Geometric Problems Associated to a Geometric Description of Arthur Packets for p-adic Groups

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Warning: The further into the talk I get the less you will likely understand about what I am saying, and the more I will understand about the things I am talking about.

If you let me get to the end of the talk you will likely be lost, so it is probably in your interest to slow me down with questions!

Langlands Correspondence

The Langlands correspondence very roughly suggests that, if we fix a field F, and a reductive group G over F, and denote by Γ_F the Galois group of F and by $\hat{G}$ the dual reductive group of G then we have a *correspondence* between

Representations of Γ_F into $\hat{G}$	$\xleftarrow{\,???\,}\!\!\rightarrow$	Representations of the group $G(F)$

Class field theory gives us a model example of such a thing happening.

But there are problems trying to generalize:

- We probably need an equivalence relation on one or both sides.
- We need to put way more adjectives on both sides to both clarify what we mean and to have even the right cardinality.
- We should specify some conditions, because the existence of a bijection between countable sets is not surprising (for example L-functions agree).

Local Langlands Correspondence

The general formulation, doesn't depend on whether the underlying field F is global ($\mathbb{Q}$) or local ($\mathbb{Q}_p$).

The local Langlands correspondence is the case of a Local field.

To make the formulation more precise in this context we:

- Replace Γ_F, by the Weil group, W_F (which is a dense subgroup of the Galois group in this context, this is done for topological compatibility reasons) and then by the local Langlands group

  $$W_F \times \mathrm{SL}_2(\mathbb{C})$$

  (This essentially just groups representations according to how dense their image is.)
- We then replace the reductive dual group $\hat{G}$, by the L-group

  $${}^L G = \hat{G} \rtimes W_F$$

  and insist on compatibility between the W_F in the maps.
  (This essentially just handles the case where the group G is not *split*).
- On the right hand side we want to consider irreducible admissible representations of $G(F)$. Importantly, the word **admissible** actually has a definition better than just the ones which should work.

What we (hope to) obtain

Remember that a correspondence isn't a bijection, what we will obtain is to decompose the left hand side

$$\left\{\begin{matrix}\text{Iso-classes of irreducible admis-} \\ \text{sible representations of } G(F)\end{matrix}\right\} = \bigsqcup_{\phi\in\Phi} \Pi_\phi$$

into (finite and non-empty) pieces based on the right hand side

$$\Phi = \{\text{Well-behaved maps } W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG\}$$

which we call Langlands parameters (the term **well-behaved** has a specific meaning). With various assertions such as:

- Properties of ϕ relate to properties of $\pi \in \Pi_\phi$, So things like L-functions are preserved.
- This decomposition is in some sense **functorial** with respect to maps between groups ${}^LG' \to {}^LG$, or rather functors between their representations (induction/restriction).

Grouping L-parameters

In order to better understand the set Φ, it is convenient to decompose it according to something called an **infinitesimal parameter**. Each $\phi \in \Phi$ has an infinitesimal parameter, $\lambda_\phi : W_F \to {}^L G$, which is essentially:

$$\lambda_\phi(w) = \phi(w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$$

(or for some people its conjugacy class).We can then write:

$$\Phi = \bigsqcup_{\lambda} \Phi_\lambda$$

and attempt to study the sets

$$\Pi_\lambda = \bigsqcup_{\phi \in \Phi_\lambda} \Pi_\phi$$

One key to our work, is that Vogan has reinterpreted the Local Langlands Correspondence in terms of statements about Π_λ.

(or rather the union $\Pi_\lambda(G')$ as we run over the set of pure inner forms of G an annoying technicallity which I will ignore).

Geometric Local Langlands Correspondence

Given an infinitesimal parameter λ we can consider:

- $V = V_\lambda$, which is the q eigenspace of $\lambda(\mathrm{Frob_q})$ acting on the Lie algebra of the centralizer of $\lambda(I_F)$ in $\hat{G}$.
- $H = H_\lambda$, which is the centralizer in $\hat{G}$ of $\lambda(\mathrm{Frob_q})$.

The group H acts on V with finitely many orbits, C_ϕ, which are in bijection with Φ_λ.

"Idea of Theorem" (Vogan)
There is a bijection between:

The set Simple H-equivarient perverse sheaves on V

and

The set: Π_λ

The actual theorem is much more specific, in particular, the subset Π_ϕ is associated naturally to a subset of sheaves associated to C_ϕ.

Note: it is also known that the graded endomorphism algebra of certain perverse sheaves is isomorphic to a certain graded Hecke algebra.

Refining L-parameters

The exact opposite of the idea of grouping is to refine the sets Π_ϕ. This can be done as follows.
Consider instead maps:

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

we will call (those of these that satisfy a couple extra properties [such as being *bounded*]) **Arthur Parameters**, and denote the set of them Ψ.

We note that each $\psi \in \Psi$ determines elements $\phi \in \Phi$, in fact it does in many different ways, however we shall consider the particular association:

$$\phi_\psi(w, x) = \phi(w, x, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2}))$$

Arthur goes on to define sets $\Pi_\psi \subset \Pi_{\phi_\psi}$, and is able to prove results about the Functoriality of these as well as results about associated distributions on $G(F)$.

This whole construction is one of the keys to Arthurs Endoscopic Classification of representations.

Natural Questions

How do Arthur's sets Π_ψ relate to the category of H-equivarient perverse sheaves on V?
The set of packets correspond to a subset of, Φ_λ, that is a subset of the orbits. (This is not obvious!)
What is in each packet is a conjecture (by either Vogan, Adams-Barbash-Vogan, or us)

How can we recover Arthur's stable distributions η_ψ from the category of H-equivarient perverse sheaves on V?
The details are a conjecture (by either Vogan, Adams-Barbash-Vogan, or us)

What evidence was there for this conjectures?
None, because previously no one was able to compute non-trivial (nor did they actually compute trivial) examples of any of this.

So how can this possibly by better than what Arthur was already doing?
It is our hope that geometric functoriality results will lead to straight forward proofs and that computations in the geometric context will be easier.

Also... what is a Perverse Sheaf?

What is an (Equivariant) Perverse Sheaf?

A Perverse Sheaf on X is:

- An element $\mathcal{P}$ of the bounded derived category of étale sheaves with constructible cohomology on X.
- For each natural number i the set of points x with $H^{-i}(j_x^*\mathcal{P}) \oplus H^i(j_x^!\mathcal{P}) \neq 0$ has dimension at most $2i$.

An equivariant perverse sheaf for the action of a group H means attaching a H action to $\mathcal{P}$ which is compatible with the action of H on X.

Observation: The above description is mostly useless.

But the above does give two major implications:

- For each orbit C of the action of H on X we have that $\mathcal{P}|_C$ restricts to a complex of equivariant (étale) local systems.
- For a connected group H irreducible equivariant (étale) local systems on C are in bijection with irreducible representations of equivariant étale fundamental group of C. And you pick them out using the decomposition theorem for proper pushforward from a finite cover.

Observation: So understanding a perverse sheaf is mostly about identifying *the set of local systems associated to its restriction to each orbit*.

Facts about Perverse Sheaves that help.

- Perverse sheaves are stable under smooth pull back [shifted by relative dimension] .
- The proper pushforward of a perverse sheaf is a direct sum of shifted perverse sheaves.
- Perverse sheaves are stable under Verdier duality, as are their pushforwards through proper maps.
- For X smooth, the sheaf $\mathbb{1}_X[\dim X]$ is a perverse sheaf on X.
- The category of perverse sheaves is abelian and semi-simple, the simple objects, $\mathcal{IC}(C, \mathcal{E})$ are in bijection with pairs consisting of a locally closed subvariety and an étale local system on it. Moreover,

$$\mathcal{IC}(C, \mathcal{E})|_C = \mathcal{E}[\dim C]$$

- If $\mathcal{P}$ is perverse, then $\mathcal{P}|_C$ is supported in degrees $\leq \dim C$.

Why these facts help

These facts tell us that in order to describe the category of equivariant perverse sheaves on X I just need to be able to explicitly construct (sufficiently many) proper covers of the closures of the orbits by smooth varieties, so:

$$\tilde{C} \xrightarrow{\pi} \overline{C} \subset X$$

where $\tilde{C}$ is smooth, π is proper and H-equivarient, and C is an orbit in X.

It turns out that once one explicitly describes the varieties X that we actually are working with, this sort of thing becomes possible.

What is a Vogan variety, really.

Some simplifying assumptions:

Theorem("Unramification")
Without loss of generality we can replace $\hat{G}$ by the centralizer of $\lambda(I_F)$ in $\hat{G}$.

(in fact one can prove that this works in terms of representations too).

Theorem("Non-elliptic")
Without loss of generality we can assume $\lambda(\mathrm{Frob}_q)$ is hyperbolic, by passing to the centralizer of its elliptic part.

(this requires allowing disconnected groups $\hat{G}$, which is problematic on the representation theory side because the phrase "connected reductive group" is pretty popular).

Assuming $\hat{G}$ is embedded in $\mathrm{GL}(W)$, then the eigenspace decomposition of

$$W = \oplus_i W_i = \oplus_i W_{q^{i/2}}$$

with respect to $\lambda(\mathrm{Frob}_q) \in \hat{G} \subset \mathrm{GL}(W)$ determines one for $\mathfrak{gl}(W)$.

Thus in the case $\hat{G} = \mathrm{GL}(W)$ we have

$$V = \bigoplus_i \mathrm{Hom}(W_i, W_{i+2}) \qquad \text{and} \qquad H = \bigtimes_i \mathrm{GL}(W_i).$$

(For other classical groups some duality considerations arise)

So what are we studying?

This naturally leads us to want to study the category of equivariant perverse sheaves on varieties of the form:

$$V = \bigoplus_i \mathrm{Hom}(W_i, W_{i+1}) \qquad \text{and} \qquad H = \bigtimes_i \mathrm{GL}(W_i).$$

or the appropriate generalization for other classical groups.

In all cases it is worth noting that the orbits are in bijection with admissible collections of ranks for all the maps and their composition.

In the case that arises for $\mathrm{GL}(W)$ a complete picture of how to find all the necessary smooth covers of strata is understood. (Blowups can be defined using generalized Grassmanians which capture the *kernels* of all of the maps and their compositions)

For other classical groups, these same ideas give us many of the covers we need, but our best strategy to get all of them is still ad-hoc explicit constructions of blowups (though I expect there is a systematic approach underlying my ad-hoc process).

In any case, we have computed these things in many examples.

ABV-Packets vs Arthur Packets?

To each orbit C of V we associate an ABV-packet which consists of the collection of simple perverse sheaves for which $[T^*_C(V)]$ is in the support of the characteristic cycles.

- $T^*_C(V)$ is the conormal bundle to the strata C, which is in particular a Lagrangian subvariety of $V \times V^*$.
- $[T^*_C(V)]$ is the cycle of this subvariety in the cohomology of $V \times V^*$.
- You can view V^* as the subvariety of the Lie algebra associated to

  $$V^* = \bigoplus_i \mathrm{Hom}(W_{i+1}, W_i)$$

  where the duality is induced by the killing form. In so doing,

  $$T^*_C(V) = \{(v, w) \in C \times V^* \mid [v, w] = 0\}$$

  (where [v,w] is the Lie bracket)

  The problem I talked about at the start of the term in this seminar is:
  Does $T^*_C(V)$ decompose into finitely many orbits?

Conjecture

Arthur packets are ABV-packets.

(The converse is false, there are in fact more strata than there are Arthur packets.)

Characteristic Cycles and Singular Support

There is a functor from perverse sheaves on X to to the middle cohomology of the cotangeant bundle to X.
It associates to a sheaf $\mathcal{F}$, the **Characteristic Cycles** of $\mathcal{F}$.
Facts about these:

- The characteristic cycles are always in the span of the cycles associated to the conormal bundles of the stratification with respect to which $\mathcal{F}$ is perverse.
- So in our context, characteristic cycles are always the conormals, $T^*_C(V)$, of our orbits.
- The coefficients associated to each orbit can be computed by computing the intersection with a sufficiently general lagrangian cycle associated to some function f (that is the lagrangian cycle which is the graph of $(x, \mathrm{df}_x) \subset V \times V^*$). For appropriate f, these intersection multiplicities are given locally by the rank of the vanishing cycles functor $R\Phi_f(\mathcal{F})$.

Upgrading Characteristic Cycles - Microlocalization

There is an exact functor Ev from perverse sheaves on V to perverse sheaves on

$$\bigsqcup_C T^*_C(V)_{reg}$$

such that:

- The rank of $\mathrm{Ev}|_C(\mathcal{F})$ is the multiplicity of $[T^*_C(V)]$ in $\mathrm{CC}(\mathcal{F})$.
- Some other desirable things happen.
- We can evaluate using vanishing cycles by the formula

  $$\mathrm{Ev}|_C(\mathcal{F}) \otimes R\Phi_{(\cdot,\cdot)}(\mathbb{1}_C \boxtimes \mathbb{1}_{C^*}) = R\Phi_{(\cdot,\cdot)}(\mathcal{F} \boxtimes \mathbb{1}_{C^*})$$

  where $(\cdot,\cdot)$ is the killing form on $V \times V^*$.

Conjecture
The characters for Arthur's stable distribution are given by the characters for $\mathrm{Ev}(\mathcal{F})$ on each strata.

What are vanishing cycles? - topological

Vanishing cycles are strongly connected to Morse theory (in nice cases they are computing Morse groups, which are allegedly things that people understand).
However, it turns out computing deformation retracts of Milnor fibers for high dimensional varieties is hard (at least for me).

Vanishing cycles can in principal be calculated using D-modules (which is another theory that people claim to understand).
However, no expert we could find was able to actually describe the D-module of the first non-trivial example we could find.

What are vanishing cycles? - schematic

Deligne gave a schematic definition of vanishing cycles as a composition of a number of derived functors.

I have been able to compute exactly one non-trivial example directly from this definition.

Combining this one example with magical functoriality results (Smooth base change, Proper Base Change, Thom-Sebastiani Isomorphism) we have built up tools to explicitly compute these for every case in all the examples we have looked at.

We now have several non-trivial examples which illustrate that our conjectures are true!!

The End.

Thank you.