Arthur packets for p-adic groups through vanishing cycles and perverse sheaves

Andrew Fiori

University of Lethbridge

2018

This work is joint with Clifton Cunningham, Ahmed Moussaoui, James Mracek and Bin Xu

A preprint of our paper:
Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples
is available: `arxiv.org/abs/1705.01885`.

Langlands Correspondence

The Langlands correspondence very roughly suggests that, if

- F is a field,
- G is a reductive group over F, and
- ${}^LG$ is the Langlands dual of G

then we have a *correspondence* between

Equivalence classes of (correctly adjectivized) Representations of Γ_F into ${}^LG$	$\longleftarrow\!-\!-\!\longrightarrow$	Equivalence classes of (correctly adjectivized) representations of the group $G(F)$

where if

$$\rho \longleftarrow\!-\!-\!\longrightarrow \pi$$

then (when appropriately defined)

$$L(s,\rho) = L(s,\pi)$$

Local Langlands Correspondence

For F a p-adic local field, we reformulate this by replacing Γ_F by W_F, a dense subgroup.
We have maps:

$$\mathrm{Hom}(\Gamma_F, {}^L G) \to \mathrm{Hom}(W_F, {}^L G) \leftarrow \mathrm{Hom}(W_F \times \mathrm{SL}_2, {}^L G)$$

and so by composition of things that aren't maps (that is correspondences), we can reformulate the correspondance in terms of some subset of

$$\mathrm{Hom}(W_F \times \mathrm{SL}_2, {}^L G).$$

We call elements $\phi \in \mathrm{Hom}(W_F \times \mathrm{SL}_2, {}^L G)$ (which satisfy a few additional technical conditions) **Langlands parameters**. We denote the set of equivalence classes Φ.

Local Langlands Correspondence

Having made this reinterpretation, the conjectured correspondence simplifies:

There is a surjective map:

$$\Phi \quad \twoheadleftarrow \quad \left\{ \begin{matrix} \text{Equivalence classes of} \\ \text{irreducible admissible} \\ \text{Representations of } G(F) \end{matrix} \right\}$$

with finite and non-empty fibers. For $\phi \in \Phi$ we denote the fiber by Π_ϕ.

Importantly, this map should be *functorial* in various ways.

Grouping L-parameters - Infinitesimal Parameters

In order to study the sets Π_ϕ, it is convenient to group them according to something called an **infinitesimal parameter**. These are maps $\lambda : W_F \to {}^LG$ (which satisfy a few technical conditions). Each $\phi \in \Phi$ has an associated infinitesimal parameter, $\lambda_\phi : W_F \to {}^LG$, which is given by:

$$\lambda_\phi(w) = \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})).$$

We can then write:

$$\Phi = \bigsqcup_{\lambda/\sim} \Phi_\lambda$$

and attempt to study the sets

$$\Pi_\lambda = \bigsqcup_{\phi \in \Phi_\lambda} \Pi_\phi$$

Note: the sets Π_λ are natural on the representation theory side, λ determines a character $\hat{\lambda}$ of $G(F)$, the set Π_λ are those representations for which $\hat{\lambda}$ is their infinitesimal character.

Moduli Space of Langlands Parameters

Given an infinitesimal parameter λ, Vogan introduced a variety with a group acting on it.

- V_λ, is the q eigenspace of $\lambda(\mathrm{Frob_q})$ acting on the Lie algebra of the centralizer of $\lambda(I_F)$ in ${}^LG$.
  This is just $\mathbb{A}^n$ for some n.
- H_λ, is the centralizer in ${}^LG$ of $\lambda(W_F)$.
  Is a reductive group which acts on V_λ through some irreducible representation.

Morally, this can be thought of as some sort of *moduli space for the set of L-parameters* which have fixed infinitesimal parameter λ. Even though it isn't actually this.

What is true: The group H_λ acts on V_λ with finitely many orbits, C_ϕ.
The set of these orbits are in bijection with Φ_λ.

Alternate parametrization for Π_λ - Arthur Parameters

Consider instead **Arthur Parameters**, that is maps:

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

(which satisfy a few technical conditions), and denote the set of them Ψ.

Each $\psi \in \Psi$ determines an element $\phi_\psi \in \Phi$, via the association:

$$\phi_\psi(w, x) = \psi\left(w, x, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})\right)$$

and also an infinitesimal parameter

$$\lambda_\psi = \lambda_{\phi_\psi} = \psi\left(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}), \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})\right).$$

From which we may define the collection Ψ_λ.

Note that the map $\Psi_\lambda \to \Phi_\lambda$ is not surject, though perhaps surprisingly, it is injective.

Geometric Interpretation of A-Parameters

Arthur parameters have a geometric interpretation in Vogan's moduli space.

Each A-parameter $\psi \in \Psi_\lambda$ naturally gives an open H_{λ_ψ}-orbit

$$\mathcal{O}_\psi \subset T^*_{C_{\phi_\psi}}(V_{\lambda_\psi})$$

in the conormal bundle, $T^*_{C_{\phi_\psi}}(V_{\lambda_\psi})$, to the strata C_{ϕ_ψ} associated to the L-parameter ϕ_ψ in V_{λ_ψ}.

Not every conormal bundle for a stratum has an associated A-parameter. Those that do satisfy some subtle geometric conditions.

Arthur Packets and Arthur's Stable Distributions

Arthur goes on to define sets, which we now call Arthur packets, Π_ψ for which we have

$$\Pi_{\phi_\psi} \subset \Pi_\psi \subset \Pi_{\lambda_\psi}$$

and to each ψ he defines an associated stable distribution

$$\Theta_\psi^G = \sum_{\pi \in \Pi_\psi} \langle \psi, \pi \rangle \Theta_\pi$$

on $G(F)$.

These packets and their distributions are crucial in Arthur's work on the endoscopic classification of unipotent representations. More specifically, establishing their functorial behavior under endoscopic transfer is essential.

The Local Langlands Correspondence according to Vogan

Vogan has shown that the conjectural local Langlands correspondence is equivalent to having a bijection between

$$\Pi_\lambda \quad \leftrightarrow \begin{Bmatrix} \text{isomorphism classes of irreducible} \\ \text{objects in } D_{H_\lambda}(V_\lambda) \text{ (up to shift)} \end{Bmatrix}$$

where

$$D_{H_\lambda}(V_\lambda) = \quad \begin{array}{l} \text{Bounded derived category of} \\ H_\lambda\text{-equivariant étale sheaves on } V_\lambda \\ \text{with constructable cohomology.} \end{array}$$

Under this bijection

$$\pi \in \Pi_\phi \quad \leftrightarrow \quad \mathcal{L}_\pi \text{ a skyscraper sheaf over } C_\phi.$$

(here Π_λ should actually be the union over pure inner forms G' of G of the sets $\Pi_\lambda(G')$)

Reinterpreted for the Grothendieck Group

We first point out that the Grothendieck group, $K\Pi_\lambda$, has two natural basis. The irreducible representations, Π_λ, and the standard modules (the induced representation which allows us to construct each $\pi \in \Pi_\lambda$).

We can reinterpret the local Langlands correspondace to say that there exists a perfect pairing between the Grothendieck groups

$$\langle \cdot, \cdot \rangle : K\Pi_\lambda \times KD_{H_\lambda}(V_\lambda) \to \mathbb{C}$$

such that the "dual basis"(up to Kottwitz sign) with respect to the basis consisting of the images of the standard modules in $K\Pi_\lambda$ are precisely the "irreducible constructable sheaves" $\mathcal{L}_\pi$ mentioned before.

and under this association each standard module $M(\pi)$ for $\pi \in \Pi_\phi$ corresponds to a sheaf $\mathcal{L}_\pi$ whose support is exactly C_ϕ.

Moral: The sets Π_ϕ are determined by the equivariant étale fundamental groups of the strata C_ϕ.

This version of LLC is known for split classical groups with no pure inner forms, as well as unitary groups, but there is ongoing work by a number of people which is expected to extend this to all classical groups.

Kazhdan-Lusztig Conjectures

If we switch the basis for $K\Pi_\lambda$ from the standard modules to, the perhaps more natural basis, Π_λ then the dual basis (under the same pairing) is precisely the set of simple H_λ-equivariant Perverse sheaves on V_λ.

(Perverse sheaves are a natural abelian subcategory of the derived category).

We note that the set of simple H_λ-equivariant Perverse sheaves on V_λ are

$$\mathrm{IC}(C_\phi, \mathcal{L}_\pi)$$

the intersection cohomology sheaves for the same collection of étale local systems on the same strata.

For $\pi \in \Pi_\phi$, if $M(\pi)$ corresponds to a sheaf $\mathcal{L}_\pi$ whose support is C_ϕ, then π corresponds to the sheaf $\mathrm{IC}(C_\phi, \mathcal{L}_\pi)$.

Moral: The opaque combinatorics that describe induction for Hecke-modules are the same as the opaque combinatorics which determine the intersection cohomology of these varieties.

This conjecture is known for GL_n and SO_{2n+1}.

Arthur Packet Conjecture

There is a "microlocalization" functor

$$\mathrm{Ev} : D_{H_\lambda}(V_\lambda) \to D_{H_\lambda}(T^*_{H_\lambda}(V_\lambda))$$

which is defined in terms of vanishing cycles.

Here $T^*_{H_\lambda}(V_\lambda)$ is the subset of $V_\lambda \times V^*_\lambda$ consisting of pairs (x, ξ) where if $x \in C$, then ξ is in the conormal bundle to C at x.
Recall: Arthur paramaters give an open orbits $\mathcal{O}_\psi \subset T^*_H(V_\lambda)$.

The conjecture is that given an Arthur parameter ψ we have the Arthur packet is

$$\Pi_\psi = \{\pi \in \Pi_\lambda \mid \mathrm{Ev}(\mathrm{IC}(C_{\phi_\pi}, \mathcal{L}_\pi))|_{\mathcal{O}_\psi} \neq 0\}$$

Moreover, the coefficients $\langle \psi, \pi \rangle$ appearing in Arthur's distribution are determined by the traces of the equivariant fundamental group of $\mathcal{O}_\psi$ acting on the stalks of $\mathrm{Ev}(\mathrm{IC}(C_\phi, \mathcal{L}_\pi))|_{\mathcal{O}_\psi}$.

Moral The local structure of the singularities along C_ϕ determines both an Arthur packet, and Arthur's distributions.

We have in mind a proof of (a more precise version of) this conjecture, and are currently working out details for the case of odd orthogonal groups.

Aubert Involution/Fourier Transform Conjecture

Aubert defined an involution on $K\Pi$ which takes $K\Pi_\lambda$ to $K\Pi_\lambda$.

Additionally, there is an involution on $D_{H_\lambda}(V_\lambda)$ given by the Fourier-Delign transform.

It is conjectured that these are adjoint under the pairing, that is:

$$\langle A(\pi), \mathcal{E} \rangle = \langle \pi, \mathrm{Ft}(\mathcal{E}) \rangle$$

Moral This geometric description is Functorial with respect to some natural Functors.

We have in mind a proof of this conjecture, and are currently working out details.

Induction/Restriction Conjecture

Given a Levi subgroup L of ${}^LG$ through which λ factors, and a parabolic subgroup P for L there are *induction and restriction* operations between the sets $\Pi_\lambda(L)$ and $\Pi_\lambda(G)$.

There are also geometric induction/restriction functors between the categories $D_{H_L}(V_L)$ and $D_{H_G}(V_G)$.
It is conjectured that these are adjoints, under the pairing, that is:

$$\langle \mathrm{Res}_P(\pi_G), \mathcal{L}_L \rangle = \langle \pi_G, \mathrm{Ind}_P(\mathcal{L}_L) \rangle$$

and

$$\langle \mathrm{Ind}_P(\pi_L), \mathcal{L}_G \rangle = \langle \pi_L, \mathrm{Res}_P(\mathcal{L}_G) \rangle$$

Moral This geometric description is Functorial with respect to some other very natural functors on the categories.
One can formulate similar claims for more general cases of transfer (Langlands-Shelstad/Kottwitz-Shelstad transfer), some cases of (some) these results are known (in some cases only to experts), other cases we are working on.

What does our paper do?

A preprint of our paper: *Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples* is available: `arxiv.org/abs/1705.01885`.

In the paper we spend around 75 pages introducing enough background and develop enough of the theory in order to define all of the objects and functors that allow us to make precise conjectures.
That is we make precise a number of implicit conjectures of Vogan (and Adams-Barbash-Vogan).

In the remaining 124 pages we work out explicit examples of the various conjectures for various infinitesimal parameters for a collection of groups. Demonstrating that it is actually possible to do explicit computations on the geometric side, $D_H(V)$, (something Adams-Barbash-Vogan did not do) and give real evidence for all of the conjectures by comparing against explicit calculations done using other techniques on the representation side, Π_λ side.

(We thought the paper was maybe long enough already and so did not actually define any of Ft, Ind_P, or Res_P in the paper.)

The End.

Thank you.