $\text{disc}(q_v) = \text{disc}(E_v)$.*
- (c) *If v is a real place, then (E_v, q_v) is realizable if and only if the signature of q_v is of the form $(2r' + \rho_v, 2s' + \rho_v)$ for some $r', s' \geq 0$.*

Proof. (a) This is [BF14, Lemma 2.1.1].

(b) If (E_v, q_v) is realizable, then $\text{disc}(q_v) = \text{disc}(E_v)$ by [BF14, Lemma 1.3.2]. Assume that $\text{disc}(q_v) = \text{disc}(E_v)$. By [BF14, Proposition 2.2.1] there exists $\alpha \in (E^\sigma)^\times$ such that $w(q_\alpha) = w(q_v)$. By [BF14, Lemma 1.3.2], we have $\text{disc}(q_\alpha) = \text{disc}(E_v)$, hence $\text{disc}(q_\alpha) = \text{disc}(q_v)$. It follows that q_v and q_α have the same rank, discriminant and Hasse invariant, hence they are isometric. By [BF14, Proposition 1.3.1] (E_v, q_α) is realizable, hence so is (E_v, q_v) .

(c) This is [BF14, Proposition 2.3.2]. \square

Let (E, σ) be an étale algebra of degree $2n$ with involution over k , and write $E = K_1 \times \cdots \times K_r$, where K_i/k is a field extension, and set $I := \{1, \dots, r\}$. We start by assuming that $\sigma(K_i) = K_i$ for every $i \in I$. For every $i \in I$, we let F_i be the fixed field of σ on K_i , so that $E^\sigma = F_1 \times \cdots \times F_r$ and $[K_i : F_i] = 2$.³ As in [BF14, §1.3], we say that (E, q) is realizable if $O(V, q)$ contains a maximal torus of the form $U(E, \sigma)$.

Following [BF14, §3.2], we let $\mathcal{C}(E, q)$ be the set of collections (q_i^v) of quadratic spaces over k_v satisfying conditions (i) - (iii) of [BF14, Proposition 3.1.3]:

- (i) for all $i \in I$ and $v \in \Sigma_k$, (K_v, q_i^v) is realizable;
- (ii) for all $v \in \Sigma_k$, $q_v \cong q_1^v \oplus \cdots \oplus q_r^v$;
- (iii) for all $i \in I$, $w(q_i^v) = 0$ for all but finitely many $v \in \Sigma_k$.

For $C = (q_i^v) \in \mathcal{C}(E, q)$ and $i \in I$, set

$$S_i(C) := \{v \in \Sigma_k : w(q_i^v) = 1\}.$$
⁴

We say that $(q_i^v) \in \mathcal{C}(E, q)$ is connected if for all $i \in I$ such that $|S_i(C)|$ is odd, there exist $j \in I$ such that $j \neq i$ and $|S_j(C)|$ is odd, a sequence $i = i_1, \dots, i_m = j$ in I such that for every $t = 1, \dots, m - 1$ there exists $v \in \Sigma_k$ that does not split in K_{i_t} and in $K_{i_{t+1}}$. We say that $\mathcal{C}(E, q)$ is connected if it contains a connected element.

The following is [BF14, Theorem 3.2.1]. It plays an important role in the proof of Theorem 1.1, and in particular in the proof of Proposition 10.1.

Theorem 7.3 (Bayer-Fluckiger). *Let k be a global field of characteristic different from 2, let (V, q) be a $2n$ -dimensional quadratic space over k , and let (E, σ) be an étale algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$. Then:*

- (a) (E, q) is realizable everywhere locally if and only if $\mathcal{C}(E, q)$ is non-empty;
- (b) (E, q) is realizable if and only if $\mathcal{C}(E, q)$ is connected.

8. CLASSIFICATION OVER LOCAL FIELDS

Let k be a local field (of characteristic $\neq 2, 3$), and let G be a semisimple k -group of type F_4 . We have $G = \text{Aut}(J)$ for some Albert k -algebra J . In this section we classify the maximal k -tori in G .

³This is not the most general case, as it implies that E has no factors $F \times F$, where σ acts by switching the two factors. However, thanks to Lemma 10.2, considering this case will be enough for our purposes.

⁴In [BF14, §3.2], the symbol Σ'_k is used instead of Σ_k . This is just a typographical error.

8.1. Non-archimedean fields. Assume first that k is a local non-archimedean field. By [SV00, §5.8 (iv)], there is a unique Albert algebra over k , that is, $J = J_s$ and $G = G_s$. In particular, $J = J(L, M)$ for every twisted composition (L, M, q, \cdot^{*2}) over k . The following proposition is now an immediate consequence of Proposition 6.4, but we record it for future reference.

Proposition 8.1. *Let k be a local non-archimedean field, let G be the unique k -group of type F_4 , and let α be a datum. Then (G, α) is realizable.*

8.2. The field of real numbers. Assume now that $k = \mathbb{R}$. We start by recalling the classification of twisted compositions over \mathbb{R} . The group $H^3(\mathbb{R}, \mathbb{Z}/3\mathbb{Z})$ is 3-torsion because so is $\mathbb{Z}/3\mathbb{Z}$, and is 2-torsion because so is $\text{Gal}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, hence $H^3(\mathbb{R}, \mathbb{Z}/3\mathbb{Z}) = 0$. By [KMRT98, Proposition 40.16], every twisted composition arises, up to similitude, from a symmetric composition of dimension 8. By [SV00, §1.10 (i)], the only two symmetric compositions of dimension 8 are the split octonions \mathbb{O} and the non-split octonions (Cayley numbers) C .

Twisted compositions over \mathbb{R}^3 and $\mathbb{R} \times \mathbb{C}$ are easily classified using [KMRT98, §36.29]. The classification has been written down explicitly in [Fio17, Example 3.23]. If $L = \mathbb{R}^3$, then $M = (\mathbb{O}, \mathbb{O}, \mathbb{O})$ or (C, C, C) . The quadratic form on M is defined as the usual quadratic form on every factor \mathbb{O} or C . If $L = \mathbb{R} \times \mathbb{C}$, then $M = V \times (V \otimes \mathbb{C})$, where V is a quadratic space of signature $(1, 7)$ or $(5, 3)$.⁵

There are three isomorphism classes of Albert \mathbb{R} -algebras: the split algebra J_s defined in Section 2,

$$J_1 := \left\{ \begin{pmatrix} \xi_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \xi_2 & c_1 \\ c_2 & \overline{c_1} & \xi_3 \end{pmatrix} : \xi_i \in \mathbb{R}, c_i \in C \right\},$$

and

$$J_c := \left\{ \begin{pmatrix} \xi_1 & c_3 & -\overline{c_2} \\ \overline{c_3} & \xi_2 & c_1 \\ c_2 & -\overline{c_1} & \xi_3 \end{pmatrix} : \xi_i \in \mathbb{R}, c_i \in C \right\};$$

see [SV00, §5.8 (ii)]. If $J = J(L, M)$ the quadratic form Q on J is related to the quadratic form $q : M \rightarrow L$ by the formula

$$(8.1) \quad Q(l, m) = \text{Tr}_{L/k}(l^2) + \text{Tr}_{L/k}(q(m));$$

see [KMRT98, Theorem 38.6]. When $L = \mathbb{R}^3$, the previous formula reduces to [SV00, (5.3)]. In particular, using the above presentations of J_s, J_1 and J_c , we see that the signatures of the quadratic forms on J_s, J_1 and J_c are $(15, 12)$, $(10, 17)$ and $(27, 0)$, respectively.

Let $G_s := \text{Aut}(J_s)$, $G_1 := \text{Aut}(J_1)$ and $G_c := \text{Aut}(J_c)$. The groups G_s, G_1 and G_c are representatives of the three isomorphism classes of \mathbb{R} -forms of type F_4 . The group G_s is split, G_1 has real rank 1 (i.e. a maximal \mathbb{R} -split subtorus of G_1 has rank 1), and G_c is anisotropic (i.e. it does not contain any nontrivial \mathbb{R} -split subtorus).

Lemma 8.2. *Let $\Gamma = (L, M, q, \cdot^{*2})$ be a twisted composition over \mathbb{R} , and let $J := J(L, M)$. Assume first that $L = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.*

- (a) *If $M = (\mathbb{O}, \mathbb{O}, \mathbb{O})$, then $J = J_s$.*
- (b) *If $M = (C, C, C)$ and q is positive definite, then $J = J_c$.*

⁵In [Fio17], the signatures are written with the negative part first, and the positive part last.

(c) If $M = (C, C, C)$ and q is positive definite on one factor and negative definite in the other two, then $J = J_1$.

Assume now that $L = \mathbb{R} \times \mathbb{C}$, and write $M = V \times W$, where V is over \mathbb{R} and W is over \mathbb{C} .

(a') If $(V, q|_V)$ has signature $(5, 3)$, then $J = J_s$.

(b') If $(V, q|_V)$ has signature $(7, 1)$, then $J = J_1$.

Proof. The proof proceeds by comparing the signatures of (M, q) and (J, Q) using (8.1).

If $L = \mathbb{R}^3$, the signature of $(L, Q|_L)$ is $(3, 0)$.

(a) If Γ is split, so is $J(L, M)$.

(b) The signature of (M, q) is $(24, 0)$, hence the signature of (J, Q) is $(27, 0)$, and so $J = J_c$.

(c) The signature of (M, q) is $(8, 16)$, hence the signature of (J, Q) is $(10, 17)$ and $J = J_1$.

If (U, q') is a complex quadratic space of complex dimension d , then $(U, \text{Tr}_{\mathbb{C}/\mathbb{R}}(q'))$ is a real quadratic space of signature (d, d) . It follows that when $L = \mathbb{R} \times \mathbb{C}$, the signature of $(L, Q|_L)$ is $(2, 1)$, and the signature of $(W, Q|_W)$ is $(8, 8)$.

(a') If the signature of $(V, q|_V)$ is $(5, 3)$, the signature of (J, Q) is $(2, 1) + (5, 3) + (8, 8) = (15, 12)$, so $J = J_s$.

(b') If the signature of $(V, q|_V)$ is $(1, 7)$, the signature of (J, Q) is $(10, 17)$, so $J = J_1$. \square

Recall that we denote by ρ_v the number of field factors of E^σ above v that ramify in E . If (E, σ) is an étale k -algebra with involution such that $\text{rank}_k E = \text{rank}_k E^\sigma = 2n$, (V, q) is a quadratic space of dimension $2n$ and v real place of k , by Proposition 7.2(c) (E_v, q_v) is realizable if and only if the signature of q_v is of the form $(\rho_v + 2s, \rho_v + 2r)$ for some $r, s \geq 0$. We want to apply this result to $k = L$. Note that L_v is never a field, but we may apply the result on each factor of L_v separately.

Proposition 8.3. *Let k be a global field, let v be a real place of k , let G be a k -group of type F_4 , and let $\alpha = (L, E, \sigma, \beta)$ be a datum. Then (G_v, α_v) is realizable over k_v if and only if one of the following is true:*

- (i) G_v is split and either $L_v = \mathbb{R}^3$ and ρ_v is even, or $L_v = \mathbb{R} \times \mathbb{C}$ and ρ_v is odd;
- (ii) G_v is anisotropic, $L_v = \mathbb{R}^3$ and $\rho_v = 0$;
- (iii) G_v has real rank 1 and either $L_v = \mathbb{R}^3$ and ρ_v is even, or $L_v = \mathbb{R} \times \mathbb{C}$ and $\rho_v = 1$.

Proof. (i) When G_v is split, by Lemma 8.2(a), SO_8^3 is a subgroup of G_v . Since $0 \leq \rho_v \leq 4$ and the quadratic form defining SO_8 has signature $(4, 4)$, it follows from Proposition 7.2(c) that (G_v, α_v) is realizable if and only if ρ_v is even.

(ii) When G_v is anisotropic, by Lemma 8.2 $L_v = \mathbb{R}^3$. Furthermore, the signature of (M_v, q_v) is $(24, 0)$, hence by Proposition 7.2(c) (G_v, α_v) is realizable if and only if $\rho_v = 0$.

(iii) When G_v has real rank 1, by Lemma 8.2 either $L_v = \mathbb{R}^3$ and (M, q) has signature $(8, 16)$, or $L_v = \mathbb{R} \times \mathbb{C}$ and the real factor of (M_v, q_v) has signature $(7, 1)$. By Proposition 7.2(c), in the first case (G_v, α_v) is realizable if and only if ρ_v is even, and in the second (G_v, α_v) is realizable if and only if $\rho_v \leq 1$.

By Lemma 8.2, we have $L_v = \mathbb{R}^3$ when $H_v = \text{Spin}_8$ or $\text{Spin}(8)$, and we have $L_v = \mathbb{R} \times \mathbb{C}$ when $H_v = \text{Spin}(7, 1)$ or $\text{Spin}(3, 5)$. \square

We conclude the section by describing the maximal tori in real forms of type F_4 explicitly, that is, without reference to a datum α . We will not use this result in the sequel.

Proposition 8.4. *Let $k = \mathbb{R}$, let G be a simple \mathbb{R} -group of type F_4 , and let T be an \mathbb{R} -torus of rank 4. Then T embeds in G if and only if $\text{rank}_{\mathbb{R}} T \leq \text{rank}_{\mathbb{R}} G$, that is, if and only if one of the following is true:*

- (a) $G = G_s$;
- (b) $G = G_c$ and $T = R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^4$;
- (c) $G = G_1$ and T is isomorphic to one of

$$\mathbb{G}_m \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^3, \quad R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^2, \quad R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^4.$$

Proof. (a) We know that G_s contains a subgroup isomorphic to Spin_8 . We have an embedding $\text{Spin}_4 \times \text{Spin}_4 \hookrightarrow \text{Spin}_8$. Recall the exceptional isomorphism $\text{Spin}_4 \cong \text{SL}_2 \times \text{SL}_2$. It follows that every torus of the form $\mathbb{G}_m^a \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^{4-a}$ arises as a maximal torus of Spin_8 .

Consider the embedding $\text{Spin}(3, 1) \times \text{Spin}(1, 3) \hookrightarrow \text{Spin}_8$. We have the isomorphisms $\text{Spin}(3, 1) \cong \text{Spin}(1, 3) \cong R_{\mathbb{C}/\mathbb{R}}(\text{SL}_2)$, hence $\text{Spin}(3, 1)$ and $\text{Spin}(1, 3)$ contain subtori isomorphic to $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. It follows that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)^2$ is a subtorus of Spin_8 .

Consider the embedding $\text{Spin}_2 \times \text{Spin}_6 \hookrightarrow \text{Spin}_8$. We have $\text{Spin}_2 \cong \mathbb{G}_m$ and $\text{Spin}_6 \cong \text{SL}_4$. The maximal tori in SL_4 have the form $R_{E/\mathbb{R}}^{(1)}(\mathbb{G}_m)$, where E is an étale algebra of degree 4 over \mathbb{R} . The subtori of SL_4 corresponding to $E = \mathbb{C} \times \mathbb{C}$ and $E = \mathbb{C} \times \mathbb{R}^2$ are $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)$ and $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times \mathbb{G}_m$, respectively.

It follows that Spin_8 contains subtori isomorphic to $\mathbb{G}_m \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)$ and $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times \mathbb{G}_m^2$.

By Lemma 8.2(a'), G_s contains a subgroup isomorphic to $\text{Spin}(3, 5)$. We have an embedding $\text{Spin}(3, 1) \times \text{Spin}(0, 4) \hookrightarrow \text{Spin}(3, 5)$. We have seen that $\text{Spin}(3, 1)$ contains a subtorus isomorphic to $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$. Since $\text{Spin}(0, 4)$ is anisotropic, all its maximal subtori are isomorphic to $R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^2$. It follows that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^2$ is a subgroup of $\text{Spin}(3, 5)$.

(b) Since G_c is anisotropic, T is also anisotropic, hence $T \cong R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^4$.

(c) Since G_1 has real rank 1, only the listed possibilities can occur. Let H be the connected component of the automorphism group of the twisted composition of Lemma 8.2(c). Then H embeds in the anisotropic group $R_{\mathbb{R}^3/\mathbb{R}}(\text{SO}(8)) = \text{SO}(8)^3$, hence it is anisotropic. It follows that $R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^4$ embeds in G_1 .

By Lemma 8.2(b'), G_1 contains a subgroup isomorphic to $\text{Spin}(7, 1)$. We have an embedding $\text{Spin}(3, 1) \times \text{Spin}(4, 0) \hookrightarrow \text{Spin}(7, 1)$. We have seen that $\text{Spin}(3, 1)$ contains a subtorus isomorphic to $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, and that all maximal tori of $\text{Spin}(4, 0)$ are isomorphic to $R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^2$. It follows that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^2$ is a subgroup of $\text{Spin}(7, 1)$, hence of G_1 .

We have an embedding $\text{Spin}(1, 1) \times \text{Spin}(6, 0) \hookrightarrow \text{Spin}(7, 1)$. Since $\text{Spin}(1, 1) = \text{Spin}_2 \cong \mathbb{G}_m$ and $\text{Spin}(6, 0)$ is anisotropic, it follows that $\text{Spin}(7, 1)$ contains a subtorus isomorphic to $\mathbb{G}_m \times R_{\mathbb{C}/\mathbb{R}}^{(1)}(\mathbb{G}_m)^3$. \square

9. LOCAL-GLOBAL PROPERTIES OF TWISTED COMPOSITIONS

The purpose of this section is to prove two auxiliary results on twisted compositions over global fields.

Proposition 9.1. *Let k be a global field, let J be an Albert algebra over k , let (L, M, q, \cdot^{*2}) be a twisted composition over k . If $J(L_v, M_v) \cong J_v$ for every $v \in \Sigma_k$, then $J(L, M) \cong J$.*

Proof. For every field extension K/k , isomorphism classes of twisted compositions over K correspond to elements of $H^1(K, \text{Spin}_8 \rtimes S_3)$; see [KMRT98, Proposition 36.7]. Isomorphism classes of Jordan algebras over K correspond to elements of $H^1(K, G_s)$, where G_s is the split group of type F_4 . Under these identifications, the map $H^1(K, \text{Spin}_8 \rtimes S_3) \rightarrow H^1(K, G_s)$ induced by the inclusion $\text{Spin}_8 \rtimes S_3 \hookrightarrow G_s$ sends a twisted composition over K to its Springer construction; see [KMRT98, Corollary 38.7].

Applying the Galois cohomology functor to the inclusion $\text{Spin}_8 \rtimes S_3 \hookrightarrow G_s$ for $K = k$ and $K = k_v$ for every $v \in \Sigma_k$ yields a commutative diagram

$$(9.1) \quad \begin{array}{ccc} H^1(k, \text{Spin}_8 \rtimes S_3) & \longrightarrow & H^1(k, G_s) \\ \downarrow & & \downarrow j \\ \prod_{v \in \Sigma_k} H^1(k_v, \text{Spin}_8 \rtimes S_3) & \longrightarrow & \prod_{v \in \Sigma_k} H^1(k_v, G_s). \end{array}$$

Since G_s is simply connected, by the Hasse principle the map j is injective. The conclusion follows from the commutativity of the diagram. \square

Proposition 9.2. *Let k be a global field, let L be a cubic étale algebra over k , and let (M, q) be a quadratic space of dimension 8 over L . Assume that for every place v of k , (L_v, M_v, q_v) admits the structure of a twisted composition $(L_v, M_v, q_v, \cdot_v^{*2})$ over k_v . Then (L, M, q) admits a unique structure of twisted composition (L, M, q, \cdot^{*2}) over k whose base change to k_v is isomorphic $(L_v, M_v, q_v, \cdot_v^{*2})$.*

Proof. We start by showing uniqueness. For every field extension K/k , twisted compositions over K are classified by $H^1(K, \text{Spin}_8 \rtimes S_3)$, hence we need to show that the natural map

$$H^1(k, \text{Spin}_8 \rtimes S_3) \xrightarrow{\partial_v} \prod_v H^1(k_v, \text{Spin}_8 \rtimes S_3)$$

is injective. Let $[\Gamma], [\Gamma'] \in H^1(k, \text{Spin}_8 \rtimes S_3)$ be such that $\partial_v([\Gamma]) = \partial_v([\Gamma'])$ for every $v \in \Sigma_k$. We have a commutative diagram

$$(9.2) \quad \begin{array}{ccc} H^1(k, \text{Spin}_8 \rtimes S_3) & \xrightarrow{\pi} & H^1(k, S_3) \\ \downarrow \partial_v & & \downarrow \\ \prod_v H^1(k_v, \text{Spin}_8 \rtimes S_3) & \xrightarrow{\pi_v} & \prod_v H^1(k_v, S_3). \end{array}$$

The vertical map on the right is injective. It follows that there exists $[L] \in H^1(k, S_3)$ such that $\pi([\Gamma]) = \pi([\Gamma']) = [L]$. Then $[\Gamma], [\Gamma'] \in \pi^{-1}([L]) = H^1(k, {}^L\text{Spin}_8)$ and $[\Gamma_v], [\Gamma'_v] \in \pi^{-1}([L_v]) = H^1(k_v, {}^{L_v}\text{Spin}_8)$ for every $v \in \Sigma_k$. Here ${}^L\text{Spin}_8$ and ${}^{L_v}\text{Spin}_8$ denote the twist of Spin_8 by the S_3 -torsor corresponding to the extension L/k and L_v/k_v , respectively. Since ${}^L\text{Spin}_8$ is simply connected, by the Hasse

principle for torsors the map

$$H^1(k, {}^L\mathrm{Spin}_8) \rightarrow \prod_v H^1(k_v, {}^{L_v}\mathrm{Spin}_8)$$

is injective. This map is just the restriction to $H^1(k, {}^L\mathrm{Spin}_8)$ of ∂_v , hence it sends $[\Gamma]$ and $[\Gamma']$ to the same element. We conclude that $[\Gamma] = [\Gamma']$, that is, $\Gamma \cong \Gamma'$.

We now turn to existence. By [SV00, (ii) p. 108], every twisted composition over a global field is reduced (the result is stated only for number fields, but the proof works equally well in the function field case). This means that, up to similitude, every twisted composition over a global field arises from a symmetric composition. Let r be the number of real places of k . By [SV00, (v) p. 22], every r -tuple of symmetric compositions at the real places of k comes from a symmetric composition over k . This gives a twisted composition Γ over k such that Γ_v is similar to $(L_v, M_v, q_v, \cdot^{*2})$ for every real place v . This means that $\Gamma_v \cong (L_v, M_v, \epsilon_v q_v, \delta_v \cdot^{*2})$, where $\epsilon_v \in L_v^\times / L_v^{\times 2}$ and $\delta_v \in L_v^\times / L_v^{\times 2}$. Since v is real, we have $k_v \cong \mathbb{R}$ and either $L_v = \mathbb{R}^3$ or $L_v = \mathbb{R} \times \mathbb{C}$. By the Weak Approximation Theorem, it is not difficult to find $\epsilon \in L$ and $\delta \in k$ such that the ϵ_v and δ_v come from ϵ and δ . If we set $\Gamma' := (L, M, \epsilon q, \delta \cdot^{*2})$, we have $\Gamma_v \cong \Gamma'_v$ for every real place v . On the other hand, if v is a finite place, then $H^1(k_v, {}^{L_v}\mathrm{Spin}_8)$ is trivial because ${}^L\mathrm{Spin}_8$ is simply connected. This means that the map $H^1(k_v, \mathrm{Spin}_8 \rtimes S_3) \rightarrow H^1(k_v, S_3)$ is a bijection, that is, a twisted composition over k_v is uniquely determined by the associated cubic étale k_v -algebra. We conclude that $\Gamma_v \cong \Gamma'_v$ for every $v \in \Sigma_k$, hence $\Gamma \cong \Gamma'$, by the uniqueness part. \square

10. LOCAL-GLOBAL PRINCIPLE FOR MAXIMAL TORI IN ORTHOGONAL GROUPS WITH TRIVIAL CLIFFORD INVARIANT

The purpose of this section is to prove the following proposition, which will be used in the proof of Theorem 1.1 and is also of independent interest.

Proposition 10.1. *Let k be a global field, let (V, q) be a quadratic space of dimension $2n$, let (E, σ) be an étale k -algebra with involution such that $\mathrm{rank}_k E = 2 \mathrm{rank}_k E^\sigma = 2n$. Assume that q has trivial Clifford invariant. If (E_v, q_v) is realizable for every $v \in \Sigma_k$, then (E, q) is realizable.*

Let (V, q) be a quadratic space of dimension $2n$. The condition that q has trivial Clifford invariant (i.e. the Clifford algebra $C(V, q)$ is split), is equivalent to the following condition on the Hasse invariant of q :

$$(10.1) \quad w(q) = \begin{cases} (-1, \mathrm{disc}(q)), & \text{if } n \equiv 0 \pmod{4} \\ 0, & \text{if } n \equiv 1 \pmod{4} \\ (-1, -\mathrm{disc}(q)), & \text{if } n \equiv 2 \pmod{4} \\ (-1, -1), & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

In particular, we see that if q and q' are $2n$ -dimensional quadratic forms with trivial Clifford invariant and equal discriminant, then $w(q) = w(q')$. These formulas are proved in [Lam05, Chapter V, 3.20], where they are expressed in terms of $\det(q)$ instead of $\mathrm{disc}(q)$. However, $\det(q)$ appears only when n is even (i.e. $\dim(q)$ is divisible by 4), and in that case $\mathrm{disc}(q) = \det(q)$.

Lemma 10.2. *In the course of proving Proposition 10.1, we may assume that $E = K_1 \times \cdots \times K_r$, where every K_i is a field and is σ -stable.*

Proof. Let (E, σ) be an étale algebra with involution over k , and write

$$E = E_1 \times E_2, \quad E_1 := K_1 \times \cdots \times K_s, \quad E_2 := K_{s+1}^2 \times \cdots \times K_r^2,$$

where every K_i is a field, K_i is σ -invariant when $1 \leq i \leq s$ and every σ acts by switching the two factors K_i when $s+1 \leq i \leq r$. We let $\text{rank}_k E_1 = 2n_1$ and $\text{rank}_k E_2 = 2n_2$.

Let (V, q) be a quadratic space of dimension $2n$, and assume that (E, q) is realizable everywhere locally. By [BF14, Theorem 3.2.1(a)], the set $\mathcal{C}(E, q)$ is not empty, and by [BF14, Theorem 3.2.1(b)] and the assumptions $\mathcal{C}(E_1, q')$ is connected for every quadratic space (V', q') of dimension $2n_1$ and trivial Clifford invariant.

We have $\Sigma_k^{\text{split}}(E_2) = \Sigma_k$. By Proposition 7.2(a), for every collection $(q_i^v) \in \mathcal{C}(E, q)$, we have that q_i^v is hyperbolic for every $v \in \Sigma_k$ and every $s+1 \leq i \leq r$. It follows that for every place v of k , we have a decomposition $q_v = q'_v \perp h_{2n_2}$ for some q'_v of dimension $2n_1$. By Witt's Decomposition Theorem [Lam05, Chapter I, 4.1], we may write $q = q_a \perp h_{2m}$, where q_a is anisotropic and h_{2m} is hyperbolic of dimension $2m$; the quadratic forms q_a and h are uniquely determined up to isometry. By the Hasse-Minkowski Principle [Lam05, Chapter VI, 3.1] there exists a place v such that $(q_a)_v$ is anisotropic. We have $(q_a)_v \perp h_{2m} \cong q_v \cong q'_v \perp h_{2n_2}$, hence by Witt's Cancellation Theorem [Lam05, Chapter I, 4.2] we must have $m \geq n_2$ (and $q'_v \cong (q_a)_v \perp h_{2(m-n_2)}$).

Since $m \geq n_2$, there exists a quadratic form p such that $q \cong p \perp h_{2n_2}$. We have $p_v \perp h_{2n_2} \cong q \cong q_v^1 \perp \cdots \perp q_v^s \perp h_{2n_2}$, hence by Witt's Cancellation Theorem $p_v \cong q_v^1 \perp \cdots \perp q_v^s$. This shows that $\mathcal{C}(E_1, p)$ is non-empty, i.e., that (E_1, p) is realizable everywhere locally. Since $\dim(p)$ and $\dim(h_{2n_2})$ are even and q has trivial Clifford invariant, by [Lam05, Chapter V, (3.13)], the Clifford invariant of p equals $(\text{disc}(p), 1)$, and so is also trivial. By assumption, this means that (E_1, p) is realizable. By Lemma 7.1, (E_2, h_{2n_2}) is realizable. Since $(E, \sigma) = (E_1, \sigma|_{E_1}) \times (E_2, \sigma|_{E_2})$, and $q \cong p \perp h_{2n_2}$, we conclude that (E, q) is realizable, as desired. \square

Lemma 10.3. *Let k be an arbitrary field, let q_1 and q_2 be quadratic forms of dimension $2n_1$ and $2n_2$. Assume that $\text{disc}(q_1)$ is a square and that q_1 and q_2 have trivial Clifford invariant. Then $q_1 \perp q_2$ has trivial Clifford invariant.*

Proof. Since $\dim(q_1)$ and $\dim(q_2)$ are even, by [Lam05, Chapter V, (3.13)] the Clifford invariant of $q_1 \perp q_2$ equals $(\text{disc}(q_1), \text{disc}(q_2))$. Since $\text{disc}(q_1)$ is a square, $(\text{disc}(q_1), \text{disc}(q_2)) = 0$, and the conclusion follows. \square

Lemma 10.4. *Let q_1 and q_2 be quadratic forms over \mathbb{R} , and let (r_1, s_1) and (r_2, s_2) be their signatures. Assume that $s_1 \equiv s_2 \pmod{2}$ and that $w(q_1) = w(q_2)$. Then $s_1 \equiv s_2 \pmod{4}$.*

Proof. We have $w(q_i) = \frac{1}{2}s_i(s_i - 1)(-1, -1)$, hence $w(q_1) = w(q_2)$ is equivalent to $s_1^2 - s_1 \equiv s_2^2 - s_2 \pmod{4}$, i.e. $(s_1 - s_2)(s_1 + s_2 - 1) \equiv 0 \pmod{4}$. By assumption $s_1 \equiv s_2 \pmod{2}$, hence $s_1 + s_2 - 1$ is odd. We conclude that 4 divides $s_1 - s_2$. \square

Lemma 10.5. *Let (E, σ) be an étale k -algebra with involution such that $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$. There exists a quadratic form q of dimension $2n$ and trivial Clifford invariant such that (E, q) is realizable.*

Proof. We may write $E^\sigma = k[t]/(f(t))$, for some separable polynomial $f(t) \in k[t]$, in such a way that $E = E^\sigma(\sqrt{t})$ and $\sigma(\sqrt{t}) = -\sqrt{t}$. Let $q : E \rightarrow k$ be the quadratic form defined by $q(x) := \text{Tr}_{E/K}(\alpha x \sigma(x))$, where $\alpha = \frac{1}{2f'(t)}$. By [BF14,

Proposition 1.3.1], (E, q) is realizable. By [Fio12, Lemma 3.6], q has trivial Clifford invariant.⁶ \square

Proof of Proposition 10.1. By Lemma 10.2, we may assume that $E = K_1 \times \cdots \times K_r$, where every K_i is a field and is σ -stable, and $\text{rank}_k E = 2 \text{rank}_k E^\sigma = 2n$. Let $I := \{1, \dots, r\}$. If I is connected, then the claim follows from [BF14, Theorem 3.2.1], so assume that I is not connected. Rearranging the K_i if necessary, we may then find some $1 < s < r$ such that if $i \leq s$ and $j \geq s + 1$, then i and j are not connected. Write

$$E = E_1 \times E_2, \quad E_1 := K_1 \times \cdots \times K_s, \quad E_2 := K_{s+1} \times \cdots \times K_r.$$

We have $\text{rank}_k E_i = 2 \text{rank}_k E_i^\sigma = 2n_i$, for some $n_i \geq 1$. For every real place v and $i = 1, 2$, we denote by ρ_v^i the number of real places of E_i^σ over v which do not ramify in E_i . We have $\rho_v = \rho_v^1 + \rho_v^2$ for every real place v .

By Lemma 10.5, there exist quadratic forms q_i over k of dimension $2n_i$ and with trivial Clifford invariant such that (E_i, q_i) are realizable, for $i = 1, 2$. By [BF14, Proposition 2.4.1] we have $\text{disc}(q_i) = \text{disc}(E_i) \in k^\times / k^{\times 2}$, and for every real place v the signature of q_i^v is of the form $(2r_v^i + \rho_v^i, 2s_v^i + \rho_v^i)$, where $r_v^i, s_v^i \geq 0$.

By [BF14, Proposition 2.4.1] we also have $\text{disc}(q) = \text{disc}(E)$, $w(q_v) = w(h_{2n})$ when $v \in \Sigma_k^{\text{split}}$, and for every real place v the signature of q_v is $(2r_v + \rho_v, 2s_v + \rho_v)$, for some $r_v, s_v \geq 0$.

Let v be a place of k . Assume that neither $\text{disc}(q_1)$ nor $\text{disc}(q_2)$ are squares in k_v^\times . We have

$$\text{disc}(q_1) = \prod_{i=1}^s \text{disc}(K_i), \quad \text{disc}(q_2) = \prod_{j=s+1}^r \text{disc}(K_j).$$

It follows that there exist $1 \leq i \leq s$ and $s + 1 \leq j \leq r$ such that $\text{disc}(K_i)$ and $\text{disc}(K_j)$ are not squares in k_v^\times . This means that i and j are connected, a contradiction.

We have just shown that for every place v of k , either $\text{disc}(q_1)$ or $\text{disc}(q_2)$ is a square in k_v^\times . By Lemma 10.3, this implies that $q_1 \perp q_2$ has trivial Clifford invariant. Since q_1 and q_2 are even-dimensional, we have $\text{disc}(q) = \text{disc}(q_1) \text{disc}(q_2) = \text{disc}(q_1 \perp q_2)$. It follows from (10.1) that $w(q_v) = w((q_1 \perp q_2)_v)$ for every place (finite or archimedean) v .

What we have shown so far is that q and $q_1 \perp q_2$ have the same rank, discriminant and Hasse invariant. It is not necessarily the case that the signatures of q and $q_1 \perp q_2$ coincide. Our strategy is to modify q_i so that (E_i, q_i) remains realizable for $i = 1, 2$, discriminant and Hasse invariant remain the same, but the signatures at real places also agree.

Let v be a real place. We have $\rho_v = \rho_v^1 + \rho_v^2$, hence the signature of $q_1 \perp q_2$ is $(2r_v^1 + 2r_v^2 + \rho_v, 2s_v^1 + 2s_v^2 + \rho_v)$. In particular, the signatures of q and $q_1 \perp q_2$ are congruent modulo 2. By Lemma 10.3, the negative parts of the signatures of q and $q_1 \perp q_2$ are congruent modulo 4, and it clearly follows that the same is true for the positive parts. In other words, for every real place v we may write

$$\text{sign}(q_v) = 2(r_v, s_v) + (\rho_v, \rho_v), \quad \text{sign}((q_1 \perp q_2)_v) = 2(r'_v, s'_v) + (\rho_v, \rho_v),$$

where $r_v \equiv r'_v \pmod{2}$ and $s_v \equiv s'_v \pmod{2}$. Note that whenever two real quadratic forms have signatures that differ by multiples of 4, their discriminants are

⁶In [Fio12], the Clifford invariant of q is called the Witt invariant, and is denoted by $W(q)$.

equal. Since the two signatures of q_v and $(q_1 \perp q_2)_v$ differ by multiples of 4, we can modify the signature of q_1 and q_2 by multiples of 4 to obtain the signature of q , leaving the discriminant untouched. Here are the details.

For every real place v , we construct two pairs (a_v^1, b_v^1) and (a_v^2, b_v^2) of non-negative integers such that $a_v^i + b_v^i = n_i - \rho_v^i$ and such that $a_v^1 + a_v^2 = 2r_v - \rho_v$, $b_v^1 + b_v^2 = 2s_v - \rho_v$, $a_v^1 + a_v^2 \equiv r_v \pmod{2}$ and $b_v^1 + b_v^2 \equiv s_v \pmod{2}$. We start by setting $a_v^i := r_v^i$ and $b_v^i := s_v^i$, and then we modify them according to the following procedure. If $a_v^1 + a_v^2 = r_v$ and $b_v^1 + b_v^2 = s_v$, then $\text{sign}(q_v) = \text{sign}((q_1 \perp q_2)_v)$ and we stop. Suppose that $a_v^1 + a_v^2 > r_v$ (if $a_v^1 + a_v^2 > r_v$, then $b_v^1 + b_v^2 > s_v$, and the procedure is entirely analogous). Since $a_v^1 + a_v^2 \equiv r_v \pmod{2}$, we have $a_v^1 + a_v^2 - 2 \geq r_v$. In particular, either (i) $a_v^1 \geq 2$, (ii) $a_v^2 \geq 2$ or (iii) $a_v^1 = a_v^2 = 1$. In case (i) we modify (a_v^1, b_v^1) into $(a_v^1 - 2, b_v^1 + 2)$, in case (ii) we modify (a_v^2, b_v^2) into $(a_v^2 - 2, b_v^2 + 2)$ and in case (iii) we modify (a_v^1, b_v^1) into $(a_v^1 - 1, b_v^1 + 1)$ and (a_v^2, b_v^2) into $(a_v^2 - 1, b_v^2 + 1)$, respectively. This process eventually stops, because the difference $a_v^1 + a_v^2 - r_v$ is non-negative, even, and decreases by 2 at each step. At each step of the process the sums $a^i + b^i$ remain invariant, hence $a_v^i + b_v^i = r_v^i + s_v^i = n_i - \rho_v^i$. At the end of the process we have $a_v^1 + a_v^2 = r_v$, hence $b_v^1 + b_v^2 = s_v$ also holds.

For every place v and $i = 1, 2$, consider the $2n_i$ -dimensional quadratic form p_i^v defined as follows: if v is finite then $p_i^v := (q_i)_v$, and if v is real then p_i^v has signature $2(a_v^i, b_v^i) + (\rho_v^i, \rho_v^i)$. By [Sch85, Theorem 6.10], there exist quadratic forms p_1 and p_2 such that $(p_i)_v = p_i^v$ for every place v . The torus (E_i, p_i) is realizable everywhere locally: if v is finite then this is because $((E_i)_v, (q_i)_v)$ is realizable, and if v is real it follows from [BF14, Proposition 2.3.1]. By induction, we deduce that (E_i, p_i) is realizable over k . If v is a finite place, all invariants of p_i and q_i agree, hence $q_v \cong (q_1 \perp q_2)_v \cong (p_1 \perp p_2)_v$. If v is a real place, then $q_v \cong (p_1 \perp p_2)_v$ because they have the same signature. By the weak Hasse-Minkowski principle [Lam05, Chapter VI, 3.3], it follows that $q \cong p_1 \perp p_2$. Since (E_i, p_i) are realizable, $(E, \sigma) = (E_1, \sigma) \times (E_2, \sigma)$, and $q \cong p_1 \perp p_2$, we conclude that (E, q) is realizable. \square

11. PROOFS OF THEOREM 1.1 AND THEOREM 1.2

Proof of Theorem 1.1. Let k be the a global field, let J be an Albert algebra over k , let $G := \text{Aut}(J)$ and let $\alpha = (L, E, \sigma, \beta)$ be a datum. It is clear that if (G, α) is realizable over k , then (G_v, α_v) is realizable over k_v for every place v of k .

Assume that (G_v, α_v) is realizable for every $v \in \Sigma_k$. By definition, for every $v \in \Sigma_k$ there exists a maximal torus T_v of G_v of type α_v . Let $\Gamma_v := (L_v, M_v, q_v, \cdot_v^{*2})$ be the twisted composition associated to T_v by Proposition 3.1. Note that $L_v = L \otimes_k k_v$ for every $v \in \Sigma_k$, but M_v, q_v, \cdot_v^{*2} are not assumed to be defined over k .

By Remark 6.6, (E_v, q_v) is realizable for every $v \in \Sigma_k$. By Proposition 7.2 and Lemma 6.7(i), we have $\text{disc}(q_v) = \text{disc}_{k_v}(L_v)$ in $L_v^\times / L_v^{\times 2}$. Since L is defined over k , we deduce that the collection $(\text{disc}(q_v))$ comes from a global discriminant $\delta = \text{disc}_k(L)$. By Lemma 6.7(ii), the Clifford invariant of $\text{O}(M_v, q_v)$ is trivial for every v . By (10.1) we see that $w(q_v) = (-1, \text{disc}(q_v)) = (-1, \delta)$. In particular, the number of places v such that $w(q_v) \neq 0$ is finite and of even cardinality, by the Hilbert reciprocity law. By [Sch85, Theorem 6.10], there exists a quadratic form q over L such that q_v is the completion of q at v for every place v . It follows that the triples (L_v, M_v, q_v) all arise as base change of a single (L, M, q) . By Proposition 9.2, there exists a unique twisted composition $\Gamma = (L, M, q, \cdot^{*2})$ over k such that $\Gamma \times_k k_v \cong \Gamma_v$ for every $v \in \Sigma_k$.

We have $J_v \cong J(L_v, M_v)$ for every $v \in \Sigma_k$, hence by Proposition 9.1 we deduce that $J \cong J(L, M)$. It follows from Remark 3.3 that $H := \text{Aut}(\Gamma)^\circ$ embeds in G .

By construction, for every $v \in \Sigma_k$ the pair (E_v, q_v) is realizable: there is a unique τ_q -stable $E'_v \subseteq \text{Aut}(L_v, M_v)$ such that $(E'_v, \tau_{q_v}) \cong (E_v, \sigma_v)$ and $T_v = U(E'_v, \tau_{q_v})$. By Lemma 6.7(ii), the L -space (M, q) has trivial Clifford invariant. By Proposition 10.1, this implies that (E, q) is realizable over k : there exists $E' \subseteq \text{Aut}_L(M)$ that is τ_q -stable and such that $(E', \tau_q) \cong (E, \sigma)$. Let $\Lambda = (\text{End}_L(M), L, \tau_q, \eta)$ be the trialitarian algebra given by applying [Fio17, Lemma 4.35] to $D = \text{End}_L(M)$ and $\tau = \tau_q$. Note that by [KMRT98, Proposition 44.16], Λ arises from a twisted composition $\Gamma' = (L, M, q, \cdot^{*2})$.

By [Fio17, Lemma 4.35], there exists a maximal torus of $\text{Spin}(\Lambda)$ such that the unique maximal L -torus of $R_{L/k}(\text{O}(\text{End}_L(M), \tau_q))$ containing the image of U under the trialitarian embedding is of type (E, σ) . Let $\beta' : (E, \sigma) \otimes L \rightarrow (E \times E^\Phi, \sigma \times \sigma)$ be the induced isomorphism.

Since k is a global field, by [Fio17, Corollary 3.30, Lemma 4.35], $\text{End}(\Gamma) \cong \text{End}(\Gamma')$, that is, Λ is isomorphic to the trialitarian algebra $\text{End}(\Gamma)$ associated to Γ . By [Fio17, Lemma 4.32], β' is determined by Γ and (E, σ) , hence $\beta = \beta'$. By Remark 6.3, we conclude that there exists a maximal torus U of $\text{Aut}(\Gamma)^\circ$ such that the unique maximal L -torus of $R_{L/k}(\text{O}(M, q))$ containing the image of U under the trialitarian embedding is of type (E, σ) , and such that the induced isomorphism $(E, \sigma) \otimes L \rightarrow (E \times E^\Phi, \sigma \times \sigma)$ is β . By definition, this means that G contains a maximal torus of type α , as desired. \square

Proof of Theorem 1.2. If (G, α) is realizable, then (G_v, α_v) is realizable for every $v \in \Sigma_k$. The condition on real places is then satisfied by Proposition 8.3.

Assume that the conditions at real places are satisfied. By Theorem 1.1, it suffices to show that (G_v, α_v) is realizable for every $v \in \Sigma_k$. If v is a finite place, (G_v, α_v) is realizable by Proposition 8.1. If v is a real place, (G_v, α_v) is realizable by Proposition 8.3. \square

12. CONJUGACY CLASSES OF MAXIMAL TORI

We conclude this work with a study of the conjugacy classes of maximal tori in groups of type F_4 over k . This is not used anywhere else in the paper.

Proposition 12.1. *Let k be a field of characteristic different from 2 and 3, let G be a k -group of type F_4 , let T_1, T_2 be two maximal k -tori in G , and let H_i be the unique subgroup of type D_4 containing T_i given by Proposition 3.1, for $i = 1, 2$. Then T_1 and T_2 are conjugate in G if and only if there exists an isomorphism $\varphi : H_1 \rightarrow H_2$ sending T_1 to T_2 .*

Proof. If there is $g \in G(k)$ such that $gT_1g^{-1} = T_2$, then by Proposition 3.1(c) we have $gH_1g^{-1} = H_2$, hence the automorphism of G given by conjugation by g sends H_1 to H_2 and T_1 to T_2 .

Assume that there exists an isomorphism $\varphi : H_1 \rightarrow H_2$ sending T_1 to T_2 . We claim that φ extends to an automorphism of G . By Proposition 3.1(c), we may write $H_i = \text{Aut}(\Gamma_i)^\circ$ for some twisted compositions $\Gamma_i = (L_i, M_i, q_i, \cdot_i^{*2})$ and $i = 1, 2$. By [KMRT98, Theorem 44.8], φ induces an isomorphism $f : \Gamma_1 \xrightarrow{\sim} \Gamma_2$. If we let J be the Albert algebra such that $G = \text{Aut}(J)$, we have $J(L_1, M_1) \cong J(L_2, M_2) \cong J$. By [KMRT98, Proof of Corollary 38.7], f extends to an automorphism of J , which by [KMRT98, Theorem 44.8] induces an automorphism of G extending φ , as claimed.

By [KMRT98, Theorem 25.16], every automorphism of F_4 is inner, hence T_1 and T_2 are conjugate. \square

Let k be a global field of characteristic different from 2 and 3, let J be an Albert algebra over k , let $G := \text{Aut}(J)$ and let T_1 and T_2 be maximal tori of G . For $i = 1, 2$, let $\alpha_i = (L_i, E_i, \tau_{q_i}, \beta_i)$ be the associated datum of T_i , and let $\Gamma_i = (L_i, M_i, q_i, \cdot_i^{*2})$ be the twisted composition associated to T_i in Proposition 3.1, so that we have the Springer decompositions $J = L_i \perp M_i$. Combining Proposition 12.1 with [Fio17, Corollary 4.63], we conclude that T_1 and T_2 are conjugate in G if and only if:

- (a) there exists an isomorphism $\varphi : \Gamma_1 \xrightarrow{\sim} \Gamma_2$;
- (b) the L -torus $\varphi_1(U(E_1, \tau_{q_1}))$ is conjugate to $U(E_2, \tau_{q_2})$ in $\text{SO}(M_2, q_2)$, where $\varphi_1 : \text{SO}(M_1, q_1) \rightarrow \text{SO}(M_2, q_2)$ is the isomorphism induced by φ ;
- (c) the L -torus $\varphi_2(U(E_1^{\Phi_1}, \tau_{q_1}))$ is conjugate to $U(E_2^{\Phi_2}, \tau_{q_2})$, where we denote by $\varphi_2 : \text{Spin}(M_1, q_1) \rightarrow \text{Spin}(M_2, q_2)$ the isomorphism induced by φ .

Condition (b) is characterized in terms of (L_i, E_i, τ_{q_i}) in [Fio17, Theorem 4.14], and condition (c) is characterized in [Fio17, Theorem 4.29]. Note that under the projections $\text{Spin}(M_i, q_i) \rightarrow \text{SO}(M_i, q_i)$, the L_i -tori $U(E_i^{\Phi_i}, \tau_{q_i})$ map onto $U(E_i, \tau_{q_i})$; see [Fio17, Theorem 4.24].

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