# Maximal Tori in $G_2$ and $F_4$

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# The General Type of Problem Being Considered

Throughout this talk $k$ will be a field and $\overline{k}$ its algebraic closure. (Some statements have not been checked in characteristic 2.)

If we want to understand the category of reductive groups over $k$, a natural question to ask is:
Given a triple:

$$(\tilde{H}, \tilde{G}, \tilde{f} : \tilde{H} \to \tilde{G})$$

all defined over $\overline{k}$ can we describe all of the triples

$$(H, G, f : H \to G)$$

over $k$ which become equivalent to the first over $\overline{k}$.

# More Specifically in This Talk

In this talk we will be primarily motivated by the case of maximal tori in groups of type $G_2$ or $F_4$, that is:

$$T \to G$$

for $T$ a maximal torus and $G$ a group of type $G_2$ (resp. $F_4$).

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for $T$ a maximal torus and $G$ a group of type $G_2$ (resp. $F_4$).

However, we will study this problem by looking at several other cases, namely:

$$T \to H \qquad \text{and} \qquad H \to G$$

where $T$ is still a maximal torus, $G$ is type $G_2$ (resp. $F_4$) and now $H$ is a group of type $A_2$ (resp. $D_4$). These will fit together to give a picture:

$$T \to H \to G.$$

# Changing The Problem

The reason this is natural is the following well-known result:

**Theorem**

*Let $G$ be a group of type $G_2$ (resp. $F_4$) then every maximal torus $T$ of $G$ factors through a unique simply connected subgroup $H$ of $G$ of type $A_2$ (resp. $D_4$).*
*Moreover, rational conjugacy classes of $T$ in $G$ are in bijection with pairs $(\overline{T}, \overline{H})$ where $\overline{H}$ is a rational conjugacy class of $H$ in $G$ and $T$ is a rational conjugacy class of $T$ in $H$.*

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## Proof.

The proof of this is "simple:" given $T$, take $H$ to be the long root subgroup of $G$. □

# First Problem

The difficulty with this proof is that it tells us virtually nothing about how to find $H$, understand the conjugacy classes, or which forms of these groups might even appear at all.

To concretely understand these subgroups $H$, we will first need to actually understand $G$.

# The Groups $G_2$ and $F_4$

Each group $G$ under consideration is the automorphism group of a certain non-associative algebra which we shall denote by $J$ in either case.

- In the case of $G_2$, the algebra $J$ is an octonion algebra. In particular, it is rank 8, non-commutative, non-associative, and alternative.
- In the case of $F_4$, the algebra $J$ is an exceptional Jordan algebra (Albert algebra). It is rank 27, non-associative, and commutative.

# Constructive Proof of Theorem

Given a maximal $k$-torus $T$ in $G$, it acts on $J$ and we gives the following $k$-rational decomposition:

$$J = L \oplus M$$

where $T$ acts trivially on $L$ and $M$ has no $T$-fixed vectors.

Multiplication in $J$ and projection lets us define 4 maps:

$$L \times L \to L \qquad L \times M \to M \qquad M \times M \to L \qquad M \times M \to M.$$

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- The $L$-valued pairing on $M$ gives an $L$-Hermitian bilinear form (resp. allows us to define a symmetric $L$-bilinear form).
- The multiplication on $M$ defines an $L$-determinant (resp. gives the quadratic space $M/L$ the structure of a twisted composition over $L$, i.e. if $Q$ is the associated quadratic form $\beta$ is a map such that $Q(x)Q(\beta(x,x)) = N_{L/k}(Q(x))$ and $\lambda\beta(\lambda x, \lambda x) = N_{L/k}(\lambda)\beta(x,x)$ ).

# Explicit Construction of $H$

We have just shown that

$$T \subset \mathrm{Aut}(L, J)^0 \subset H$$

where $H$ is the automorphism group of the Hermitian space (resp. of the twisted composition). Since these automorphisms groups are of type $A_2$ (resp. $D_4$) these are the same groups as obtained previously.

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Conveniently, this construction can be reversed. Given an étale algebra $L$ of rank 2 (resp. 3) and maps:

$$L \times M \to M \qquad M \times M \to L \qquad M \times M \to M$$

which come from an $L$-Hermitian space of trivial discriminant (resp. come from a twisted composition over $L$) we can define an algebra $J' = L \oplus M$.

# How is That Useful?

It follows immediately that

**Theorem**

*The set of triples* $(H, G, f : H \to G)$ *for* $H$ *of type* $A_2$ *(resp.* $D_4$*) and* $G$ *of type* $G_2$ *(resp.* $D_4$*) is in bijection with the collection of all triples* $(L, V, Q)$ *(resp.* $(L, V, Q, \beta)$*) consisting of a quadratic étale algebra and a rank 3 Hermitian space of trivial discriminant (resp. a cubic étale algebra and a twisted composition over it).*

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To study the collection of $H$ for a fixed $G$, we must ask for which of these data $(L, V, Q)$ (resp. $(L, V, Q, \beta)$) we reconstruct the original $J$. For local fields and global fields this is very easy to do. For more general fields, the condition $L \hookrightarrow J$ is clear, but there are higher cohomological invariants to consider. (This has been done for $G_2$ (Hooda) but not $F_4$ (as far as I know)).

# Maximal Tori in These Groups.

We actually know a great deal about classifying maximal tori in classical groups. If $H(R) = \{g \in A^\times \mid g\tau(g) = 1\}$ (where $(A, \tau)$ is a central simple algebra with involution) then

$$T \hookrightarrow H \qquad \leftrightarrow \qquad (E, \sigma) \to (A, \tau)$$

and we can write:

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Unfortunately, $D_4$ isn't classical for two reasons:

1. We need $\mathrm{Spin}_8$ rather than $\mathrm{SO}_8$, which does not directly come from the $(A, \tau)$ construction.
2. We need to understand certain triality forms of $D_4$.

# Tori in Spin Groups

If we are dealing with an $H$ which is a non-triality spin group, even thought $H$ isn't defined by some $(A, \tau)$ in the same way as classical groups, there are in fact two different ways to get a central simple algebra with involution out of it:

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- Firstly, it has have a quotient which does corresponds to an $(A, \tau)$; the orthogonal group.
  Consequently, given $T \subset H$ we obtain its image $T' \subset O(A, \tau)$ and we have $T' \leftrightarrow (E, \sigma) \subset (A, \tau)$. Moreover, there is a surjective map $T \to T_{E,\sigma} = T'$.

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- Secondly, the even Clifford algebra $(C, \tau)$ of $(A, \tau)$ actually contains $H$.
  Consequently, we have $T \to H \to \mathrm{SU}(C, \tau)$ and there is a unique maximal torus of $\mathrm{SU}(C, \tau)$ which contains $T$. We obtain another algebra $(\tilde{E}, \sigma)$ and we have $T \subset T_{\tilde{E},\sigma}$.

# From $E$ to $\tilde{E}$

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Recall that there is a bijection between étale algebras and finite sets with a Galois action, so to specify $E$ is to specify a Galois set:

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Let us make a definition:
Given $(E, \sigma)$, we say that $\phi \subset \mathrm{Hom}(E, \overline{k})$ is a $\sigma$-reflex type if:

$$\mathrm{Hom}(E, \overline{k}) = \phi \sqcup \sigma(\phi).$$

Let $\Phi = \{\phi \mid \sigma - \text{reflex type of } E\}$. Then $\Phi$ is a Galois set with a canonical involution, and we obtain $(E^{\Phi}, \sigma)$ .

# About $E^\Phi$

One of the results from my thesis was $(\tilde{E}, \sigma) \simeq (E^\Phi, \sigma)$, that is, if $(E, \sigma)$ is associated to a torus in an orthogonal group, the canonical algebra $(E^\Phi, \sigma)$ is the only possible thing which can be associated to a corresponding torus in the spin group!
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- The existence of an embedding $(E^\Phi, \sigma) \to (C, \tau)$ over the center implies Hasse invariant and discriminant conditions for the existence of an embedding $(E, \sigma) \to (A, \tau)$.
- There exists a canonical multiplicative map $N_{E/E^\Phi} : E \to E^\Phi$, whose restriction (almost) gives a section of the map from the spin group to the orthogonal group in that it maps the torus in the orthogonal group to its preimage in the spin group.

# Back to Tori in Spin

We can understand the image of $N_{E/E^{\Phi}} : E \to E^{\Phi}$ as follows. There is a natural multiplicative map:

$$\Psi : E^{\Phi} \to E \otimes_k E^{\Phi}$$

such that $\Psi \circ N_{E/E^{\Phi}}$ takes $x$ to $x^2 \otimes 1$. Moreover, $\Psi$ naturally restricts to give the covering map from the spin group to the orthogonal group.

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In particular, we can describe all the possible tori in (non-triality) spin groups:

$$T(R) = \{(g_1, g_2) \in ((E \oplus E^\Phi) \otimes R)^\times \mid g_i\sigma(g_i) = 1 \text{ and } \Psi(g_2) = g_1\}.$$

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$$T(R) = \{(g_1, g_2) \in ((E \oplus E^\Phi) \otimes R)^\times \mid g_i\sigma(g_i) = 1 \text{ and } \Psi(g_2) = g_1\}.$$

Understanding which $(E, \sigma, E^\Phi, \sigma)$ can correspond to a torus in a given spin group reduces to the question for orthogonal groups.

# Triality Groups

We are interested in triality groups attached to $(L, V, Q, \beta)$ for $L$ a cubic étale algebra, $(V, Q)$ a quadratic space, and $\beta$ a twisted composition. We have the following two ways to attach $(E, \sigma)$ to tori in these groups:

- Firstly,
$$\mathrm{Spin}(L, V, Q, \beta) \to \mathrm{Res}_{L/k}(O(V, Q)).$$
Hence, tori in spin give tori in $O(V, Q)$ which give $(E, \sigma)$ an étale algebra over $L$.

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- Secondly, after base change to $L$ we obtain the "classical" spin group associated to $(V, Q)$. Moreover, as a representation, $V \otimes_k L = V \oplus S$ decomposes as a direct sum of the vector representation and the spin representation.
Thus, given a torus in $\mathrm{Spin}(L, V, Q, \beta)_L$, we obtain $(E, \sigma) \times (E^{\Phi}, \sigma)$, where $(E, \sigma)$ is an étale algebra over $L$.

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It seems there is a conflict in our notation, the theorem on the next slide will resolve this.

# Tori in Triality Groups

Note: proofs for these results have only been sketched.

**Theorem**

*Consider a twisted composition* $(L, V, Q, \beta)$, *and a torus* $T$ *in* $\mathrm{Spin}(L, V, Q, \beta)$. *Then* $T$ *is associated to* $(E, \sigma)$, *an étale algebra with involution over* $L$, *arising from a torus of* $O(V, Q)$. *Moreover,* $(E, \sigma)$ *satisfies* $(E, \sigma) \otimes L \simeq (E, \sigma) \oplus (E^{\Phi}, \sigma)$.

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**Theorem**

*Given $(L, V, Q)$ for which $\beta$ exists, suppose $T_{E,\sigma}$ is a torus of $O(V, Q)$ and that $(E, \sigma) \otimes L \simeq (E, \sigma) \oplus (E^{\Phi}, \sigma)$. One can construct $\beta$ such that $(E, \sigma)$ will be associated to a torus in $\mathrm{Spin}(L, V, Q, \beta)$.*

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One should be able to describe all possible $\beta$ for any given $(E, \sigma)$. For local/global fields this is straightforward; for more general fields I do not have a solid grasp on the classification of twisted compositions.

# Conclusions and Further Remarks

Not all triality groups are from twisted compositions. The others come from data $(L, A, \tau, \beta)$, and analogous theorems should still hold.

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With all we have discussed today we obtain a fairly complete picture of the structure of tori in all these groups. In particular we hopefully now have have a much better understanding of the diagrams:

$$T \to A_2 \to G_2$$

and

$$T \to D_4 \to F_4$$

# The End.

Thank you.