THE UNBOUNDED DENOMINATOR CONJECTURE FOR THE NONCONGRUENCE SUBGROUPS OF INDEX 7

ANDREW FIORI AND CAMERON FRANC

ABSTRACT. We study modular forms for the minimal index noncongruence subgroups of the modular group. Our main theorem is a proof of the unbounded denominator conjecture for these groups, and we also provide a study of the Fourier coefficients of Eisenstein series for one of these minimal groups.

CONTENTS

1. INTRODUCTION

Finite index subgroups of the modular group $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ play an important role in the study of algebraic curves thanks to a Theorem of Belyi [4], which implies that curves of genus at least two defined over $\overline{\mathbf{Q}}$ can be uniformized by such groups. The congruence subgroups correspond to the well-studied and fundamentally important modular curves, whereas the vast majority of curves correspond to finite index subgroups of Γ that are *not* defined by congruence conditions. Such noncongruence subgroups and their corresponding modular forms are much less well-understood than congruence groups and forms.

To date much of the work on noncongruence modular forms has focused on the following topics: Galois representations and congruences with congruence modular forms [3], [28], [31], [32], [12], [18], [1], [11], [20], [2], [19]; the unbounded denominator conjecture [3], [28], [17], [15], [16], [8], [9], [10]; moduli interpretations [7]; spectral results [23], [24], [25] ; algebraic properties of Eisenstein series [13], [29], [21]; computation of scattering matrices [14], [26], [5]. Given the vast generality inherent in the study of noncongruence modular forms, most papers in the subject have made progress by restricting to classes of groups that are more amenable to study than a general subgroup — for example, one could consider the kernel of a

The authors gratefully acknowledge financial support received from NSERC through their respective Discovery Grants, the financial support of the University of Lethbridge, and the use of computational resources made available through WestGrid and Compute Canada.

character of a congruence subgroup, so that some finite power of each noncongruence form is a congruence modular form.

In the present paper our aim is to study some noncongruence subgroups that have not yet received particular focus and, from this perspective, it is natural to focus on subgroups of small index in Γ. The minimal index of a noncongruence subgroup of Γ is known to be seven, and there are twenty-eight such subgroups of index seven that fall into four conjugacy classes. Our main theorem is a proof of the unbounded denominator conjecture for these groups:

Theorem 1. *Let G be any of the noncongruence subgroups of Γ of index seven, and let $f \in M_k(G) \setminus M_k(\Gamma)$ have algebraic Fourier coefficients at the cusp ∞. Then f has unbounded denominators.*

Our proof of Theorem 1 proceeds as follows:

(a) solve for a hauptmodul (or Belyi map);
(b) establish unbounded denominators at the prime $p = 7$ for this hauptmodul[1];
(c) use this result to prove Theorem 1 in general.

This argument can be adapted to many other groups, but it does not seem suited to generalization for at least two independent reasons: first, the diophantine problem involved in solving for a hauptmodul can be somewhat tricky in general and, second, it is not clear how the proof of unbounded denominators would generalize (in this paper we are aided by the fact that the index $[\Gamma\colon G] = 7$ is prime).

After we complete the proof of Theorem 1 we turn to the study Eisenstein series. As in [29], [21] we are able to establish the algebraicity of the Eisenstein series of weight two for more or less trivial reasons, but in higher weights we are only able to determine the complex phase of the Fourier coefficients. Knowledge of this phase follows from the study of the outer automorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

of Γ acting on the subgroups of index seven. We end the paper with some numerical computations that indicate that there is more that one might be able to say about these Fourier coefficients in general, although we make no precise conjectures along these lines.

1.1. **Acknowledgments.** The authors thank Jenna Rajchgot for some help in using *Macaulay 2* to solve the j-equations in Sections 4 and 5. We also thank *WestGrid* and *Compute Canada* for some computational support.

1.2. **Notation.** Throughout the paper we use the following notation:

- $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, the modular group;
- $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $R = ST$;
- if $G \subseteq \Gamma$ is of finite index, then $M(G) = \bigoplus_{k\geq 0} M_k(G)$ denotes the graded module of modular forms for G;
- $\zeta_n = e^{2\pi i/n}$ and $q_n = e^{2\pi i\tau/n}$;
- $E_k \in M_k(\Gamma)$ denotes the Eisenstein series of weight k with constant term normalized to equal 1;
- E_2 denotes the normalized quasi-modular Eisenstein series of weight 2;
- j is the usual j-function, with constant term 744.

[1]In two cases, called U_1 and U_6 below, we must use a prime over 7 in a quadratic extension of $\mathbf{Q}$.

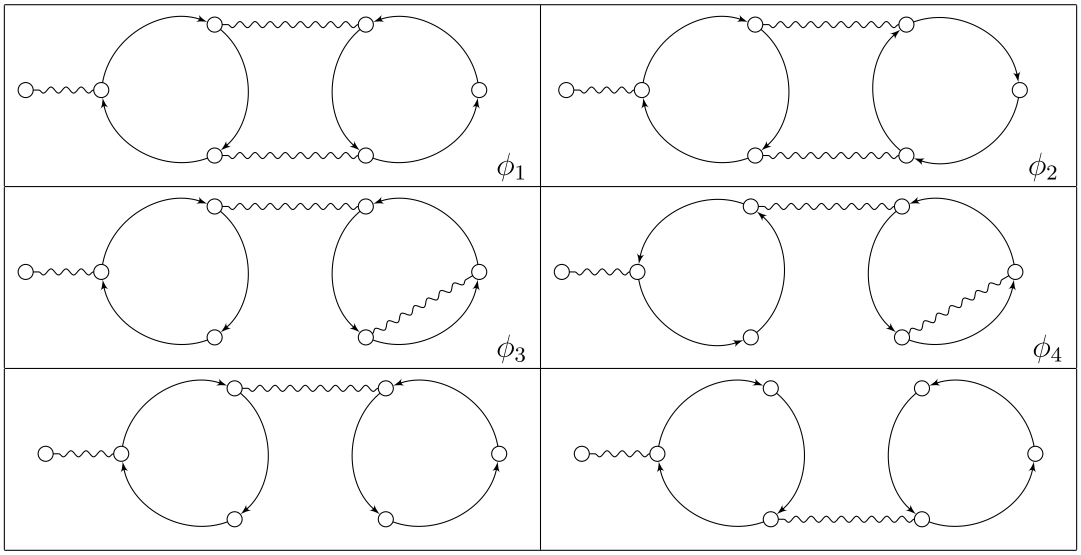

FIGURE 1. Cycle types for the homomorphisms $\Gamma \to S_7$ with transitive image. The squiggly lines correspond to the image of S and the arrows correspond to the image of R. The graphs are unlabeled since we are interested in homomorphisms up to conjugation in S_7. The last row corresponds to congruence subgroups.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
$S \mapsto$	$(12)(34)(56)$	$(12)(34)(56)$	$(12)(34)(67)$	$(12)(34)(67)$
$R \mapsto$	$(235)(467)$	$(235)(764)$	$(235)(467)$	$(253)(467)$
$T \mapsto$	$(1245)(367)$	$(12475)(36)$	(124735)	(125473)
$\lvert \mathrm{im}\,\phi_j \rvert$	$7!$	$7!$	42	42

TABLE 1. Data for the noncongruence homomorphisms $\Gamma \to S_7$ with transitive image.

2. NONCONGRUENCE SUBGROUPS OF INDEX 7

If $G \subseteq \Gamma$ is of index 7, then the action of G on the cosets defines a homomorphism $\Gamma \to S_7$ with transitive image. Since $\Gamma = \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$ is freely generated by order 2 and 3 elements S and R, respectively, there are six such homomorphisms up to conjugation. They are depicted in Figure 1, where the squiggly lines describe the image of S, and the arcs describe the image of R. The last row in Figure 1 corresponds to congruence subgroups of level 7, and so we shall only focus on the first two rows. Call the corresponding (conjugacy classes) of homomorphisms ϕ_j as in Figure 1. To be more precise, we pick the representatives for the conjugacy classes as in Table 1.

Note that ϕ_1 and ϕ_2 are surjective onto S_7, while ϕ_3 and ϕ_4 have images of order 42. Let

$$G_j = \phi_1^{-1}(\mathrm{Stab}_{\mathrm{im}\,\phi_1}(j)), \qquad H_j = \phi_2^{-1}(\mathrm{Stab}_{\mathrm{im}\,\phi_2}(j)),$$
$$U_j = \phi_3^{-1}(\mathrm{Stab}_{\mathrm{im}\,\phi_3}(j)), \qquad V_j = \phi_4^{-1}(\mathrm{Stab}_{\mathrm{im}\,\phi_4}(j)).$$

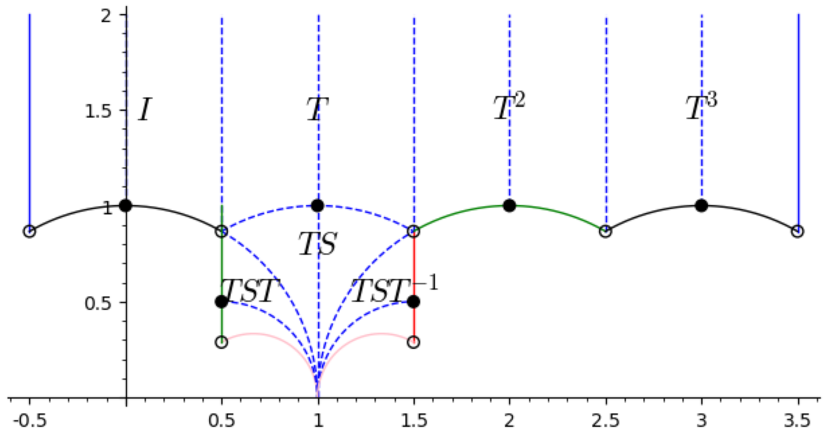

FIGURE 2. A fundamental domain for G_1. The colours of the edges describe the edge pairing.

The following result is known to experts, but we did not find a suitable reference in the literature.

Theorem 2. *The groups G_j, H_j, U_j and V_j for $j = 1, \ldots, 7$ are the noncongruence subgroups of Γ of smallest index.*

Proof. Theorem 5 of [35] shows that all subgroups of Γ of index ≤ 6 are congruence. That these groups are noncongruence will follow by our proof of unbounded denominators, but this can also be proved by more elementary means: for example, for the G_j and H_j it follows easily by the simplicity of A_7. For the U_j, note that T has order 6 in the quotient $\Gamma/\ker\phi_3$. Hence by Theorem 2 in §3.1 of [27], if U_j were congruence it would have to contain $\Gamma(6)$. But by elementary group theory one sees that there are no congruence subgroups of index 7 and level 6. Therefore the U_j (and similarly the V_j) are noncongruence. By considering the image of T, it is clear that all of these subgroups are distinct except possibly for some identity $U_i = V_j$, and without loss of generality we may assume $U_1 = V_j$ for some j. Since $R \in U_1$, and the only V_j that contains R is V_1, we thus would have $U_1 = V_1$. But one can easily check that U_1 contains $SRSR^2S$, while V_1 does not. Therefore $U_1 \neq V_1$ and hence $U_i \neq V_j$ for all i and j.

The only other subgroups of index 7 come from the last line in Figure 1, but those homomorphisms yield the congruence subgroups of level 7 and index 7. □

Remark 3. In general, the number of subgroups of index n in Γ can be counted using Exercise 5.13 of [34].

In the remainder of this section we shall describe some group theoretic data for G_1, H_1, U_1 and V_1 that will be useful for what follows. The analogous data for the other groups can be obtained by conjugation.

2.1. **Data for G_1.** A fundamental domain for G_1 is given in Figure 2 on page 4. The elliptic points are represented by ζ_3 and $\frac{1}{2}(3+i)$. The edge pairing is defined by the

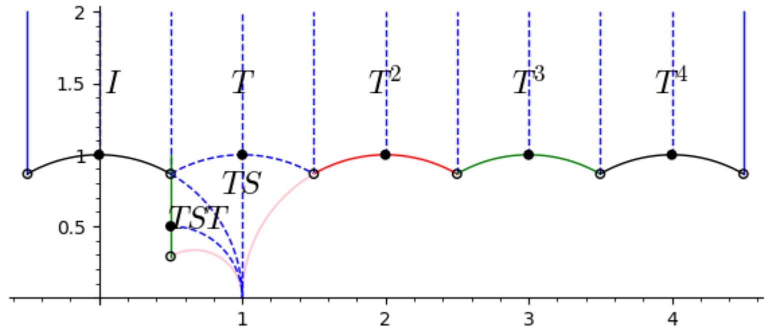

FIGURE 3. A fundamental domain for H_1. The colours of the edges describe the edge pairing.

following matrices:

$$T^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}.$$

Here T^4 identifies the vertical blue sides, C_1 identifies the pink edges, C_2 identifies the green and black edges with their pairs, and E_1 identifies the red edge with itself. Note that $R = C_2^{-1}C_1$, so that if we use R as a generator, we can dispense with C_2. Similarly, $C_1 = E_1^{-1}CR^{-1}$, so that we can also dispense with C_1. By standard results on Fuchsian groups, one obtains the following presentation for G_1:

$$G_1 = \langle T^4, E_1, R \mid E_1^2 = R^3 = 1 \rangle.$$

2.2. **Data for H_1.** A fundamental domain for H_1 is given in Figure 3 on page 5. The elliptic points are represented by ζ_3 and $2 + i$. The edge pairing is defined by the following matrices:

$$T^5 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}.$$

Here T^5 identifies the vertical blue sides, D_1 identifies the pink edges, D_2 identifies the green and black edges with their pairs, and E_2 identifies the red edge with itself. Similarly to above we obtain a presentation

$$H_1 = \langle T^5, E_2, R \mid E_2^2 = R^3 = 1 \rangle.$$

In particular $G_1 \cong H_1$ but this will play no role in what follows.

2.3. **Data for U_1.** A fundamental domain for U_1 is given in Figure 4 on page 6. The elliptic points are represented by ζ_3 and $1 + i$. The edge pairing is defined by the following matrices:

$$T^6 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -5 & 1 \\ -1 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -2 & 9 \\ -1 & 4 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

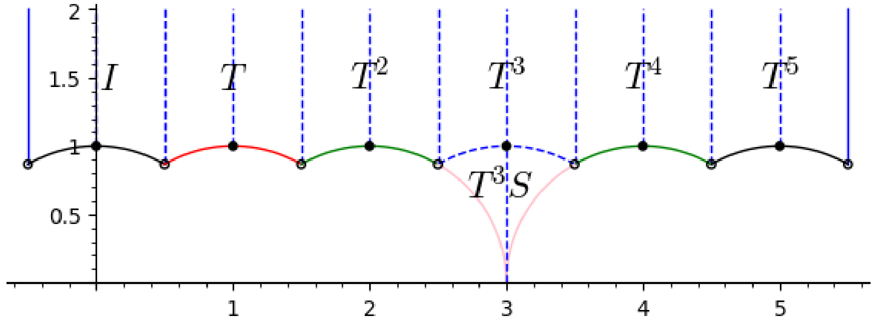

FIGURE 4. A fundamental domain for U_1. The colours of the edges describe the edge pairing.

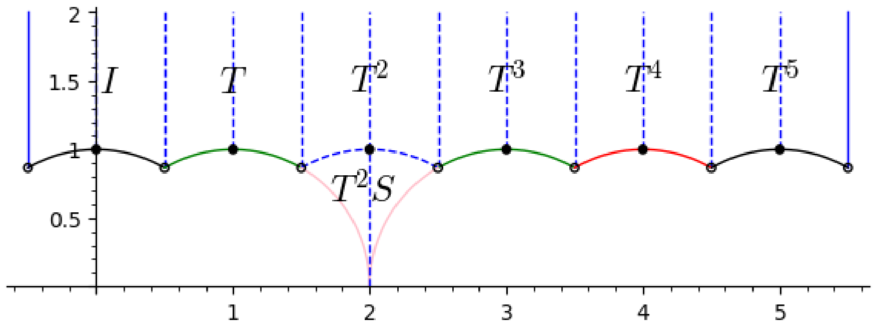

FIGURE 5. A fundamental domain for V_1. The colours of the edges describe the edge pairing.

Here T^6 identifies the vertical blue sides, A_1 identifies the pink and green edges with their pairs, A_2 identifies the black edges, and E_3 identifies the red edge with itself. Similarly to above we obtain a presentation

$$U_1 = \langle T^6, E_3, R \mid E_3^2 = R^3 = 1 \rangle.$$

2.4. **Data for V_1.** A fundamental domain for V_1 is given in Figure 5 on page 6. The elliptic points are represented by ζ_3 and $4+i$. The edge pairing is defined by the following matrices:

$$T^6 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -5 & 1 \\ -1 & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 4 & -17 \\ 1 & -4 \end{pmatrix}.$$

Here T^6 identifies the vertical blue sides, B_1 identifies the pink and green edges with their pairs, B_2 identifies the black edges, and E_4 identifies the red edge with itself. Similarly to above we obtain a presentation

$$V_1 = \langle T^6, E_4, R \mid E_4^2 = R^3 = 1 \rangle.$$

3. OUTLINE OF PROOF OF THEOREM 1

We begin the proof of Theorem 1 with an elementary reduction.

k	$\dim M_k(G)$
0	1
2	1
4	3
6	4
8	5
10	6
$k \geq 12$	$\dim M_{k-12}(G) + 7$

TABLE 2. Dimension of $M_k(G)$ for the noncongruence subgroups G of index 7.

Lemma 4. *To prove Theorem 1 it suffices to treat the cases $G = G_1$, G_3, H_1, H_3, U_1, U_6, V_1 and V_6.*

Proof. We first consider the case of G_1, which is conjugate with G_2, G_4 and G_5 via powers of T. Therefore, if $f \in M_k(G_1)$ has Fourier expansion $f = \sum_{n\geq 0} a_n q_4^n$ where $q_4 = e^{2\pi i\tau/4}$, then the forms $f(\tau + m) = \sum_{n\geq 0} a_n i^{mn} q_4^n$ are forms on the various conjugate groups, and vice versa. Hence Theorem 1 holds for G_1 if and only if it holds for any one of G_1, G_2, G_4 or G_5. An identical arguments applies to the other cases. □

While not strictly necessary, we can describe the structure of $M(G)$ as an $M(\Gamma)$-module, which then immediately gives the dimensions of the graded pieces of $M(G)$ (see Table 2).

Lemma 5. *Let G be any of the noncongruence subgroups of Γ of index 7. Then there exists a free-basis for $M(G)$ as an $M(\Gamma)$-module with generators in weights 0, 2, 4, 4, 6, 6 and 8 such that the generators have algebraic Fourier coefficients.*

Proof. Let ρ be the representation of Γ obtained from the permutation representation of G on its cosets in Γ, so that if $M(\rho)$ is the corresponding space of vector-valued modular forms, then there is an isomorphism $M(G) \cong M(\rho)$. Since ρ decomposes as the trivial representation plus an even 6-dimensional irreducible representation for each of these G, which at the level of scalar forms corresponds to the decomposition $M(G) = M(\Gamma) \oplus \ker \mathrm{Tr}$ where $\mathrm{Tr}\colon M(G) \to M(\Gamma)$ is the congruence trace, one can use Riemann-Roch to find the weights of a free-basis for $M(\rho)$ of the desired form — see Example 7.4 of [6] where ϕ_1 and ϕ_2 are treated explicitly. Both ϕ_3 and ϕ_4 are analogous to that case, as the local monodromies around elliptic points are conjugate in all cases, and the sum of the exponents of $\rho(T)$ is $5/2$ in each of these four cases, as can be easily read off from the cycle type of $\psi_j(T)$ (not that this does not hold for the congruence subgroups of index 7 where T acts as a seven-cycle).

To obtain a free-basis with algebraic Fourier coefficients, one could for example diagonalize $\rho(T)$ and use the Frobenius method (in fact, in the course of the proof of Theorem 1 we will write down an explicit free-basis with algebraic Fourier coefficients). Since $\rho(T)$ can diagonalized over a finite extension of $\mathbf{Q}$, algebraicity of Fourier coefficients will be preserved under this operation. □

Remark 6. One can easily use the hauptmoduls described below to compute explicit free-bases for $M(G)$ as in Lemma 5, through taking derivatives and products of forms, but we have no need for such a free-basis in this paper.

In light of Lemmas 4 and 5, we can now proceed as follows:

(1) describe a hauptmodul for each of the genus 0 groups G_1, G_3, H_1, H_3, U_1, U_6, V_1 and V_6;
(2) establish unbounded denominators for each hauptmodul;
(3) express modular forms in $M(G)$ as forms of level one times rational functions in the hauptmodul, and deduce unbouded denominators as a result.

We shall give all of the details for the group G_1 in Section 4, and then in Section 5 we shall summarize the key facts that allow one to carry out the same argument for the other groups.

4. Details for the group G_1

Following Atkin–Swinnerton-Dyer [3], we compute a hauptmodul for the genus 0 group G_1. Note that this hauptmodul is an example of a uniformizing Belyi map, and there exists an extensive literature on computing Belyi maps – see [22], [33] and the references contained therein for more information.

Recall that $j = \frac{1}{q} + 744 + 196884q + \cdots$. In the setting of G_1, ASD solve for a hauptmodul $z = q_4^{-1} + 0 + O(q_4)$ (so that $q_4 = \xi$ in the ASD notation) by introducing polynomials:

$$
\begin{aligned}
a_1 &= z + c_1,\\
f_3 &= z^2 + c_2 z + c_3,\\
e_3 &= z + c_4,\\
f_2 &= z^3 + c_5 z^2 + c_6 z + c_7,\\
e_2 &= z + c_8,
\end{aligned}
$$

and then solving for the unknown c_j's via the j-equations:

$$
\begin{aligned}
j a_1^3 &= f_3^3 e_3,\\
(j - 1728) a_1^3 &= f_2^2 e_2.
\end{aligned}
$$

Eliminating j from these two equations gives the system of nonlinear equations:

$$
\begin{aligned}
0 &= -c_3^3 c_4 + 1728 c_1^3 + c_7^2 c_8,\\
0 &= -3c_2 c_3^2 c_4 - c_3^3 + 2c_6 c_7 c_8 + 5184 c_1^2 + c_7^2,\\
0 &= -3c_2^2 c_3 c_4 - 3c_2 c_3^2 - 3c_3^2 c_4 + c_6^2 c_8 + 2c_5 c_7 c_8 + 2c_6 c_7 + 5184 c_1,\\
0 &= -c_2^3 c_4 - 3c_2^2 c_3 - 6c_2 c_3 c_4 + 2c_5 c_6 c_8 - 3c_3^2 + c_6^2 + 2c_5 c_7 + 2c_7 c_8 + 1728,\\
0 &= -c_2^3 - 3c_2^2 c_4 + c_5^2 c_8 - 6c_2 c_3 - 3c_3 c_4 + 2c_5 c_6 + 2c_6 c_8 + 2c_7,\\
0 &= -3c_2^2 - 3c_2 c_4 + c_5^2 + 2c_5 c_8 - 3c_3 + 2c_6,\\
0 &= -3c_2 - c_4 + 2c_5 + c_8.
\end{aligned}
$$

Furthermore, if we expand the j-equations in z and compare constant terms, we get an additional linear equation $3c_1 - 3c_2 - c_4 = 0$. Finally, we must insist that a_1, f_3, e_3, f_2

n	a_n/u^{n+1}
-1	1
0	0
1	$2^2 \cdot 3 \cdot 7 \cdot 173$
2	$2^{11} \cdot 7 \cdot 43$
3	$-1 \cdot 2 \cdot 3 \cdot 7 \cdot 173 \cdot 199$
4	$-1 \cdot 2^{14} \cdot 3^9 \cdot 7$
5	$-1 \cdot 2^3 \cdot 7 \cdot 17 \cdot 89 \cdot 1969543$
6	$-1 \cdot 2^{13} \cdot 3 \cdot \cdot 5273 \cdot 47339$
7	$3^2 \cdot 7 \cdot 11 \cdot 19 \cdot 26353729$
8	$2^{17} \cdot 3^{10} \cdot 7 \cdot 31 \cdot 67 \cdot 131$
9	$2^2 \cdot 3 \cdot 7 \cdot 11869625271733553$
10	$2^{12} \cdot 3 \cdot 7 \cdot 17 \cdot 1579 \cdot 36677 \cdot 385321$
11	$2 \cdot 3 \cdot 7^2 \cdot 7204271 \cdot 2154711443$

TABLE 3. Normalized Fourier coefficients of the hauptmodul for G_1.

and e_2 have distinct roots as polynomials in z. This means we have the nonequalities:

$$\begin{aligned}
(c_1 - c_4)(c_1 - c_8)(c_4 - c_8) &\neq 0, \\
c_j^2 + c_2 c_j + c_3 &\neq 0 \qquad (j = 1, 4, 8), \\
c_j^3 + c_5 c_j^2 + c_6 c_j + c_7 &\neq 0 \qquad (j = 1, 4, 8), \\
c_2^2 - 4c_3 &\neq 0, \\
c_5^2 c_6^2 - 4c_6^3 - 4c_5^3 c_7 - 27c_7^2 + 18c_5 c_6 c_7 &\neq 0.
\end{aligned}$$

We put these into *Macaulay 2* and performed a Grobner basis computation to find the following substitutions:

$$\begin{aligned}
c_1 &= \tfrac{2}{3}c_5 + \tfrac{1}{3}c_8, \\
c_3 &= 2c_2^2 - 2c_2c_5 + \tfrac{1}{3}c_5^2 - c_2c_8 + \tfrac{2}{3}c_5c_8 + \tfrac{2}{3}c_6, \\
c_4 &= -3c_2 + 2c_5 + c_8, \\
c_7 &= -\tfrac{3}{7}\big(c_2^3 + \tfrac{3}{2}c_2c_5^2 - \tfrac{119}{81}c_5^3 + \tfrac{2}{3}c_2^2c_8 + c_2c_5c_8 - \tfrac{70}{27}c_5^2c_8 + \tfrac{7}{6}c_2c_8^2 - \tfrac{52}{81}c_8^3 \\
&\qquad - \tfrac{11}{3}c_2c_6 + 3c_5c_6 + \tfrac{32}{9}c_6c_8\big).
\end{aligned}$$

This leaves a system of 5 equations in the unknowns c_2, c_5, c_6 and c_8. With some effort involving saturating with respect to the last nonequality condition, we managed to find the following solution: if $u = \sqrt[4]{-7}/7^2$ then

$$\begin{aligned}
c_1 &= 168u, & c_2 &= 256u, \\
c_3 &= 10869u^2, & c_4 &= -264u, \\
c_5 &= 160u, & c_6 &= -28968u^2, \\
c_7 &= -5900544u^3, & c_8 &= 184u.
\end{aligned}$$

Given this, one can recursively solve for the q_4-expansion coefficients of z using the j-equations. In Table 3 on page 9 we list the rational part of the Fourier coefficients of the hauptmodul z.

Lemma 7. *The hauptmodul has unbounded denominators.*

Proof. We note the minimal polynomial for the hauptmodul, z, as it defines a finite extension of $\overline{\mathbf{Q}}(j)$ is

$$(z^2+c_2z+c_3)^3(z+c_4)-j(z+c_1)^3.$$

We now perform the changes of variables

$$z = u\hat{z}$$

where $u=\sqrt[4]{-7}/7^2$, as above. We see that $\hat{z}$ satisfies

$$\begin{aligned}\hat{z}^7+504\hat{z}^6&+26544\hat{z}^5-27020672\hat{z}^4-6349147392\hat{z}^3\\&-568400910336\hat{z}^2-22777684586496\hat{z}-341511404027904\\&-7^7j(\hat{z}^3+504\hat{z}^2+84672\hat{z}+4741632).\end{aligned}$$

To better understand the denominators in the Laurent expansion we shall formally substitute

$$q_4 = u^{-1}\hat{q}$$

and study $\hat{z}$ as a Laurent series in $\hat{q}$. Additionally, we renormalize $j=-\hat{j}/7^7$ so that $\hat{j}=\frac{1}{\hat{q}^4} \pmod 7$. We then have

$$\begin{aligned}\hat{z}^7+504\hat{z}^6&+26544\hat{z}^5-27020672\hat{z}^4-6349147392\hat{z}^3\\&-568400910336\hat{z}^2-22777684586496\hat{z}-341511404027904\\&+\hat{j}(\hat{z}^3+504\hat{z}^2+84672\hat{z}+4741632).\end{aligned}$$

By an application of Hensel's lemma we can conclude that the coefficients of $\hat{z}$, as a Laurent series in $\hat{q}$, are integers.

We may thus reduce the series $\hat{z}$ modulo 7 and notice that the result satisfies the minimal polynomial

$$x^7+\hat{j}x^3+2 \pmod 7$$

over the function field $\mathbf{F}_7(\hat{j})$. Solutions to this equation in Laurent series $\mathbf{F}_7(\!(\hat{q})\!)$ must have infinitely many non-zero coefficients. Indeed, a non-constant polynomial cannot satisfy a polynomial equation of degree greater than 0.

This implies that $\hat{z}$, and hence z, has unbounded denominators as a Laurent series when expressed in the variable q_4. □

Corollary 8. *Any element of $\overline{\mathbf{Q}}(z)$ not in $\overline{\mathbf{Q}}(j)$ has unbounded denominators.*

Proof. If $f\in\overline{\mathbf{Q}}(z)\setminus\overline{\mathbf{Q}}(j)$ then the field extension it generates satisfies

$$\overline{\mathbf{Q}}(f)/\overline{\mathbf{Q}}(j)=\overline{\mathbf{Q}}(z)/\overline{\mathbf{Q}}(j)$$

and hence it follows that z can be expressed as $P(f)$ where P is a polynomial in $\overline{\mathbf{Q}}(j)$. By clearing denominators from the coefficients of P we can write

$$zR_1(j)=P_2(f)$$

where now $R_1\in\overline{\mathbf{Q}}[j]$ and P_2 has coefficients in $\overline{\mathbf{Q}}[j]$. As the left hand side, $zR_1(j)$, has unbounded denominators, so too must f. □

Completion of proof of Theorem 1 for G_1. Every form $f\in M_k(G)\setminus M_k(\Gamma)$ of weight at least 4 with algebraic Fourier coefficients can be expressed as $f=E_kP(z)$ for $P(z)\in\overline{\mathbf{Q}}(z)\setminus\overline{\mathbf{Q}}(j)$. By Corollary 8 $P(z)$ has unbounded denominators, and hence so does f. If $f\in M_2(G)\setminus M_2(\Gamma)$ then instead write $f=(E_6/E_4)P(z)$ for $P(z)\in\overline{\mathbf{Q}}(z)\setminus\overline{\mathbf{Q}}(j)$ and then the same argument applies. □

5. Some details for the other groups

Since the data about elliptic points for all the index 7 subgroups agree, the degrees of a_1, f_3, e_3, f_2 and e_2 are the same as for the group G_1. Therefore in each case below we retain our notation from Section 4 for these polynomials in terms of unknowns c_1 through c_8. For all but H_1 we were able to solve the equations through a mixture of saturating with respect to the ASD nonequalities for the j-equations, as well as using Grobner bases. Unfortunately H_1 does not meet the locus defined by the nonequalities and this strategy did not help there. For H_1 we instead performed a sequence of projections all the way down to the variable c_8 and then we were able to decompose the ideal. This yielded two additional spurious components, as well as the unique correct solution below.

Let $\psi\colon \Gamma \to \Gamma$ be the outer automorphism given by conjugation with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that $\psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right)$. Notice that $\psi(T) = T^{-1}$ and $\psi(S) = S^{-1}$, and that ψ maps congruence subgroups to congruence subgroups. Since the j-equations do not distinguish between complex conjugate subgroups, we shall describe how complex conjugation acts on our groups.

Lemma 9. *The outer automorphism ψ permutes the G_j among themselves, and likewise for the H_j. In both cases the action is given by the permutation $(12)(36)(45)$. On the other hand, ψ maps the U_j groups to the V_j groups as follows:*

$$\begin{array}{llll} \psi(U_1) = V_2, & \psi(U_2) = V_1, & \psi(U_3) = V_4, & \psi(U_4) = V_3, \\ \psi(U_5) = V_5, & \psi(U_6) = V_6, & \psi(U_7) = V_7. & \end{array}$$

Proof. This can be proved using the presentations for each of these groups. Conjugate the generators of each subgroup and test what group contains the result. This is a finite computation that is easily performed on a computer. □

What this lemma means in practical terms is that we only need to solve the j-equations for U_1 and U_6, as the Galois orbits of these solutions will contain the hauptmoduls for all of the U_j and V_j.

The details in establishing unbounded denominators are the same as for G_1, save that certain constants change, and so we shall omit them.

5.1. The conjugates of G_3.

In this case the j-equations read

$$\begin{aligned} j a_1^4 &= f_3^3 e_3, \\ (j-1728) a_1^4 &= f_2^2 e_2. \end{aligned}$$

If $u = \sqrt[3]{-2/7}/7^2$, then the solution to the j-equations is:

$$\begin{aligned} c_1 &= -462u, & c_2 &= -444u, \\ c_3 &= -148284u^2, & c_4 &= -516u, \\ c_5 &= -1422u, & c_6 &= 822204u^2, \\ c_7 &= -185029704u^3, & c_8 &= 996u. \end{aligned}$$

The first few terms of the Fourier expansion of the hauptmodul are given in Tables 4.

Remark 10. Note that the apparent powers of 2 in the denominators are not necessary in most of the terms displayed. Though if the table were extended with the same pattern there would be infinitely many terms where the numerator is odd. However,

n	a_n/u^{n+1}
-1	1
0	0
1	148932
2	$-71333864/2$
3	$14784602112/2$
4	$-2720037481056/2$
5	$926140535244764/2^2$
6	$-147594381291749376/2^2$
7	$22341564325891713168/2^2$
8	$-6482694981105850075968/2^3$
9	$907550467150406926565376/2^3$
10	$-123344662799290912907945472/2^3$
11	$32655462531659638680360877638/2^4$

TABLE 4. Normalized Fourier coefficients of the hauptmodul for G_3

these powers of 2 cancel with those from the power of u and are not unbounded denominators in the actual q-expansion.

5.2. **The conjugates of H_1.** In this case the j-equations read

$$ja_1^2 = f_3^3 e_3,$$
$$(j - 1728)a_1^2 = f_2^2 e_2.$$

If $u = \sqrt[5]{-7^3}/7^2$, then the solution to the j-equations is:

$$c_1 = 28u, \qquad c_2 = 51u,$$
$$c_3 = -636u^2, \qquad c_4 = -97u,$$
$$c_5 = -18u, \qquad c_6 = -2979u^2,$$
$$c_7 = -111348u^3, \qquad c_8 = 92u.$$

The first few terms of the Fourier expansion of the hauptmodul are given in Table 5.

n	a_n/u^{n+1}
-1	1
0	0
1	1946
2	17780
3	813295
4	-20472508
5	-194969600
6	-21590535732
7	-86533770365
8	-5540827925500
9	121544077700080
10	954435095756800
11	97227702559110739

TABLE 5. Normalized Fourier coefficients of the hauptmodul for H_1

n	a_n/u^{n+1}
-1	1
0	0
1	7583156
2	-8915200000
3	25855539541090
4	-38753899878400000
5	59853295754680171800
6	$-107814623754600729600000$
7	$130691527974826975392903135$
8	$-229196454200112641389772800000$
9	$294346563065808045129145192319236$
10	$-427644716636763893188085418688000000$
11	$606586125578466006634487839969153168734$

TABLE 6. Normalized Fourier coefficients of the hauptmodul for H_3

5.3. **The conjugates of H_3.** In this case the j-equations read

$$\begin{aligned} ja_1^5 &= f_3^3 e_3, \\ (j-1728)a_1^5 &= f_2^2 e_2. \end{aligned}$$

If $u = \sqrt{-7}/7^4$, then the solution to the j-equations is:

$$\begin{aligned} c_1 &= -952u, & c_2 &= 96u, \\ c_3 &= -205797696u^2, & c_4 &= -5048u, \\ c_5 &= -5904u, & c_6 &= 426314304u^2, \\ c_7 &= -2498515200000u^3, & c_8 &= 7048u. \end{aligned}$$

The first few terms of the Fourier expansion of the hauptmodul are given in Table 6.

5.4. **The conjugates of U_1.** In this case the j-equations read

$$\begin{aligned} ja_1 &= f_3^3 e_3, \\ (j-1728)a_1 &= f_2^2 e_2. \end{aligned}$$

Let ζ_3 denote a third root of unity and set

$$u = \left((1763\zeta_3 + 1255)2^2 3/7^7\right)^{(1/6)}.$$

Note that the minimal polynomial of u over $\mathbf{Q}$ is $823543X^{12} - 8964X^6 + 432$. The solution to the j-equations is then:

$$\begin{aligned} c_1 &= (-8\zeta_3 - 10)u, & c_2 &= (-6\zeta_3 - 6)u, \\ c_3 &= (-28\zeta_3 - 20)u^2, & c_4 &= (10\zeta_3 + 8)u, \\ c_5 &= (-4\zeta_3 - 8)u, & c_6 &= (-60\zeta_3 + 12)u^2, \\ c_7 &= (60\zeta_3 - 276)u^3, & c_8 &= 6u. \end{aligned}$$

The first few terms of the Fourier expansion of the hauptmodul are given in Table 7.

Remark 11. In $\mathbf{Q}(\sqrt{-3})$ the element $(1763\zeta_3 + 1255)2^2 3/7^7$ has a non-trivial valuation at 2 of 2, at 3 of 3 and at one of the two primes dividing 7 of -7. The valuation is 0 at

n	a_n/u^{n+1}
-1	1
0	0
1	$20\zeta_3+4$
2	$60\zeta_3+12$
3	$-96\zeta_3+48$
4	$432\zeta_3+288$
5	$-(3893/9)\zeta_3-1060/9$
6	$576\zeta_3-576$
7	$(13952/3)\zeta_3+7372$
8	$7168\zeta_3+18312$
9	$-45200\zeta_3-33568$
10	$-4160\zeta_3+93248$
11	$-(3412747/72)\zeta_3-22548985/216$

TABLE 7. Normalized Fourier coefficients of the hauptmodul for U_1

all other primes of $\mathbf{Q}(\sqrt{-3})$. Consequently, the denominators of u are at exactly one of the two primes over 7.

The 2's and 3's appearing in the denominators are cancelled by those appearing in the numerator in the power of u.

5.5. **The conjugates of U_6.** In this case the j-equations read

$$ja_1^6 = f_3^3 e_3,$$
$$(j-1728)a_1^6 = f_2^2 e_2.$$

As above let ζ_3 denote a third root of unity and set

$$u = ((3\zeta_3+1)/7)^7 .$$

The solution to the j-equations is:

$c_1 = (-1368\zeta_3 - 4944)u,$ $c_2 = (59472\zeta_3 + 238944)u,$

$c_3 = (738742464\zeta_3 + 1457337024)u^2,$ $c_4 = (-1368\zeta_3 + 1968)u,$

$c_5 = (-128520\zeta_3 - 512496)u,$ $c_6 = (-5453272512\zeta_3 - 13411016640)u^2,$

$c_7 = (-8345692154880\zeta_3 - 38174900673024)u^3,$ $c_8 = (3816\zeta_3 + 5424)u.$

The first few terms of the Fourier expansion of the hauptmodul are given in Table 8.

Remark 12. The only prime of $\mathbf{Q}(\sqrt{-3})$ at which u has non-trivial valuation is one of the two primes over 7. It is has valuation -7 at this prime.

6. SOME RESULTS ON EISENSTEIN SERIES

One of our original aims was to see how much one could say about the Eisenstein series associated to these minimal noncongruence subgroups, but as pointed out by Philips-Sarnak [23], it is difficult if not impossible to say too much about their Fourier coefficients in general. See also [29], [30] and [21] where it is observed that even the *algebraicity* of Fourier coefficients of Eisenstein series can be a thorny question. For example, by the main theorem of [29], the holomorphic Eisenstein series of weight

n	a_n/u^{n+1}
-1	1
0	0
1	$3195612\zeta_3 + 4653180$
2	$2113007616\zeta_3 + 7901431808$
3	$-5777884753902\zeta_3 - 11584189398816$
4	$3171254057975808\zeta_3 + 3027156411138048$
5	$-2039191564783610822 4\zeta_3 - 380081990673348532 0$
6	$1547825541807078376 2432\zeta_3 - 1080327659012898457 1904$
7	$2659116112895547884 4327729\zeta_3 + 2490879492609671882 3786001$
8	$-261819115586763533 82430801920\zeta_3 - 127277979777275746 91751002112$
9	$266040877484779825 57834447865556\zeta_3 - 159294367897426924 51659751424160$

TABLE 8. Normalized Fourier coefficients of the hauptmodul for U_6

2 associated to the noncongruence subgroup discussed in [5] has infinitely many transcendental Fourier coefficients.

Remark 13. Note that by [29], the Eisenstein series of weight 2 for the noncongruence subgroups of index 7 have algebraic Fourier coefficients, since the Manin-Drinfeld condition is trivially satisfied in these cases: the Picard group of degree zero divisor classes on the compactified curve associated to each of these groups is cyclic of order 12. More simply, algebraicity follows in weight 2 because the space of forms of weight 2 is one-dimensional in each case.

We begin by recalling a standard computation for the Fourier coefficients of Eisenstein series on any subgroup of Γ of finite index; see [13] for more details. For simplicity in this section we focus solely on the group G_1. For even integers $k \geq 4$ define
$$g_k(\tau) := g_k^{(\infty)}(\tau) = \sum_{\langle \pm T^4 \rangle \backslash G_1} \frac{1}{(c\tau + d)^k},$$
which converges absolutely for $k \geq 4$. Observe that elements of $\langle \pm T^4 \rangle \backslash G_1$ are in one-to-one correspondence with the equivalence classes of elements in G_1 with the same bottom row (up to sign). Therefore we define
$$\chi(c, d) = \begin{cases} 1 & \exists \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in G_1, \\ 0 & \text{otherwise}, \end{cases}$$
and we find that
$$g_k^{(\infty)}(\tau) = 1 + \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} \frac{\chi(c, d)}{(c\tau + d)^k}.$$

Remark 14. The above defines the Eisenstein series for the cusp at infinity, $g_k^{(\infty)}$. We can analogously define an Eisenstein series, $g_k^{(1)}$, for the cusp at one. It is elementary to verify that $g_k^{(\infty)} + g_k^{(1)} = E_k$. As such in what follows we will not typically consider $g_k^{(1)}$ and so we define $g_k = g_k^{(\infty)}$ for $k \geq 4$. For $k = 2$ these series are not holomorphic modular forms and we instead consider $g_2 = g_2^{(\infty)} - \frac{4}{3} g_2^{(1)}$.

Proposition 15. *The indicator function $\chi(c, d)$ satisfies the following properties:*

(1) $\chi(c, d) = \chi(c, d + 4c)$ *and* $\chi(c, d) = \chi(c + 4d, d)$*;*

(2) $\chi(c,d) = \chi(-c,-d)$;
(3) $\chi(c,d) = \chi(d, d-c)$;
(4) $\chi(c,d) = \chi(3c+2d, -5c-3d)$;
(5) $\chi(c,d) = \chi(-c, d-c)$;
(6) $\chi(c,d) = \chi(d,c)$.

Proof. The first identity in Property (1) follows from the simple observation that

$$\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) T^4 = \left(\begin{smallmatrix} * & * \\ c & d+4c \end{smallmatrix}\right)$$

and $T^4 \in G_1$. The second follows likewise using $U = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, where $U^4 \in G_1$ is the minimal power in G_1. Properties (2), (3) and (4) are equivalent with $-1, R, E \in G_1$, respectively. For (5) we can use Lemma 9 and the fact that $T^{-1}G_2T = G_1$, so that $T^{-1}\psi T$ fixes G_1. Since $T^{-1}\psi T$ acts on bottom rows as $(c,d) \mapsto (-c, d-c)$, Property (5) follows. Now we can show that Property (6) is a consequence of the other properties:

$$\chi(c,d) = \chi(-c, d-c) = \chi(c, c-d) = \chi(d,c). \qquad \square$$

Remark 16. We make no use of Property (4) stated above, but we include it for completeness, as Properties (1) through (4) in Proposition 15 encode the action of the generators of G_1. Properties (5) and (6) are somewhat less trivial, as they utilize the symmetry of the outer automorphism discussed above.

Given Proposition 15, we can simplify the expression for $g_k^{(\infty)}$:

$$\begin{aligned}
g_k^{(\infty)}(\tau) &= 1 + \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty} \frac{\chi(c,d)}{(c\tau+d)^k} \\
&= 1 + \sum_{c=1}^{\infty} \sum_{d=1}^{4c} \sum_{t=-\infty}^{\infty} \frac{\chi(c, d+4ct)}{(c\tau+d+4ct)^k} \\
&= 1 + \sum_{c=1}^{\infty} \frac{1}{(4c)^k} \sum_{d=1}^{4c} \chi(c,d) \sum_{t=-\infty}^{\infty} \frac{1}{(\frac{c\tau+d}{4c}+t)^k} \\
&= 1 + \frac{(2\pi i)^k}{(k-1)!} \sum_{c=1}^{\infty} \frac{1}{(4c)^k} \sum_{d=1}^{4c} \chi(c,d) \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n(c\tau+d)/4c} \\
&= 1 + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sum_{c=1}^{\infty} n^{k-1} \frac{1}{(4c)^k} \left(\sum_{d=1}^{4c} \chi(c,d) e^{2\pi i n d/4c} \right) e^{2\pi i n\tau/4}
\end{aligned}$$

Thus we obtain the Fourier expansion:

$$g_k^{(\infty)}(\tau) = 1 + \frac{(2\pi i)^k}{4^k(k-1)!} \sum_{n=1}^{\infty} n^{k-1} \left(\sum_{c=1}^{\infty} \left(\sum_{d=1}^{4c} \chi(c,d) e^{2\pi i n d/4c} \right) \frac{1}{c^k} \right) q_4^n. \tag{1}$$

In particular, if we define

$$\begin{aligned}
X(n,c) &:= \sum_{d=1}^{4c} \chi(c,d) e^{2\pi i n d/4c}, \\
D(n,s) &:= \sum_{c=1}^{\infty} \frac{X(n,c)}{c^s},
\end{aligned}$$

then the Fourier coefficients are:

$$a_n = \left(\frac{n\pi i}{2}\right)^k \frac{D(n,k)}{n(k-1)!}.$$

Thus, computation of Fourier coefficients is reduced to the evaluation of special values of the Dirichlet series $D(n,s)$. To aid in evaluating such series numerically we provide an algorithm for computing $\chi(c,d)$:

(1) lift (c,d) to a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ using the extended Euclidean algorithm;
(2) test if g, Tg , T^2g or T^3g is in G_1; if so, $\chi(c,d) = 1$ and if not then $\chi(c,d) = 0$.
(3) To test if a matrix $h \in \Gamma$ lies in G_1, write it as a word in S and T, then obtain the analogous word in $\phi_1(S), \phi_1(T) \in S_7$, and check whether the resulting permutation fixes 1. If so $h \in G_1$, and not otherwise.

See Appendix A for a *Pari* code implementation.

Remark 17. To obtain a very rough estimate for $D(n,k)$ computed using the values $c \leq N$, call this approximation S_N, observe that

$$|D(n,k) - S_N| \leq \sum_{c>N} \frac{4}{c^{k-1}} < 4\int_N^\infty \frac{dx}{x^{k-1}} = \frac{4}{(k-2)N^{k-2}}.$$

In particular, the number of digits of accuracy in this approximation is at least $k-2$ times the number of digits in N. In practice it appears that $X(n,c) \ll c$, see Figure 6. This results that is slightly better than this. This also leads to the apparent absolute convergence of the series for $k=2$. None the less the naive method for evaluating the Fourier coefficients is in general quite inefficient for small k.

Remark 18. Observe that

$$X(n,c) = \sum_{d=1}^{c} \left(\chi(c,d) + i^n\chi(c,d+c) + (-1)^n\chi(c,d+2c) + (-i)^n\chi(c,d+3c)\right) e^{\frac{2\pi i n d}{4c}}.$$

Evaluating $\chi(c,d)$ involves solving a word problem, and the solution to that word problem can be used to solve the corresponding word problems involved in evaluating $\chi(c,d+c)$, $\chi(c,d+2c)$ and $\chi(c,d+3c)$. In this way one can speed up the evaluation of the approximation to $D(n,k)$ by precomputing values $\chi(c,d)$ four at a time using this optimization.

Remark 19. For each group being considered there is a unique (normalized) modular form in weight 2. A simple computation with the divisors reveals that in every case it will be given precisely by $g_2 = (E_6 \cdot e_3 \cdot f_3)/(E_4 \cdot f_2)$. In Table 9 we give the first few Fourier coefficients as a series in respectively q_4, q_3, q_5, and q_2.

Assuming for a moment the convergence of $D(n,2)$ then with $g_2^{(\infty)}$ the function formally defined by Equation (1) and $g_2^{(1)}$ the analogous function defined for the cusp 1 one has that $g_2^{(\infty)} - \frac{4}{3}g_2^{(1)}$ will define a holomorphic modular form for G_1, and consequently $g_2 = g_2^{(\infty)} - \frac{4}{3}g_2^{(1)}$. Noting that $g_2^{(\infty)} + g_2^{(1)} = E_2$ this would allow us to immediately deduce that the special value $D(n,2)$ is in fact algebraic. This of course agrees with the expectations from [29] and is analogous to the Proposition on page 260 of [21].

Although it is generally nontrivial to determine the field of definition of Eisenstein series, or even if they are algebraic, in this case we can at least determine the phase of the Fourier coefficients:

n	G_1: a_n/u^n, $u=\sqrt[4]{-7}/7^2$	G_3: a_n/u^n, $u=\sqrt[3]{-2/7}/7^2$
0	1	1
1	-168	462
2	-840	-84420
3	733152	-807828
4	-1615656	-891458736
5	1179184272	82305718992
6	-5780133408	5155138704870
7	-1097701319232	807981764899218
8	20620554819480	-57396539567144736
9	-1310614136578824	829520378016134700
10	-14959868841286320	-368800915551641445600

n	H_1: a_n/u^n, $u=\sqrt[5]{-7^3}/7^2$	H_3: a_n/u^n, $u=\sqrt[2]{-7}/7^4$
0	1	1
1	-28	952
2	-3108	-14260008
3	88172	5950907872
4	824012	18866241755032
5	-14260008	14858201843068752
6	352362948	-29392973490650091168
7	13569079384	18769317912571342452672
8	-195382795860	$26663537479505346618394392$
9	-1200557668744	$127133105049733771815754544552$
10	18866241755032	$-3619424077855847163524499059940 8$

n	U_1: a_n/u^n, $u=((1763\zeta_3+1255)2^2 3/7^7)^{(1/6)}$
0	1
1	$8\zeta_3+10$
2	$56\zeta_3+28$
3	$84\zeta_3-84$
4	-336
5	$-1008\zeta_3-1008$
6	$-710/3\zeta_3-184/3$
7	$-9566/3\zeta_3+9088/3$
8	$4256\zeta_3+30016/3$
9	$15624\zeta_3+19404$
10	$139552/3\zeta_3-25984/3$

n	U_6: a_n/u^n, $u=((3\zeta_3+1)/7)^7$
0	1
1	$1368\zeta_3+4944$
2	$5264136\zeta_3+13265352$
3	$12839470272\zeta_3+16044542112$
4	$22545390152664\zeta_3+27018559576704$
5	$9748947084182352\zeta_3+9649676839772016$
6	$3471897202643819 7504\zeta_3+16480296599809346784$
7	$9778372812649484494272\zeta_3+9122178274543742453376$
8	$3520767486662051378584 3560\zeta_3+3599167618394097606994536$
9	$3521279102586782142823326 1296\zeta_3-1534671671263749769838754840$
10	$1985843820948831885269745820 5264\zeta_3-842403636372392338719784706 7264$

TABLE 9. Normalized Fourier coefficients of the forms g_2.

Proposition 20. *For all $c \geq 1$ we have $X(n,c) \in (\mathbf{R} \cap \bar{\mathbf{Q}}) \cdot \zeta_8^n$, and so $a_n \in \mathbf{R} \cdot \zeta_8^n$.*

Proof. Take a complex conjugate of the identity in Remark 18 and use the properties in Proposition 15 to obtain:

$$\begin{aligned}
&\overline{\sum_{d=1}^{c} \left(\chi(c,d) + i^n \chi(c,d+c) + (-1)^n \chi(c,d+2c) + (-i)^n \chi(c,d+3c)\right) e^{\frac{2\pi i n d}{4c}}} \\
&= \sum_{d=1}^{c} \left(\chi(c,d) + (-i)^n \chi(c,d+c) + (-1)^n \chi(c,d+2c) + i^n \chi(c,d+3c)\right) e^{\frac{2\pi i n(3c+c-d)}{4c}} \\
&= \sum_{d=1}^{c} \left(\chi(c,c-d) + (-i)^n \chi(c,2c-d) + (-1)^n \chi(c,3c-d) + i^n \chi(c,4c-d)\right) e^{\frac{2\pi i n(3c+d)}{4c}} \\
&= \sum_{d=1}^{c} \left(\chi(-c,d+3c) + (-i)^n \chi(-c,d+2c) + (-1)^n \chi(-c,d+c) + i^n \chi(-c,d)\right) (-i)^n e^{\frac{2\pi i n d}{4c}} \\
&= \sum_{d=1}^{c} \left(\chi(c,d) + (-i)^n \chi(c,d+3c) + (-1)^n \chi(c,d+2c) + i^n \chi(c,d+c)\right) (-i)^n e^{\frac{2\pi i n d}{4c}}
\end{aligned}$$

which shows that $\overline{X(n,c)} = (-i)^n X(n,c)$. Thus,

$$\overline{X(n,c)(1+i)^{-n}} = (-i)^n X(n,c)(1-i)^{-n} = X(n,c)(1+i)^{-n}.$$

Hence $X(n,c) \in (\mathbf{R} \cap \overline{\mathbf{Q}}) \cdot \zeta_8^n$ as claimed. The second claim follows immediately from this. □

The Dirichlet series $D(n,s)$ are quite mysterious, as their coefficients $X(n,s)$ lie in increasingly large number fields, as opposed to more typical Dirichlet L-series, or Dedekind ζ-functions, and so many standard techniques cannot be brought to bear on $D(n,s)$. It appears that perhaps $X(n,c) = O(c^{5/7})$, and Figure 6 on page 20 shows a plot of some values that supports this. More precisely, when $n = 1$, we have $|X(1,c)| < c^{5/7}$ for all $32,769 < c < 2,000,000$ (this bound fails for 15 values below $32,769$). Likewise for $n = 2, \ldots, 11$ with $c < 300,000$ and $n = 12, \ldots, 50$ with $c < 100,000$ the only values with $X(n,c) > c^{5/7}$ come from small values c. Experimentally, see again Figure 6 but also Figures 7 and 8, it is evident that the distribution of the values $X(n,c)$ along the line u^n is broadly controlled by the congruence $c \pmod{12}$. The exact distributions appear to depend on n: the cases for $X(1,c)$ are illustrated in Figures 7 and 8. We note that the exponent $2/7$ on c is selected to make the graphs appear approximately normal, we have no evidence this is the correct exponent, nor that these distributions should be normal.

As with g_2 it is an exercise to conclude that $g_4 = (E_4/f_3) \cdot a_1 \cdot (z - Cu)$ for some $C \in \mathbb{R}$. The algebraicity of C is equivalent to that of both the divisor of g_4 as well as that of its Fourier coefficients. We have included for the curious reader our computations of the first few Fourier coefficients of g_4 in Table 10 on page 20. The computations for a_1 used terms with c up to $2,000,000$, while for $a_2, \ldots, a_{10}$, we used c up to $300,000$. The computations for a_1 took over a month of CPU time using resources from Compute Canada. Note that the final digits may not be accurate, as we have provided one digit beyond the apparent precision, and the actual precision may be less still (see Remark 17). We have not been able to identify any apparent algebraic dependency for the higher coefficients, but this may be a simple reflection of a lack of sufficient precision to detect dependency relations.

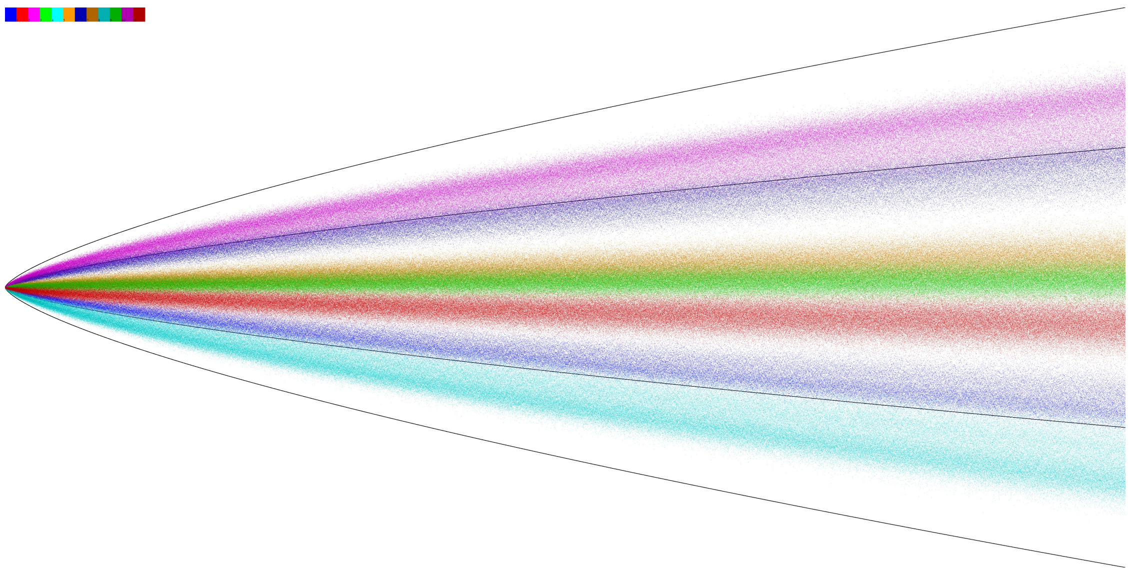

FIGURE 6. $X(1,c)/e^{(\pi i/4)}$ for $c \leq 2,000,000$. The outer black curve is $\pm c^{5/7}$, the inner is $\pm(1/2)c^{5/7}$. From top to bottom the colored bands are $\pm 2 \pmod{12}$, $6 \pmod{12}$, $\pm 5 \pmod{12}$, $\pm 3 \pmod{12}$, $\pm 1 \pmod{12}$, $0 \pmod{12}$, and $\pm 4 \pmod{12}$.

n	$a_n/u^n, \quad u = (-7)^{(1/4)}/7^2$
0	1
1	$40.7303189636318364926\cdots$
2	$303.7319312003984\cdots$
3	$-1113445.924994532325\cdots$
4	$-101378021.6026120116\cdots$
5	$-4677356098.49752275\cdots$
6	$110516113983.5601513\cdots$
7	$10622672944963.34244\cdots$
8	$703827515349172.972\cdots$
9	$20587451911329502.7\cdots$
10	$549857713550018050.6\cdots$

TABLE 10. Approximate values of normalized Fourier coefficients for g_4 for the group G_1

APPENDIX A. CODE

The following PARI/GP code computes $\chi(c,d)$

```
Chi(c,d) = {
   my(s=Vecsmall([2,1,4,3,6,5,7])); /* Permutation Phi(S) */
   my(t=Vecsmall([2,4,6,5,1,7,3])); /* Permutation Phi(T) */
   my(res=Vecsmall([1,2,3,4,5,6,7]));
   my(q);
   while( c != 0,  /* Compute Phi(M) for M with bottom row c,d */
      if( abs(d) >= abs(c) ,
          q=d\c;
          d=d-q*c;
```

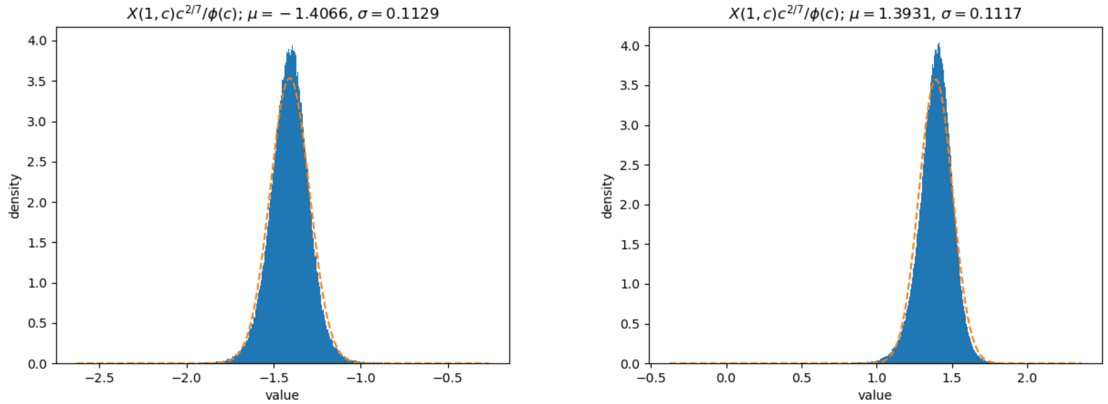

FIGURE 7. Normalized histograms for $X(1,c)c^{2/7}/(\phi(c)e^{\pi i/4})$ for $2||c$, $4|c$ with c up to 2,000,000. Curve is a normal distribution with given paramters.

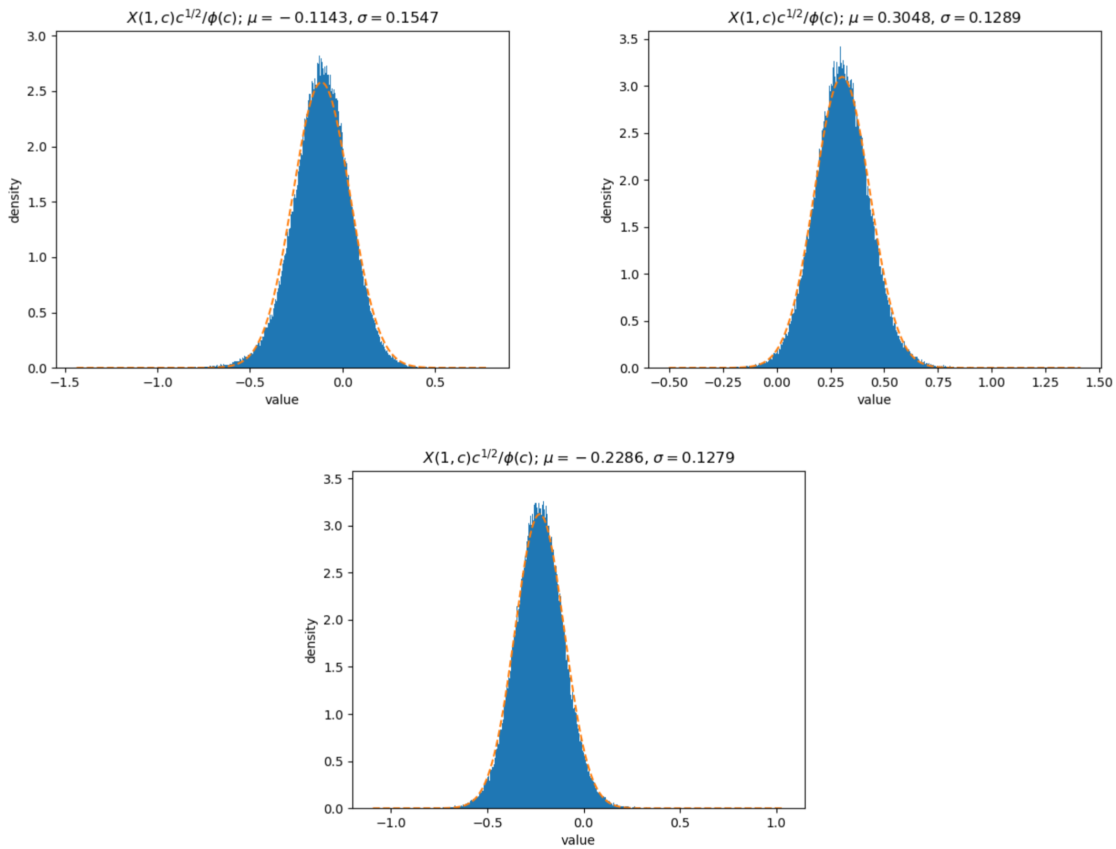

FIGURE 8. Normalized histograms for $X(1,c)c^{1/2}/(\phi(c)e^{\pi i/4})$ for (clockwise from top left) $c = \pm 3 \pmod{12}$, $c = \pm 1 \pmod{12}$, and $c = \pm 5 \pmod{12}$ with c up to 2,000,000. Curve is a normal distribution with given paramters.

```
            res = (t^(q%12))*res;
         ,
            res = s*res;
            q=c;
            c=d;
            d=-q;
        );
    );
    if( abs(d) != 1, return(0)); /* c,d  not relatively prime */
    if( res[1]==1||res[1]==t[1]||res[1]==(t*t)[1]||res[1]==(t*t*t)[i],
            return(1);
    );
    return(0);
}
```

References

[1] A. O. L. Atkin, Wen-Ching Winnie Li, Tong Liu, and Ling Long. Galois representations with quaternion multiplication associated to noncongruence modular forms. *Trans. Amer. Math. Soc.*, 365(12):6217–6242, 2013.

[2] A. O. L. Atkin, Wen-Ching Winnie Li, and Ling Long. On Atkin and Swinnerton-Dyer congruence relations. II. *Math. Ann.*, 340(2):335–358, 2008.

[3] A. O. L. Atkin and H. P. F. Swinnerton-Dyer. Modular forms on noncongruence subgroups. In *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968)*, pages 1–25, 1971.

[4] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[5] Vincenz Busch, Ulf Kuehn, and Anna Posingies. On scattering constants for a non-congruence subgroup. *arXiv preprint arXiv:0906.1466*, 2009.

[6] Luca Candelori and Cameron Franc. Vector-valued modular forms and the modular orbifold of elliptic curves. *Int. J. Number Theory*, 13(1):39–63, 2017.

[7] William Yun Chen. Moduli interpretations for noncongruence modular curves. *Math. Ann.*, 371(1-2):41–126, 2018.

[8] Cameron Franc and Geoffrey Mason. Fourier coefficients of vector-valued modular forms of dimension 2. *Canad. Math. Bull.*, 57(3):485–494, 2014.

[9] Cameron Franc and Geoffrey Mason. Three-dimensional imprimitive representations of the modular group and their associated modular forms. *J. Number Theory*, 160:186–214, 2016.

[10] Richard Gottesman. The arithmetic of vector-valued modular forms on $\Gamma_0(2)$. *Int. J. Number Theory*, 16(2):241–289, 2020.

[11] Jerome William Hoffman, Ling Long, and Helena Verrill. On ℓ-adic representations for a space of noncongruence cuspforms. *Proc. Amer. Math. Soc.*, 140(5):1569–1584, 2012.

[12] Matija Kazalicki and Anthony J. Scholl. Modular forms, de Rham cohomology and congruences. *Trans. Amer. Math. Soc.*, 368(10):7097–7117, 2016.

[13] Tomio Kubota. *Elementary theory of Eisenstein series*. Kodansha Ltd., Tokyo; Halsted Press [John Wiley & Sons], New York-London-Sydney, 1973.

[14] Ulf Kühn. Néron-Tate heights on algebraic curves and subgroups of the modular group. *Manuscripta Math.*, 116(4):401–419, 2005.

[15] Chris Kurth and Ling Long. On modular forms for some noncongruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. II. *Bull. Lond. Math. Soc.*, 41(4):589–598, 2009.

[16] Chris A. Kurth and Ling Long. On modular forms for some noncongruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. *J. Number Theory*, 128(7):1989–2009, 2008.

[17] Wen-Ching Winnie Li and Ling Long. Fourier coefficients of noncongruence cuspforms. *Bull. Lond. Math. Soc.*, 44(3):591–598, 2012.

[18] Wen-Ching Winnie Li and Ling Long. Atkin and Swinnerton-Dyer congruences and noncongruence modular forms. In *Algebraic number theory and related topics 2012*, RIMS Kôkyûroku Bessatsu, B51, pages 269–299. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.

[19] Wen-Ching Winnie Li, Ling Long, and Zifeng Yang. On Atkin-Swinnerton-Dyer congruence relations. *J. Number Theory*, 113(1):117–148, 2005.

[20] Ling Long. On Atkin and Swinnerton-Dyer congruence relations. III. *J. Number Theory*, 128(8):2413–2429, 2008.

[21] V. Kumar Murty and Dinakar Ramakrishnan. The Manin-Drinfeld theorem and Ramanujan sums. *Proc. Indian Acad. Sci. Math. Sci.*, 97(1-3):251–262 (1988), 1987.

[22] Michael Musty, Sam Schiavone, Jeroen Sijsling, and John Voight. A database of Belyi maps. In *Proceedings of the Thirteenth Algorithmic Number Theory Symposium*, volume 2 of *Open Book Ser.*, pages 375–392. Math. Sci. Publ., Berkeley, CA, 2019.

[23] R. Phillips and P. Sarnak. The spectrum of Fermat curves. *Geom. Funct. Anal.*, 1(1):80–146, 1991.

[24] R. Phillips and P. Sarnak. Cusp forms for character varieties. *Geom. Funct. Anal.*, 4(1):93–118, 1994.

[25] R. S. Phillips and P. Sarnak. On cusp forms for co-finite subgroups of $\mathrm{PSL}(2, \mathbf{R})$. *Invent. Math.*, 80(2):339–364, 1985.

[26] Anna Posingies. Kronecker limit formulas and scattering constants for fermat curves. *arXiv preprint arXiv:1111.1132*, 2011.

[27] Bruno Schoeneberg. *Elliptic modular functions: an introduction*, volume 203. Springer Science & Business Media, 2012.

[28] A. J. Scholl. Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences. *Invent. Math.*, 79(1):49–77, 1985.

[29] A. J. Scholl. Fourier coefficients of Eisenstein series on noncongruence subgroups. *Math. Proc. Cambridge Philos. Soc.*, 99(1):11–17, 1986.

[30] A. J. Scholl. Modular forms on noncongruence subgroups. In *Séminaire de Théorie des Nombres, Paris 1985–86*, volume 71 of *Progr. Math.*, pages 199–206. Birkhäuser Boston, Boston, MA, 1987.

[31] A. J. Scholl. The l-adic representations attached to a certain noncongruence subgroup. *J. Reine Angew. Math.*, 392:1–15, 1988.

[32] A. J. Scholl. On some l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to noncongruence subgroups. *Bull. London Math. Soc.*, 38(4):561–567, 2006.

[33] J. Sijsling and J. Voight. On computing Belyi maps. In *Numéro consacré au trimestre "Méthodes arithmétiques et applications", automne 2013*, volume 2014/1 of *Publ. Math. Besançon Algèbre Théorie Nr.*, pages 73–131. Presses Univ. Franche-Comté, Besançon, 2014.

[34] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[35] Klaus Wohlfahrt. An extension of F. Klein's level concept. *Illinois J. Math.*, 8:529–535, 1964.

UNIVERSITY OF LETHBRIDGE
Email address: andrew.fiori@uleth.ca

MCMASTER UNIVERSITY
Email address: franc@math.mcmaster.ca