MATH 1010
INTRODUCTION TO CALCULUS

Summer 2016 Edition, University of Lethbridge

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Contributing Textbooks

Precalculus, Version $\lfloor \pi \rfloor = 3$
Carl Stitz and Jeff Zeager
www.stitz-zeager.com

APEX *Calculus*
Gregory Hartman et al
apexcalculus.com

This version of the text assembled and edited by Sean Fitzpatrick, University of Lethbridge, May, 2016.

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Preface

One of the challenges with a new course like Math 1010 is finding a suitable textbook for the course. This is made additionally difficult for a course that covers two topics – Precalculus and Calculus – that are usually offered as separate courses, with separate texts. I reviewed a number of commercially available options, but these all had two things in common: they did not quite meet our needs, and they were all very expensive (some were as much as $400).

Since writing a new textbook from scratch is a huge undertaking, requiring resources (like time) we simply did not have, I chose to explore non-commercial options. This took a bit of searching, since non-commercial texts, while inexpensive (or free), are of varying quality. Fortunately, there are some decent texts out there. Unfortunately, I couldn't find a single text that covered all of the material we need for Math 1010.

To get around this problem, I have selected two textbooks as our primary sources for the course. The first is *Precalculus*, version 3, by Carl Stitz and Jeff Zeager. The second is *APEX Calculus I*, version 3.0, by Hartman et al. Both texts have two very useful advantages. First, they're both free (as in beer): you can download either text in PDF format from the authors' web pages. Second, they're also *open source* texts (that is, free, as in speech). Both books are written using the LaTeXmarkup language, as is typical in mathematics publishing. What is not typical is that the authors of both texts make their source code freely available, allowing others (such as myself) to edit and customize the books as they see fit.

In the first iteration of this project (Fall 2015), I was only able to edit each text individually for length and content, resulting in two separate textbooks for Math 1010. This time around, I've had enough time to take the content of the Precalculus textbook and adapt its source code to be compatible with the formatting of the Calculus textbook, allowing me to produce a single textbook for all of Math 1010.

The book is very much a work in progress, and I will be editing it regularly. Feedback is always welcome.

Acknowledgements

First and foremost, I need to thank the authors of the two textbooks that provide the source material for this text. Without their hard work, and willingness to make their books (and the source code) freely available, it would not have been possible to create an affordable textbook for this course. You can find the original textbooks at their websites:

www.stitz-zeager.com, for the *Precalculus* textbook, by Stitz and Zeager, and

apexcalculus.com, for the APEX *Calculus* textbook, by Hartman et al.

I'd also like to thank Dave Morris for help with converting the graphics in the Precalculus textbook to work with the formatting code of the APEX text, Howard Cheng for providing some C++ code to convert the exercises, and the other faculty members involved with this course — Alia Hamieh, David Kaminsky, and Nicole Wilson — for their input on the content of the text.

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May, 2016

1: The Real Numbers

1.1 Some Basic Set Theory Notions

While the authors would like nothing more than to delve quickly and deeply into the sheer excitement that is *Precalculus*, experience has taught us that a brief refresher on some basic notions is welcome, if not completely necessary, at this stage. To that end, we present a brief summary of 'set theory' and some of the associated vocabulary and notations we use in the text. Like all good Math books, we begin with a definition.

Definition 1 Set

A **set** is a well-defined collection of objects which are called the 'elements' of the set. Here, 'well-defined' means that it is possible to determine if something belongs to the collection or not, without prejudice.

For example, the collection of letters that make up the word "pronghorns" is well-defined and is a set, but the collection of the worst math teachers in the world is **not** well-defined, and so is **not** a set. In general, there are three ways to describe sets. They are

One thing that student evaluations teach us is that any given Mathematics instructor can be simultaneously the best and worst teacher ever, depending on who is completing the evaluation.

Key Idea 1 Ways to Describe Sets

1. **The Verbal Method:** Use a sentence to define a set.
2. **The Roster Method:** Begin with a left brace '$\{$', list each element of the set *only once* and then end with a right brace '$\}$'.
3. **The Set-Builder Method:** A combination of the verbal and roster methods using a "dummy variable" such as x.

For example, let S be the set described *verbally* as the set of letters that make up the word "pronghorns". A **roster** description of S would be $\{p, r, o, n, g, h, s\}$. Note that we listed 'r', 'o', and 'n' only once, even though they appear twice in "pronghorns." Also, the *order* of the elements doesn't matter, so $\{o, n, p, r, g, s, h\}$ is also a roster description of S. A **set-builder** description of S is:

$$\{x \mid x \text{ is a letter in the word "pronghorns".}\}$$

The way to read this is: 'The set of elements x <u>such that</u> x is a letter in the word "pronghorns."' In each of the above cases, we may use the familiar equals sign '$=$' and write $S = \{p, r, o, n, g, h, s\}$ or $S = \{x \mid x \text{ is a letter in the word "pronghorns".}\}$. Clearly r is in S and q is not in S. We express these sentiments mathematically by writing $r \in S$ and $q \notin S$.

More precisely, we have the following.

Definition 2 Notation for set inclusion

Let A be a set.

- If x is an element of A then we write $x \in A$ which is read 'x is in A'.
- If x is *not* an element of A then we write $x \notin A$ which is read 'x is not in A'.

Now let's consider the set $C = \{x \mid x$ is a consonant in the word "pronghorns"$\}$. A roster description of C is $C = \{p, r, n, g, h, s\}$. Note that by construction, every element of C is also in S. We express this relationship by stating that the set C is a **subset** of the set S, which is written in symbols as $C \subseteq S$. The more formal definition is given below.

Definition 3 Subset

Given sets A and B, we say that the set A is a **subset** of the set B and write '$A \subseteq B$' if every element in A is also an element of B.

Note that in our example above $C \subseteq S$, but not vice-versa, since $o \in S$ but $o \notin C$. Additionally, the set of vowels $V = \{a, e, i, o, u\}$, while it does have an element in common with S, is not a subset of S. (As an added note, S is not a subset of V, either.) We could, however, *build* a set which contains both S and V as subsets by gathering all of the elements in both S and V together into a single set, say $U = \{p, r, o, n, g, h, s, a, e, i, u\}$. Then $S \subseteq U$ and $V \subseteq U$. The set U we have built is called the **union** of the sets S and V and is denoted $S \cup V$. Furthermore, S and V aren't completely *different* sets since they both contain the letter 'o.' (Since the word 'different' could be ambiguous, mathematicians use the word *disjoint* to refer to two sets that have no elements in common.) The **intersection** of two sets is the set of elements (if any) the two sets have in common. In this case, the intersection of S and V is $\{o\}$, written $S \cap V = \{o\}$. We formalize these ideas below.

Definition 4 Intersection and Union

Suppose A and B are sets.

- The **intersection** of A and B is $A \cap B = \{x \mid x \in A$ and $x \in B\}$
- The **union** of A and B is $A \cup B = \{x \mid x \in A$ or $x \in B$ (or both)$\}$

The key words in Definition 4 to focus on are the conjunctions: 'intersection' corresponds to 'and' meaning the elements have to be in *both* sets to be in the intersection, whereas 'union' corresponds to 'or' meaning the elements have to be in one set, or the other set (or both). In other words, to belong to the union of two sets an element must belong to *at least one* of them.

Returning to the sets C and V above, $C \cup V = \{p, r, n, g, h, s, a, e, i, o, u\}$. When it comes to their intersection, however, we run into a bit of notational awkwardness since C and V have no elements in common. While we could write $C \cap V = \{\}$, this sort of thing happens often enough that we give the set with no elements a name.

Definition 5 Empty set

The **Empty Set** $\emptyset$ is the set which contains no elements. That is,

$$\emptyset = \{\} = \{x \mid x \neq x\}.$$

The full extent of the empty set's role will not be explored in this text, but it is of fundamental importance in Set Theory. In fact, the empty set can be used to generate numbers - mathematicians can create something from nothing! If you're interested, read about the von Neumann construction of the natural numbers or consider signing up for Math 2000.

As promised, the empty set is the set containing no elements since no matter what 'x' is, '$x = x$.' Like the number '0,' the empty set plays a vital role in mathematics. We introduce it here more as a symbol of convenience as opposed to a contrivance. Using this new bit of notation, we have for the sets C and V above that $C \cap V = \emptyset$. A nice way to visualize relationships between sets and set operations is to draw a **Venn Diagram**. A Venn Diagram for the sets S, C and V is drawn in Figure 1.1.

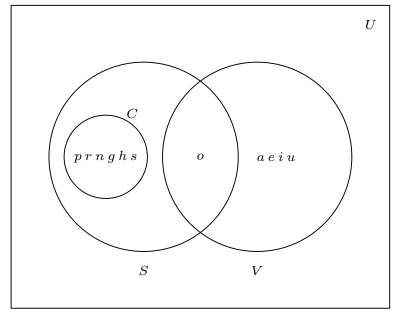
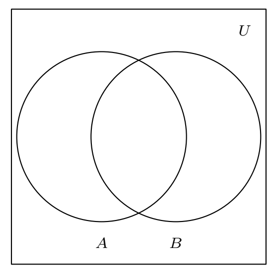


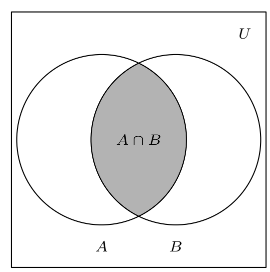
Figure 1.1: A Venn diagram for C, S, and V

In Figure 1.1 we have three circles - one for each of the sets C, S and V. We visualize the area enclosed by each of these circles as the elements of each set. Here, we've spelled out the elements for definitiveness. Notice that the circle representing the set C is completely inside the circle representing S. This is a geometric way of showing that $C \subseteq S$. Also, notice that the circles representing S and V overlap on the letter 'o'. This common region is how we visualize $S \cap V$. Notice that since $C \cap V = \emptyset$, the circles which represent C and V have no overlap whatsoever.

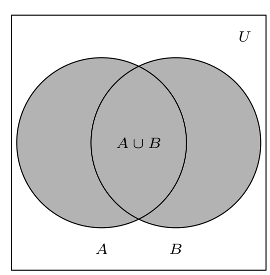
All of these circles lie in a rectangle labelled U (for 'universal' set). A universal set contains all of the elements under discussion, so it could always be taken as the union of all of the sets in question, or an even larger set. In this case, we could take $U = S \cup V$ or U as the set of letters in the entire alphabet. The usual triptych of Venn Diagrams indicating generic sets A and B along with $A \cap B$ and $A \cup B$ is given below.



Sets A and B.



$A \cap B$ is shaded.



$A \cup B$ is shaded.

Figure 1.2: Venn diagrams for intersection and union

(The reader may well wonder if there is an ultimate universal set which contains *everything*. The short answer is 'no'. Our definition of a set turns out to be overly simplistic, but correcting this takes us well beyond the confines of this course. If you want the longer answer, you can begin by reading about Russell's Paradox on Wikipedia.)

1.1.1 Sets of Real Numbers

The playground for most of this text is the set of **Real Numbers**. Many quantities in the 'real world' can be quantified using real numbers: the temperature at a given time, the revenue generated by selling a certain number of products and the maximum population of Sasquatch which can inhabit a particular region are just three basic examples. A succinct, but nonetheless incomplete definition of a real number is given below.

Definition 6 The real numbers

A **real number** is any number which possesses a decimal representation. The set of real numbers is denoted by the character $\mathbb{R}$.

Certain subsets of the real numbers are worthy of note and are listed below. In more advanced courses like Analysis, you learn that the real numbers can be *constructed* from the rational numbers, which in turn can be constructed from the integers (which themselves come from the natural numbers, which in turn can be defined as sets...).

Definition 7 Sets of Numbers

1. The **Empty Set**: $\emptyset = \{\} = \{x \mid x \neq x\}$. This is the set with no elements. Like the number '0,' it plays a vital role in mathematics.
2. The **Natural Numbers**: $\mathbb{N} = \{1, 2, 3, \ldots\}$ The periods of ellipsis here indicate that the natural numbers contain 1, 2, 3, 'and so forth'.
3. The **Integers**: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. The **Rational Numbers**: $\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\right\}$. Rational numbers are the ratios of integers (provided the denominator is not zero!) It turns out that another way to describe the rational numbers is:

   $$\mathbb{Q} = \{x \mid x \text{ possesses a repeating or terminating decimal representation.}\}$$

5. The **Real Numbers**: $\mathbb{R} = \{x \mid x \text{ possesses a decimal representation.}\}$
6. The **Irrational Numbers**: Real numbers that are not rational are called **irrational**. As a set, we have $\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$. (There is no standard symbol for this set.) Every irrational number has a decimal expansion which neither repeats nor terminates.
7. The **Complex Numbers**: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (We will not deal with complex numbers in Math 1010, although they usually make an appearance in Math 1410.)

An example of a number with a repeating decimal expansion is $a = 2.13234234234\ldots$. This is rational since $100a = 213.2342342342\ldots$, and $100000a = 213234.234234\ldots$ so $99900a = 100000a - 100a = 213021$. This gives us the rational expression $a = \dfrac{213021}{99900}$.

The classic example of an irrational number is the number π (See Section 8.1), but numbers like $\sqrt{2}$ and $0.101001000100001\ldots$ are other fine representatives.

It is important to note that every natural number is a whole number is an integer. Each integer is a rational number (take $b = 1$ in the above definition for $\mathbb{Q}$) and the rational numbers are all real numbers, since they possess decimal representations (via long division!). If we take $b = 0$ in the above definition of $\mathbb{C}$, we see that every real number is a complex number. In this sense, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are 'nested' like Matryoshka dolls. More formally, these sets form a subset chain: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. The reader is encouraged to sketch a Venn Diagram depicting $\mathbb{R}$ and all of the subsets mentioned above. It is time for an example.

Example 1 Sets of real numbers

1. Write a roster description for $P = \{2^n \mid n \in \mathbb{N}\}$ and $E = \{2n \mid n \in \mathbb{Z}\}$.
2. Write a verbal description for $S = \{x^2 \mid x \in \mathbb{R}\}$.
3. Let $A = \{-117, \frac{4}{5}, 0.202\overline{002}, 0.202002000200002\ldots\}$.

   Which elements of A are natural numbers? Rational numbers? Real numbers?

Solution

1. To find a roster description for these sets, we need to list their elements. Starting with $P = \{2^n \mid n \in \mathbb{N}\}$, we substitute natural number values n into the formula 2^n. For $n = 1$ we get $2^1 = 2$, for $n = 2$ we get $2^2 = 4$, for $n = 3$ we get $2^3 = 8$ and for $n = 4$ we get $2^4 = 16$. Hence P describes the powers of 2, so a roster description for P is $P = \{2, 4, 8, 16, \ldots\}$ where the '...' indicates the that pattern continues.

   Proceeding in the same way, we generate elements in $E = \{2n \mid n \in \mathbb{Z}\}$ by plugging in integer values of n into the formula $2n$. Starting with $n = 0$ we obtain $2(0) = 0$. For $n = 1$ we get $2(1) = 2$, for $n = -1$ we get $2(-1) = -2$ for $n = 2$, we get $2(2) = 4$ and for $n = -2$ we get $2(-2) = -4$. As n moves through the integers, $2n$ produces all of the *even* integers. A roster description for E is $E = \{0, \pm 2, \pm 4, \ldots\}$.

2. One way to verbally describe S is to say that S is the 'set of all squares of real numbers'. While this isn't incorrect, we'd like to take this opportunity to delve a little deeper. What makes the set $S = \{x^2 \mid x \in \mathbb{R}\}$ a little trickier to wrangle than the sets P or E above is that the dummy variable here, x, runs through all *real* numbers. Unlike the natural numbers or the integers, the real numbers cannot be listed in any methodical way. Nevertheless, we can select some real numbers, square them and get a sense of what kind of numbers lie in S. For $x = -2$, $x^2 = (-2)^2 = 4$ so 4 is in S, as are $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$ and $(\sqrt{117})^2 = 117$. Even things like $(-\pi)^2$ and $(0.101001000100001\ldots)^2$ are in S.

   So suppose $s \in S$. What can be said about s? We know there is some real number x so that $s = x^2$. Since $x^2 \geq 0$ for any real number x, we know $s \geq 0$. This tells us that everything in S is a non-negative real number. This begs the question: are <u>all</u> of the non-negative real numbers in S? Suppose n is a non-negative real number, that is, $n \geq 0$. If n were in S, there would be a real number x so that $x^2 = n$. As you may recall, we can solve $x^2 = n$ by 'extracting square roots': $x = \pm\sqrt{n}$. Since $n \geq 0$, $\sqrt{n}$ is a real number. Moreover, $(\sqrt{n})^2 = n$ so n is the square of a real number which means $n \in S$. Hence, S is the set of non-negative real numbers.

3. The set A contains no natural numbers. Clearly, $\frac{4}{5}$ is a rational number as is -117 (which can be written as $\frac{-117}{1}$). It's the last two numbers listed in A, $0.20\overline{2002}$ and $0.202002000200002\ldots$, that warrant some discussion. First, recall that the 'line' over the digits 2002 in $0.20\overline{2002}$ (called the vinculum) indicates that these digits repeat, so it is a rational number. As for the number $0.202002000200002\ldots$, the '...' indicates the pattern of adding an extra '0' followed by a '2' is what defines this real number. Despite the fact there is a *pattern* to this decimal, this decimal is *not repeating*, so it is not a rational number - it is, in fact, an irrational number. All of the elements of A are real numbers, since all of them can be expressed as decimals (remember that $\frac{4}{5} = 0.8$). □

This isn't the most *precise* way to describe this set - it's always dangerous to use '...' since we assume that the pattern is clearly demonstrated and thus made evident to the reader. Formulas are more precise because the pattern is clear.

It shouldn't be too surprising that E is the set of all even integers, since an even integer is *defined* to be an integer multiple of 2.

The fact that the real numbers cannot be listed is a nontrivial statement. Interested readers are directed to a discussion of <u>Cantor's Diagonal Argument</u>.

As you may recall, we often visualize the set of real numbers $\mathbb{R}$ as a line where each point on the line corresponds to one and only one real number. Given two different real numbers a and b, we write $a < b$ if a is located to the left of b on the number line, as shown in Figure 1.3.

While this notion seems innocuous, it is worth pointing out that this convention is rooted in two deep properties of real numbers. The first property is that

$\mathbb{R}$ is complete. This means that there are no 'holes' or 'gaps' in the real number line. (This intuitive feel for what it means to be 'complete' is as good as it gets at this level. Completeness does get a much more precise meaning later in courses like Analysis and Topology.) Another way to think about this is that if you choose any two distinct (different) real numbers, and look between them, you'll find a solid line segment (or interval) consisting of infinitely many real numbers.

The next result tells us what types of numbers we can expect to find.

Theorem 1 Density Property of $\mathbb{Q}$ in $\mathbb{R}$

Between any two distinct real numbers, there is at least one rational number and irrational number. It then follows that between any two distinct real numbers there will be infinitely many rational and irrational numbers.

The root word 'dense' here communicates the idea that rationals and irrationals are 'thoroughly mixed' into $\mathbb{R}$. The reader is encouraged to think about how one would find both a rational and an irrational number between, say, 0.9999 and 1. Once you've done that, ask yourself whether there is any difference between the numbers $0.\overline{9}$ and 1.

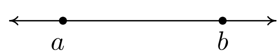


Figure 1.3: The real number line with two numbers *a* and *b*, where $a < b$.

The second property $\mathbb{R}$ possesses that lets us view it as a line is that the set is totally ordered. This means that given any two real numbers *a* and *b*, either $a < b$, $a > b$ or $a = b$ which allows us to arrange the numbers from least (left) to greatest (right). You may have heard this property given as the 'Law of Trichotomy'.

The Law of Trichotomy, strictly speaking, is an *axiom* of the real numbers: a basic requirement that we assume to be true. However, in any *construction* of the real, such as the method of Dedekind cuts, it is necessary to *prove* that the Law of Trichotomy is satisfied.

Definition 8 Law of Trichotomy

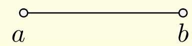
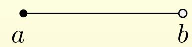
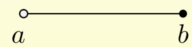
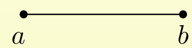
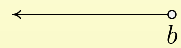
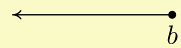
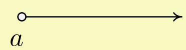
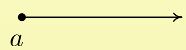
If *a* and *b* are real numbers then **exactly one** of the following statements is true:

$a < b$ $a > b$ $a = b$

Segments of the real number line are called **intervals** of numbers. Below is a summary of the so-called **interval notation** associated with given sets of numbers. For intervals with finite endpoints, we list the left endpoint, then the right endpoint. We use square brackets, '[' or ']', if the endpoint is included in the interval and use a filled-in or 'closed' dot to indicate membership in the interval. Otherwise, we use parentheses, '(' or ')' and an 'open' circle to indicate that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbols $-\infty$ to indicate that the interval extends indefinitely to the left and ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use an appropriate arrow to indicate that the interval extends indefinitely in one (or both) directions.

Definition 9 Interval Notation

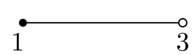
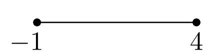
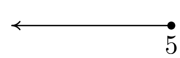
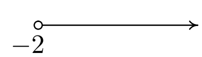
Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

The importance of understanding interval notation in Calculus cannot be overstated. If you don't find yourself getting the hang of it through repeated use, you may need to take the time to just memorize this chart.

As you can glean from the table, for intervals with finite endpoints we start by writing 'left endpoint, right endpoint'. We use square brackets, '[' or ']', if the endpoint is included in the interval. This corresponds to a 'filled-in' or 'closed' dot on the number line to indicate that the number is included in the set. Otherwise, we use parentheses, '(' or ')' that correspond to an 'open' circle which indicates that the endpoint is not part of the set. If the interval does not have finite endpoints, we use the symbol $-\infty$ to indicate that the interval extends indefinitely to the left and the symbol ∞ to indicate that the interval extends indefinitely to the right. Since infinity is a concept, and not a number, we always use parentheses when using these symbols in interval notation, and use the appropriate arrow to indicate that the interval extends indefinitely in one or both directions.

Let's do a few examples to make sure we have the hang of the notation:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid 1 \leq x < 3\}$	$[1, 3)$	
$\{x \mid -1 \leq x \leq 4\}$	$[-1, 4]$	
$\{x \mid x \leq 5\}$	$(-\infty, 5]$	
$\{x \mid x > -2\}$	$(-2, \infty)$	

We defined the intersection and union of arbitrary sets in Definition 4. Recall that the union of two sets consists of the totality of the elements in each of the sets, collected together. For example, if $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$ and $A \cup B = \{1, 2, 3, 4, 6\}$. If $A = [-5, 3)$ and $B = (1, \infty)$, then we can find $A \cap B$ and $A \cup B$ graphically. To find $A \cap B$, we shade the overlap of the two and obtain $A \cap B = (1, 3)$. To find $A \cup B$, we shade each of A and B and describe the resulting shaded region to find $A \cup B = [-5, \infty)$.

While both intersection and union are important, we have more occasion to use union in this text than intersection, simply because most of the sets of real numbers we will be working with are either intervals or are unions of intervals, as the following example illustrates.

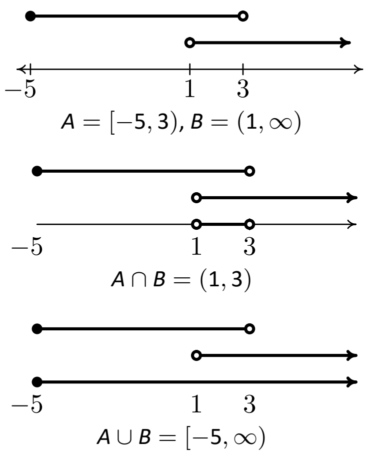


Figure 1.4: Union and intersection of intervals

Example 2 **Expressing sets as unions of intervals**

Express the following sets of numbers using interval notation.

1. $\{x \,|\, x \leq -2 \text{ or } x \geq 2\}$
2. $\{x \,|\, x \neq 3\}$
3. $\{x \,|\, x \neq \pm 3\}$
4. $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$

Solution

1. The best way to proceed here is to graph the set of numbers on the number line and glean the answer from it. The inequality $x \leq -2$ corresponds to the interval $(-\infty, -2]$ and the inequality $x \geq 2$ corresponds to the interval $[2, \infty)$. Since we are looking to describe the real numbers x in one of these *or* the other, we have $\{x \,|\, x \leq -2 \text{ or } x \geq 2\} = (-\infty, -2] \cup [2, \infty)$.

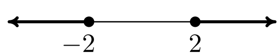


Figure 1.5: The set $(-\infty, -2] \cup [2, \infty)$

2. For the set $\{x \,|\, x \neq 3\}$, we shade the entire real number line except $x = 3$, where we leave an open circle. This divides the real number line into two intervals, $(-\infty, 3)$ and $(3, \infty)$. Since the values of x could be in either one of these intervals *or* the other, we have that $\{x \,|\, x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

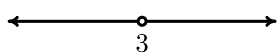


Figure 1.6: The set $(-\infty, 3) \cup (3, \infty)$

3. For the set $\{x \,|\, x \neq \pm 3\}$, we proceed as before and exclude both $x = 3$ and $x = -3$ from our set. This breaks the number line into *three* intervals, $(-\infty, -3)$, $(-3, 3)$ and $(3, \infty)$. Since the set describes real numbers which come from the first, second *or* third interval, we have $\{x \,|\, x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

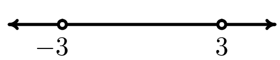


Figure 1.7: The set $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

4. Graphing the set $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\}$, we get one interval, $(-1, 3]$ along with a single number, or point, $\{5\}$. While we *could* express the latter as $[5, 5]$ (Can you see why?), we choose to write our answer as $\{x \,|\, -1 < x \leq 3 \text{ or } x = 5\} = (-1, 3] \cup \{5\}$.

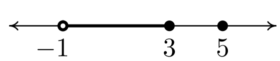
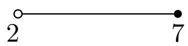
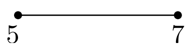
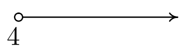


Figure 1.8: The set $(-1, 3] \cup \{5\}$

Exercises 1.1

Problems

1. Fill in the chart below:

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$		
	$[0, 3)$	
		
$\{x \mid -5 < x \leq 0\}$		
	$(-3, 3)$	
		
$\{x \mid x \leq 3\}$		
	$(-\infty, 9)$	
		
$\{x \mid x \geq -3\}$		

In Exercises 2 – 7, find the indicated intersection or union and simplify if possible. Express your answers in interval notation.

2. $(-1, 5] \cap [0, 8)$
3. $(-1, 1) \cup [0, 6]$
4. $(-\infty, 4] \cap (0, \infty)$
5. $(-\infty, 0) \cap [1, 5]$
6. $(-\infty, 0) \cup [1, 5]$
7. $(-\infty, 5] \cap [5, 8)$

In Exercises 8 – 19, write the set using interval notation.

8. $\{x \mid x \neq 5\}$
9. $\{x \mid x \neq -1\}$
10. $\{x \mid x \neq -3,\ 4\}$
11. $\{x \mid x \neq 0,\ 2\}$
12. $\{x \mid x \neq 2,\ -2\}$
13. $\{x \mid x \neq 0,\ \pm 4\}$
14. $\{x \mid x \leq -1 \text{ or } x \geq 1\}$
15. $\{x \mid x < 3 \text{ or } x \geq 2\}$
16. $\{x \mid x \leq -3 \text{ or } x > 0\}$
17. $\{x \mid x \leq 5 \text{ or } x = 6\}$
18. $\{x \mid x > 2 \text{ or } x = \pm 1\}$
19. $\{x \mid -3 < x < 3 \text{ or } x = 4\}$

1.2 Real Number Arithmetic

In this section we list the properties of real number arithmetic. This is meant to be a succinct, targeted review so we'll resist the temptation to wax poetic about these axioms and their subtleties and refer the interested reader to a more formal course in Abstract Algebra. There are two (primary) operations one can perform with real numbers: addition and multiplication.

Definition 10 Properties of Real Number Addition

- **Closure:** For all real numbers a and b, $a + b$ is also a real number.
- **Commutativity:** For all real numbers a and b, $a + b = b + a$.
- **Associativity:** For all real numbers a, b and c, $a + (b + c) = (a + b) + c$.
- **Identity:** There is a real number '0' so that for all real numbers a, $a + 0 = a$.
- **Inverse:** For all real numbers a, there is a real number $-a$ such that $a + (-a) = 0$.
- **Definition of Subtraction:** For all real numbers a and b, $a - b = a + (-b)$.

Next, we give real number multiplication a similar treatment. Recall that we may denote the product of two real numbers a and b a variety of ways: ab, $a \cdot b$, $a(b)$, $(a)(b)$ and so on. We'll refrain from using $a \times b$ for real number multiplication in this text.

Definition 11 Properties of Real Number Multiplication

- **Closure:** For all real numbers a and b, ab is also a real number.
- **Commutativity:** For all real numbers a and b, $ab = ba$.
- **Associativity:** For all real numbers a, b and c, $a(bc) = (ab)c$.
- **Identity:** There is a real number '1' so that for all real numbers a, $a \cdot 1 = a$.
- **Inverse:** For all real numbers $a \neq 0$, there is a real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = 1$.
- **Definition of Division:** For all real numbers a and $b \neq 0$, $a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right)$.

While most students (and some faculty) tend to skip over these properties or give them a cursory glance at best, it is important to realize that the properties stated above are what drive the symbolic manipulation for all of Algebra. When listing a tally of more than two numbers, $1+2+3$ for example, we don't need to specify the order in which those numbers are added. Notice though, try as we might, we can add only two numbers at a time and it is the associative property of addition which assures us that we could organize this sum as $(1+2)+3$ or $1+(2+3)$. This brings up a note about 'grouping symbols'. Recall that parentheses and brackets are used in order to specify which operations are to be performed first. In the absence of such grouping symbols, multiplication (and hence division) is given priority over addition (and hence subtraction). For example, $1+2\cdot 3 = 1+6 = 7$, but $(1+2)\cdot 3 = 3\cdot 3 = 9$. As you may recall, we can 'distribute' the 3 across the addition if we really wanted to do the multiplication first: $(1+2)\cdot 3 = 1\cdot 3 + 2\cdot 3 = 3+6 = 9$. More generally, we have the following.

Definition 12 The Distributive Property and Factoring

For all real numbers *a*, *b* and *c*:

- **Distributive Property:** $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$.
- **Factoring:** $ab+ac = a(b+c)$ and $ac+bc = (a+b)c$.

Warning: A common source of errors for beginning students is the misuse (that is, lack of use) of parentheses. When in doubt, more is better than less: redundant parentheses add clutter, but do not change meaning, whereas writing $2x+1$ when you meant to write $2(x+1)$ is almost guaranteed to cause you to make a mistake. (Even if you're able to proceed correctly in spite of your lack of proper notation, this is the sort of thing that will get you on your grader's bad side, so it's probably best to avoid the problem in the first place.)

It is worth pointing out that we didn't really need to list the Distributive Property both for $a(b+c)$ (distributing from the left) and $(a+b)c$ (distributing from the right), since the commutative property of multiplication gives us one from the other. Also, 'factoring' really is the same equation as the distributive property, just read from right to left. These are the first of many redundancies in this section, and they exist in this review section for one reason only - in our experience, many students *see* these things differently so we will list them as such.

It is hard to overstate the importance of the Distributive Property. For example, in the expression $5(2+x)$, without knowing the value of x, we cannot perform the addition inside the parentheses first; we must rely on the distributive property here to get $5(2+x) = 5\cdot 2 + 5\cdot x = 10+5x$. The Distributive Property is also responsible for combining 'like terms'. Why is $3x+2x = 5x$? Because $3x+2x = (3+2)x = 5x$.

We continue our review with summaries of other properties of arithmetic, each of which can be derived from the properties listed above. First up are properties of the additive identity 0.

The Zero Product Property drives most of the equation solving algorithms in Algebra because it allows us to take complicated equations and reduce them to simpler ones. For example, you may recall that one way to solve $x^2+x-6=0$ is by factoring the left hand side of this equation to get $(x-2)(x+3)=0$. From here, we apply the Zero Product Property and set each factor equal to zero. This yields $x-2=0$ or $x+3=0$ so $x=2$ or $x=-3$. This application to solving equations leads, in turn, to some deep and profound structure theorems in Chapter 4.

The expression $\frac{0}{0}$ is technically an 'indeterminate form' as opposed to being strictly 'undefined' meaning that with Calculus we can make some sense of it in certain situations. We'll talk more about this in Chapter 5.

Theorem 2 Properties of Zero

Suppose a and b are real numbers.

- **Zero Product Property:** $ab=0$ if and only if $a=0$ or $b=0$ (or both)

  Note: This not only says that $0\cdot a=0$ for any real number a, it also says that the *only* way to get an answer of '0' when multiplying two real numbers is to have one (or both) of the numbers be '0' in the first place.

- **Zeros in Fractions:** If $a\neq 0$, $\dfrac{0}{a}=0\cdot\left(\dfrac{1}{a}\right)=0$.

  Note: The quantity $\dfrac{a}{0}$ is undefined.

We now continue with a review of arithmetic with fractions.

It's always worth remembering that division is the same as multiplication by the reciprocal. You'd be surprised how often this comes in handy.

Note: A common denominator is **not** required to **multiply** or **divide** fractions!

Note: A common denominator **is** required to **add or subtract** fractions!

Note: The *only* way to change the denominator is to multiply both it and the numerator by the same nonzero value because we are, in essence, multiplying the fraction by 1.

We reduce fractions by 'cancelling' common factors - this is really just reading the previous property 'from right to left'.**Caution:** We may only cancel common **factors** from both numerator and denominator.

Key Idea 2 Properties of Fractions

Suppose a, b, c and d are real numbers. Assume them to be nonzero whenever necessary; for example, when they appear in a denominator.

- **Identity Properties:** $a=\dfrac{a}{1}$ and $\dfrac{a}{a}=1$.
- **Fraction Equality:** $\dfrac{a}{b}=\dfrac{c}{d}$ if and only if $ad=bc$.
- **Multiplication of Fractions:** $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd}$. In particular: $\dfrac{a}{b}\cdot c=\dfrac{a}{b}\cdot\dfrac{c}{1}=\dfrac{ac}{b}$
- **Division of Fractions:** $\dfrac{a}{b}\Big/\dfrac{c}{d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$.

  In particular: $1\Big/\dfrac{a}{b}=\dfrac{b}{a}$ and $\dfrac{a}{b}/c=\dfrac{a}{b}\Big/\dfrac{c}{1}=\dfrac{a}{b}\cdot\dfrac{1}{c}=\dfrac{a}{bc}$
- **Addition and Subtraction of Fractions:** $\dfrac{a}{b}\pm\dfrac{c}{b}=\dfrac{a\pm c}{b}$.
- **Equivalent Fractions:** $\dfrac{a}{b}=\dfrac{ad}{bd}$, since $\dfrac{a}{b}=\dfrac{a}{b}\cdot 1=\dfrac{a}{b}\cdot\dfrac{d}{d}=\dfrac{ad}{bd}$
- **'Reducing' Fractions:** $\dfrac{a\not{d}}{b\not{d}}=\dfrac{a}{b}$, since $\dfrac{ad}{bd}=\dfrac{a}{b}\cdot\dfrac{d}{d}=\dfrac{a}{b}\cdot 1=\dfrac{a}{b}$.

  In particular, $\dfrac{ab}{b}=a$ since $\dfrac{ab}{b}=\dfrac{ab}{1\cdot b}=\dfrac{a\not{b}}{1\cdot\not{b}}=\dfrac{a}{1}=a$ and $\dfrac{b-a}{a-b}=\dfrac{(-1)\cancel{(a-b)}}{\cancel{(a-b)}}=-1$.

Next up is a review of the arithmetic of 'negatives'. On page 10 we first introduced the dash which we all recognize as the 'negative' symbol in terms of the additive inverse. For example, the number -3 (read 'negative 3') is defined so that $3 + (-3) = 0$. We then defined subtraction using the concept of the additive inverse again so that, for example, $5 - 3 = 5 + (-3)$.

It might be junior high (elementary?) school material, but arithmetic with fractions is one of the most common sources of errors among university students. If you're not comfortable working with fractions, we strongly recommend seeing your instructor (or a tutor) to go over this material until you're completely confident that you understand it. Experience (and even formal educational studies) suggest that your success handling fractions corresponds pretty well with your overall success in passing your Mathematics courses.

Key Idea 3 Properties of Negatives

Given real numbers a and b we have the following.

- **Additive Inverse Properties:** $-a = (-1)a$ and $-(-a) = a$
- **Products of Negatives:** $(-a)(-b) = ab$.
- **Negatives and Products:** $-ab = -(ab) = (-a)b = a(-b)$.
- **Negatives and Fractions:** If b is nonzero, $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$.
- **'Distributing' Negatives:** $-(a + b) = -a - b$ and $-(a - b) = -a + b = b - a$.
- **'Factoring' Negatives:** $-a - b = -(a + b)$ and $b - a = -(a - b)$.

In this text we do not distinguish typographically between the dashes in the expressions '$5 - 3$' and '-3' even though they are mathematically quite different. In the expression '$5 - 3$,' the dash is a *binary* operation (that is, an operation requiring *two* numbers) whereas in '-3', the dash is a *unary* operation (that is, an operation requiring only one number). You might ask, 'Who cares?' Your calculator does - that's who! In the text we can write $-3 - 3 = -6$ but that will not work in your calculator. Instead you'd need to type $^{-}3 - 3$ to get -6 where the first dash comes from the '$+/-$' key.

An important point here is that when we 'distribute' negatives, we do so across addition or subtraction only. This is because we are really distributing a factor of -1 across each of these terms: $-(a+b) = (-1)(a+b) = (-1)(a) + (-1)(b) = (-a)+(-b) = -a-b$. Negatives do not 'distribute' across multiplication: $-(2\cdot3) \neq (-2)\cdot(-3)$. Instead, $-(2\cdot3) = (-2)\cdot(3) = (2)\cdot(-3) = -6$. The same sort of thing goes for fractions: $-\frac{3}{5}$ can be written as $\frac{-3}{5}$ or $\frac{3}{-5}$, but not $\frac{-3}{-5}$. It's about time we did a few examples to see how these properties work in practice.

Example 3 Arithmetic with fractions

Perform the indicated operations and simplify. By 'simplify' here, we mean to have the final answer written in the form $\frac{a}{b}$ where a and b are integers which have no common factors. Said another way, we want $\frac{a}{b}$ in 'lowest terms'.

1. $\frac{1}{4} + \frac{6}{7}$
2. $\frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right)$
3. $\dfrac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)}$
4. $\dfrac{(2(2)+1)(-3-(-3)) - 5(4-7)}{4-2(3)}$
5. $\left(\frac{3}{5}\right)\left(\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)$

Solution

1. It may seem silly to start with an example this basic but experience has taught us not to take much for granted. We start by finding the lowest common denominator and then we rewrite the fractions using that new denominator. Since 4 and 7 are **relatively prime**, meaning they have no

factors in common, the lowest common denominator is $4 \cdot 7 = 28$.

$$\begin{aligned} \frac{1}{4} + \frac{6}{7} &= \frac{1}{4} \cdot \frac{7}{7} + \frac{6}{7} \cdot \frac{4}{4} && \text{Equivalent Fractions} \\ &= \frac{7}{28} + \frac{24}{28} && \text{Multiplication of Fractions} \\ &= \frac{31}{28} && \text{Addition of Fractions} \end{aligned}$$

The result is in lowest terms because 31 and 28 are relatively prime so we're done.

We could have used $12 \cdot 30 \cdot 3 = 1080$ as our common denominator but then the numerators would become unnecessarily large. It's best to use the *lowest* common denominator.

2. We could begin with the subtraction in parentheses, namely $\frac{47}{30} - \frac{7}{3}$, and then subtract that result from $\frac{5}{12}$. It's easier, however, to first distribute the negative across the quantity in parentheses and then use the Associative Property to perform all of the addition and subtraction in one step. The lowest common denominator for all three fractions is 60.

$$\begin{aligned} \frac{5}{12} - \left(\frac{47}{30} - \frac{7}{3}\right) &= \frac{5}{12} - \frac{47}{30} + \frac{7}{3} && \text{Distribute the Negative} \\ &= \frac{5}{12} \cdot \frac{5}{5} - \frac{47}{30} \cdot \frac{2}{2} + \frac{7}{3} \cdot \frac{20}{20} && \text{Equivalent Fractions} \\ &= \frac{25}{60} - \frac{94}{60} + \frac{140}{60} && \text{Multiplication of Fractions} \\ &= \frac{71}{60} && \text{Addition and Subtraction of Fractions} \end{aligned}$$

The numerator and denominator are relatively prime so the fraction is in lowest terms and we have our final answer.

3. What we are asked to simplify in this problem is known as a 'complex' or 'compound' fraction. Simply put, we have fractions within a fraction. The longest division line (also called a 'vinculum') acts as a grouping symbol, quite literally dividing the compound fraction into a numerator (containing fractions) and a denominator (which in this case does not contain fractions):

$$\frac{\frac{12}{5} - \frac{7}{24}}{1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)} = \frac{\left(\frac{12}{5} - \frac{7}{24}\right)}{\left(1 + \left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)}$$

The first step to simplifying a compound fraction like this one is to see if you can simplify the little fractions inside it. There are two ways to proceed. One is to simplify the numerator and denominator separately, and then use the fact that division is the same thing as multiplication by the reciprocal, as follows:

$$\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} = \frac{\left(\frac{12}{5}\cdot\frac{24}{24}-\frac{7}{24}\cdot\frac{5}{5}\right)}{\left(1\cdot\frac{120}{120}+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} \quad \text{Equivalent Fractions}$$

$$= \frac{288/120-35/120}{120/120+84/120} \quad \text{Multiplication of fractions}$$

$$= \frac{253/120}{204/120} \quad \text{Addition and subtraction of fractions}$$

$$= \frac{253}{\cancel{120}}\cdot\frac{\cancel{120}}{204} \quad \text{Division of fractions and cancellation}$$

$$= \frac{253}{204}$$

Since $253 = 11\cdot 23$ and $204 = 2\cdot 2\cdot 3\cdot 17$ have no common factors our result is in lowest terms which means we are done.

While there is nothing wrong with the above approach, we can also use our Equivalent Fractions property to rid ourselves of the 'compound' nature of this fraction straight away. The idea is to multiply both the numerator and denominator by the lowest common denominator of each of the 'smaller' fractions - in this case, $24\cdot 5 = 120$.

$$\frac{\left(\frac{12}{5}-\frac{7}{24}\right)}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)} = \frac{\left(\frac{12}{5}-\frac{7}{24}\right)\cdot 120}{\left(1+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)\right)\cdot 120} \quad \text{Equivalent Fractions}$$

$$= \frac{\left(\frac{12}{5}\right)(120)-\left(\frac{7}{24}\right)(120)}{(1)(120)+\left(\frac{12}{5}\right)\left(\frac{7}{24}\right)(120)} \quad \text{Distributive Property}$$

$$= \frac{\frac{12\cdot 120}{5}-\frac{7\cdot 120}{24}}{120+\frac{12\cdot 7\cdot 120}{5\cdot 24}} \quad \text{Multiply fractions}$$

$$= \frac{\frac{12\cdot 24\cdot\cancel{5}}{\cancel{5}}-\frac{7\cdot 5\cdot\cancel{24}}{\cancel{24}}}{120+\frac{12\cdot 7\cdot\cancel{5}\cdot\cancel{24}}{\cancel{5}\cdot\cancel{24}}} \quad \text{Factor and cancel}$$

$$= \frac{(12\cdot 24)-(7\cdot 5)}{120+(12\cdot 7)}$$

$$= \frac{288-35}{120+84} = \frac{253}{204},$$

which is the same as we obtained above.

4. This fraction may look simpler than the one before it, but the negative signs and parentheses mean that we shouldn't get complacent. Again we note that the division line here acts as a grouping symbol. That is,

$$\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} = \frac{((2(2)+1)(-3-(-3))-5(4-7))}{(4-2(3))}$$

This means that we should simplify the numerator and denominator first, then perform the division last. We tend to what's in parentheses first, giving multiplication priority over addition and subtraction.

$$\begin{aligned}\frac{(2(2)+1)(-3-(-3))-5(4-7)}{4-2(3)} &= \frac{(4+1)(-3+3)-5(-3)}{4-6}\\ &= \frac{(5)(0)+15}{-2}\\ &= \frac{15}{-2}\\ &= -\frac{15}{2} \quad \text{Properties of Negatives}\end{aligned}$$

Since $15 = 3 \cdot 5$ and 2 have no common factors, we are done.

5. In this problem, we have multiplication and subtraction. Multiplication takes precedence so we perform it first. Recall that to multiply fractions, we do *not* need to obtain common denominators; rather, we multiply the corresponding numerators together along with the corresponding denominators. Like the previous example, we have parentheses and negative signs for added fun!

$$\begin{aligned}\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) &= \frac{3\cdot 5}{5\cdot 13}-\frac{4\cdot(-12)}{5\cdot 13} \quad \text{Multiply fractions}\\ &= \frac{15}{65}-\frac{-48}{65}\\ &= \frac{15}{65}+\frac{48}{65} \quad \text{Properties of Negatives}\\ &= \frac{15+48}{65} \quad \text{Add numerators}\\ &= \frac{63}{65}\end{aligned}$$

Since $64 = 3 \cdot 3 \cdot 7$ and $65 = 5 \cdot 13$ have no common factors, our answer $\frac{63}{65}$ is in lowest terms and we are done.

Of the issues discussed in the previous set of examples none causes students more trouble than simplifying compound fractions. We presented two different methods for simplifying them: one in which we simplified the overall numerator and denominator and then performed the division and one in which we removed the compound nature of the fraction at the very beginning. We encourage the reader to go back and use both methods on each of the compound fractions presented. Keep in mind that when a compound fraction is encountered in the rest of the text it will usually be simplified using only one method and we may not choose your favourite method. Feel free to use the other one in your notes.

Next, we review exponents and their properties. Recall that $2 \cdot 2 \cdot 2$ can be written as 2^3 because exponential notation expresses repeated multiplication. In the expression 2^3, 2 is called the **base** and 3 is called the **exponent**. In order to generalize exponents from natural numbers to the integers, and eventually to rational and real numbers, it is helpful to think of the exponent as a count of the number of factors of the base we are multiplying by 1. For instance,

$$2^3 = 1 \cdot (\text{three factors of two}) = 1 \cdot (2 \cdot 2 \cdot 2) = 8.$$

From this, it makes sense that

$$2^0 = 1 \cdot (\text{zero factors of two}) = 1.$$

What about 2^{-3}? The '$-$' in the exponent indicates that we are 'taking away' three factors of two, essentially dividing by three factors of two. So,

$$2^{-3} = 1 \div (\text{three factors of two}) = 1 \div (2 \cdot 2 \cdot 2) = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8}.$$

We summarize the properties of integer exponents below.

Definition 13 Properties of Integer Exponents

Suppose a and b are nonzero real numbers and n and m are integers.

- **Product Rules:** $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$.
- **Quotient Rules:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\frac{a^n}{a^m} = a^{n-m}$.
- **Power Rule:** $(a^n)^m = a^{nm}$.
- **Negatives in Exponents:** $a^{-n} = \frac{1}{a^n}$.

  In particular, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.
- **Zero Powers:** $a^0 = 1$.

  Note: The expression 0^0 is an indeterminate form.[a]
- **Powers of Zero:** For any *natural* number n, $0^n = 0$.

  Note: The expression 0^n for integers $n \leq 0$ is not defined.

[a] See the comment regarding '$\frac{0}{0}$' on page 12.

While it is important the state the Properties of Exponents, it is also equally important to take a moment to discuss one of the most common errors in Algebra. It is true that $(ab)^2 = a^2 b^2$ (which some students refer to as 'distributing' the exponent to each factor) but you **cannot** do this sort of thing with addition. That is, in general, $(a + b)^2 \neq a^2 + b^2$. (For example, take $a = 3$ and $b = 4$.) The same goes for any other powers.

With exponents now in the mix, we can now state the Order of Operations Agreement.

Definition 14 Order of Operations Agreement

When evaluating an expression involving real numbers:

1. Evaluate any expressions in **p**arentheses (or other grouping symbols.)
2. Evaluate **e**xponents.
3. Evaluate **d**ivision and **m**ultiplication as you read from left to right.
4. Evaluate **a**ddition and **s**ubtraction as you read from left to right.

For example, $2+3\cdot 4^2 = 2+3\cdot 16 = 2+48 = 50$. Where students get into trouble is with things like -3^2. If we think of this as $0-3^2$, then it is clear that we evaluate the exponent first: $-3^2 = 0-3^2 = 0-9 = -9$. In general, we interpret $-a^n = -(a^n)$. If we want the 'negative' to also be raised to a power, we must write $(-a)^n$ instead. To summarize, $-3^2 = -9$ but $(-3)^2 = 9$.

Of course, many of the 'properties' we've stated in this section can be viewed as ways to circumvent the order of operations. We've already seen how the distributive property allows us to simplify $5(2+x)$ by performing the indicated multiplication **before** the addition that's in parentheses. Similarly, consider trying to evaluate $2^{30172}\cdot 2^{-30169}$. The Order of Operations Agreement demands that the exponents be dealt with first, however, trying to compute 2^{30172} is a challenge, even for a calculator. One of the Product Rules of Exponents, however, allow us to rewrite this product, essentially performing the multiplication first, to get: $2^{30172-30169} = 2^3 = 8$.

Order of operations follows the "PEDMAS" rule some of you may have encountered.

Example 4 Operations with exponents

Perform the indicated operations and simplify.

1. $\dfrac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2}$
2. $12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5}$
3. $\dfrac{\left(\dfrac{5\cdot 3^{51}}{4^{36}}\right)}{\left(\dfrac{5\cdot 3^{49}}{4^{34}}\right)}$
4. $\dfrac{2\left(\dfrac{5}{12}\right)^{-1}}{1-\left(\dfrac{5}{12}\right)^{-2}}$

Solution

1. We begin working inside parentheses then deal with the exponents before working through the other operations. As we saw in Example 3, the division here acts as a grouping symbol, so we save the division to the end.

$$\begin{aligned}\frac{(4-2)(2\cdot 4)-(4)^2}{(4-2)^2} &= \frac{(2)(8)-(4)^2}{(2)^2} = \frac{(2)(8)-16}{4}\\ &= \frac{16-16}{4} = \frac{0}{4} = 0\end{aligned}$$

2. As before, we simplify what's in the parentheses first, then work our way

through the exponents, multiplication, and finally, the addition.

$$\begin{aligned}
12(-5)(-5+3)^{-4} + 6(-5)^2(-4)(-5+3)^{-5} \\
&= 12(-5)(-2)^{-4} + 6(-5)^2(-4)(-2)^{-5} \\
&= 12(-5)\left(\frac{1}{(-2)^4}\right) + 6(-5)^2(-4)\left(\frac{1}{(-2)^5}\right) \\
&= 12(-5)\left(\frac{1}{16}\right) + 6(25)(-4)\left(\frac{1}{-32}\right) \\
&= (-60)\left(\frac{1}{16}\right) + (-600)\left(\frac{1}{-32}\right) \\
&= \frac{-60}{16} + \left(\frac{-600}{-32}\right) \\
&= \frac{-15 \cdot \cancel{4}}{4 \cdot \cancel{4}} + \frac{-75 \cdot \cancel{8}}{-4 \cdot \cancel{8}} \\
&= \frac{-15}{4} + \frac{-75}{-4} \\
&= \frac{-15}{4} + \frac{75}{4} \\
&= \frac{-15+75}{4} \\
&= \frac{60}{4} \\
&= 15
\end{aligned}$$

3. The Order of Operations Agreement mandates that we work within each set of parentheses first, giving precedence to the exponents, then the multiplication, and, finally the division. The trouble with this approach is that the exponents are so large that computation becomes a trifle unwieldy. What we observe, however, is that the bases of the exponential expressions, 3 and 4, occur in both the numerator and denominator of the compound fraction, giving us hope that we can use some of the Properties of Exponents (the Quotient Rule, in particular) to help us out. Our first step here is to invert and multiply. We see immediately that the 5's cancel after which we group the powers of 3 together and the powers of 4 together and apply the properties of exponents.

$$\begin{aligned}
\frac{\left(\frac{5 \cdot 3^{51}}{4^{36}}\right)}{\left(\frac{5 \cdot 3^{49}}{4^{34}}\right)} &= \frac{5 \cdot 3^{51}}{4^{36}} \cdot \frac{4^{34}}{5 \cdot 3^{49}} &&= \frac{\cancel{5} \cdot 3^{51} \cdot 4^{34}}{\cancel{5} \cdot 3^{49} \cdot 4^{36}} &&= \frac{3^{51}}{3^{49}} \cdot \frac{4^{34}}{4^{36}} \\
&= 3^{51-49} \cdot 4^{34-36} &&= 3^2 \cdot 4^{-2} &&= 3^2 \cdot \left(\frac{1}{4^2}\right) \\
&= 9 \cdot \left(\frac{1}{16}\right) &&= \frac{9}{16}
\end{aligned}$$

4. We have yet another instance of a compound fraction so our first order of business is to rid ourselves of the compound nature of the fraction like we did in Example 3. To do this, however, we need to tend to the exponents first so that we can determine what common denominator is needed to

simplify the fraction.

$$\frac{2\left(\frac{5}{12}\right)^{-1}}{1-\left(\frac{5}{12}\right)^{-2}} = \frac{2\left(\frac{12}{5}\right)}{1-\left(\frac{12}{5}\right)^{2}} = \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{12^2}{5^2}\right)}$$
$$= \frac{\left(\frac{24}{5}\right)}{1-\left(\frac{144}{25}\right)} = \frac{\left(\frac{24}{5}\right)\cdot 25}{\left(1-\frac{144}{25}\right)\cdot 25}$$
$$= \frac{\left(\frac{24\cdot 5\cdot \cancel{5}}{\cancel{5}}\right)}{\left(1\cdot 25 - \frac{144\cdot \cancel{25}}{\cancel{25}}\right)} = \frac{120}{25-144}$$
$$= \frac{120}{-119} = -\frac{120}{119}$$

Since 120 and 119 have no common factors, we are done.

It's important that you understand the difference between the statements $y = \sqrt{x}$ and $y^2 = x$. As we'll discuss in Chapter 2, the equation $y = \sqrt{x}$ defines y as a **function** of x, which means that for each value of $x \geq 0$ there is only one value of y such that $y = \sqrt{x}$. For example, $y = \sqrt{4}$ is equivalent to $y = 2$. On the other hand, there are **two** solutions to $y^2 = x$; namely, $y = \sqrt{x}$ and $y = -\sqrt{x}$. For example, the equation $y^2 = 4$ is equivalent to the two equations $y = 2$ and $y = -2$ (or, more concisely, $y = \pm 2$). Since these two equations are closely related, it's easy to mix them up. The main thing to remember is that $\sqrt{x}$ always denotes the *positive* square root of x.

We close our review of real number arithmetic with a discussion of roots and radical notation. Just as subtraction and division were defined in terms of the inverse of addition and multiplication, respectively, we define roots by undoing natural number exponents.

Definition 15 The principal n^{th} root

Let a be a real number and let n be a natural number. If n is odd, then the **principal n^{th} root** of a (denoted $\sqrt[n]{a}$) is the unique real number satisfying $\left(\sqrt[n]{a}\right)^n = a$. If n is even, $\sqrt[n]{a}$ is defined similarly provided $a \geq 0$ and $\sqrt[n]{a} \geq 0$. The number n is called the **index** of the root and the the number a is called the **radicand**. For $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.

The reasons for the added stipulations for even-indexed roots in Definition 15 can be found in the Properties of Negatives. First, for all real numbers, $x^{\text{even power}} \geq 0$, which means it is never negative. Thus if a is a *negative* real number, there are no real numbers x with $x^{\text{even power}} = a$. This is why if n is even, $\sqrt[n]{a}$ only exists if $a \geq 0$. The second restriction for even-indexed roots is that $\sqrt[n]{a} \geq 0$. This comes from the fact that $x^{\text{even power}} = (-x)^{\text{even power}}$, and we require $\sqrt[n]{a}$ to have just one value. So even though $2^4 = 16$ and $(-2)^4 = 16$, we require $\sqrt[4]{16} = 2$ and ignore -2.

Dealing with odd powers is much easier. For example, $x^3 = -8$ has one and only one real solution, namely $x = -2$, which means not only does $\sqrt[3]{-8}$ exist, there is only one choice, namely $\sqrt[3]{-8} = -2$. Of course, when it comes to solving $x^{5213} = -117$, it's not so clear that there is one and only one real solution, let alone that the solution is $\sqrt[5213]{-117}$. Such pills are easier to swallow once we've thought a bit about such equations graphically, (see Chapter 4) and ultimately, these things come from the completeness property of the real numbers mentioned earlier.

We list properties of radicals below as a 'theorem' since they can be justified using the properties of exponents.

Theorem 3 Properties of Radicals

Let a and b be real numbers and let m and n be natural numbers. If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$
- **Quotient Rule:** $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, provided $b \neq 0$.
- **Power Rule:** $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The proof of Theorem 3 is based on the definition of the principal n^{th} root and the Properties of Exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{ab}$ is the unique real number such that $(\sqrt[n]{ab})^n = ab$. Given that $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab$ as well, it must be the case that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. If n is even, then $\sqrt[n]{ab}$ is the unique non-negative real number such that $(\sqrt[n]{ab})^n = ab$. Note that since n is even, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also non-negative thus $\sqrt[n]{a}\sqrt[n]{b} \geq 0$ as well. Proceeding as above, we find that $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{a}$ is a real number to start with. We leave that as an exercise as well.

We pause here to point out one of the most common errors students make when working with radicals. Obviously $\sqrt{9} = 3$, $\sqrt{16} = 4$ and $\sqrt{9+16} = \sqrt{25} = 5$. Thus we can clearly see that $5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ because we all know that $5 \neq 7$. The authors urge you to **never consider 'distributing' roots or exponents**. It's wrong and no good will come of it because in general $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Since radicals have properties inherited from exponents, they are often written as such. We define rational exponents in terms of radicals in the box below.

Definition 16 Rational exponents

Let a be a real number, let m be an integer and let n be a natural number.

- $a^{\frac{1}{n}} = \sqrt[n]{a}$ whenever $\sqrt[n]{a}$ is a real number. (If n is even we need $a \geq 0$.)
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ whenever $\sqrt[n]{a}$ is a real number.

Things get more complicated once complex numbers are involved. Fortunately (disappointingly?), that's not a can of worms we'll be opening in this course.

It would make life really nice if the rational exponents defined in Definition 16 had all of the same properties that integer exponents have as listed on page 17 - but they don't. Why not? Let's look at an example to see what goes wrong. Consider the Product Rule which says that $(ab)^n = a^n b^n$ and let $a = -16$, $b = -81$ and $n = \frac{1}{4}$. Plugging the values into the Product Rule yields the equation $((-16)(-81))^{1/4} = (-16)^{1/4}(-81)^{1/4}$. The left side of this equation is $1296^{1/4}$ which equals 6 but the right side is undefined because neither root is a real number. Would it help if, when it comes to even roots (as signified by even denominators in the fractional exponents), we ensure that everything they apply to is non-negative? That works for some of the rules - we leave it as an exercise to see which ones - but does not work for the Power Rule.

Consider the expression $\left(a^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(a^{2/3}\right)^{3/2} = a^{\frac{2}{3}\cdot\frac{3}{2}} = a^1 = a$. However, if we substitute $a = -1$ and apply Definition 16, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. Thus in this case we have $\left(a^{2/3}\right)^{3/2} \neq a$ even though all of the roots were defined. It is true, however, that $\left(a^{3/2}\right)^{2/3} = a$ and we leave this for the reader to show. The moral of the story is that when simplifying powers of rational exponents where the base is negative or worse, unknown, it's usually best to rewrite them as radicals.

Example 5 Combining operations

Perform the indicated operations and simplify.

1. $\dfrac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$

2. $\dfrac{2\left(\dfrac{\sqrt{3}}{3}\right)}{1 - \left(\dfrac{\sqrt{3}}{3}\right)^2}$

3. $(\sqrt[3]{-2} - \sqrt[3]{-54})^2$

4. $2\left(\dfrac{9}{4} - 3\right)^{1/3} + 2\left(\dfrac{9}{4}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{9}{4} - 3\right)^{-2/3}$

Solution

1. We begin in the numerator and note that the radical here acts a grouping symbol,[1] so our first order of business is to simplify the radicand.

$$\begin{aligned}
\frac{-(-4) - \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} &= \frac{-(-4) - \sqrt{16 - 4(2)(-3)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - 4(-6)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 - (-24)}}{2(2)} \\
&= \frac{-(-4) - \sqrt{16 + 24}}{2(2)} \\
&= \frac{-(-4) - \sqrt{40}}{2(2)}
\end{aligned}$$

As you may recall, 40 can be factored using a perfect square as $40 = 4 \cdot 10$ so we use the product rule of radicals to write $\sqrt{40} = \sqrt{4 \cdot 10} =$

[1]The line extending horizontally from the square root symbol '$\sqrt{}$' is, you guessed it, another vinculum.

$\sqrt{4}\sqrt{10} = 2\sqrt{10}$. This lets us factor a '2' out of both terms in the numerator, eventually allowing us to cancel it with a factor of 2 in the denominator.

$$\begin{aligned}
\frac{-(-4)-\sqrt{40}}{2(2)} &= \frac{-(-4)-2\sqrt{10}}{2(2)} &&= \frac{4-2\sqrt{10}}{2(2)} \\
&= \frac{2\cdot 2-2\sqrt{10}}{2(2)} &&= \frac{2(2-\sqrt{10})}{2(2)} \\
&= \frac{\cancel{2}(2-\sqrt{10})}{\cancel{2}(2)} &&= \frac{2-\sqrt{10}}{2}
\end{aligned}$$

Since the numerator and denominator have no more common factors,[2] we are done.

2. Once again we have a compound fraction, so we first simplify the exponent in the denominator to see which factor we'll need to multiply by in order to clean up the fraction.

$$\begin{aligned}
\frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2} &= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{(\sqrt{3})^2}{3^2}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{3}{9}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1\cdot\cancel{3}}{3\cdot\cancel{3}}\right)} &&= \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{1}{3}\right)} \\
&= \frac{2\left(\frac{\sqrt{3}}{3}\right)\cdot 3}{\left(1-\left(\frac{1}{3}\right)\right)\cdot 3} &&= \frac{\frac{2\cdot\sqrt{3}\cdot\cancel{3}}{\cancel{3}}}{1\cdot 3-\frac{1\cdot\cancel{3}}{\cancel{3}}} \\
&= \frac{2\sqrt{3}}{3-1} &&= \frac{\cancel{2}\sqrt{3}}{\cancel{2}} = \sqrt{3}
\end{aligned}$$

3. Working inside the parentheses, we first encounter $\sqrt[3]{-2}$. While the -2 isn't a perfect cube,[3] we may think of $-2 = (-1)(2)$. Since $(-1)^3 = -1$, -1 *is* a perfect cube, and we may write $\sqrt[3]{-2} = \sqrt[3]{(-1)(2)} = \sqrt[3]{-1}\sqrt[3]{2} = -\sqrt[3]{2}$. When it comes to $\sqrt[3]{54}$, we may write it as $\sqrt[3]{(-27)(2)} = \sqrt[3]{-27}\sqrt[3]{2} = -3\sqrt[3]{2}$. So,

$$\sqrt[3]{-2} - \sqrt[3]{-54} = -\sqrt[3]{2} - (-3\sqrt[3]{2}) = -\sqrt[3]{2} + 3\sqrt[3]{2}.$$

At this stage, we can simplify $-\sqrt[3]{2} + 3\sqrt[3]{2} = 2\sqrt[3]{2}$. You may remember this as being called 'combining like radicals,' but it is in fact just another application of the distributive property:

$$-\sqrt[3]{2} + 3\sqrt[3]{2} = (-1)\sqrt[3]{2} + 3\sqrt[3]{2} = (-1+3)\sqrt[3]{2} = 2\sqrt[3]{2}.$$

Putting all this together, we get:

$$\begin{aligned}
(\sqrt[3]{-2} - \sqrt[3]{-54})^2 &= (-\sqrt[3]{2} + 3\sqrt[3]{2})^2 &&= (2\sqrt[3]{2})^2 \\
&= 2^2(\sqrt[3]{2})^2 = 4\sqrt[3]{2^2} &&= 4\sqrt[3]{4}
\end{aligned}$$

Since there are no perfect integer cubes which are factors of 4 (apart from 1, of course), we are done.

[2] Do you see why we aren't 'cancelling' the remaining 2's?

[3] Of an integer, that is!

4. We start working in parentheses and get a common denominator to subtract the fractions:

$$\frac{9}{4} - 3 = \frac{9}{4} - \frac{3 \cdot 4}{1 \cdot 4} = \frac{9}{4} - \frac{12}{4} = \frac{-3}{4}$$

Since the denominators in the fractional exponents are odd, we can proceed using the properties of exponents:

$$\begin{aligned}
2\left(\frac{9}{4} - 3\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{9}{4} - 3\right)^{-2/3}
&= 2\left(\frac{-3}{4}\right)^{1/3} + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)^{-2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{-3}\right)^{2/3} \\
&= 2\left(\frac{(-3)^{1/3}}{(4)^{1/3}}\right) + 2\left(\frac{9}{4}\right)\left(\frac{1}{3}\right)\left(\frac{(4)^{2/3}}{(-3)^{2/3}}\right) \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{2 \cdot 9 \cdot 1 \cdot 4^{2/3}}{4 \cdot 3 \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{\cancel{2} \cdot 3 \cdot \cancel{3} \cdot 4^{2/3}}{2 \cdot \cancel{2} \cdot \cancel{3} \cdot (-3)^{2/3}} \\
&= \frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}}
\end{aligned}$$

At this point, we could start looking for common denominators but it turns out that these fractions reduce even further. Since $4 = 2^2$, $4^{1/3} = (2^2)^{1/3} = 2^{2/3}$. Similarly, $4^{2/3} = (2^2)^{2/3} = 2^{4/3}$. The expressions $(-3)^{1/3}$ and $(-3)^{2/3}$ contain negative bases so we proceed with caution and convert them back to radical notation to get: $(-3)^{1/3} = \sqrt[3]{-3} = -\sqrt[3]{3} = -3^{1/3}$ and $(-3)^{2/3} = (\sqrt[3]{-3})^2 = (-\sqrt[3]{3})^2 = (\sqrt[3]{3})^2 = 3^{2/3}$. Hence:

$$\begin{aligned}
\frac{2 \cdot (-3)^{1/3}}{4^{1/3}} + \frac{3 \cdot 4^{2/3}}{2 \cdot (-3)^{2/3}} &= \frac{2 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3 \cdot 2^{4/3}}{2 \cdot 3^{2/3}} \\
&= \frac{2^1 \cdot (-3^{1/3})}{2^{2/3}} + \frac{3^1 \cdot 2^{4/3}}{2^1 \cdot 3^{2/3}} \\
&= 2^{1-2/3} \cdot (-3^{1/3}) + 3^{1-2/3} \cdot 2^{4/3-1} \\
&= 2^{1/3} \cdot (-3^{1/3}) + 3^{1/3} \cdot 2^{1/3} \\
&= -2^{1/3} \cdot 3^{1/3} + 3^{1/3} \cdot 2^{1/3} \\
&= 0
\end{aligned}$$

Exercises 1.2

Problems

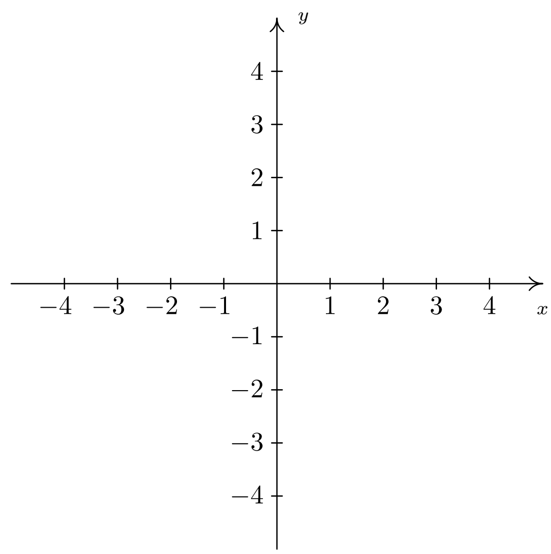
In Exercises 1 – 33, perform the indicated operations and simplify.

1. $5-2+3$
2. $5-(2+3)$
3. $\dfrac{2}{3}-\dfrac{4}{7}$
4. $\dfrac{3}{8}+\dfrac{5}{12}$
5. $\dfrac{5-3}{-2-4}$
6. $\dfrac{2(-3)}{3-(-3)}$
7. $\dfrac{2(3)-(4-1)}{2^2+1}$
8. $\dfrac{4-5.8}{2-2.1}$
9. $\dfrac{1-2(-3)}{5(-3)+7}$
10. $\dfrac{5(3)-7}{2(3)^2-3(3)-9}$
11. $\dfrac{2((-1)^2-1)}{((-1)^2+1)^2}$
12. $\dfrac{(-2)^2-(-2)-6}{(-2)^2-4}$
13. $\dfrac{3-\frac{4}{9}}{-2-(-3)}$
14. $\dfrac{\frac{2}{3}-\frac{4}{5}}{4-\frac{7}{10}}$
15. $\dfrac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}$
16. $\dfrac{1-\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}{1+\left(\frac{5}{3}\right)\left(\frac{3}{5}\right)}$
17. $\left(\dfrac{2}{3}\right)^{-5}$
18. $3^{-1}-4^{-2}$
19. $\dfrac{1+2^{-3}}{3-4^{-1}}$
20. $\dfrac{3\cdot 5^{100}}{12\cdot 5^{98}}$
21. $\sqrt{3^2+4^2}$
22. $\sqrt{12}-\sqrt{75}$
23. $(-8)^{2/3}-9^{-3/2}$
24. $\left(-\frac{32}{9}\right)^{-3/5}$
25. $\sqrt{(3-4)^2+(5-2)^2}$
26. $\sqrt{(2-(-1))^2+\left(\frac{1}{2}-3\right)^2}$
27. $\sqrt{(\sqrt{5}-2\sqrt{5})^2+(\sqrt{18}-\sqrt{8})^2}$
28. $\dfrac{-12+\sqrt{18}}{21}$
29. $\dfrac{-2-\sqrt{(2)^2-4(3)(-1)}}{2(3)}$
30. $\dfrac{-(-4)+\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$
31. $2(-5)(-5+1)^{-1}+(-5)^2(-1)(-5+1)^{-2}$
32. $3\sqrt{2(4)+1}+3(4)\left(\frac{1}{2}\right)(2(4)+1)^{-1/2}(2)$
33. $2(-7)\sqrt[3]{1-(-7)}+(-7)^2\left(\frac{1}{3}\right)(1-(-7))^{-2/3}(-1)$

1.3 The Cartesian Coordinate Plane

The Cartesian Plane is named in honour of René Descartes.

In order to visualize the pure excitement that is Precalculus, we need to unite Algebra and Geometry. Simply put, we must find a way to draw algebraic things. Let's start with possibly the greatest mathematical achievement of all time: the **Cartesian Coordinate Plane**. Imagine two real number lines crossing at a right angle at 0 as drawn below.

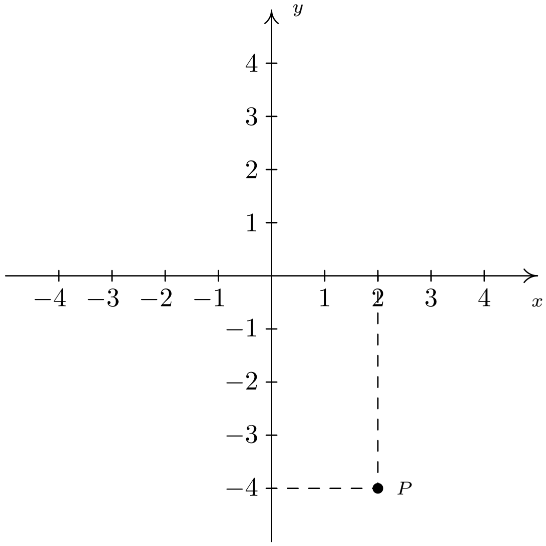


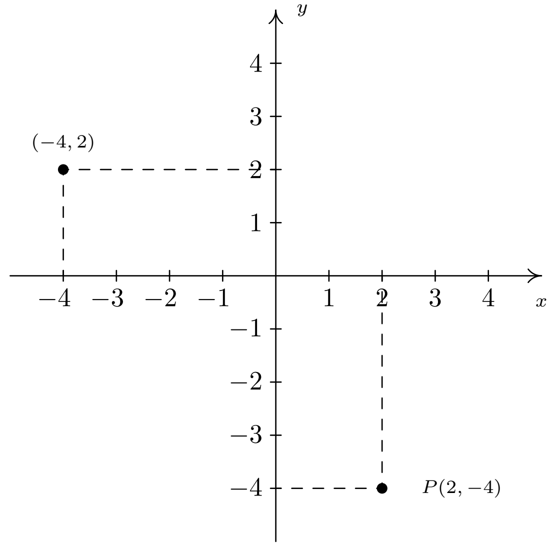
Usually extending off towards infinity is indicated by arrows, but here, the arrows are used to indicate the *direction* of increasing values of *x* and *y*.

The horizontal number line is usually called the ***x*-axis** while the vertical number line is usually called the ***y*-axis**. As with the usual number line, we imagine these axes extending off indefinitely in both directions. Having two number lines allows us to locate the positions of points off of the number lines as well as points on the lines themselves.

The names of the coordinates can vary depending on the context of the application. If, for example, the horizontal axis represented time we might choose to call it the *t*-axis. The first number in the ordered pair would then be the *t*-coordinate.

For example, consider the point *P* on the next page. To use the numbers on the axes to label this point, we imagine dropping a vertical line from the *x*-axis to *P* and extending a horizontal line from the *y*-axis to *P*. This process is sometimes called 'projecting' the point *P* to the *x*- (respectively *y*-) axis. We then describe the point *P* using the **ordered pair** $(2, -4)$. The first number in the ordered pair is called the **abscissa** or ***x*-coordinate** and the second is called the **ordinate** or ***y*-coordinate**. Taken together, the ordered pair $(2, -4)$ comprise the **Cartesian coordinates** of the point *P*. In practice, the distinction between a point and its coordinates is blurred; for example, we often speak of 'the point $(2, -4)$.' We can think of $(2, -4)$ as instructions on how to reach *P* from the **origin** $(0, 0)$ by moving 2 units to the right and 4 units downwards. Notice that the order in the ordered pair is important — if we wish to plot the point $(-4, 2)$, we would move to the left 4 units from the origin and then move upwards 2 units, as below on the right.





When we speak of the Cartesian Coordinate Plane, we mean the set of all possible ordered pairs (x, y) as x and y take values from the real numbers. Below is a summary of important facts about Cartesian coordinates.

Key Idea 4 Important Facts about the Cartesian Coordinate Plane

- (a, b) and (c, d) represent the same point in the plane if and only if $a = c$ and $b = d$.
- (x, y) lies on the x-axis if and only if $y = 0$.
- (x, y) lies on the y-axis if and only if $x = 0$.
- The origin is the point $(0, 0)$. It is the only point common to both axes.

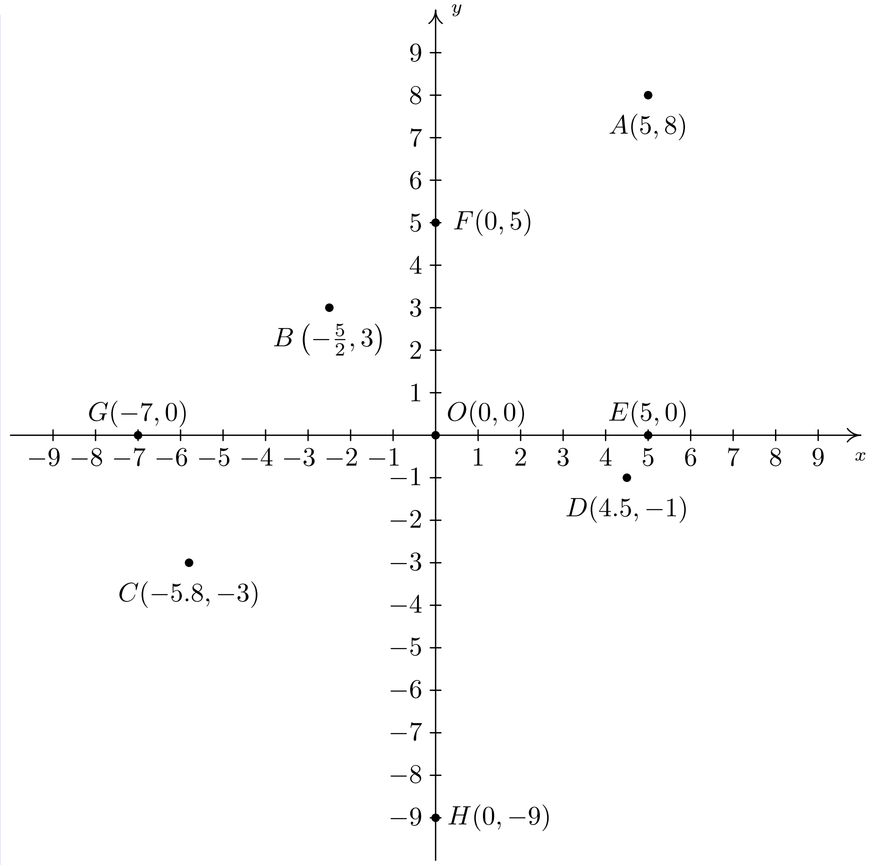
Cartesian coordinates are sometimes referred to as *rectangular coordinates*, to distinguish them from other coordinate systems such as *polar coordinates*.

Example 6 Plotting points in the Cartesian Plane

Plot the following points: $A(5, 8)$, $B\left(-\frac{5}{2}, 3\right)$, $C(-5.8, -3)$, $D(4.5, -1)$, $E(5, 0)$, $F(0, 5)$, $G(-7, 0)$, $H(0, -9)$, $O(0, 0)$.

The letter *O* is almost always reserved for the origin.

Solution To plot these points, we start at the origin and move to the right if the x-coordinate is positive; to the left if it is negative. Next, we move up if the y-coordinate is positive or down if it is negative. If the x-coordinate is 0, we start at the origin and move along the y-axis only. If the y-coordinate is 0 we move along the x-axis only.



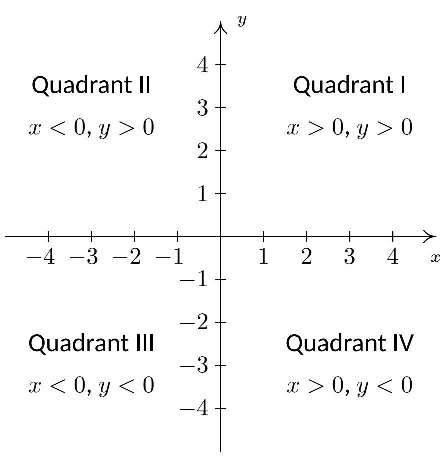


Figure 1.9: The four quadrants of the Cartesian plane

The axes divide the plane into four regions called **quadrants**. They are labelled with Roman numerals and proceed counterclockwise around the plane: see Figure 1.9.

For example, $(1, 2)$ lies in Quadrant I, $(-1, 2)$ in Quadrant II, $(-1, -2)$ in Quadrant III and $(1, -2)$ in Quadrant IV. If a point other than the origin happens to lie on the axes, we typically refer to that point as lying on the positive or negative *x*-axis (if $y = 0$) or on the positive or negative *y*-axis (if $x = 0$). For example, $(0, 4)$ lies on the positive *y*-axis whereas $(-117, 0)$ lies on the negative *x*-axis. Such points do not belong to any of the four quadrants.

One of the most important concepts in all of Mathematics is **symmetry**. There are many types of symmetry in Mathematics, but three of them can be discussed easily using Cartesian Coordinates.

Definition 17 Symmetry in the Cartesian Plane

Two points (a, b) and (c, d) in the plane are said to be

- **symmetric about the *x*-axis** if $a = c$ and $b = -d$
- **symmetric about the *y*-axis** if $a = -c$ and $b = d$
- **symmetric about the origin** if $a = -c$ and $b = -d$

In Figure 1.10, *P* and *S* are symmetric about the *x*-axis, as are *Q* and *R*; *P* and *Q* are symmetric about the *y*-axis, as are *R* and *S*; and *P* and *R* are symmetric about the origin, as are *Q* and *S*.

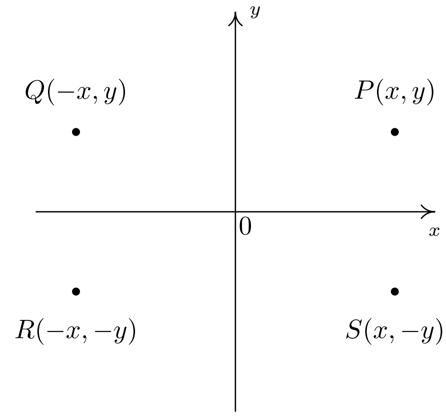


Figure 1.10: The three types of symmetry in the plane

Example 7 **Finding points exhibiting symmetry**

Let *P* be the point $(-2, 3)$. Find the points which are symmetric to *P* about the:

1. *x*-axis
2. *y*-axis
3. origin

Check your answer by plotting the points.

Solution The figure after Definition 17 gives us a good way to think about finding symmetric points in terms of taking the opposites of the *x*- and/or *y*-coordinates of $P(-2, 3)$.

1. To find the point symmetric about the *x*-axis, we replace the *y*-coordinate with its opposite to get $(-2, -3)$.
2. To find the point symmetric about the *y*-axis, we replace the *x*-coordinate with its opposite to get $(2, 3)$.
3. To find the point symmetric about the origin, we replace the *x*- and *y*-coordinates with their opposites to get $(2, -3)$.

   The points are plotted in Figure 1.11.

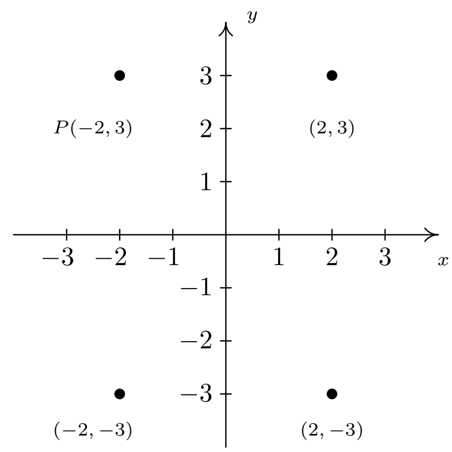


Figure 1.11: The point $P(-2, 3)$ and its three reflections

One way to visualize the processes in the previous example is with the concept of a **reflection**. If we start with our point $(-2, 3)$ and pretend that the *x*-axis is a mirror, then the reflection of $(-2, 3)$ across the *x*-axis would lie at $(-2, -3)$. If we pretend that the *y*-axis is a mirror, the reflection of $(-2, 3)$ across that axis would be $(2, 3)$. If we reflect across the *x*-axis and then the *y*-axis, we would go from $(-2, 3)$ to $(-2, -3)$ then to $(2, -3)$, and so we would end up at the point symmetric to $(-2, 3)$ about the origin. We summarize and generalize this process below.

Key Idea 5 **Reflections in the Cartesian Plane**

To reflect a point (x, y) about the:

- *x*-axis, replace *y* with $-y$.
- *y*-axis, replace *x* with $-x$.
- origin, replace *x* with $-x$ and *y* with $-y$.

1.3.1 Distance in the Plane

Another important concept in Geometry is the notion of length. If we are going to unite Algebra and Geometry using the Cartesian Plane, then we need to develop an algebraic understanding of what distance in the plane means. Suppose we have two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, in the plane. By the **distance** *d* between *P* and *Q*, we mean the length of the line segment joining *P* with *Q*. (Remember, given any two distinct points in the plane, there is a unique line containing both points.) Our goal now is to create an algebraic formula to compute the distance between these two points. Consider the generic situation in Figure 1.12.

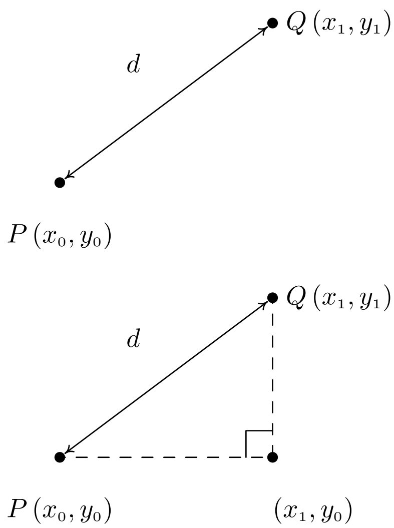


Figure 1.12: Distance between P and Q

With a little more imagination, we can envision a right triangle whose hypotenuse has length d as drawn above on the right. From the latter figure, we see that the lengths of the legs of the triangle are $|x_1 - x_0|$ and $|y_1 - y_0|$ so the Pythagorean Theorem gives us

$$|x_1 - x_0|^2 + |y_1 - y_0|^2 = d^2$$

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = d^2$$

(Do you remember why we can replace the absolute value notation with parentheses?) By extracting the square root of both sides of the second equation and using the fact that distance is never negative, we get

Key Idea 6 The Distance Formula

The distance d between the points $P\,(x_0, y_0)$ and $Q\,(x_1, y_1)$ is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

It is not always the case that the points P and Q lend themselves to constructing such a triangle. If the points P and Q are arranged vertically or horizontally, or describe the exact same point, we cannot use the above geometric argument to derive the distance formula. It is left to the reader in Exercise 16 to verify Equation 6 for these cases.

Example 8 Distance between two points

Find and simplify the distance between $P(-2, 3)$ and $Q(1, -3)$.

Solution

$$\begin{aligned} d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\ &= \sqrt{(1-(-2))^2 + (-3-3)^2} \\ &= \sqrt{9+36} \\ &= 3\sqrt{5} \end{aligned}$$

So the distance is $3\sqrt{5}$.

Example 9 Finding points at a given distance

Find all of the points with x-coordinate 1 which are 4 units from the point $(3, 2)$.

Solution We shall soon see that the points we wish to find are on the line $x = 1$, but for now we'll just view them as points of the form $(1, y)$.

We require that the distance from $(3, 2)$ to $(1, y)$ be 4. The Distance Formula, Equation 6, yields

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \\
4 &= \sqrt{(1-3)^2 + (y-2)^2} \\
4 &= \sqrt{4 + (y-2)^2} \\
4^2 &= \left(\sqrt{4 + (y-2)^2}\right)^2 && \text{squaring both sides} \\
16 &= 4 + (y-2)^2 \\
12 &= (y-2)^2 \\
(y-2)^2 &= 12 \\
y - 2 &= \pm\sqrt{12} && \text{extracting the square root} \\
y - 2 &= \pm 2\sqrt{3} \\
y &= 2 \pm 2\sqrt{3}
\end{aligned}
$$

We obtain two answers: $(1, 2+2\sqrt{3})$ and $(1, 2-2\sqrt{3})$. The reader is encouraged to think about why there are two answers.

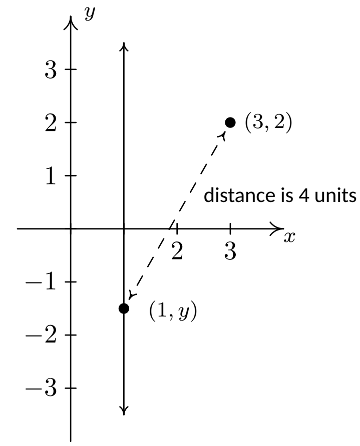


Figure 1.13: Diagram for Example 9

Related to finding the distance between two points is the problem of finding the **midpoint** of the line segment connecting two points. Given two points, $P(x_0, y_0)$ and $Q(x_1, y_1)$, the **midpoint** M of P and Q is defined to be the point on the line segment connecting P and Q whose distance from P is equal to its distance from Q.

If we think of reaching M by going 'halfway over' and 'halfway up' we get the following formula.

Key Idea 7 The Midpoint Formula

The midpoint M of the line segment connecting $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$M = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

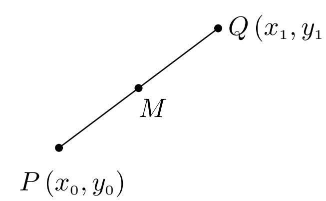


Figure 1.14: The midpoint of a line segment

If we let d denote the distance between P and Q, we leave it as Exercise 17 to show that the distance between P and M is $d/2$ which is the same as the distance between M and Q. This suffices to show that Key Idea 7 gives the coordinates of the midpoint.

Example 10 Finding the midpoint of a line segment

Find the midpoint of the line segment connecting $P(-2, 3)$ and $Q(1, -3)$.

SOLUTION

$$
\begin{aligned}
M &= \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \\
&= \left(\frac{(-2)+1}{2}, \frac{3+(-3)}{2}\right) = \left(-\frac{1}{2}, \frac{0}{2}\right) \\
&= \left(-\frac{1}{2}, 0\right)
\end{aligned}
$$

The midpoint is $\left(-\frac{1}{2}, 0\right)$.

We close with a more abstract application of the Midpoint Formula. We will revisit the following example in Exercise 72 in Section 3.1.

Example 11 **An abstract midpoint problem**

If $a \neq b$, prove that the line $y = x$ equally divides the line segment with endpoints (a, b) and (b, a).

Solution To prove the claim, we use Equation 7 to find the midpoint

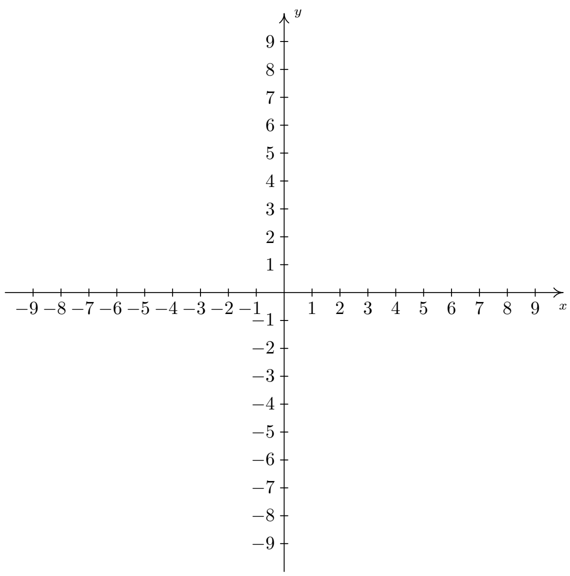
$$\begin{aligned} M &= \left(\frac{a+b}{2}, \frac{b+a}{2}\right) \\ &= \left(\frac{a+b}{2}, \frac{a+b}{2}\right) \end{aligned}$$

Since the x and y coordinates of this point are the same, we find that the midpoint lies on the line $y = x$, as required.

Exercises 1.3

Problems

1. Plot and label the points $A(-3,-7)$, $B(1.3,-2)$, $C(\pi,\sqrt{10})$, $D(0,8)$, $E(-5.5,0)$, $F(-8,4)$, $G(9.2,-7.8)$ and $H(7,5)$ in the Cartesian Coordinate Plane given below.



2. For each point given in Exercise 1 above
   - Identify the quadrant or axis in/on which the point lies.
   - Find the point symmetric to the given point about the x-axis.
   - Find the point symmetric to the given point about the y-axis.
   - Find the point symmetric to the given point about the origin.

In Exercises 3 – 10, find the distance d between the points and the midpoint M of the line segment which connects them.

3. $(1,2)$, $(-3,5)$

4. $(3,-10)$, $(-1,2)$

5. $\left(\frac{1}{2},4\right)$, $\left(\frac{3}{2},-1\right)$

6. $\left(-\frac{2}{3},\frac{3}{2}\right)$, $\left(\frac{7}{3},2\right)$

7. $\left(\frac{24}{5},\frac{6}{5}\right)$, $\left(-\frac{11}{5},-\frac{19}{5}\right)$.

8. $\left(\sqrt{2},\sqrt{3}\right)$, $\left(-\sqrt{8},-\sqrt{12}\right)$

9. $\left(2\sqrt{45},\sqrt{12}\right)$, $\left(\sqrt{20},\sqrt{27}\right)$.

10. $(0,0)$, (x,y)

11. Find all of the points of the form $(x,-1)$ which are 4 units from the point $(3,2)$.

12. Find all of the points on the y-axis which are 5 units from the point $(-5,3)$.

13. Find all of the points on the x-axis which are 2 units from the point $(-1,1)$.

14. Find all of the points of the form $(x,-x)$ which are 1 unit from the origin.

15. Let's assume for a moment that we are standing at the origin and the positive y-axis points due North while the positive x-axis points due East. Our Sasquatch-o-meter tells us that Sasquatch is 3 miles West and 4 miles South of our current position. What are the coordinates of his position? How far away is he from us? If he runs 7 miles due East what would his new position be?

16. Verify the Distance Formula 6 for the cases when:
    (a) The points are arranged vertically. (Hint: Use $P(a,y_0)$ and $Q(a,y_1)$.)
    (b) The points are arranged horizontally. (Hint: Use $P(x_0,b)$ and $Q(x_1,b)$.)
    (c) The points are actually the same point. (You shouldn't need a hint for this one.)

17. Verify the Midpoint Formula by showing the distance between $P(x_1,y_1)$ and M and the distance between M and $Q(x_2,y_2)$ are both half of the distance between P and Q.

18. Show that the points A, B and C below are the vertices of a right triangle.
    (a) $A(-3,2)$, $B(-6,4)$, and $C(1,8)$
    (b) $A(-3,1)$, $B(4,0)$ and $C(0,-3)$

19. Find a point $D(x,y)$ such that the points $A(-3,1)$, $B(4,0)$, $C(0,-3)$ and D are the corners of a square. Justify your answer.

20. Discuss with your classmates how many numbers are in the interval $(0,1)$.

21. The world is not flat. (There are those who disagree with this statement. Look them up on the Internet some time when you're bored.) Thus the Cartesian Plane cannot possibly be the end of the story. Discuss with your classmates how you would extend Cartesian Coordinates to represent the three dimensional world. What would the Distance and Midpoint formulas look like, assuming those concepts make sense at all?

1.4 Complex Numbers

We conclude our first chapter with a review the set of **Complex Numbers**. As you may recall, the complex numbers fill an algebraic gap left by the real numbers. There is no real number x with $x^2 = -1$, since for any real number $x^2 \geq 0$. However, we could formally extract square roots and write $x = \pm\sqrt{-1}$. We build the complex numbers by relabelling the quantity $\sqrt{-1}$ as i, the unfortunately misnamed **imaginary unit**.[4] The number i, while not a real number, is defined so that it plays along well with real numbers and acts very much like any other radical expression. For instance, $3(2i) = 6i$, $7i - 3i = 4i$, $(2-7i)+(3+4i) = 5-3i$, and so forth. The key properties which distinguish i from the real numbers are listed below.

Historically, the lack of solutions to the equation $x^2 = -1$ had nothing to do with the development of the complex numbers. Until the 19th century, equations such as $x^2 = -1$ would have been considered in the context of the analytic geometry of Descartes. The lack of solutions simply indicated that the graph $y = x^2$ did not intersect the line $y = -1$. The more remarkable case was that of *cubic* equations, of the form $x^3 = ax + b$. In this case a **real** solution is *guaranteed*, but there are cases where one needs **complex** numbers to find it! For details, see the excellent book *Visual Complex Analysis*, by Tristan Needham.

Definition 18 The imaginary unit

The imaginary unit i satisfies the two following properties:

1. $i^2 = -1$
2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$

Note the use of the indefinite article 'a'. Whatever beast is chosen to be i, $-i$ is the other square root of -1.

Property 1 in Definition 18 establishes that i does act as a square root of -1, and property 2 establishes what we mean by the 'principal square root' of a negative real number. In property 2, it is important to remember the restriction on c. For example, it is perfectly acceptable to say $\sqrt{-4} = i\sqrt{4} = i(2) = 2i$. However, $\sqrt{-(-4)} \neq i\sqrt{-4}$, otherwise, we'd get

$$2 = \sqrt{4} = \sqrt{-(-4)} = i\sqrt{-4} = i(2i) = 2i^2 = 2(-1) = -2,$$

which is unacceptable. The moral of this story is that the general properties of radicals do not apply for even roots of negative quantities. With Definition 18 in place, we are now in position to define the **complex numbers**.

Definition 19 Complex number

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is the imaginary unit. The set of complex numbers is denoted $\mathbb{C}$.

To use the language of Section 1.1.1, $\mathbb{R} \subseteq \mathbb{C}$.

Complex numbers include things you'd normally expect, like $3 + 2i$ and $\frac{2}{5} - i\sqrt{3}$. However, don't forget that a or b could be zero, which means numbers like $3i$ and 6 are also complex numbers. In other words, don't forget that the complex numbers *include* the real numbers, so 0 and $\pi - \sqrt{21}$ are both considered complex numbers. The arithmetic of complex numbers is as you would expect. The only things you need to remember are the two properties in Definition 18. The next example should help recall how these animals behave.

Example 12 Arithmetic with complex numbers

Perform the indicated operations.

[4] Some Technical Mathematics textbooks label it 'j'. While it carries the adjective 'imaginary', these numbers have essential real-world implications. For example, every electronic device owes its existence to the study of 'imaginary' numbers.

1. $(1-2i)-(3+4i)$
2. $(1-2i)(3+4i)$
3. $\dfrac{1-2i}{3-4i}$
4. $\sqrt{-3}\sqrt{-12}$
5. $\sqrt{(-3)(-12)}$
6. $(x-[1+2i])(x-[1-2i])$

Solution

1. As mentioned earlier, we treat expressions involving i as we would any other radical. We distribute and combine like terms:

$$\begin{aligned} (1-2i)-(3+4i) &= 1-2i-3-4i && \text{Distribute} \\ &= -2-6i && \text{Gather like terms} \end{aligned}$$

   Technically, we'd have to rewrite our answer $-2-6i$ as $(-2)+(-6)i$ to be (in the strictest sense) 'in the form $a+bi$'. That being said, even pedants have their limits, and we'll consider $-2-6i$ good enough.

2. Using the Distributive Property (a.k.a. F.O.I.L.), we get

$$\begin{aligned} (1-2i)(3+4i) &= (1)(3)+(1)(4i)-(2i)(3)-(2i)(4i) && \text{F.O.I.L.} \\ &= 3+4i-6i-8i^2 \\ &= 3-2i-8(-1) && i^2=-1 \\ &= 3-2i+8 \\ &= 11-2i \end{aligned}$$

3. How in the world are we supposed to simplify $\frac{1-2i}{3-4i}$? Well, we deal with the denominator $3-4i$ as we would any other denominator containing two terms, one of which is a square root: we and multiply both numerator and denominator by $3+4i$, the (complex) conjugate of $3-4i$. Doing so produces

$$\begin{aligned} \frac{1-2i}{3-4i} &= \frac{(1-2i)(3+4i)}{(3-4i)(3+4i)} && \text{Equivalent Fractions} \\ &= \frac{3+4i-6i-8i^2}{9-16i^2} && \text{F.O.I.L.} \\ &= \frac{3-2i-8(-1)}{9-16(-1)} && i^2=-1 \\ &= \frac{11-2i}{25} \\ &= \frac{11}{25}-\frac{2}{25}i \end{aligned}$$

4. We use property 2 of Definition 18 first, then apply the rules of radicals applicable to real numbers to get $\sqrt{-3}\sqrt{-12} = \left(i\sqrt{3}\right)\left(i\sqrt{12}\right) = i^2\sqrt{3\cdot 12} = -\sqrt{36} = -6$.

5. We adhere to the order of operations here and perform the multiplication before the radical to get $\sqrt{(-3)(-12)} = \sqrt{36} = 6$.

6. We can brute force multiply using the distributive property and see that

$$\begin{aligned} (x - [1+2i])(x - [1-2i]) &= x^2 - x[1-2i] - x[1+2i] + [1-2i][1+2i] && \text{F.O.I.L.} \\ &= x^2 - x + 2ix - x - 2ix + 1 - 2i + 2i - 4i^2 && \text{Distribute} \\ &= x^2 - 2x + 1 - 4(-1) && \text{Gather like terms} \\ &= x^2 - 2x + 5 && i^2 = -1 \end{aligned}$$

This type of factoring will be revisited in Section 4.4.

In the previous example, we used the idea of a 'conjugate' to divide two complex numbers. (You may recall using conjugates to rationalize expressions involving square roots.) More generally, the **complex conjugate** of a complex number $a + bi$ is the number $a - bi$. The notation commonly used for complex conjugation is a 'bar': $\overline{a + bi} = a - bi$. For example, $\overline{3 + 2i} = 3 - 2i$ and $\overline{3 - 2i} = 3 + 2i$. To find $\overline{6}$, we note that $\overline{6} = \overline{6 + 0i} = 6 - 0i = 6$, so $\overline{6} = 6$. Similarly, $\overline{4i} = -4i$, since $\overline{4i} = \overline{0 + 4i} = 0 - 4i = -4i$. Note that $\overline{3 + \sqrt{5}} = 3 + \sqrt{5}$, not $3 - \sqrt{5}$, since $\overline{3 + \sqrt{5}} = \overline{3 + \sqrt{5} + 0i} = 3 + \sqrt{5} - 0i = 3 + \sqrt{5}$. Here, the conjugation specified by the 'bar' notation involves reversing the sign before $i = \sqrt{-1}$, not before $\sqrt{5}$. The properties of the conjugate are summarized in the following theorem.

Theorem 4 Properties of the Complex Conjugate

Let z and w be complex numbers.

- $\overline{\overline{z}} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $\overline{z^n} = (\overline{z})^n$, for any natural number n
- z is a real number if and only if $\overline{z} = z$.

Essentially, Theorem 4 says that complex conjugation works well with addition, multiplication and powers. The proofs of these properties can best be achieved by writing out $z = a + bi$ and $w = c + di$ for real numbers a, b, c and d. Next, we compute the left and right sides of each equation and verify that they are the same.

The proof of the first property is a very quick exercise. To prove the second property, we compare $\overline{z + w}$ with $\overline{z} + \overline{w}$. We have $\overline{z} + \overline{w} = \overline{a + bi} + \overline{c + di} = a - bi + c - di$. To find $\overline{z + w}$, we first compute

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$

so

$$\overline{z + w} = \overline{(a + c) + (b + d)i} = (a+c)-(b+d)i = a+c-bi-di = a-bi+c-di = \overline{z}+\overline{w}$$

As such, we have established $\overline{z+w} = \overline{z} + \overline{w}$. The proof for multiplication works similarly. The proof that the conjugate works well with powers can be viewed as a repeated application of the product rule, and is best proved using a technique called Mathematical Induction. The last property is a characterization of real numbers. If z is real, then $z = a + 0i$, so $\overline{z} = a - 0i = a = z$. On the other hand, if $z = \overline{z}$, then $a + bi = a - bi$ which means $b = -b$ so $b = 0$. Hence, $z = a + 0i = a$ and is real.

Proof by Mathematical Induction is usually taught in Math 2000.

We now consider the problem of solving quadratic equations. Consider $x^2 - 2x + 5 = 0$. The discriminant $b^2 - 4ac = -16$ is negative, so we know by Theorem 17 there are no *real* solutions, since the Quadratic Formula would involve the term $\sqrt{-16}$. Complex numbers, however, are built just for such situations, so we can go ahead and apply the Quadratic Formula to get:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

We're assuming some prior familiarity on the part of the reader where quadratic equations are concerned. If you feel that it would be unfair to tackle quadratic equations with complex solutions before the case of real solutions has been properly addressed, you may want to briefly skip ahead to Section 3.3.

Example 13 Finding complex solutions

Find the complex solutions to the following equations.

1. $\dfrac{2x}{x+1} = x + 3$
2. $2t^4 = 9t^2 + 5$
3. $z^3 + 1 = 0$

Solution

Remember, all real numbers are complex numbers, so 'complex solutions' means both real and non-real answers.

1. Clearing fractions yields a quadratic equation so we collect all terms on one side and apply the Quadratic Formula.

$$\begin{aligned} \frac{2x}{x+1} &= x+3 \\ 2x &= (x+3)(x+1) && \text{Multiply by } (x+1) \text{ to clear denominators} \\ 2x &= x^2 + x + 3x + 3 && \text{F.O.I.L.} \\ 2x &= x^2 + 4x + 3 && \text{Gather like terms} \\ 0 &= x^2 + 2x + 3 && \text{Subtract } 2x \end{aligned}$$

From here, we apply the Quadratic Formula

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} && \text{Quadratic Formula} \\ &= \frac{-2 \pm \sqrt{-8}}{2} && \text{Simplify} \\ &= \frac{-2 \pm i\sqrt{8}}{2} && \text{Definition of } i \\ &= \frac{-2 \pm i2\sqrt{2}}{2} && \text{Product Rule for Radicals} \\ &= \frac{\cancel{2}(-1 \pm i\sqrt{2})}{\cancel{2}} && \text{Factor and reduce} \\ &= -1 \pm i\sqrt{2} \end{aligned}$$

We get two answers: $x = -1 + i\sqrt{2}$ and its conjugate $x = -1 - i\sqrt{2}$. Checking both of these answers reviews all of the salient points about complex number arithmetic and is therefore strongly encouraged.

2. Since we have three terms, and the exponent on one term ('4' on t^4) is exactly twice the exponent on the other ('2' on t^2), we have a Quadratic in Disguise. We proceed accordingly.

$$\begin{array}{rcll} 2t^4 &=& 9t^2+5 & \\ 2t^4-9t^2-5 &=& 0 & \text{Subtract } 9t^2 \text{ and } 5 \\ (2t^2+1)(t^2-5) &=& 0 & \text{Factor} \\ 2t^2+1=0 & \text{or} & t^2=5 & \text{Zero Product Property} \end{array}$$

From $2t^2 + 1 = 0$ we get $2t^2 = -1$, or $t^2 = -\frac{1}{2}$. We extract square roots as follows:

$$t = \pm\sqrt{-\frac{1}{2}} = \pm i\sqrt{\frac{1}{2}} = \pm i\frac{\sqrt{1}}{\sqrt{2}} = \pm i\frac{1}{\sqrt{2}} = \pm\frac{i\sqrt{2}}{2},$$

where we have rationalized the denominator per convention. From $t^2 = 5$, we get $t = \pm\sqrt{5}$. In total, we have four complex solutions - two real: $t = \pm\sqrt{5}$ and two non-real: $t = \pm\frac{i\sqrt{2}}{2}$.

3. To find the *real* solutions to $z^3 + 1 = 0$, we can subtract the 1 from both sides and extract cube roots: $z^3 = -1$, so $z = \sqrt[3]{-1} = -1$. It turns out there are two more non-real complex number solutions to this equation. To get at these, we factor:

$$\begin{array}{rcll} z^3+1 &=& 0 & \\ (z+1)(z^2-z+1) &=& 0 & \text{Factor (Sum of Two Cubes)} \\ z+1=0 & \text{or} & z^2-z+1=0 & \end{array}$$

From $z + 1 = 0$, we get our real solution $z = -1$. From $z^2 - z + 1 = 0$, we apply the Quadratic Formula to get:

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus we get *three* solutions to $z^3 + 1 = 0$ - one real: $z = -1$ and two non-real: $z = \frac{1 \pm i\sqrt{3}}{2}$. As always, the reader is encouraged to test their algebraic mettle and check these solutions.

It is no coincidence that the non-real solutions to the equations in Example 13 appear in complex conjugate pairs. Any time we use the Quadratic Formula to solve an equation with real coefficients, the answers will form a complex conjugate pair owing to the $\pm$ in the Quadratic Formula. This leads us to a generalization of Theorem 17 which we state on the next page.

Theorem 5 Discriminant Theorem

Given a Quadratic Equation $AX^2 + BX + C = 0$, where A, B and C are real numbers, let $D = B^2 - 4AC$ be the discriminant.

- If $D > 0$, there are two distinct real number solutions to the equation.
- If $D = 0$, there is one (repeated) real number solution.

  Note: 'Repeated' here comes from the fact that 'both' solutions $\frac{-B\pm 0}{2A}$ reduce to $-\frac{B}{2A}$.
- If $D < 0$, there are two non-real solutions which form a complex conjugate pair.

We will have much more to say about complex solutions to equations in Section 4.4 and we will revisit Theorem 5 then.

Exercises 1.4

Problems

In Exercises 1 – 10, use the given complex numbers z and w to find and simplify the following:

- $z+w$
- zw
- z^2
- $\frac{1}{z}$
- $\frac{z}{w}$
- $\frac{w}{z}$
- $\bar{z}$
- $z\bar{z}$
- $(\bar{z})^2$

1. $z=2+3i$, $w=4i$
2. $z=1+i$, $w=-i$
3. $z=i$, $w=-1+2i$
4. $z=4i$, $w=2-2i$
5. $z=3-5i$, $w=2+7i$
6. $z=-5+i$, $w=4+2i$
7. $z=\sqrt{2}-i\sqrt{2}$, $w=\sqrt{2}+i\sqrt{2}$
8. $z=1-i\sqrt{3}$, $w=-1-i\sqrt{3}$
9. $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $w=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$
10. $z=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $w=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$

In Exercises 11 – 20, simplify the quantity.

11. $\sqrt{-49}$
12. $\sqrt{-9}$
13. $\sqrt{-25}\sqrt{-4}$
14. $\sqrt{(-25)(-4)}$
15. $\sqrt{-9}\sqrt{-16}$
16. $\sqrt{(-9)(-16)}$
17. $\sqrt{-(-9)}$
18. $-\sqrt{(-9)}$

We know that $i^2=-1$ which means $i^3=i^2\cdot i=(-1)\cdot i=-i$ and $i^4=i^2\cdot i^2=(-1)(-1)=1$. In Exercises 19 – 28, use this information to simplify the given power of i.

19. i^5
20. i^6
21. i^7
22. i^8
23. i^{15}
24. i^{26}
25. i^{117}
26. i^{304}

In Exercises 27 – 35, find all complex solutions.

27. $3x^2+6=4x$
28. $15t^2+2t+5=3t(t^2+1)$
29. $3y^2+4=y^4$
30. $\frac{2}{1-w}=w$
31. $\frac{y}{3}-\frac{3}{y}=y$
32. $\frac{x^3}{2x-1}=\frac{x}{3}$
33. $x=\frac{2}{\sqrt{5}-x}$
34. $\frac{5y^4+1}{y^2-1}=3y^2$
35. $z^4=16$
36. Multiply and simplify: $\left(x-[3-i\sqrt{23}]\right)\left(x-[3+i\sqrt{23}]\right)$

2: Relations and Functions

2.1 Relations

From one point of view, all of Precalculus can be thought of as studying sets of points in the plane. With the Cartesian Plane now fresh in our memory we can discuss those sets in more detail and as usual, we begin with a definition.

> **Definition 20 Relations in the Cartesian Plane**
>
> A **relation** is a set of points in the plane.

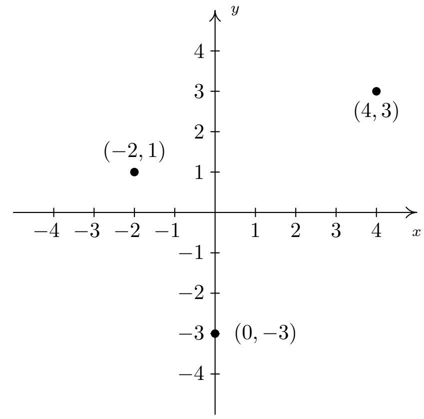


Figure 2.1: The graph of the relation $R = \{(-2,1),(4,3),(0,-3)\}$

Since relations are sets, we can describe them using the techniques presented in Section 1.1. That is, we can describe a relation verbally, using the roster method, or using set-builder notation. Since the elements in a relation are points in the plane, we often try to describe the relation graphically or algebraically as well. Depending on the situation, one method may be easier or more convenient to use than another. As an example, consider the relation $R = \{(-2,1),(4,3),(0,-3)\}$. As written, R is described using the roster method. Since R consists of points in the plane, we follow our instinct and plot the points. Doing so produces the **graph** of R: see Figure 2.1.

In the following example, we graph a variety of relations.

Example 14 Graphing relations

Graph the following relations.

1. $A = \{(0,0),(-3,1),(4,2),(-3,2)\}$
2. $HLS_1 = \{(x,3) \mid -2 \leq x \leq 4\}$
3. $HLS_2 = \{(x,3) \mid -2 \leq x < 4\}$
4. $V = \{(3,y) \mid y \text{ is a real number}\}$
5. $H = \{(x,y) \mid y = -2\}$
6. $R = \{(x,y) \mid 1 < y \leq 3\}$

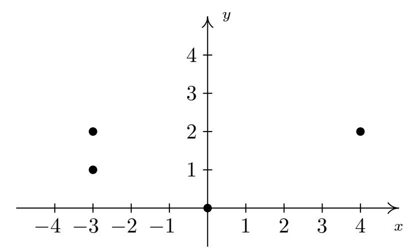


Figure 2.2: The graph of A

Listing the points in a line segment is *really* impossible. The interested reader is encouraged to research **countable** versus **uncountable** sets.

Solution

1. To graph A, we simply plot all of the points which belong to A, as shown below on the left.

2. Don't let the notation in this part fool you. The name of this relation is HLS_1, just like the name of the relation in number 1 was A. The letters and numbers are just part of its name, just like the numbers and letters of the phrase 'King George III' were part of George's name. In words, $\{(x,3) \mid -2 \leq x \leq 4\}$ reads 'the set of points $(x,3)$ such that $-2 \leq x \leq 4$.' All of these points have the same y-coordinate, 3, but the x-coordinate is allowed to vary between -2 and 4, inclusive. Some of the points which belong to HLS_1 include some friendly points like: $(-2,3)$, $(-1,3)$, $(0,3)$, $(1,3)$, $(2,3)$, $(3,3)$, and $(4,3)$. However, HLS_1 also contains the points $(0.829,3)$, $\left(-\frac{5}{6},3\right)$, $(\sqrt{\pi},3)$, and so on. It is impossible to list all of these points, which is why the variable x is used. Plotting several friendly representative points should convince you that HLS_1 describes the horizontal line segment from the point $(-2,3)$ up to and including the point $(4,3)$.

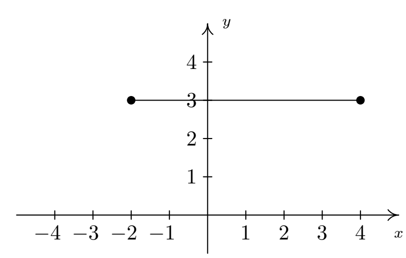


Figure 2.3: The graph of HLS_1

3. HLS_2 is hauntingly similar to HLS_1. In fact, the only difference between the two is that instead of '$-2 \leq x \leq 4$' we have '$-2 \leq x < 4$'. This means that we still get a horizontal line segment which includes $(-2,3)$ and extends to $(4,3)$, but we do *not* include $(4,3)$ because of the strict

inequality $x < 4$. How do we denote this on our graph? It is a common mistake to make the graph start at $(-2, 3)$ end at $(3, 3)$ as pictured below on the left. The problem with this graph is that we are forgetting about the points like $(3.1, 3)$, $(3.5, 3)$, $(3.9, 3)$, $(3.99, 3)$, and so forth. There is no real number that comes 'immediately before' 4, so to describe the set of points we want, we draw the horizontal line segment starting at $(-2, 3)$ and draw an open circle at $(4, 3)$ as depicted below on the right.

4. Next, we come to the relation *V*, described as the set of points $(3, y)$ such that *y* is a real number. All of these points have an *x*-coordinate of 3, but the *y*-coordinate is free to be whatever it wants to be, without restriction. Plotting a few 'friendly' points of *V* should convince you that all the points of *V* lie on the vertical line $x = 3$. Since there is no restriction on the *y*-coordinate, we put arrows on the end of the portion of the line we draw to indicate it extends indefinitely in both directions. The graph of *V* is below on the left.

5. Though written slightly differently, the relation $H = \{(x, y) \mid y = -2\}$ is similar to the relation *V* above in that only one of the coordinates, in this case the *y*-coordinate, is specified, leaving *x* to be 'free'. Plotting some representative points gives us the horizontal line $y = -2$.

6. For our last example, we turn to $R = \{(x, y) \mid 1 < y \leq 3\}$. As in the previous example, *x* is free to be whatever it likes. The value of *y*, on the other hand, while not completely free, is permitted to roam between 1 and 3 excluding 1, but including 3. After plotting some friendly elements of *R*, it should become clear that *R* consists of the region between the horizontal lines $y = 1$ and $y = 3$. Since *R* requires that the *y*-coordinates be greater than 1, but not equal to 1, we dash the line $y = 1$ to indicate that those points do not belong to *R*.

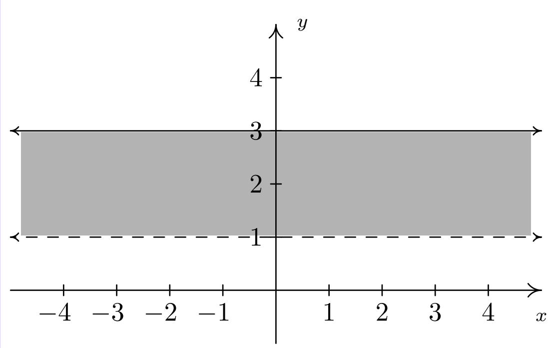
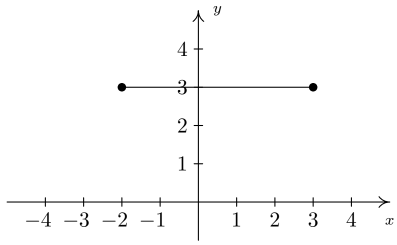
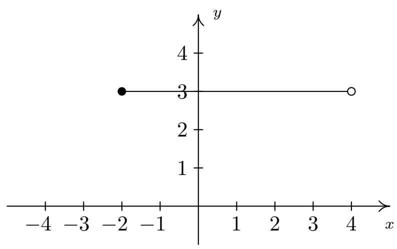


Figure 2.4: The graph of *R*



This is NOT the correct graph of HLS_2



The graph of HLS_2

Figure 2.5: Getting the right graph for HLS_2

When we say you should plot some points in the relation *H*, the word 'some' is a relative term. It may take 5, 10, or 50 points until you see the pattern, depending on the relation.

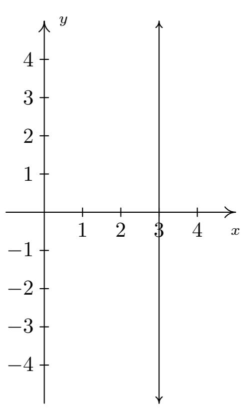


Figure 2.6: The graph of *V*

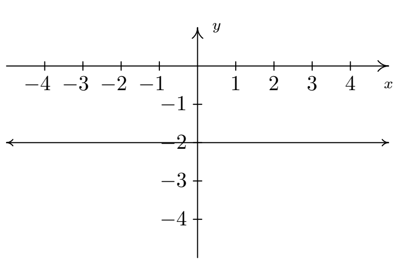


Figure 2.7: The graph of *H*

The relations *V* and *H* in the previous example lead us to our final way to describe relations: **algebraically**. We can more succinctly describe the points in *V* as those points which satisfy the equation '$x = 3$'. Most likely, you have seen equations like this before. Depending on the context, '$x = 3$' could mean we have solved an equation for *x* and arrived at the solution $x = 3$. In this case, however, '$x = 3$' describes a set of points in the plane whose *x*-coordinate is 3. Similarly, the relation *H* above can be described by the equation '$y = -2$'. At some point in your mathematical upbringing, you probably learned the following.

> **Key Idea 8 Equations of Vertical and Horizontal Lines**
>
> - The graph of the equation $x = a$ is a **vertical line** through $(a, 0)$.
> - The graph of the equation $y = b$ is a **horizontal line** through $(0, b)$.

Given that the very simple equations $x = a$ and $y = b$ produced lines, it's natural to wonder what shapes other equations might yield. Thus our next objective is to study the graphs of equations in a more general setting as we continue to unite Algebra and Geometry.

2.1.1 Graphs of Equations

In this section, we delve more deeply into the connection between Algebra and Geometry by focusing on graphing relations described by equations. The main idea of this section is the following.

> **Key Idea 9 The Fundamental Graphing Principle**
>
> The graph of an equation is the set of points which satisfy the equation. That is, a point (x, y) is on the graph of an equation if and only if x and y satisfy the equation.

Here, 'x and y satisfy the equation' means 'x and y make the equation true'. It is at this point that we gain some insight into the word 'relation'. If the equation to be graphed contains both x and y, then the equation itself is what is relating the two variables. More specifically, in the next two examples, we consider the graph of the equation $x^2 + y^3 = 1$. Even though it is not specifically spelled out, what we are doing is graphing the relation $R = \{(x, y) \,|\, x^2 + y^3 = 1\}$. The points (x, y) we graph belong to the *relation* R and are necessarily *related* by the equation $x^2 + y^3 = 1$, since it is those pairs of x and y which make the equation true.

Example 15 Checking to see if a point lies on a graph
Determine whether or not $(2, -1)$ is on the graph of $x^2 + y^3 = 1$.

Solution We substitute $x = 2$ and $y = -1$ into the equation to see if the equation is satisfied.

$$\begin{aligned} (2)^2 + (-1)^3 &\stackrel{?}{=} 1 \\ 3 &\neq 1 \end{aligned}$$

Hence, $(2, -1)$ is **not** on the graph of $x^2 + y^3 = 1$. □

We could spend hours randomly guessing and checking to see if points are on the graph of the equation. A more systematic approach is outlined in the following example.

Example 16 **Determining points on a graph systematically**

Graph $x^2 + y^3 = 1$.

SOLUTION To efficiently generate points on the graph of this equation, we first solve for y

$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

We now substitute a value in for x, determine the corresponding value y, and plot the resulting point (x, y). For example, substituting $x = -3$ into the equation yields

$$y = \sqrt[3]{1 - x^2} = \sqrt[3]{1 - (-3)^2} = \sqrt[3]{-8} = -2,$$

so the point $(-3, -2)$ is on the graph. Continuing in this manner, we generate a table of points which are on the graph of the equation. These points are then plotted in the plane as shown in Figure 2.8.

x	y	(x, y)
-3	-2	$(-3, -2)$
-2	$-\sqrt[3]{3}$	$(-2, -\sqrt[3]{3})$
-1	0	$(-1, 0)$
0	1	$(0, 1)$
1	0	$(1, 0)$
2	$-\sqrt[3]{3}$	$(2, -\sqrt[3]{3})$
3	-2	$(3, -2)$

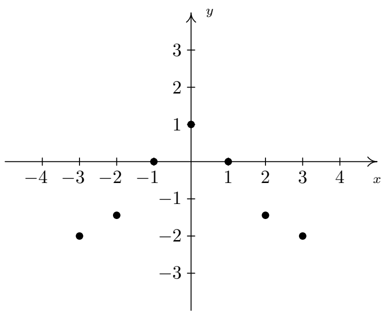


Figure 2.8: Points on the curve $x^2+y^3 = 1$

Remember, these points constitute only a small sampling of the points on the graph of this equation. To get a better idea of the shape of the graph, we could plot more points until we feel comfortable 'connecting the dots'. Doing so would result in a curve similar to the one pictured in Figure 2.9.

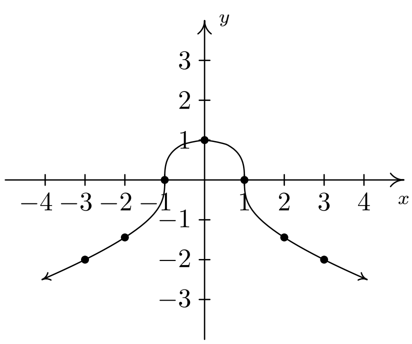


Figure 2.9: The completed graph of $x^2 + y^3 = 1$

Don't worry if you don't get all of the little bends and curves just right — Calculus is where the art of precise graphing takes center stage. For now, we will settle with our naive 'plug and plot' approach to graphing. If you feel like all of this tedious computation and plotting is beneath you, then you can try inputting the equation into a graphing calculator or an online tool such as Wolfram Alpha.

Of all of the points on the graph of an equation, the places where the graph crosses or touches the axes hold special significance. These are called the **intercepts** of the graph. Intercepts come in two distinct varieties: x-intercepts and y-intercepts. They are defined below.

Definition 21 ***x*- and *y*-intercepts**

Suppose the graph of an equation is given.

- A point on a graph which is also on the x-axis is called an **x-intercept** of the graph.
- A point on a graph which is also on the y-axis is called an **y-intercept** of the graph.

In our previous example the graph had two x-intercepts, $(-1, 0)$ and $(1, 0)$, and one y-intercept, $(0, 1)$. The graph of an equation can have any number of intercepts, including none at all! Since x-intercepts lie on the x-axis, we can find them by setting $y = 0$ in the equation. Similarly, since y-intercepts lie on the y-axis, we can find them by setting $x = 0$ in the equation. Keep in mind, intercepts are *points* and therefore must be written as ordered pairs. To summarize,

Key Idea 10 Finding the Intercepts of the Graph of an Equation

Given an equation involving x and y, we find the intercepts of the graph as follows:

- x-intercepts have the form $(x, 0)$; set $y = 0$ in the equation and solve for x.
- y-intercepts have the form $(0, y)$; set $x = 0$ in the equation and solve for y.

Another fact which you may have noticed about the graph in the previous example is that it seems to be symmetric about the y-axis. To actually prove this analytically, we assume (x, y) is a generic point on the graph of the equation. That is, we assume $x^2 + y^3 = 1$ is true. As we learned in Section 1.3, the point symmetric to (x, y) about the y-axis is $(-x, y)$. To show that the graph is symmetric about the y-axis, we need to show that $(-x, y)$ satisfies the equation $x^2 + y^3 = 1$, too. Substituting $(-x, y)$ into the equation gives

$$\begin{aligned} (-x)^2 + (y)^3 &\stackrel{?}{=} 1 \\ x^2 + y^3 &\stackrel{\checkmark}{=} 1 \end{aligned}$$

Since we are assuming the original equation $x^2 + y^3 = 1$ is true, we have shown that $(-x, y)$ satisfies the equation (since it leads to a true result) and hence is on the graph. In this way, we can check whether the graph of a given equation possesses any of the symmetries discussed in Section 1.3. We summarize the procedure in the following result.

Key Idea 11 Testing the Graph of an Equation for Symmetry

To test the graph of an equation for symmetry

- about the y-axis $-$ substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the x-axis – substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the origin - substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

Intercepts and symmetry are two tools which can help us sketch the graph of an equation analytically, as demonstrated in the next example.

Example 17 Finding intercepts and testing for symmetry

Find the x- and y-intercepts (if any) of the graph of $(x - 2)^2 + y^2 = 1$. Test for symmetry. Plot additional points as needed to complete the graph.

Solution To look for x-intercepts, we set $y = 0$ and solve

$$
\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(x-2)^2 + 0^2 &= 1 \\
(x-2)^2 &= 1 \\
\sqrt{(x-2)^2} &= \sqrt{1} \qquad \text{extract square roots} \\
x - 2 &= \pm 1 \\
x &= 2 \pm 1 \\
x &= 3, 1
\end{aligned}
$$

We get two answers for x which correspond to two x-intercepts: $(1,0)$ and $(3,0)$. Turning our attention to y-intercepts, we set $x = 0$ and solve

$$
\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(0-2)^2 + y^2 &= 1 \\
4 + y^2 &= 1 \\
y^2 &= -3
\end{aligned}
$$

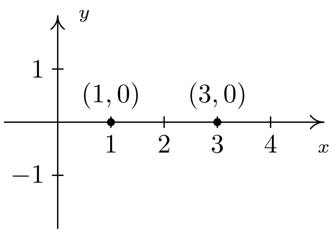


Figure 2.10: Plotting the data so far

Since there is no real number which squares to a negative number (Do you remember why?), we are forced to conclude that the graph has no y-intercepts. We plot our results so far in Figure 2.10.

Moving along to symmetry, we can immediately dismiss the possibility that the graph is symmetric about the y-axis or the origin. If the graph possessed either of these symmetries, then the fact that $(1,0)$ is on the graph would mean $(-1,0)$ would have to be on the graph. (Why?) Since $(-1,0)$ would be another x-intercept (and we've found all of these), the graph can't have y-axis or origin symmetry. The only symmetry left to test is symmetry about the x-axis. To that end, we substitute $(x, -y)$ into the equation and simplify

$$
\begin{aligned}
(x-2)^2 + y^2 &= 1 \\
(x-2)^2 + (-y)^2 &\stackrel{?}{=} 1 \\
(x-2)^2 + y^2 &\stackrel{\checkmark}{=} 1
\end{aligned}
$$

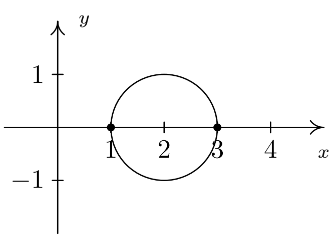


Figure 2.11: The final result

Since we have obtained our original equation, we know the graph is symmetric about the x-axis. This means we can cut our 'plug and plot' time in half: whatever happens below the x-axis is reflected above the x-axis, and vice-versa. Proceeding as we did in the previous example, we obtain the plot shown in Figure 2.11.

A couple of remarks are in order. First, it is entirely possible to choose a value for x which does not correspond to a point on the graph. For example, in the previous example, if we solve for y as is our custom, we get

$$y = \pm\sqrt{1 - (x-2)^2}.$$

Upon substituting $x = 0$ into the equation, we would obtain

$$y = \pm\sqrt{1 - (0-2)^2} = \pm\sqrt{1-4} = \pm\sqrt{-3},$$

which is not a real number. This means there are no points on the graph with an x-coordinate of 0. When this happens, we move on and try another point. This is another drawback of the 'plug-and-plot' approach to graphing equations. Luckily, we will devote much of the remainder of this book to developing techniques which allow us to graph entire families of equations quickly. Second, it is instructive to show what would have happened had we tested the equation

By the end of this course, you'll be able to accurately graph a wide variety of equations, without the use of a calculator, if you can believe it!

in the last example for symmetry about the y-axis. Substituting $(-x, y)$ into the equation yields

$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ (-x-2)^2 + y^2 &\stackrel{?}{=} 1 \\ ((-1)(x+2))^2 + y^2 &\stackrel{?}{=} 1 \\ (x+2)^2 + y^2 &\stackrel{?}{=} 1. \end{aligned}$$

This last equation does not *appear* to be equivalent to our original equation. However, to actually prove that the graph is not symmetric about the y-axis, we need to find a point (x, y) on the graph whose reflection $(-x, y)$ is not. Our x-intercept $(1, 0)$ fits this bill nicely, since if we substitute $(-1, 0)$ into the equation we get

$$\begin{aligned} (x-2)^2 + y^2 &\stackrel{?}{=} 1 \\ (-1-2)^2 + 0^2 &\neq 1 \\ 9 &\neq 1. \end{aligned}$$

This proves that $(-1, 0)$ is not on the graph.

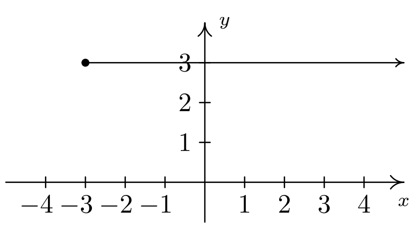
Exercises 2.1

Problems

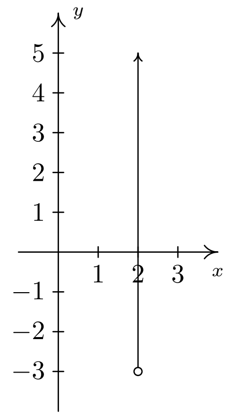
In Exercises 1 – 20, graph the given relation.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$
3. $\{(m,2m)\ |\ m=0,\pm1,\pm2\}$
4. $\left\{\left(\frac{6}{k},k\right)\ |\ k=\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\right\}$
5. $\left\{\left(n,4-n^2\right)\ |\ n=0,\pm1,\pm2\right\}$
6. $\left\{\left(\sqrt{j},j\right)\ |\ j=0,1,4,9\right\}$
7. $\{(x,-2)\ |\ x>-4\}$
8. $\{(x,3)\ |\ x\leq 4\}$
9. $\{(-1,y)\ |\ y>1\}$
10. $\{(2,y)\ |\ y\leq 5\}$
11. $\{(-2,y)\ |\ -3<y\leq 4\}$
12. $\{(3,y)\ |\ -4\leq y<3\}$
13. $\{(x,2)\ |\ -2\leq x<3\}$
14. $\{(x,-3)\ |\ -4<x\leq 4\}$
15. $\{(x,y)\ |\ x>-2\}$
16. $\{(x,y)\ |\ x\leq 3\}$
17. $\{(x,y)\ |\ y<4\}$
18. $\{(x,y)\ |\ x\leq 3,\ y<2\}$
19. $\{(x,y)\ |\ x>0,\ y<4\}$
20. $\{(x,y)\ |\ -\sqrt{2}\leq x\leq \frac{2}{3},\ \pi<y\leq \frac{9}{2}\}$

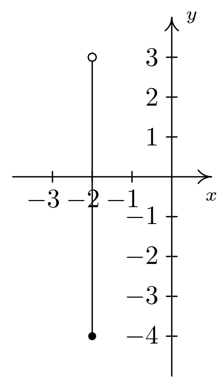
In Exercises 21 – 30, describe the given relation using either the roster or set-builder method.

21. 

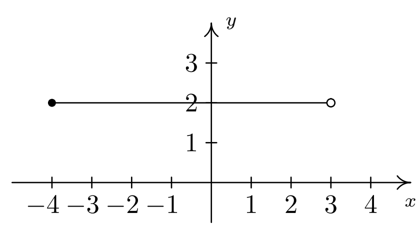
Relation *A*

22. 

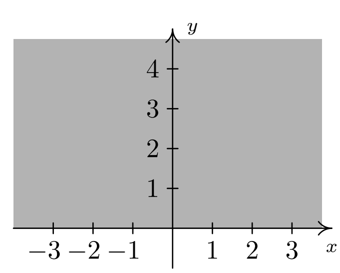
Relation *B*

23. 

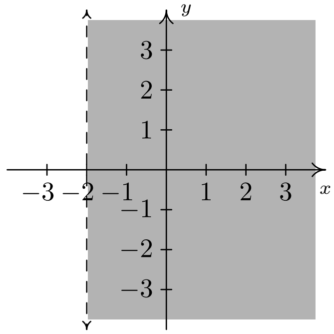
Relation *C*

24. 

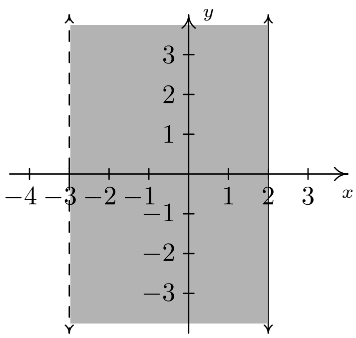
Relation *D*

25. 

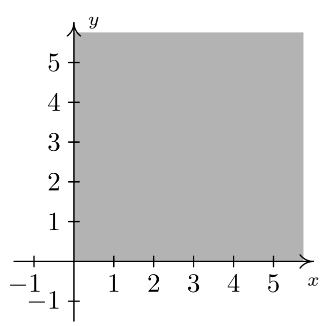
Relation *E*

26. 

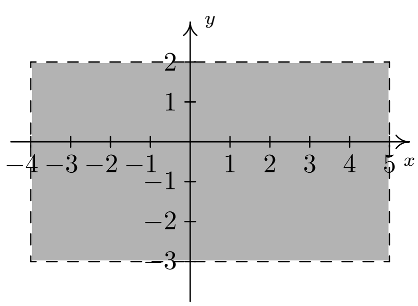
Relation *F*

27. 

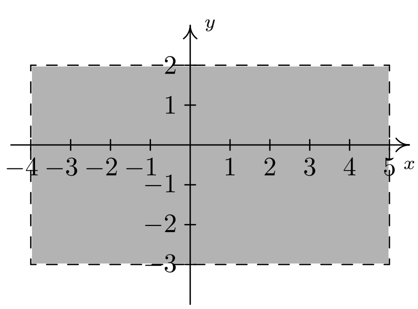
Relation *G*

28. 

Relation *H*

29. 

Relation *I*

30. 

Relation *J*

In Exercises 31 – 36, graph the given line.

31. $x = -2$

32. $x = 3$

33. $y = 3$

34. $y = -2$

35. $x = 0$

36. $y = 0$

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 37 – 40, discuss with your classmates how you might graph the given relation.

37. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

38. $\{(x, 1) \mid x$ is an irrational number $\}$

39. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$

40. $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

For each equation given in Exercises 41 – 52, (a) Find the *x* and *y* intercepts of the graph, if any exist; (b) Follow the procedure in Example 16 to create a table of sample points on the graph of the equation; (c) Plot the sample points and create a rough sketch of the graph of the equation; Test for symmetry. If the equation appears to fail any of the symmetry tests, find a point on the graph of the equation whose reflection fails to be on the graph as was done at the end of Example 17.

41. $y = x^2 + 1$

42. $y = x^2 - 2x - 8$

43. $y = x^3 - x$

44. $y = \frac{x^3}{4} - 3x$

45. $y = \sqrt{x - 2}$

46. $y = 2\sqrt{x + 4} - 2$

47. $3x - y = 7$

48. $3x - 2y = 10$

49. $(x + 2)^2 + y^2 = 16$

50. $x^2 - y^2 = 1$

51. $4y^2 - 9x^2 = 36$

52. $x^3y = -4$

53. With the help of your classmates, find examples of equations whose graphs possess

- symmetry about the *x*-axis only
- symmetry about the *y*-axis only
- symmetry about the origin only
- symmetry about the *x*-axis, *y*-axis, and origin

Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

2.2 Introduction to Functions

One of the core concepts in College Algebra is the **function**. There are many ways to describe a function and we begin by defining a function as a special kind of relation.

Definition 22 Function

A relation in which each x-coordinate is matched with only one y-coordinate is said to describe y as a **function** of x.

Example 18 Determining if a relation is a function

Which of the following relations describe y as a function of x?

1. $R_1 = \{(-2,1),(1,3),(1,4),(3,-1)\}$
2. $R_2 = \{(-2,1),(1,3),(2,3),(3,-1)\}$

We will have occasion later in the text to concern ourselves with the concept of x being a function of y. In this case, R_1 represents x as a function of y; R_2 does not.

Solution A quick scan of the points in R_1 reveals that the x-coordinate 1 is matched with two *different* y-coordinates: namely 3 and 4. Hence in R_1, y is not a function of x. On the other hand, every x-coordinate in R_2 occurs only once which means each x-coordinate has only one corresponding y-coordinate. So, R_2 does represent y as a function of x.

Note that in the previous example, the relation R_2 contained two different points with the same y-coordinates, namely $(1,3)$ and $(2,3)$. Remember, in order to say y is a function of x, we just need to ensure the same x-coordinate isn't used in more than one point.

To see what the function concept means geometrically, we graph R_1 and R_2 in the plane.

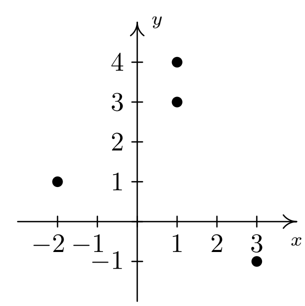


Figure 2.12: The graph of R_1

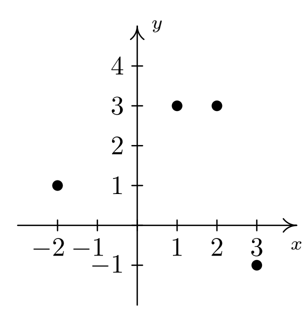


Figure 2.13: The graph of R_2

The fact that the x-coordinate 1 is matched with two different y-coordinates in R_1 presents itself graphically as the points $(1,3)$ and $(1,4)$ lying on the same vertical line, $x = 1$. If we turn our attention to the graph of R_2, we see that no two points of the relation lie on the same vertical line. We can generalize this idea as follows

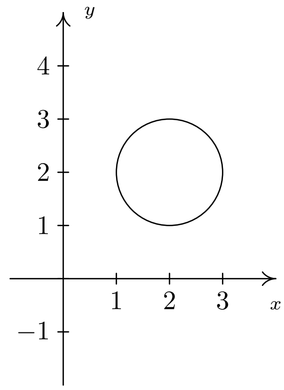
Theorem 6 The Vertical Line Test

A set of points in the plane represents y as a function of x if and only if no two points lie on the same vertical line.

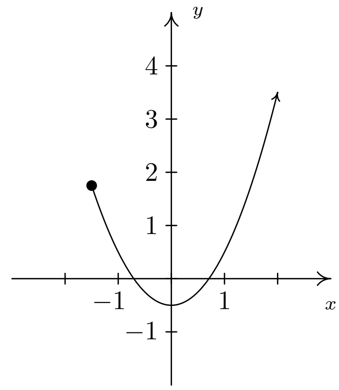
It is worth taking some time to meditate on the Vertical Line Test; it will check to see how well you understand the concept of 'function' as well as the concept of 'graph'.

Example 19 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of R



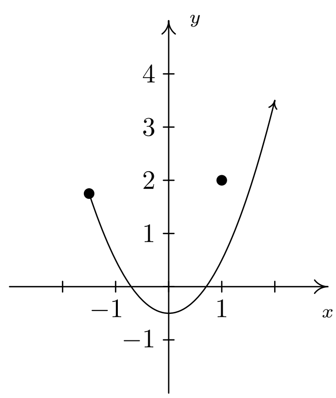
The graph of S

SOLUTION Looking at the graph of R, we can easily imagine a vertical line crossing the graph more than once. Hence, R does not represent y as a function of x. However, in the graph of S, every vertical line crosses the graph at most once, so S does represent y as a function of x. □

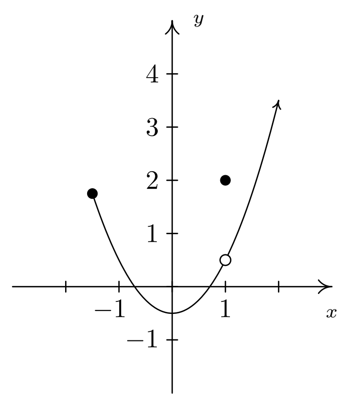
In the previous test, we say that the graph of the relation R **fails** the Vertical Line Test, whereas the graph of S **passes** the Vertical Line Test. Note that in the graph of R there are infinitely many vertical lines which cross the graph more than once. However, to fail the Vertical Line Test, all you need is one vertical line that fits the bill, as the next example illustrates.

Example 20 Using the Vertical Line Test

Use the Vertical Line Test to determine which of the following relations describes y as a function of x.



The graph of S_1



The graph of S_2

SOLUTION Both S_1 and S_2 are slight modifications to the relation S in the previous example whose graph we determined passed the Vertical Line Test. In both S_1 and S_2, it is the addition of the point $(1, 2)$ which threatens to cause trouble. In S_1, there is a point on the curve with x-coordinate 1 just below $(1, 2)$, which means that both $(1, 2)$ and this point on the curve lie on the vertical line $x = 1$. (See the picture below and the left.) Hence, the graph of S_1 fails the Vertical Line Test, so y is not a function of x here. However, in S_2 notice that the point with x-coordinate 1 on the curve has been omitted, leaving an 'open circle' there. Hence, the vertical line $x = 1$ crosses the graph of S_2 only at the point $(1, 2)$. Indeed, any vertical line will cross the graph at most once, so we

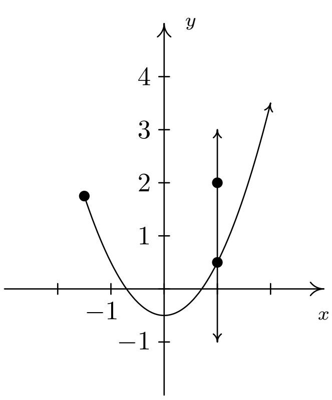


Figure 2.14: S_1 and the line $x = 1$

have that the graph of S_2 passes the Vertical Line Test. Thus it describes *y* as a function of *x*.

Suppose a relation *F* describes *y* as a function of *x*. The sets of *x*- and *y*-coordinates are given special names which we define below.

Definition 23 Domain and range

Suppose *F* is a relation which describes *y* as a function of *x*.

- The set of the *x*-coordinates of the points in *F* is called the **domain** of *F*.
- The set of the *y*-coordinates of the points in *F* is called the **range** of *F*.

We demonstrate finding the domain and range of functions given to us either graphically or via the roster method in the following example.

Example 21 Finding domain and range

Find the domain and range of the function $F = \{(-3, 2), (0, 1), (4, 2), (5, 2)\}$ and of the function *G* whose graph is given in Figure 2.15.

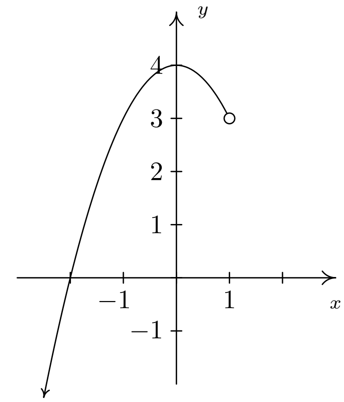


Figure 2.15: The graph of *G* for Example 21

Solution The domain of *F* is the set of the *x*-coordinates of the points in *F*, namely $\{-3, 0, 4, 5\}$ and the range of *F* is the set of the *y*-coordinates, namely $\{1, 2\}$.

To determine the domain and range of *G*, we need to determine which *x* and *y* values occur as coordinates of points on the given graph. To find the domain, it may be helpful to imagine collapsing the curve to the *x*-axis and determining the portion of the *x*-axis that gets covered. This is called **projecting** the curve to the *x*-axis. Before we start projecting, we need to pay attention to two subtle notations on the graph: the arrowhead on the lower left corner of the graph indicates that the graph continues to curve downwards to the left forever more; and the open circle at $(1, 3)$ indicates that the point $(1, 3)$ isn't on the graph, but all points on the curve leading up to that point are.

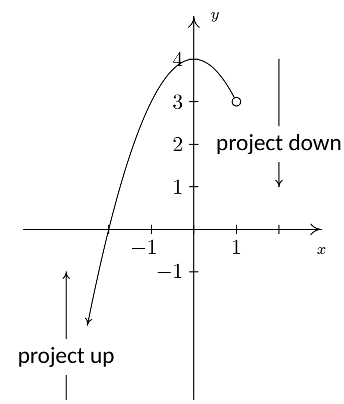


Figure 2.16: Projecting the graph onto the *x*-axis in Example 21

We see from Figures 2.16 and 2.17 that if we project the graph of *G* to the *x*-axis, we get all real numbers less than 1. Using interval notation, we write the domain of *G* as $(-\infty, 1)$. To determine the range of *G*, we project the curve to the *y*-axis as follows:

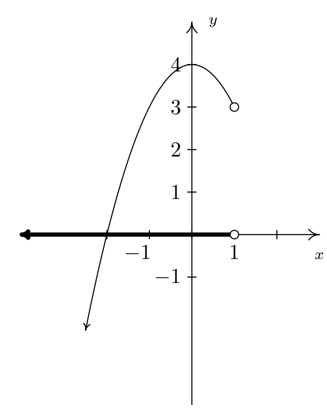


Figure 2.17: The domain of *G* in Example 21

Note that even though there is an open circle at $(1, 3)$, we still include the *y* value of 3 in our range, since the point $(-1, 3)$ is on the graph of *G*. Referring to Figures 2.18 and 2.19, we see that the range of *G* is all real numbers less than or equal to 4, or, in interval notation, $(-\infty, 4]$.

All functions are relations, but not all relations are functions. Thus the equations which described the relations in Section 2.1 may or may not describe *y* as a function of *x*. The algebraic representation of functions is possibly the most important way to view them so we need a process for determining whether or not an equation of a relation represents a function. (We delay the discussion of finding the domain of a function given algebraically until Section 2.3.)

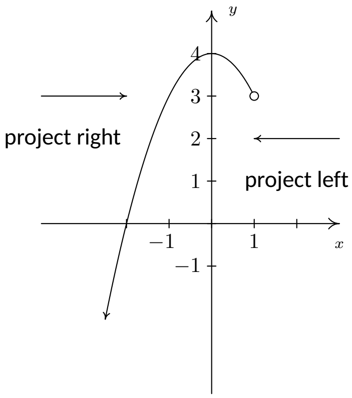


Figure 2.18: Projecting the graph onto the y-axis in Example 21

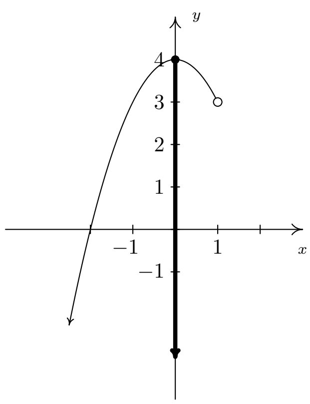


Figure 2.19: The range of G in Example 21

Example 22 **Functions defined by equations**

Determine which equations represent y as a function of x.

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $x^2y = 1 - 3y$

Solution For each of these equations, we solve for y and determine whether each choice of x will determine only one corresponding value of y.

1.
$$\begin{aligned} x^3 + y^2 &= 1 \\ y^2 &= 1 - x^3 \\ \sqrt{y^2} &= \sqrt{1 - x^3} \quad \text{extract square roots} \\ y &= \pm\sqrt{1 - x^3} \end{aligned}$$

   If we substitute $x = 0$ into our equation for y, we get $y = \pm\sqrt{1 - 0^3} = \pm 1$, so that $(0, 1)$ and $(0, -1)$ are on the graph of this equation. Hence, this equation does not represent y as a function of x.

2.
$$\begin{aligned} x^2 + y^3 &= 1 \\ y^3 &= 1 - x^2 \\ \sqrt[3]{y^3} &= \sqrt[3]{1 - x^2} \\ y &= \sqrt[3]{1 - x^2} \end{aligned}$$

   For every choice of x, the equation $y = \sqrt[3]{1 - x^2}$ returns only **one** value of y. Hence, this equation describes y as a function of x.

3.
$$\begin{aligned} x^2y &= 1 - 3y \\ x^2y + 3y &= 1 \\ y\left(x^2 + 3\right) &= 1 \quad \text{factor} \\ y &= \frac{1}{x^2 + 3} \end{aligned}$$

For each choice of x, there is only one value for y, so this equation describes y as a function of x.

Exercises 2.2

Problems

In Exercises 1 – 12, determine whether or not the relation represents *y* as a function of *x*. Find the domain and range of those relations which are functions.

1. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$

2. $\{(-3,0),(1,6),(2,-3),(4,2),(-5,6),(4,-9),(6,2)\}$

3. $\{(-3,0),(-7,6),(5,5),(6,4),(4,9),(3,0)\}$

4. $\{(1,2),(4,4),(9,6),(16,8),(25,10),(36,12),\ldots\}$

5. $\{(x,y)\mid x$ is an odd integer, and y is an even integer$\}$

6. $\{(x,1)\mid x$ is an irrational number$\}$

7. $\{(1,0),(2,1),(4,2),(8,3),(16,4),(32,5),\ldots\}$

8. $\{\ldots,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$

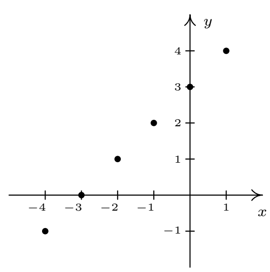
9. $\{(-2,y)\mid -3<y<4\}$

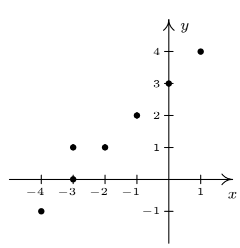
10. $\{(x,3)\mid -2\le x<4\}$

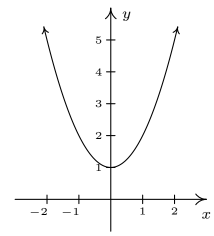
11. $\{(x,x^2)\mid x$ is a real number$\}$

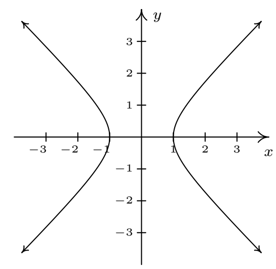
12. $\{(x^2,x)\mid x$ is a real number$\}$

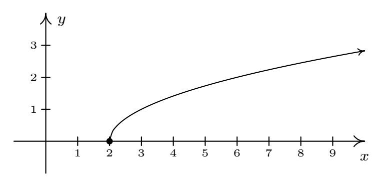
In Exercises 13 – 32, determine whether or not the relation represents *y* as a function of *x*. Find the domain and range of those relations which are functions.

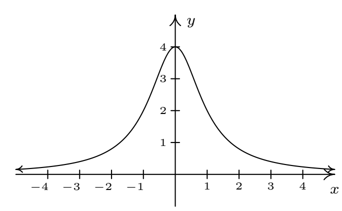
13. 

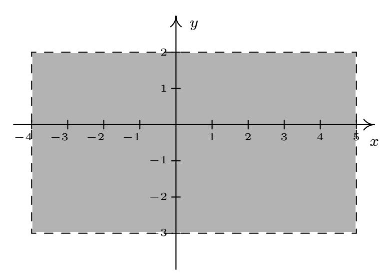
14. 

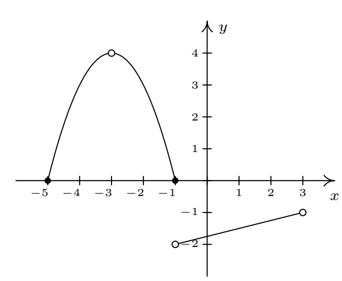
15. 

16. 

17. 

18. 

19. 

20. 

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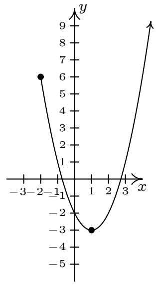

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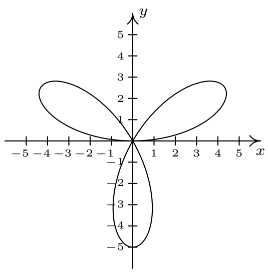

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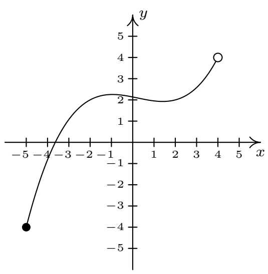

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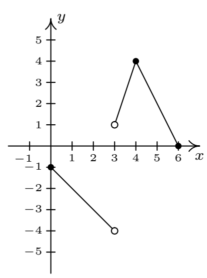

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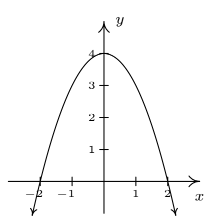

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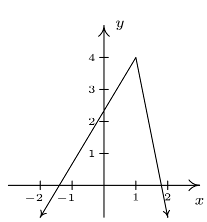

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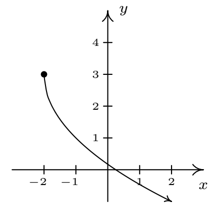

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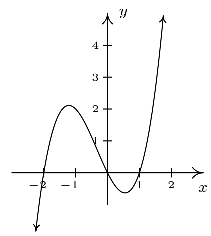

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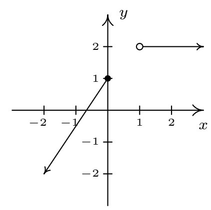

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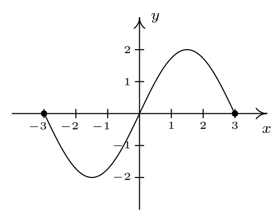

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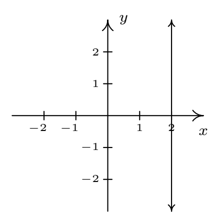

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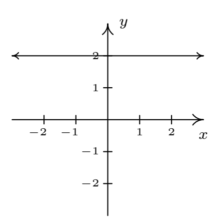


In Exercises 33 – 47, determine whether or not the equation represents *y* as a function of *x*.

33. $y = x^3 - x$

34. $y = \sqrt{x - 2}$

35. $x^3y = -4$

36. $x^2 - y^2 = 1$

37. $y = \dfrac{x}{x^2 - 9}$

38. $x = -6$

39. $x = y^2 + 4$

40. $y = x^2 + 4$

41. $x^2 + y^2 = 4$

42. $y = \sqrt{4 - x^2}$

43. $x^2 - y^2 = 4$

44. $x^3 + y^3 = 4$

45. $2x + 3y = 4$

46. $2xy = 4$

47. $x^2 = y^2$

48. Explain why the population P of Sasquatch in a given area is a function of time t. What would be the range of this function?

49. Explain why the relation between your classmates and their email addresses may not be a function. What about phone numbers and Social Security Numbers?

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. For Exercises 50 – 53, discuss with your classmates how you might graph the given relation.

50. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

51. $\{(x, 1) \mid x$ is an irrational number $\}$

52. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), \ldots\}$

53. $\{\ldots, (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), \ldots\}$

2.3 Function Notation

In Definition 22, we described a function as a special kind of relation — one in which each x-coordinate is matched with only one y-coordinate. In this section, we focus more on the **process** by which the x is matched with the y. If we think of the domain of a function as a set of **inputs** and the range as a set of **outputs**, we can think of a function f as a process by which each input x is matched with only one output y. Since the output is completely determined by the input x and the process f, we symbolize the output with **function notation**: '$f(x)$', read 'f **of** x.' In other words, $f(x)$ is the output which results by applying the process f to the input x. In this case, the parentheses here do not indicate multiplication, as they do elsewhere in Algebra. This can cause confusion if the context is not clear, so you must read carefully. This relationship is typically visualized using a diagram similar to the one in Figure 2.20.

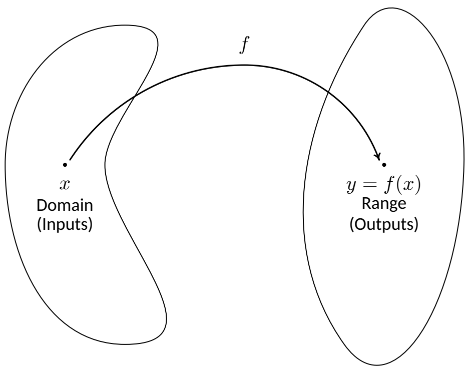


Figure 2.20: Graphical depiction of a function

The value of y is completely dependent on the choice of x. For this reason, x is often called the **independent variable**, or **argument** of f, whereas y is often called the **dependent variable**.

As we shall see, the process of a function f is usually described using an algebraic formula. For example, suppose a function f takes a real number and performs the following two steps, in sequence

1. Multiply by 3
2. Add 4

If we choose 5 as our input, in Step 1 we multiply by 3 to get $(5)(3) = 15$. In Step 2, we add 4 to our result from Step 1 which yields $15 + 4 = 19$. Using function notation, we would write $f(5) = 19$ to indicate that the result of applying the process f to the input 5 gives the output 19. In general, if we use x for the input, applying Step 1 produces $3x$. Following with Step 2 produces $3x + 4$ as our final output. Hence for an input x, we get the output $f(x) = 3x + 4$. Notice that to check our formula for the case $x = 5$, we replace the occurrence of x in the formula for $f(x)$ with 5 to get $f(5) = 3(5) + 4 = 15 + 4 = 19$, as required.

Example 23 Finding a forumula for a function

Suppose a function g is described by applying the following steps, in sequence

1. add 4
2. multiply by 3

Determine $g(5)$ and find an expression for $g(x)$.

SOLUTION Starting with 5, Step 1 gives $5 + 4 = 9$. Continuing with Step 2, we get $(3)(9) = 27$. To find a formula for $g(x)$, we start with our input x. Step 1 produces $x + 4$. We now wish to multiply this entire quantity by 3, so we use a parentheses: $3(x + 4) = 3x + 12$. Hence, $g(x) = 3x + 12$. We can check our formula by replacing x with 5 to get $g(5) = 3(5) + 12 = 15 + 12 = 27$ ✓.

Most of the functions we will encounter in Math 1010 will be described using formulas like the ones we developed for $f(x)$ and $g(x)$ above. Evaluating formulas using this function notation is a key skill for success in this and many other Math courses.

Example 24 Using function notation

Let $f(x) = -x^2 + 3x + 4$

1. Find and simplify the following.
   (a) $f(-1), f(0), f(2)$
   (b) $f(2x)$, $2f(x)$
   (c) $f(x+2), f(x)+2, f(x)+f(2)$
2. Solve $f(x) = 4$.

Solution

1. (a) To find $f(-1)$, we replace every occurrence of x in the expression $f(x)$ with -1

$$\begin{aligned} f(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0 \end{aligned}$$

Similarly, $f(0) = -(0)^2+3(0)+4 = 4$, and $f(2) = -(2)^2+3(2)+4 = -4+6+4 = 6$.

(b) To find $f(2x)$, we replace every occurrence of x with the quantity $2x$

$$\begin{aligned} f(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4 \end{aligned}$$

The expression $2f(x)$ means we multiply the expression $f(x)$ by 2

$$\begin{aligned} 2f(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8 \end{aligned}$$

(c) To find $f(x+2)$, we replace every occurrence of x with the quantity $x+2$

$$\begin{aligned} f(x+2) &= -(x+2)^2 + 3(x+2) + 4 \\ &= -\left(x^2 + 4x + 4\right) + (3x+6) + 4 \\ &= -x^2 - 4x - 4 + 3x + 6 + 4 \\ &= -x^2 - x + 6 \end{aligned}$$

To find $f(x) + 2$, we add 2 to the expression for $f(x)$

$$\begin{aligned} f(x) + 2 &= \left(-x^2 + 3x + 4\right) + 2 \\ &= -x^2 + 3x + 6 \end{aligned}$$

From our work above, we see $f(2) = 6$ so that

$$\begin{aligned} f(x) + f(2) &= \left(-x^2 + 3x + 4\right) + 6 \\ &= -x^2 + 3x + 10 \end{aligned}$$

2. Since $f(x) = -x^2 + 3x + 4$, the equation $f(x) = 4$ is equivalent to $-x^2 + 3x + 4 = 4$. Solving we get $-x^2 + 3x = 0$, or $x(-x+3) = 0$. We get $x = 0$ or $x = 3$, and we can verify these answers by checking that $f(0) = 4$ and $f(3) = 4$.

A few notes about Example 24 are in order. First note the difference between the answers for $f(2x)$ and $2f(x)$. For $f(2x)$, we are multiplying the *input* by 2; for $2f(x)$, we are multiplying the *output* by 2. As we see, we get entirely different results. Along these lines, note that $f(x+2)$, $f(x)+2$ and $f(x)+f(2)$ are three *different* expressions as well. Even though function notation uses parentheses, as does multiplication, there is *no* general 'distributive property' of function notation. Finally, note the practice of using parentheses when substituting one algebraic expression into another; we highly recommend this practice as it will reduce careless errors.

Suppose now we wish to find $r(3)$ for $r(x) = \dfrac{2x}{x^2 - 9}$. Substitution gives

$$r(3) = \frac{2(3)}{(3)^2 - 9} = \frac{6}{0},$$

which is undefined. (Why is this, again?) The number 3 is not an allowable input to the function r; in other words, 3 is not in the domain of r. Which other real numbers are forbidden in this formula? We think back to arithmetic. The reason $r(3)$ is undefined is because substitution results in a division by 0. To determine which other numbers result in such a transgression, we set the denominator equal to 0 and solve

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ \sqrt{x^2} &= \sqrt{9} \quad \text{extract square roots} \\ x &= \pm 3 \end{aligned}$$

As long as we substitute numbers other than 3 and -3, the expression $r(x)$ is a real number. Hence, we write our domain in interval notation (see the Exercises for Section 1.3) as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. When a formula for a function is given, we assume that the function is valid for all real numbers which make arithmetic sense when substituted into the formula. This set of numbers is often called the **implied domain** (or 'implicit domain') of the function. At this stage, there are only two mathematical sins we need to avoid: division by 0 and extracting even roots of negative numbers. The following example illustrates these concepts.

Example 25 Determining an implied domain

Find the domain of the following functions.

1. $g(x) = \sqrt{4 - 3x}$
2. $h(x) = \sqrt[5]{4 - 3x}$
3. $f(x) = \dfrac{2}{1 - \dfrac{4x}{x - 3}}$
4. $F(x) = \dfrac{\sqrt[4]{2x + 1}}{x^2 - 1}$
5. $r(t) = \dfrac{4}{6 - \sqrt{t + 3}}$
6. $I(x) = \dfrac{3x^2}{x}$

The 'radicand' is the expression 'inside' the radical.

SOLUTION

1. The potential disaster for g is if the radicand is negative. To avoid this, we set $4-3x \geq 0$. From this, we get $3x \leq 4$ or $x \leq \frac{4}{3}$. What this shows is that as long as $x \leq \frac{4}{3}$, the expression $4-3x \geq 0$, and the formula $g(x)$ returns a real number. Our domain is $\left(-\infty, \frac{4}{3}\right]$.

2. The formula for $h(x)$ is hauntingly close to that of $g(x)$ with one key difference $-$ whereas the expression for $g(x)$ includes an even indexed root (namely a square root), the formula for $h(x)$ involves an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.

3. In the expression for f, there are two denominators. We need to make sure neither of them is 0. To that end, we set each denominator equal to 0 and solve. For the 'small' denominator, we get $x-3=0$ or $x=3$. For the 'large' denominator

$$\begin{aligned} 1-\frac{4x}{x-3} &= 0 \\ 1 &= \frac{4x}{x-3} \\ (1)(x-3) &= \left(\frac{4x}{\cancel{x-3}}\right)\cancel{(x-3)} \quad \text{clear denominators} \\ x-3 &= 4x \\ -3 &= 3x \\ -1 &= x \end{aligned}$$

So we get two real numbers which make denominators 0, namely $x=-1$ and $x=3$. Our domain is all real numbers except -1 and 3: $(-\infty,-1) \cup (-1,3) \cup (3,\infty)$.

4. In finding the domain of F, we notice that we have two potentially hazardous issues: not only do we have a denominator, we have a fourth (even-indexed) root. Our strategy is to determine the restrictions imposed by each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, we require $2x+1 \geq 0$. From this we get $2x \geq -1$ or $x \geq -\frac{1}{2}$. Next, we round up the values of x which could cause trouble in the denominator by setting the denominator equal to 0. We get $x^2-1=0$, or $x=\pm 1$. Hence, in order for a real number x to be in the domain of F, $x \geq -\frac{1}{2}$ but $x \neq \pm 1$. In interval notation, this set is $\left[-\frac{1}{2}, 1\right) \cup (1, \infty)$.

5. Don't be put off by the 't' here. It is an independent variable representing a real number, just like x does, and is subject to the same restrictions. As in the previous problem, we have double danger here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set $t+3 \geq 0$ to get $t \geq -3$. Setting the denominator equal to zero gives $6-\sqrt{t+3}=0$, or $\sqrt{t+3}=6$. Squaring both sides gives $t+3=36$, or $t=33$. Since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $t=33$, $6-\sqrt{t+3}=6-\sqrt{36}=0$, so $t=33$ will cause problems in the denominator. At last we can find the domain of r: we need $t \geq -3$, but $t \neq 33$. Our final answer is $[-3,33) \cup (33,\infty)$.

Squaring both sides of an equation can introduce *extraneous solutions*. Do you remember why? Consider squaring both sides to 'solve' $\sqrt{t+1}=-2$.

6. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$, and, since there are no longer any denominators, claim that there are no longer any restrictions. However, in simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined. Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$.

It is worth reiterating the importance of finding the domain of a function *before* simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$. It would be analogous to not reporting taxable income or some other sin of omission.

2.3.1 Modelling with Functions

The importance of Mathematics to our society lies in its value to approximate, or **model** real-world phenomenon. Whether it be used to predict the high temperature on a given day, determine the hours of daylight on a given day, or predict population trends of various and sundry real and mythical beasts, Mathematics is second only to literacy in the importance humanity's development.

It is important to keep in mind that any time Mathematics is used to approximate reality, there are always limitations to the model. For example, suppose grapes are on sale at the local market for $\$1.50$ per pound. Then one pound of grapes costs $\$1.50$, two pounds of grapes cost $\$3.00$, and so forth. Suppose we want to develop a formula which relates the cost of buying grapes to the amount of grapes being purchased. Since these two quantities vary from situation to situation, we assign them variables. Let c denote the cost of the grapes and let g denote the amount of grapes purchased. To find the cost c of the grapes, we multiply the amount of grapes g by the price $\$1.50$ dollars per pound to get

$$c = 1.5g$$

In order for the units to be correct in the formula, g must be measured in *pounds* of grapes in which case the computed value of c is measured in *dollars*. Since we're interested in finding the cost c given an amount g, we think of g as the independent variable and c as the dependent variable. Using the language of function notation, we write

$$c(g) = 1.5g$$

where g is the amount of grapes purchased (in pounds) and $c(g)$ is the cost (in dollars). For example, $c(5)$ represents the cost, in dollars, to purchase 5 pounds of grapes. In this case, $c(5) = 1.5(5) = 7.5$, so it would cost $\$7.50$. If, on the other hand, we wanted to find the *amount* of grapes we can purchase for $\$5$, we would need to set $c(g) = 5$ and solve for g. In this case, $c(g) = 1.5g$, so solving $c(g) = 5$ is equivalent to solving $1.5g = 5$ Doing so gives $g = \frac{5}{1.5} = 3.\overline{3}$. This means we can purchase exactly $3.\overline{3}$ pounds of grapes for $\$5$. Of course, you would be hard-pressed to buy exactly $3.\overline{3}$ pounds of grapes, (you could get close... within a certain specified margin of error, perhaps) and this leads us to our next topic of discussion, the **applied domain**, or 'explicit domain' of a function.

Even though, mathematically, $c(g) = 1.5g$ has no domain restrictions (there are no denominators and no even-indexed radicals), there are certain values of g that don't make any physical sense. For example, $g = -1$ corresponds to 'purchasing' -1 pounds of grapes. (Maybe this means *returning* a pound of grapes?)

Also, unless the 'local market' mentioned is the State of California (or some other exporter of grapes), it also doesn't make much sense for $g = 500{,}000{,}000$, either. So the reality of the situation limits what g can be, and these limits determine the applied domain of g. Typically, an applied domain is stated explicitly. In this case, it would be common to see something like $c(g) = 1.5g$, $0 \leq g \leq 100$, meaning the number of pounds of grapes purchased is limited from 0 up to 100. The upper bound here, 100 may represent the inventory of the market, or some other limit as set by local policy or law. Even with this restriction, our model has its limitations. As we saw above, it is virtually impossible to buy exactly $3.\overline{3}$ pounds of grapes so that our cost is exactly \$5. In this case, being sensible shoppers, we would most likely 'round down' and purchase 3 pounds of grapes or however close the market scale can read to $3.\overline{3}$ without being over. It is time for a more sophisticated example.

Example 26 Height of a model rocket

The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if } \quad 0 \leq t \leq 20 \\ 0, & \text{if } \quad t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.

2. Solve $h(t) = 375$ and interpret your answers.

Solution

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.

2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t-5)(t-15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude *twice* – once 5 seconds after launch, and again 15 seconds after launch.

The type of function in the previous example is called a **piecewise-defined** function, or 'piecewise' function for short. Many real-world phenomena, income tax formulas for example, are modelled by such functions.

By the way, if we wanted to avoid using a piecewise function in Example 26, we could have used $h(t) = -5t^2 + 100t$ on the explicit domain $0 \leq t \leq 20$ because after 20 seconds, the rocket is on the ground and stops moving. In many cases, though, piecewise functions are your only choice, so it's best to understand them well.

Mathematical modelling is not a one-section topic. It's not even a one-*course* topic as is evidenced by undergraduate and graduate courses in mathematical modelling being offered at many universities. Thus our goal in this section cannot possibly be to tell you the whole story. What we can do is get you started. As we study new classes of functions, we will see what phenomena they can be used to model. In that respect, mathematical modelling cannot be a topic in a book, but rather, must be a theme of the book. For now, we have you explore some very basic models in the Exercises because you need to crawl to walk to run. As we learn more about functions, we'll help you build your own models and get you on your way to applying Mathematics to your world.

Exercises 2.3

Problems

In Exercises 1 – 10, find an expression for $f(x)$ and state its domain.

1. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) divide by 4.

2. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) divide by 4.

3. f is a function that takes a real number x and performs the following three steps in the order given: (1) divide by 4; (2) add 3; (3) multiply by 2.

4. f is a function that takes a real number x and performs the following three steps in the order given: (1) multiply by 2; (2) add 3; (3) take the square root.

5. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) multiply by 2; (3) take the square root.

6. f is a function that takes a real number x and performs the following three steps in the order given: (1) add 3; (2) take the square root; (3) multiply by 2.

7. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) subtract 13; (3) make the quantity the denominator of a fraction with numerator 4.

8. f is a function that takes a real number x and performs the following three steps in the order given: (1) subtract 13; (2) take the square root; (3) make the quantity the denominator of a fraction with numerator 4.

9. f is a function that takes a real number x and performs the following three steps in the order given: (1) take the square root; (2) make the quantity the denominator of a fraction with numerator 4; (3) subtract 13.

10. f is a function that takes a real number x and performs the following three steps in the order given: (1) make the quantity the denominator of a fraction with numerator 4; (2) take the square root; (3) subtract 13.

In Exercises 11 – 18, use the given function f to find and simplify the following:

- $f(3)$
- $f(-1)$
- $f\left(\frac{3}{2}\right)$
- $f(4x)$
- $4f(x)$
- $f(-x)$
- $f(x-4)$
- $f(x)-4$
- $f\left(x^2\right)$

11. $f(x) = 2x + 1$

12. $f(x) = 3 - 4x$

13. $f(x) = 2 - x^2$

14. $f(x) = x^2 - 3x + 2$

15. $f(x) = \dfrac{x}{x-1}$

16. $f(x) = \dfrac{2}{x^3}$

17. $f(x) = 6$

18. $f(x) = 0$

In Exercises 19 – 26, use the given function f to find and simplify the following:

- $f(2)$
- $f(-2)$
- $f(2a)$
- $2f(a)$
- $f(a+2)$
- $f(a) + f(2)$
- $f\left(\frac{2}{a}\right)$
- $\frac{f(a)}{2}$
- $f(a+h)$

19. $f(x) = 2x - 5$

20. $f(x) = 5 - 2x$

21. $f(x) = 2x^2 - 1$

22. $f(x) = 3x^2 + 3x - 2$

23. $f(x) = \sqrt{2x+1}$

24. $f(x) = 117$

25. $f(x) = \dfrac{x}{2}$

26. $f(x) = \dfrac{2}{x}$

In Exercises 27 – 34, use the given function f to find $f(0)$ and solve $f(x) = 0$.

27. $f(x) = 2x - 1$

28. $f(x) = 3 - \frac{2}{5}x$

29. $f(x) = 2x^2 - 6$

30. $f(x) = x^2 - x - 12$

31. $f(x) = \sqrt{x+4}$

32. $f(x) = \sqrt{1-2x}$

33. $f(x) = \dfrac{3}{4-x}$

34. $f(x) = \dfrac{3x^2-12x}{4-x^2}$

35. Let $f(x) = \begin{cases} x+5 & \text{if} \quad x \le -3 \\ \sqrt{9-x^2} & \text{if} \quad -3 < x \le 3 \\ -x+5 & \text{if} \quad x > 3 \end{cases}$ Compute the following function values.

(a) $f(-4)$
(b) $f(-3)$
(c) $f(3)$
(d) $f(3.001)$
(e) $f(-3.001)$
(f) $f(2)$

36. Let $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ \sqrt{1-x^2} & \text{if} \quad -1 < x \le 1 \\ x & \text{if} \quad x > 1 \end{cases}$ Compute the following function values.

(a) $f(4)$
(b) $f(-3)$
(c) $f(1)$
(d) $f(0)$
(e) $f(-1)$
(f) $f(-0.999)$

In Exercises 37 – 62, find the (implied) domain of the function.

37. $f(x) = x^4 - 13x^3 + 56x^2 - 19$

38. $f(x) = x^2 + 4$

39. $f(x) = \dfrac{x-2}{x+1}$

40. $f(x) = \dfrac{3x}{x^2+x-2}$

41. $f(x) = \dfrac{2x}{x^2+3}$

42. $f(x) = \dfrac{2x}{x^2-3}$

43. $f(x) = \dfrac{x+4}{x^2-36}$

44. $f(x) = \dfrac{x-2}{x-2}$

45. $f(x) = \sqrt{3-x}$

46. $f(x) = \sqrt{2x+5}$

47. $f(x) = 9x\sqrt{x+3}$

48. $f(x) = \dfrac{\sqrt{7-x}}{x^2+1}$

49. $f(x) = \sqrt{6x-2}$

50. $f(x) = \dfrac{6}{\sqrt{6x-2}}$

51. $f(x) = \sqrt[3]{6x-2}$

52. $f(x) = \dfrac{6}{4-\sqrt{6x-2}}$

53. $f(x) = \dfrac{\sqrt{6x-2}}{x^2-36}$

54. $f(x) = \dfrac{\sqrt[3]{6x-2}}{x^2+36}$

55. $s(t) = \dfrac{t}{t-8}$

56. $Q(r) = \dfrac{\sqrt{r}}{r-8}$

57. $b(\theta) = \dfrac{\theta}{\sqrt{\theta-8}}$

58. $A(x) = \sqrt{x-7} + \sqrt{9-x}$

59. $\alpha(y) = \sqrt[3]{\dfrac{y}{y-8}}$

60. $g(v) = \dfrac{1}{4-\dfrac{1}{v^2}}$

61. $T(t) = \dfrac{\sqrt{t}-8}{5-t}$

62. $u(w) = \dfrac{w-8}{5-\sqrt{w}}$

63. The area A enclosed by a square, in square inches, is a function of the length of one of its sides x, when measured in inches. This relation is expressed by the formula $A(x) = x^2$ for $x > 0$. Find $A(3)$ and solve $A(x) = 36$. Interpret your answers to each. Why is x restricted to $x > 0$?

64. The area A enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This relation is expressed by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

65. The volume V enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides x, when measured in centimeters. This relation is expressed by the formula $V(x) = x^3$ for $x > 0$. Find $V(5)$ and solve $V(x) = 27$. Interpret your answers to each. Why is x restricted to $x > 0$?

66. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet.

This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

67. The volume V enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This relation is expressed by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

68. The height of an object dropped from the roof of an eight story building is modeled by: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

69. The temperature T in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

70. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

    (The value $C(0)$ is called the 'fixed' or 'start-up' cost. We'll revisit this concept on page 73.)

71. Using data from the Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modelled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use your calculator to find $F(0)$, $F(14)$ and $F(28)$. Round your answers to two decimal places and interpret your answers to each.

72. The population of Sasquatch in Portage County can be modeled by the function $P(t) = \frac{150t}{t+15}$, where t represents the number of years since 1803. Find and interpret $P(0)$ and $P(205)$. Discuss with your classmates what the applied domain and range of P should be.

73. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if } 1 \leq n \leq 25 \\ 13.50n & \text{if } 25 < n \leq 50 \\ 12n & \text{if } n > 50 \end{cases}$$

    (a) Find and interpret $C(20)$.
    (b) How much does it cost to order 50 copies of the book? What about 51 copies?
    (c) Your answer to 73b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

74. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if } 1 \leq n \leq 14 \\ 0 & \text{if } n \geq 15 \end{cases}$$

    where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

    (a) What is the cost to ship 10 comic books?
    (b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

75. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if } 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } m > 1000 \end{cases}$$

    (a) How much does it cost to talk 750 minutes per month with this plan?
    (b) How much does it cost to talk 20 hours a month with this plan?
    (c) Explain the terms of the plan verbally.

76. In Section 1.1.1 we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The **greatest integer of** x, denoted by $\lfloor x \rfloor$, is defined to be the largest integer k with $k \leq x$.

    Note: The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count.'

    (a) Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$, and $\lfloor \pi + 6 \rfloor$
    (b) Discuss with your classmates how $\lfloor x \rfloor$ may be described as a piecewise defined function.

        HINT: There are infinitely many pieces!
    (c) Is $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ always true? What if a or b is an integer? Test some values, make a conjecture, and explain your result.

77.

78. We have through our examples tried to convince you that, in general, $f(a+b) \neq f(a)+f(b)$. It has been our experience that students refuse to believe us so we'll try again with a different approach. With the help of your classmates, find a function f for which the following properties are always true.

    (a) $f(0) = f(-1+1) = f(-1) + f(1)$
    (b) $f(5) = f(2+3) = f(2) + f(3)$
    (c) $f(-6) = f(0-6) = f(0) - f(6)$
    (d) $f(a+b) = f(a) + f(b)$ regardless of what two numbers we give you for a and b.

    How many functions did you find that failed to satisfy the conditions above? Did $f(x) = x^2$ work? What about $f(x) = \sqrt{x}$ or $f(x) = 3x + 7$ or $f(x) = \dfrac{1}{x}$? Did you find an attribute common to those functions that did succeed? You should have, because there is only one extremely special family of functions that actually works here. Thus we return to our previous statement, **in general**, $f(a+b) \neq f(a) + f(b)$.

2.4 Function Arithmetic

In the previous section we used the newly defined function notation to make sense of expressions such as '$f(x)+2$' and '$2f(x)$' for a given function f. It would seem natural, then, that functions should have their own arithmetic which is consistent with the arithmetic of real numbers. The following definitions allow us to add, subtract, multiply and divide functions using the arithmetic we already know for real numbers.

Recall that if x is in the domains of both f and g, then we can say that x is an element of the intersection of the two domains.

Definition 24 Function Arithmetic

Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f+g$, is the function defined by the formula
$$(f+g)(x) = f(x) + g(x)$$
- The **difference** of f and g, denoted $f-g$, is the function defined by the formula
$$(f-g)(x) = f(x) - g(x)$$
- The **product** of f and g, denoted fg, is the function defined by the formula
$$(fg)(x) = f(x)g(x)$$
- The **quotient** of f and g, denoted $\dfrac{f}{g}$, is the function defined by the formula
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$
provided $g(x) \neq 0$.

In other words, to add two functions, we add their outputs; to subtract two functions, we subtract their outputs, and so on. Note that while the formula $(f+g)(x) = f(x)+g(x)$ looks suspiciously like some kind of distributive property, it is nothing of the sort; the addition on the left hand side of the equation is *function* addition, and we are using this equation to *define* the output of the new function $f+g$ as the sum of the real number outputs from f and g.

Example 27 Arithmetic with functions

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find $(f+g)(-1)$
2. Find $(fg)(2)$
3. Find the domain of $g-f$ then find and simplify a formula for $(g-f)(x)$.
4. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

Solution

1. To find $(f+g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that $(f+g)(-1) = f(-1) + g(-1) = 8 + 4 = 12$.

2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

3. One method to find the domain of $g - f$ is to find the domain of g and of f separately, then find the intersection of these two sets. Owing to the denominator in the expression $g(x) = 3 - \frac{1}{x}$, we get that the domain of g is $(-\infty, 0) \cup (0, \infty)$. Since $f(x) = 6x^2 - 2x$ is valid for all real numbers, we have no further restrictions. Thus the domain of $g - f$ matches the domain of g, namely, $(-\infty, 0) \cup (0, \infty)$.

   A second method is to analyze the formula for $(g-f)(x)$ *before simplifying* and look for the usual domain issues. In this case,

$$(g - f)(x) = g(x) - f(x) = \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right),$$

   so we find, as before, the domain is $(-\infty, 0) \cup (0, \infty)$.

   Moving along, we need to simplify a formula for $(g-f)(x)$. In this case, we get common denominators and attempt to reduce the resulting fraction. Doing so, we get

$$\begin{aligned}
(g - f)(x) &= g(x) - f(x) \\
&= \left(3 - \frac{1}{x}\right) - \left(6x^2 - 2x\right) \\
&= 3 - \frac{1}{x} - 6x^2 + 2x \\
&= \frac{3x}{x} - \frac{1}{x} - \frac{6x^3}{x} + \frac{2x^2}{x} \qquad \text{get common denominators} \\
&= \frac{3x - 1 - 6x^3 - 2x^2}{x} \\
&= \frac{-6x^3 - 2x^2 + 3x - 1}{x}
\end{aligned}$$

4. As in the previous example, we have two ways to approach finding the domain of $\frac{g}{f}$. First, we can find the domain of g and f separately, and find the intersection of these two sets. In addition, since $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$, so we need to guard against this being 0 as well. Our previous work tells us that the domain of g is $(-\infty, 0) \cup (0, \infty)$ and the domain of f is $(-\infty, \infty)$. Setting $f(x) = 0$ gives $6x^2 - 2x = 0$ or $x = 0, \frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x = 0$ and $x = \frac{1}{3}$, or $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

   Alternatively, we may proceed as above and analyze the expression $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$ *before* simplifying. In this case,

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{3 - \frac{1}{x}}{6x^2 - 2x}$$

   We see immediately from the 'little' denominator that $x \neq 0$. To keep the 'big' denominator away from 0, we solve $6x^2 - 2x = 0$ and get $x = 0$ or

$x = \frac{1}{3}$. Hence, as before, we find the domain of $\frac{g}{f}$ to be

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

Next, we find and simplify a formula for $\left(\frac{g}{f}\right)(x)$.

$$\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \\
&= \frac{3 - \frac{1}{x}}{6x^2 - 2x} \cdot \frac{x}{x} && \text{simplify compound fractions} \\
&= \frac{\left(3 - \frac{1}{x}\right)x}{(6x^2 - 2x)\,x} \\
&= \frac{3x - 1}{(6x^2 - 2x)\,x} \\
&= \frac{3x - 1}{2x^2(3x - 1)} && \text{factor} \\
&= \frac{\cancel{(3x - 1)}^{\,1}}{2x^2\cancel{(3x - 1)}} && \text{cancel} \\
&= \frac{1}{2x^2}
\end{aligned}$$

Please note the importance of finding the domain of a function *before* simplifying its expression. In number 4 in Example 27 above, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we'd just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty, 0) \cup (0, \infty)$, since the other troublesome number, $x = \frac{1}{3}$, was cancelled away.

We'll see what cancelling factors means geometrically in Chapter 5.

Next, we turn our attention to the **difference quotient** of a function.

Definition 25 Difference quotient of a function

Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h) - f(x)}{h}$$

We will revisit this concept in Section 3.1, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, 'simplifying' a difference quotient means rewriting it in a form where the 'h' in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

Example 28 Computing difference quotients

Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$
3. $r(x) = \sqrt{x}$

Solution

1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2 - x - h - 2\right) - \left(x^2 - x - 2\right)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h\,(2x + h - 1)}{h} && \text{factor} \\ &= \frac{\not{h}\,(2x + h - 1)}{\not{h}} && \text{cancel} \\ &= 2x + h - 1. \end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \frac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x+h) - g(x)}{h} &= \frac{\dfrac{3}{2x+2h+1} - \dfrac{3}{2x+1}}{h} \\ &= \frac{\dfrac{3}{2x+2h+1} - \dfrac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{3(2x+1) - 3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\
&= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \\
&= \frac{-6}{(2x+2h+1)(2x+1)}.
\end{aligned}
$$

Since we have managed to cancel the original '*h*' from the denominator, we are done.

3. For $r(x) = \sqrt{x}$, we get $r(x+h) = \sqrt{x+h}$ so the difference quotient is

$$\frac{r(x+h) - r(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

In order to cancel the '*h*' from the denominator, we rationalize the *numerator* by multiplying by its conjugate.

$$
\begin{aligned}
\frac{r(x+h) - r(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h} + \sqrt{x}\right)} && \text{Difference of Squares.} \\
&= \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}
$$

Since we have removed the original '*h*' from the denominator, we are done.

As mentioned before, we will revisit difference quotients in Section 3.1 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.

Suppose you are a manufacturer making a certain product. Let x be the **production level**, that is, the number of items produced in a given time period. It is customary to let $C(x)$ denote the function which calculates the total **cost** of producing the x items. The quantity $C(0)$, which represents the cost of producing no items, is called the **fixed** cost, and represents the amount of money required to begin production. Associated with the total cost $C(x)$ is cost per item, or **average cost**, denoted $\overline{C}(x)$ and read 'C-bar' of x. To compute $\overline{C}(x)$, we take the total cost $C(x)$ and divide by the number of items produced x to get

$$\overline{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price** p charged per item. To simplify the dialogue and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a retail perspective, it seems natural to think of the number of items sold, x, as a function of the price charged, p. After all, the retailer can easily adjust the price to sell more product. In the language of functions, x would be the *dependent* variable and p would be the *independent* variable or, using function notation, we have a function $x(p)$. While we will adopt this convention later in the text, (see Example 108 in Section 6.2) we will hold with tradition at this point and consider the price p as a function of the number of items sold, x. That is, we regard x as the independent variable and p as the dependent variable and speak of the **price-demand** function, $p(x)$. Hence, $p(x)$ returns the price charged per item when x items are produced and sold. Our next function to consider is the **revenue** function, $R(x)$. The function $R(x)$ computes the amount of money collected as a result of selling x items. Since $p(x)$ is the price charged per item, we have $R(x) = xp(x)$. Finally, the **profit** function, $P(x)$ calculates how much money is earned after the costs are paid. That is, $P(x) = (R - C)(x) = R(x) - C(x)$. We summarize all of these functions below.

Key Idea 12 Summary of Common Economic Functions

Suppose x represents the quantity of items produced and sold.

- The price-demand function $p(x)$ calculates the price per item.
- The revenue function $R(x)$ calculates the total money collected by selling x items at a price $p(x)$, $R(x) = x\,p(x)$.
- The cost function $C(x)$ calculates the cost to produce x items. The value $C(0)$ is called the fixed cost or start-up cost.
- The average cost function $\overline{C}(x) = \frac{C(x)}{x}$ calculates the cost per item when making x items. Here, we necessarily assume $x > 0$.
- The profit function $P(x)$ calculates the money earned after costs are paid when x items are produced and sold, $P(x) = (R-C)(x) = R(x) - C(x)$.

Example 29 Computing (and interpreting) cost and profit functions

Let x represent the number of dOpi media players ('dOpis' – pronounced 'dopeys' ...) produced and sold in a typical week. Suppose the cost, in dollars, to produce x dOpis is given by $C(x) = 100x + 2000$, for $x \geq 0$, and the price, in dollars per dOpi, is given by $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

1. Find and interpret $C(0)$.
2. Find and interpret $\overline{C}(10)$.
3. Find and interpret $p(0)$ and $p(20)$.
4. Solve $p(x) = 0$ and interpret the result.
5. Find and simplify expressions for the revenue function $R(x)$ and the profit function $P(x)$.
6. Find and interpret $R(0)$ and $P(0)$.
7. Solve $P(x) = 0$ and interpret the result.

Solution

1. We substitute $x = 0$ into the formula for $C(x)$ and get $C(0) = 100(0) + 2000 = 2000$. This means to produce 0 dOpis, it costs \$2000. In other words, the fixed (or start-up) costs are \$2000. The reader is encouraged to contemplate what sorts of expenses these might be.
2. Since $\overline{C}(x) = \frac{C(x)}{x}$, $\overline{C}(10) = \frac{C(10)}{10} = \frac{3000}{10} = 300$. This means when 10 dOpis are produced, the cost to manufacture them amounts to \$300 per dOpi.
3. Plugging $x = 0$ into the expression for $p(x)$ gives $p(0) = 450 - 15(0) = 450$. This means no dOpis are sold if the price is \$450 per dOpi. On the other hand, $p(20) = 450 - 15(20) = 150$ which means to sell 20 dOpis in a typical week, the price should be set at \$150 per dOpi.
4. Setting $p(x) = 0$ gives $450 - 15x = 0$. Solving gives $x = 30$. This means in order to sell 30 dOpis in a typical week, the price needs to be set to \$0. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.
5. To find the revenue, we compute $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$. Since the formula for $p(x)$ is valid only for $0 \leq x \leq 30$, our formula $R(x)$ is also restricted to $0 \leq x \leq 30$. For the profit, $P(x) = (R - C)(x) = R(x) - C(x)$. Using the given formula for $C(x)$ and the derived formula for $R(x)$, we get $P(x) = \left(450x - 15x^2\right) - (100x + 2000) = -15x^2 + 350x - 2000$. As before, the validity of this formula is for $0 \leq x \leq 30$ only.
6. We find $R(0) = 0$ which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit, $P(0) = -2000$ since $P(x) = R(x) - C(x)$ and $P(0) = R(0) - C(0) = -2000$. This means that if no dOpis are sold, more money (\$2000 to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be \$2000, so it makes sense that if we sell no dOpis, we are out those start-up costs.
7. Setting $P(x) = 0$ gives $-15x^2 + 350x - 2000 = 0$. Factoring gives $-5(x - 10)(3x - 40) = 0$ so $x = 10$ or $x = \frac{40}{3}$. What do these values mean in the context of the problem? Since $P(x) = R(x) - C(x)$, solving $P(x) = 0$ is the same as solving $R(x) = C(x)$. This means that the solutions to $P(x) = 0$ are the production (and sales) figures for which the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution $x = 10$ means 10 dOpis should be produced (and

sold) during the week to recoup the cost of production. For $x = \frac{40}{3} = 13.\overline{3}$, things are a bit more complicated. Even though $x = 13.\overline{3}$ satisfies $0 \leq x \leq 30$, and hence is in the domain of P, it doesn't make sense in the context of this problem to produce a fractional part of a dOpi. Evaluating $P(13) = 15$ and $P(14) = -40$, we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red. While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit, and we'll do just that …in Section 3.3.

Recall from Section 2.3.1 that in problems such as this, it is necessary to take the **applied domain** of the function into account.

Exercises 2.4

Problems

In Exercises 1 – 10, use the pair of functions f and g to find the following values if they exist:

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1$ and $g(x) = 4 - x$
2. $f(x) = x^2$ and $g(x) = -2x + 1$
3. $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
4. $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
5. $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$
6. $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$
7. $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
8. $f(x) = x^2$ and $g(x) = \dfrac{3}{2x-3}$
9. $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
10. $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x^2+1}$

In Exercises 11 – 20, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

11. $f(x) = 2x + 1$ and $g(x) = x - 2$
12. $f(x) = 1 - 4x$ and $g(x) = 2x - 1$
13. $f(x) = x^2$ and $g(x) = 3x - 1$
14. $f(x) = x^2 - x$ and $g(x) = 7x$
15. $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
16. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
17. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
18. $f(x) = x - 1$ and $g(x) = \dfrac{1}{x-1}$
19. $f(x) = x$ and $g(x) = \sqrt{x+1}$
20. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 21 – 45, find and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the given function.

21. $f(x) = 2x - 5$
22. $f(x) = -3x + 5$
23. $f(x) = 6$
24. $f(x) = 3x^2 - x$
25. $f(x) = -x^2 + 2x - 1$
26. $f(x) = 4x^2$
27. $f(x) = x - x^2$
28. $f(x) = x^3 + 1$
29. $f(x) = mx + b$ where $m \neq 0$
30. $f(x) = ax^2 + bx + c$ where $a \neq 0$
31. $f(x) = \dfrac{2}{x}$
32. $f(x) = \dfrac{3}{1-x}$
33. $f(x) = \dfrac{1}{x^2}$
34. $f(x) = \dfrac{2}{x+5}$
35. $f(x) = \dfrac{1}{4x-3}$
36. $f(x) = \dfrac{3x}{x+1}$
37. $f(x) = \dfrac{x}{x-9}$
38. $f(x) = \dfrac{x^2}{2x+1}$
39. $f(x) = \sqrt{x-9}$
40. $f(x) = \sqrt{2x+1}$
41. $f(x) = \sqrt{-4x+5}$
42. $f(x) = \sqrt{4-x}$

43. $f(x) = \sqrt{ax+b}$, where $a \neq 0$.

44. $f(x) = x\sqrt{x}$

45. $f(x) = \sqrt[3]{x}$. **HINT:** $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

In Exercises 46 – 50, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:

- **Find and interpret $C(0)$.**
- **Find and interpret $\overline{C}(10)$.**
- **Find and interpret $p(5)$**
- **Find and simplify $R(x)$.**
- **Find and simplify $P(x)$.**
- **Solve $P(x) = 0$ and interpret.**

46. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.

47. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.

48. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.

49. The daily cost, in dollars, to produce x Sasquatch Berry Pies $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

50. The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in hundreds of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

In Exercises 51 – 62, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Compute the indicated value if it exists.

51. $(f+g)(-3)$

52. $(f-g)(2)$

53. $(fg)(-1)$

54. $(g+f)(1)$

55. $(g-f)(3)$

56. $(gf)(-3)$

57. $\left(\dfrac{f}{g}\right)(-2)$

58. $\left(\dfrac{f}{g}\right)(-1)$

59. $\left(\dfrac{f}{g}\right)(2)$

60. $\left(\dfrac{g}{f}\right)(-1)$

61. $\left(\dfrac{g}{f}\right)(3)$

62. $\left(\dfrac{g}{f}\right)(-3)$

2.5 Graphs of Functions

In Section 2.2 we defined a function as a special type of relation; one in which each x-coordinate was matched with only one y-coordinate. We spent most of our time in that section looking at functions graphically because they were, after all, just sets of points in the plane. Then in Section 2.3 we described a function as a process and defined the notation necessary to work with functions algebraically. So now it's time to look at functions graphically again, only this time we'll do so with the notation defined in Section 2.3. We start with what should not be a surprising connection.

Key Idea 13 The Fundamental Graphing Principle for Functions

The graph of a function f is the set of points which satisfy the equation $y = f(x)$. That is, the point (x, y) is on the graph of f if and only if $y = f(x)$.

Example 30 Graphing a function

Graph $f(x) = x^2 - x - 6$.

Solution To graph f, we graph the equation $y = f(x)$. To this end, we use the techniques outlined in Section 2.1.1. Specifically, we check for intercepts, test for symmetry, and plot additional points as needed. To find the x-intercepts, we set $y = 0$. Since $y = f(x)$, this means $f(x) = 0$.

$$\begin{array}{rcll} f(x) & = & x^2 - x - 6 & \\ 0 & = & x^2 - x - 6 & \\ 0 & = & (x-3)(x+2) & \text{factor} \\ x - 3 = 0 & \text{or} & x + 2 = 0 & \\ x & = & -2, 3 & \end{array}$$

So we get $(-2, 0)$ and $(3, 0)$ as x-intercepts. To find the y-intercept, we set $x = 0$. Using function notation, this is the same as finding $f(0)$ and $f(0) = 0^2 - 0 - 6 = -6$. Thus the y-intercept is $(0, -6)$. As far as symmetry is concerned, we can tell from the intercepts that the graph possesses none of the three symmetries discussed thus far. (You should verify this.) We can make a table analogous to the ones we made in Section 2.1.1, plot the points and connect the dots in a somewhat pleasing fashion to get the graph shown in Figure 2.21.

x	$f(x)$	$(x, f(x))$
-3	6	$(-3, 6)$
-2	0	$(-2, 0)$
-1	-4	$(-1, -4)$
0	-6	$(0, -6)$
1	-6	$(1, -6)$
2	-4	$(2, -4)$
3	0	$(3, 0)$
4	6	$(4, 6)$

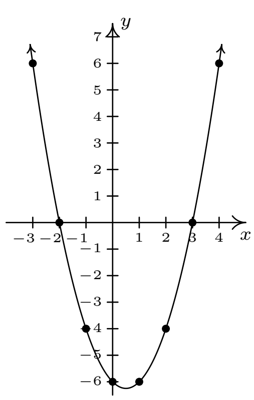


Figure 2.21: Graphing the function $f(x) = x^2 - x - 6$

Graphing piecewise-defined functions is a bit more of a challenge.

Example 31 Graphing a piecewise-defined function

Graph: $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ x - 3, & \text{if } x \geq 1 \end{cases}$

Solution We proceed as before – finding intercepts, testing for symmetry and then plotting additional points as needed. To find the x-intercepts, as before, we set $f(x) = 0$. The twist is that we have two formulas for $f(x)$. For $x < 1$, we use the formula $f(x) = 4 - x^2$. Setting $f(x) = 0$ gives $0 = 4 - x^2$, so that $x = \pm 2$. However, of these two answers, only $x = -2$ fits in the domain $x < 1$ for this piece. This means the only x-intercept for the $x < 1$ region of the x-axis is $(-2, 0)$. For $x \geq 1$, $f(x) = x - 3$. Setting $f(x) = 0$ gives $0 = x - 3$, or $x = 3$. Since $x = 3$ satisfies the inequality $x \geq 1$, we get $(3, 0)$ as another

x-intercept. Next, we seek the y-intercept. Notice that $x = 0$ falls in the domain $x < 1$. Thus $f(0) = 4 - 0^2 = 4$ yields the y-intercept $(0, 4)$. As far as symmetry is concerned, you can check that the equation $y = 4 - x^2$ is symmetric about the y-axis; unfortunately, this equation (and its symmetry) is valid only for $x < 1$. You can also verify $y = x - 3$ possesses none of the symmetries discussed in the Section 2.1.1. When plotting additional points, it is important to keep in mind the restrictions on x for each piece of the function. The sticking point for this function is $x = 1$, since this is where the equations change. When $x = 1$, we use the formula $f(x) = x - 3$, so the point on the graph $(1, f(1))$ is $(1, -2)$. However, for all values less than 1, we use the formula $f(x) = 4 - x^2$. As we have discussed earlier in Section 2.1, there is no real number which immediately precedes $x = 1$ on the number line. Thus for the values $x = 0.9$, $x = 0.99$, $x = 0.999$, and so on, we find the corresponding y values using the formula $f(x) = 4 - x^2$. Making a table as before, we see that as the x values sneak up to $x = 1$ in this fashion, the $f(x)$ values inch closer and closer to $4 - 1^2 = 3$. To indicate this graphically, we use an open circle at the point $(1, 3)$. Putting all of this information together and plotting additional points, we get the result in Figure 2.22.

x	$f(x)$	$(x, f(x))$
0.9	3.19	$(0.9, 3.19)$
0.99	≈ 3.02	$(0.99, 3.02)$
0.999	≈ 3.002	$(0.999, 3.002)$

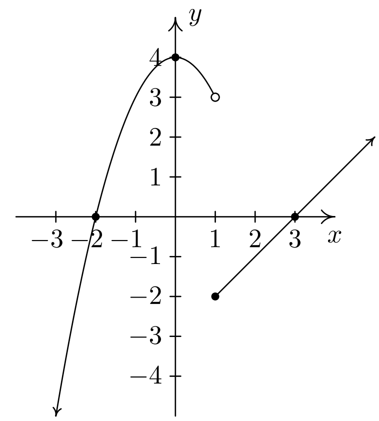


Figure 2.22: The graph of $f(x)$ from Example 31

In the previous two examples, the x-coordinates of the x-intercepts of the graph of $y = f(x)$ were found by solving $f(x) = 0$. For this reason, they are called the **zeros** of f.

Definition 26 Zeros of a function

The **zeros** of a function f are the solutions to the equation $f(x) = 0$. In other words, x is a zero of f if and only if $(x, 0)$ is an x-intercept of the graph of $y = f(x)$.

Of the three symmetries discussed in Section 2.1.1, only two are of significance to functions: symmetry about the y-axis and symmetry about the origin. Recall that we can test whether the graph of an equation is symmetric about the y-axis by replacing x with $-x$ and checking to see if an equivalent equation results. If we are graphing the equation $y = f(x)$, substituting $-x$ for x results in the equation $y = f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $f(-x) = f(x)$. In a similar fashion, we recall that to test an equation's graph for symmetry about the origin, we replace x and y with $-x$ and $-y$, respectively. Doing this substitution in the equation $y = f(x)$ results in $-y = f(-x)$. Solving the latter equation for y gives $y = -f(-x)$. In order for this equation to be equivalent to the original equation $y = f(x)$ we need $-f(-x) = f(x)$, or, equivalently, $f(-x) = -f(x)$. These results are summarized below.

Key Idea 14 Testing the Graph of a Function for Symmetry

The graph of a function f is symmetric

- about the y-axis if and only if $f(-x) = f(x)$ for all x in the domain of f.
- about the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

Note that for graphs of functions, we don't bother to discuss symmetry about the x-axis. Why do you suppose this is?

For reasons which won't become clear until we study polynomials, we call a function **even** if its graph is symmetric about the y-axis or **odd** if its graph is symmetric about the origin. Apart from a very specialized family of functions which are both even and odd, (any ideas?) functions fall into one of three distinct categories: even, odd, or neither even nor odd.

A good resource when you need to quickly check something like the graph of a function is Wolfram Alpha.
If you want a good (and free!) program you can run locally on a computer or tablet, we recommend trying Geogebra. It's free to download, works on all major operating systems, and it's pretty easy to figure out the basics.

Example 32 **Even and odd functions**

Determine analytically if the following functions are even, odd, or neither even nor odd. Verify your result with a graphing calculator or computer software.

1. $f(x) = \dfrac{5}{2-x^2}$
2. $g(x) = \dfrac{5x}{2-x^2}$
3. $h(x) = \dfrac{5x}{2-x^3}$
4. $i(x) = \dfrac{5x}{2x-x^3}$
5. $j(x) = x^2 - \dfrac{x}{100} - 1$
6. $p(x) = \begin{cases} rclx+3 & \text{if } x<0 \\ -x+3, & \text{if } x \geq 0 \end{cases}.$

Solution The first step in all of these problems is to replace x with $-x$ and simplify.

1.

$$\begin{aligned} f(x) &= \frac{5}{2-x^2} \\ f(-x) &= \frac{5}{2-(-x)^2} \\ f(-x) &= \frac{5}{2-x^2} \\ f(-x) &= f(x) \end{aligned}$$

Hence, f is **even**. A plot of $f(x)$ using GeoGebra is given in Figure 2.23.

This suggests that the graph of f is symmetric about the y-axis, as expected.

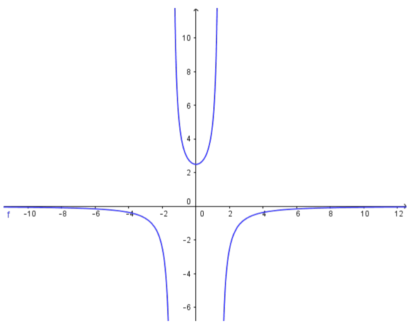


Figure 2.23: The graph of $f(x)$ in Example 32

While the plot provided by the software can provide us with visual evidence that a function is even or odd, this evidence is never conclusive. The only way to know for sure is to check analytically using the definitions of even and odd functions.

2.

$$\begin{aligned} g(x) &= \frac{5x}{2-x^2} \\ g(-x) &= \frac{5(-x)}{2-(-x)^2} \\ g(-x) &= \frac{-5x}{2-x^2} \end{aligned}$$

It doesn't appear that $g(-x)$ is equivalent to $g(x)$. To prove this, we check with an x value. After some trial and error, we see that $g(1) = 5$ whereas $g(-1) = -5$. This proves that g is not even, but it doesn't rule out the possibility that g is odd. (Why not?) To check if g is odd, we compare $g(-x)$ with $-g(x)$

$$\begin{aligned} -g(x) &= -\frac{5x}{2-x^2} \\ &= \frac{-5x}{2-x^2} \\ -g(x) &= g(-x) \end{aligned}$$

Hence, g is odd: see Figure 2.24.

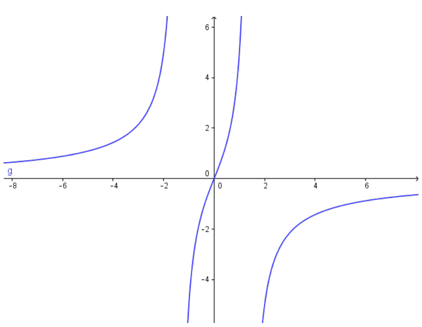


Figure 2.24: The graph of $g(x)$ in Example 32

3.

$$\begin{aligned} h(x) &= \frac{5x}{2-x^3} \\ h(-x) &= \frac{5(-x)}{2-(-x)^3} \\ h(-x) &= \frac{-5x}{2+x^3} \end{aligned}$$

Once again, $h(-x)$ doesn't appear to be equivalent to $h(x)$. We check with an x value, for example, $h(1) = 5$ but $h(-1) = -\frac{5}{3}$. This proves that h is not even and it also shows h is not odd. (Why?)

In Figure 2.25, the graph of h appears to be neither symmetric about the y-axis nor the origin.

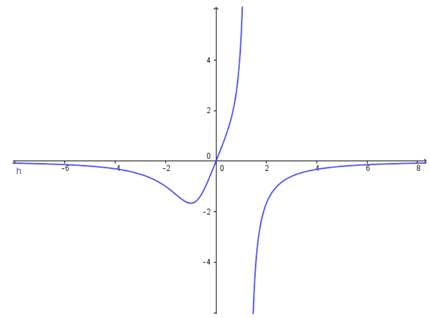


Figure 2.25: The graph of $h(x)$ in Example 32

4.

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ i(-x) &= \frac{5(-x)}{2(-x)-(-x)^3} \\ i(-x) &= \frac{-5x}{-2x+x^3} \end{aligned}$$

The expression $i(-x)$ doesn't appear to be equivalent to $i(x)$. However, after checking some x values, for example $x = 1$ yields $i(1) = 5$ and $i(-1) = 5$, it appears that $i(-x)$ does, in fact, equal $i(x)$. However, while this suggests i is even, it doesn't prove it. (It does, however, prove i is not odd.) To prove $i(-x) = i(x)$, we need to manipulate our expressions for $i(x)$ and $i(-x)$ and show that they are equivalent. A clue as to how to proceed is in the numerators: in the formula for $i(x)$, the numerator is $5x$ and in $i(-x)$ the numerator is $-5x$. To re-write $i(x)$ with a numerator of $-5x$, we need to multiply its numerator by -1. To keep the value of the fraction the same, we need to multiply the denominator by -1 as well. Thus

$$\begin{aligned} i(x) &= \frac{5x}{2x-x^3} \\ &= \frac{(-1)5x}{(-1)\left(2x-x^3\right)} \\ &= \frac{-5x}{-2x+x^3} \end{aligned}$$

Hence, $i(x) = i(-x)$, so i is even. See Figure 2.26 for the graph.

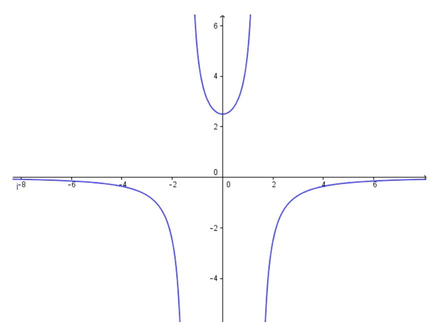


Figure 2.26: The graph of $i(x)$ in Example 32

5.

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ j(-x) &= (-x)^2 - \frac{-x}{100} - 1 \\ j(-x) &= x^2 + \frac{x}{100} - 1 \end{aligned}$$

The expression for $j(-x)$ doesn't seem to be equivalent to $j(x)$, so we check using $x = 1$ to get $j(1) = -\frac{1}{100}$ and $j(-1) = \frac{1}{100}$. This rules out j being even. However, it doesn't rule out j being odd. Examining $-j(x)$ gives

$$\begin{aligned} j(x) &= x^2 - \frac{x}{100} - 1 \\ -j(x) &= -\left(x^2 - \frac{x}{100} - 1\right) \\ -j(x) &= -x^2 + \frac{x}{100} + 1 \end{aligned}$$

The expression $-j(x)$ doesn't seem to match $j(-x)$ either. Testing $x = 2$ gives $j(2) = \frac{149}{50}$ and $j(-2) = \frac{151}{50}$, so j is not odd, either.

Notice in Figure 2.27 that the computer plot seems to suggests that the graph of j is symmetric about the y-axis which would imply that j is even. However, we have proven that is not the case. The problem is that the effect of the $x/100$ term is so small, our eyes don't detect it in the graph.

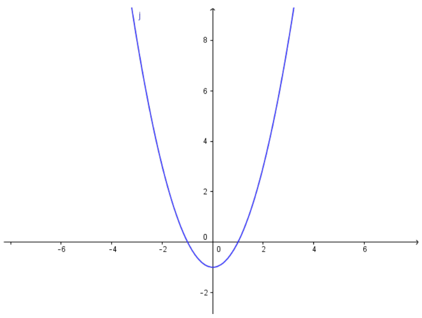


Figure 2.27: The graph of $j(x)$ in Example 32

6. Testing the graph of $y = p(x)$ for symmetry is complicated by the fact $p(x)$ is a piecewise-defined function. As always, we handle this by checking the condition for symmetry by checking it on each piece of the domain. We first consider the case when $x < 0$ and set about finding the correct expression for $p(-x)$. Even though $p(x) = x+3$ for $x < 0$, $p(-x) \neq -x+3$ here. The reason for this is that since $x < 0$, $-x > 0$ which means to find $p(-x)$, we need to use the *other* formula for $p(x)$, namely $p(x) = -x + 3$. Hence, for $x < 0$, $p(-x) = -(-x) + 3 = x + 3 = p(x)$. For $x \geq 0$, $p(x) = -x + 3$ and we have two cases. If $x > 0$, then $-x < 0$ so $p(-x) = (-x) + 3 = -x + 3 = p(x)$. If $x = 0$, then $p(0) = 3 = p(-0)$. Hence, in all cases, $p(-x) = p(x)$, so p is even. Since $p(0) = 3$ but $p(-0) = p(0) = 3 \neq -3$, we also have p is not odd.

   In Figure 2.28, we see that the graph appears to be symmetric about the y-axis.

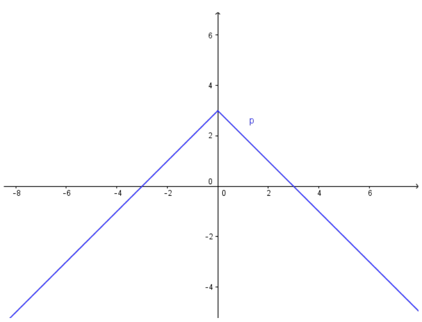


Figure 2.28: The graph of $p(x)$ in Example 32

There are two lessons to be learned from the last example. The first is that sampling function values at particular x values is not enough to prove that a function is even or odd — despite the fact that $j(-1) = -j(1)$, j turned out not to be odd. Secondly, while the calculator may *suggest* mathematical truths, it is the Algebra which *proves* mathematical truths. (Or, in other words, don't rely too heavily on the machine!)

2.5.1 General Function Behaviour

The last topic we wish to address in this section is general function behaviour. As you shall see in the next several chapters, each family of functions has its own unique attributes and we will study them all in great detail. The purpose of this section's discussion, then, is to lay the foundation for that further study by investigating aspects of function behaviour which apply to all functions. To start, we will examine the concepts of **increasing**, **decreasing** and **constant**. Before defining the concepts algebraically, it is instructive to first look at them graphically. Consider the graph of the function f in Figure 2.29.

Reading from left to right, the graph 'starts' at the point $(-4, -3)$ and 'ends' at the point $(6, 5.5)$. If we imagine walking from left to right on the graph, between $(-4, -3)$ and $(-2, 4.5)$, we are walking 'uphill'; then between $(-2, 4.5)$ and $(3, -8)$, we are walking 'downhill'; and between $(3, -8)$ and $(4, -6)$, we are walking 'uphill' once more. From $(4, -6)$ to $(5, -6)$, we 'level off', and then resume walking 'uphill' from $(5, -6)$ to $(6, 5.5)$. In other words, for the x values between -4 and -2 (inclusive), the y-coordinates on the graph are getting larger, or **increasing**, as we move from left to right. Since $y = f(x)$, the y values on the graph are the function values, and we say that the function f is **increasing** on the interval $[-4, -2]$. Analogously, we say that f is **decreasing** on the interval $[-2, 3]$ increasing once more on the interval $[3, 4]$, **constant** on $[4, 5]$, and finally increasing once again on $[5, 6]$. It is extremely important to notice that the behaviour (increasing, decreasing or constant) occurs on an interval on the x-axis. When we say that the function f is increasing on $[-4, -2]$ we do not mention the actual y values that f attains along the way. Thus, we report *where* the behaviour occurs, not to what extent the behaviour occurs. Also notice that we do not say that a function is increasing, decreasing or constant at a single x value. In fact, we would run into serious trouble in our previous example if we tried to do so because $x = -2$ is contained in an interval on which f was increasing and one on which it is decreasing. (There's more on this issue – and many others – in the Exercises.)

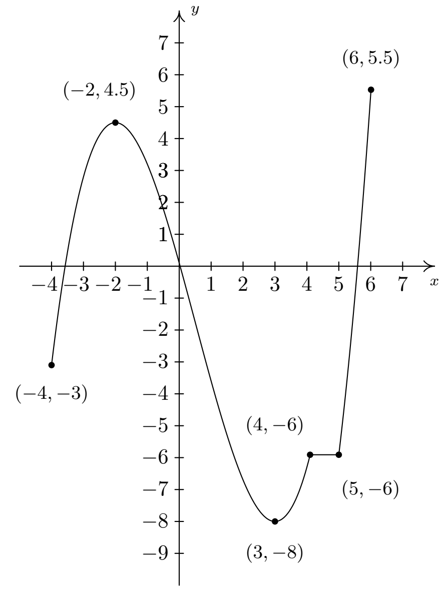


Figure 2.29: The graph $y = f(x)$

We're now ready for the more formal algebraic definitions of what it means for a function to be increasing, decreasing or constant.

Definition 27 Increasing, decreasing, and constant functions

Suppose f is a function defined on an interval I. We say f is:

- **increasing** on I if and only if $f(a) < f(b)$ for all real numbers a, b in I with $a < b$.
- **decreasing** on I if and only if $f(a) > f(b)$ for all real numbers a, b in I with $a < b$.
- **constant** on I if and only if $f(a) = f(b)$ for all real numbers a, b in I.

The notions of how quickly or how slowly a function increases or decreases are explored in Calculus.

It is worth taking some time to see that the algebraic descriptions of increasing, decreasing and constant as stated in Definition 27 agree with our graphical descriptions given earlier. You should look back through the examples and exercise sets in previous sections where graphs were given to see if you can determine the intervals on which the functions are increasing, decreasing or constant. Can you find an example of a function for which none of the concepts in Definition 27 apply?

Now let's turn our attention to a few of the points on the graph. Clearly the point $(-2, 4.5)$ does not have the largest y value of all of the points on the graph of f — indeed that honour goes to $(6, 5.5)$ — but $(-2, 4.5)$ should get some sort of consolation prize for being 'the top of the hill' between $x = -4$ and $x = 3$. We say that the function f has a **local maximum** (or **relative maximum**) at the point $(-2, 4.5)$, because the y-coordinate 4.5 is the largest y-value (hence, function value) on the curve 'near' $x = -2$. Similarly, we say that the function f has a **local minimum** (or **relative minimum**) at the point $(3, -8)$, since the y-coordinate -8 is the smallest function value near $x = 3$. Although it is tempting to say that local extrema occur when the function changes from increasing to decreasing or vice versa, it is not a precise enough way to define the concepts for the needs of Calculus. At the risk of being pedantic, we will present the traditional definitions and thoroughly vet the pathologies they induce in the Exercises. We have one

Typically, in (pre)calculus, whenever you're told that something occurs 'near' a given point, you should read this as 'on some open interval I containing that point'.

last observation to make before we proceed to the algebraic definitions and look at a fairly tame, yet helpful, example.

If we look at the entire graph, we see that the largest y value (the largest function value) is 5.5 at $x = 6$. In this case, we say the **maximum** (often called the 'absolute' or 'global' maximum) of f is 5.5; similarly, the **minimum** (again, 'absolute' or 'global' minimum can be used.) of f is -8.

'Maxima' is the plural of 'maximum' and 'mimima' is the plural of 'minimum'. 'Extrema' is the plural of 'extremum' which combines maximum and minimum.

We formalize these concepts in the following definitions.

Definition 28 Local maximum and minimum

Suppose f is a function with $f(a) = b$.

- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f' in this case.
- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f' in this case.
- The value b is called the **maximum** of f if $b \geq f(x)$ for all x in the domain of f.
- The value b is called the **minimum** of f if $b \leq f(x)$ for all x in the domain of f.

It's important to note that not every function will have all of these features. Indeed, it is possible to have a function with no local or absolute extrema at all! (Any ideas of what such a function's graph would have to look like?) We shall see examples of functions in the Exercises which have one or two, but not all, of these features, some that have instances of each type of extremum and some functions that seem to defy common sense. In all cases, though, we shall adhere to the algebraic definitions above as we explore the wonderful diversity of graphs that functions provide us.

Here is the 'tame' example which was promised earlier. It summarizes all of the concepts presented in this section as well as some from previous sections so you should spend some time thinking deeply about it before proceeding to the Exercises.

Example 33 A 'tame' example

Given the graph of $y = f(x)$ in Figure 2.30, answer all of the following questions.

1. Find the domain of f.
2. Find the range of f.
3. List the x-intercepts, if any exist.
4. List the y-intercepts, if any exist.
5. Find the zeros of f.
6. Solve $f(x) < 0$.

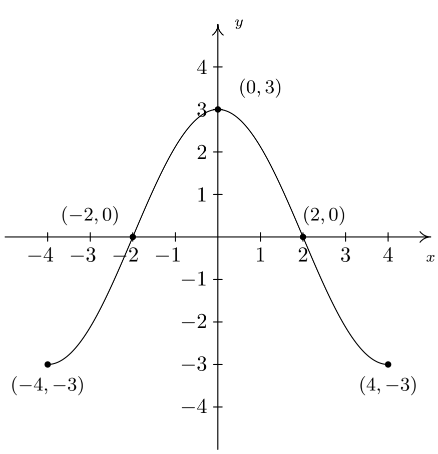


Figure 2.30: The graph for Example 33

7. Determine $f(2)$.

8. Solve $f(x) = -3$.

9. Find the number of solutions to $f(x) = 1$.

10. Does f appear to be even, odd, or neither?

11. List the intervals on which f is increasing.

12. List the intervals on which f is decreasing.

13. List the local maximums, if any exist.

14. List the local minimums, if any exist.

15. Find the maximum, if it exists.

16. Find the minimum, if it exists.

Solution

1. To find the domain of f, we proceed as in Section 2.2. By projecting the graph to the x-axis, we see that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

4. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

5. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

6. To solve $f(x) < 0$, we look for the x values of the points on the graph where the y-coordinate is less than 0. Graphically, we are looking for where the graph is below the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

7. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

8. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

9. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate is 1. Even though these points aren't specified, we see that the curve has two points with a y value of 1, as seen in the graph below. That means there are two solutions to $f(x) = 1$: see Figure 2.31.

10. The graph appears to be symmetric about the y-axis. This suggests (but does not prove) that f is even.

11. As we move from left to right, the graph rises from $(-4, -3)$ to $(0, 3)$. This means f is increasing on the interval $[-4, 0]$. (Remember, the answer here is an interval on the x-axis.)

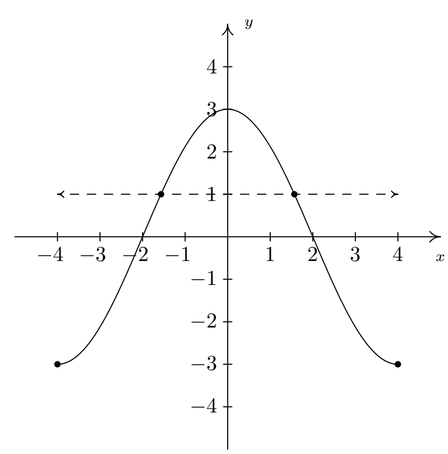


Figure 2.31: Solving $f(x) = 1$ in Example 33

12. As we move from left to right, the graph falls from $(0, 3)$ to $(4, -3)$. This means f is decreasing on the interval $[0, 4]$. (Remember, the answer here is an interval on the x-axis.)

13. The function has its only local maximum at $(0, 3)$ so $f(0) = 3$ is the local minimum value.

14. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval I which contains $x = -4$ such that $f(-4) < f(x)$ for all x in I different from -4. But if we put an open interval around $x = -4$ a portion of that interval will lie outside of the domain of f. Because we are unable to satisfy the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason — no open interval around $x = 4$ stays within the domain of f.

15. The maximum value of f is the largest y-coordinate which is 3.

16. The minimum value of f is the smallest y-coordinate which is -3.

In general, the problem of finding maximum and minimum values, requires the techniques of Calculus. We will explore this in Chapter 12. In the meantime, we'll have to rely on technology to assist us. Most graphing calculators and many mathematics software programs have 'Minimum' and 'Maximum' features which can be used to approximate these values, as we now demonstrate.

Example 34 Using the computer to find maxima and minima

Let $f(x) = \dfrac{15x}{x^2 + 3}$. Use the computer or a graphing calculator to approximate the intervals on which f is increasing and those on which it is decreasing. Approximate all extrema.

SOLUTION Using GeoGebra, we enter `f(x) = 15x/(x^2+3)` to plot the graph of f. The command `Max[f,-3,3]` then calculates the maximum value of f on the interval $[-3, 3]$. Similarly, `Min[f,-3,3]` gives the minimum value of f on the interval $[-3, 3]$. The graph of f, together with the local maximum and local minimum, are plotted in Figure 2.32.

To two decimal places, f appears to have its only local minimum at $(-1.73, -4.33)$ and its only local maximum at $(1.73, 4.33)$. Given the symmetry about the origin suggested by the graph, the relation between these points shouldn't be too surprising. The function appears to be increasing on $[-1.73, 1.73]$ and decreasing on $(-\infty, -1.73] \cup [1.73, \infty)$. This makes -4.33 the (absolute) minimum and 4.33 the (absolute) maximum.

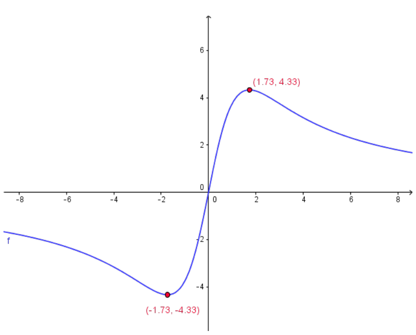


Figure 2.32: The local maximum and minimum of $f(x) = \dfrac{15x}{x^2 + 3}$ in Example 34

Example 35 Minimizing distance from a graph to the origin

Find the points on the graph of $y = (x - 3)^2$ which are closest to the origin. Round your answers to two decimal places.

SOLUTION Suppose a point (x, y) is on the graph of $y = (x - 3)^2$. Its distance to the origin $(0, 0)$ is given by

$$\begin{aligned} d &= \sqrt{(x-0)^2+(y-0)^2} \\ &= \sqrt{x^2+y^2} \\ &= \sqrt{x^2+\left[(x-3)^2\right]^2} \qquad \text{Since } y=(x-3)^2 \\ &= \sqrt{x^2+(x-3)^4} \end{aligned}$$

Given a value for x, the formula $d = \sqrt{x^2+(x-3)^4}$ is the distance from $(0,0)$ to the point (x,y) on the curve $y=(x-3)^2$. What we have defined, then, is a function $d(x)$ which we wish to minimize over all values of x. To accomplish this task analytically would require Calculus so as we've mentioned before, we can use a graphing calculator to find an approximate solution. Using Geogebra, we enter the function $d(x)$ as shown below and graph.

Using the Minimum feature, we see above on the right that the (absolute) minimum occurs near $x = 2$. Rounding to two decimal places, we get that the minimum distance occurs when $x = 2.00$. To find the y value on the parabola associated with $x = 2.00$, we substitute 2.00 into the equation to get $y = (x-3)^2 = (2.00-3)^2 = 1.00$. So, our final answer is $(2.00, 1.00)$.

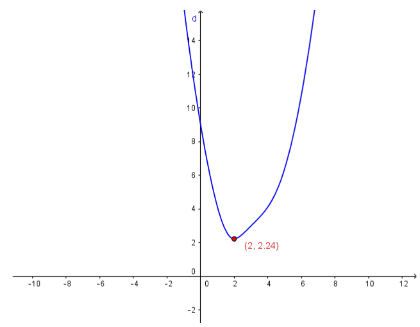


Figure 2.33: Minimizing $d(x)$ in Example 35

It seems silly to list a final answer as $(2.00, 1.00)$. Indeed, Calculus confirms that the *exact* answer to this problem is, in fact, $(2,1)$. As you are well aware by now, the authors are overly pedantic, and as such, use the decimal places to remind the reader that *any* result garnered from a calculator in this fashion is an approximation, and should be treated as such. (What does the y value calculated by GeoGebra in Figure 2.33 mean in this problem?)

Exercises 2.5

Problems

In Exercises 1 – 12, sketch the graph of the given function. State the domain of the function, identify any intercepts and test for symmetry.

1. $f(x) = 2 - x$

2. $f(x) = \dfrac{x-2}{3}$

3. $f(x) = x^2 + 1$

4. $f(x) = 4 - x^2$

5. $f(x) = 2$

6. $f(x) = x^3$

7. $f(x) = x(x-1)(x+2)$

8. $f(x) = \sqrt{x-2}$

9. $f(x) = \sqrt{5-x}$

10. $f(x) = 3 - 2\sqrt{x+2}$

11. $f(x) = \sqrt[3]{x}$

12. $f(x) = \dfrac{1}{x^2+1}$

In Exercises 13 – 20, sketch the graph of the given piecewise-defined function.

13. $f(x) = \begin{cases} 4-x & \text{if} \quad x \le 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$

14. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le 0 \\ 2x & \text{if} \quad x > 0 \end{cases}$

15. $f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x-3 & \text{if} \quad 0 \le x \le 3 \\ 3 & \text{if} \quad x > 3 \end{cases}$

16. $f(x) = \begin{cases} x^2-4 & \text{if} \quad x \le -2 \\ 4-x^2 & \text{if} \quad -2 < x < 2 \\ x^2-4 & \text{if} \quad x \ge 2 \end{cases}$

17. $f(x) = \begin{cases} -2x-4 & \text{if} \quad x < 0 \\ 3x & \text{if} \quad x \ge 0 \end{cases}$

18. $f(x) = \begin{cases} \sqrt{x+4} & \text{if} \quad -4 \le x < 5 \\ \sqrt{x-1} & \text{if} \quad x \ge 5 \end{cases}$

19. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le -2 \\ 3-x & \text{if} \quad -2 < x < 2 \\ 4 & \text{if} \quad x \ge 2 \end{cases}$

20. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad -6 < x < -1 \\ x & \text{if} \quad -1 < x < 1 \\ \sqrt{x} & \text{if} \quad 1 < x < 9 \end{cases}$

In Exercises 21 – 41, determine analytically if the following functions are even, odd or neither.

21. $f(x) = 7x$

22. $f(x) = 7x + 2$

23. $f(x) = 7$

24. $f(x) = 3x^2 - 4$

25. $f(x) = 4 - x^2$

26. $f(x) = x^2 - x - 6$

27. $f(x) = 2x^3 - x$

28. $f(x) = -x^5 + 2x^3 - x$

29. $f(x) = x^6 - x^4 + x^2 + 9$

30. $f(x) = x^3 + x^2 + x + 1$

31. $f(x) = \sqrt{1-x}$

32. $f(x) = \sqrt{1-x^2}$

33. $f(x) = 0$

34. $f(x) = \sqrt[3]{x^2}$

35. $f(x) = \sqrt[3]{x^2}$

36. $f(x) = \dfrac{3}{x^2}$

37. $f(x) = \dfrac{2x-1}{x+1}$

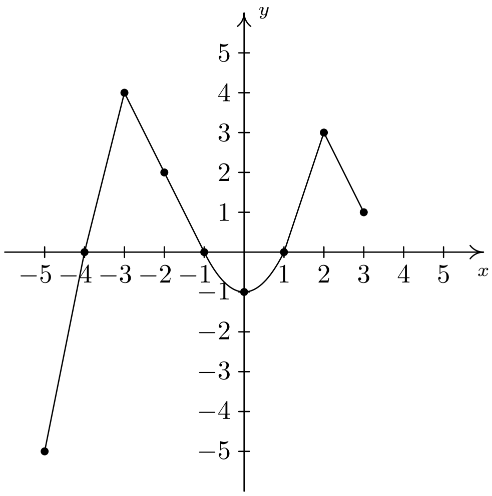
38. $f(x) = \dfrac{3x}{x^2+1}$

39. $f(x) = \dfrac{x^2-3}{x-4x^3}$

40. $f(x) = \dfrac{9}{\sqrt{4-x^2}}$

41. $f(x) = \dfrac{\sqrt[3]{x^3+x}}{5x}$

In Exercises 42 – 57, use the graph of $y = f(x)$ given below to answer the question.



42. Find the domain of f.

43. Find the range of f.

44. Determine $f(-2)$.

45. Solve $f(x) = 4$.

46. List the x-intercepts, if any exist.

47. List the y-intercepts, if any exist.

48. Find the zeros of f.

49. Solve $f(x) \geq 0$.

50. Find the number of solutions to $f(x) = 1$.

51. Does f appear to be even, odd, or neither?

52. List the intervals where f is increasing.

53. List the intervals where f is decreasing.

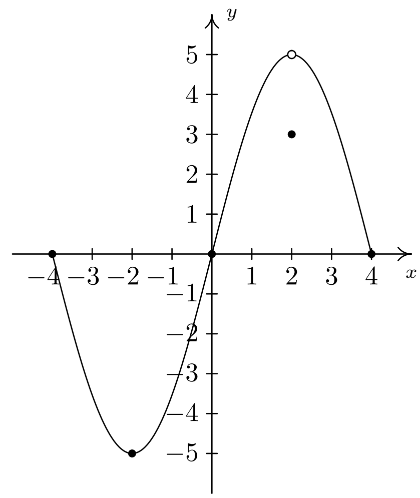
54. List the local maximums, if any exist.

55. List the local minimums, if any exist.

56. Find the maximum, if it exists.

57. Find the minimum, if it exists.

In Exercises 58 – 73, use the graph of $y = f(x)$ given below to answer the question.



58. Find the domain of f.

59. Find the range of f.

60. Determine $f(2)$.

61. Solve $f(x) = -5$.

62. List the x-intercepts, if any exist.

63. List the y-intercepts, if any exist.

64. Find the zeros of f.

65. Solve $f(x) \leq 0$.

66. Find the number of solutions to $f(x) = 3$.

67. Does f appear to be even, odd, or neither?

68. List the intervals where f is increasing.

69. List the intervals where f is decreasing.

70. List the local maximums, if any exist.

71. List the local minimums, if any exist.

72. Find the maximum, if it exists.

73. Find the minimum, if it exists.

In Exercises 74 – 77, use a graphing calculator or software (such as GeoGebra) to approximate the local and absolute extrema of the given function. Approximate the intervals on which the function is increasing and those on which it is decreasing. Round your answers to two decimal places.

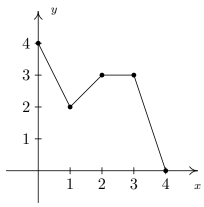
74. $f(x) = x^4 - 3x^3 - 24x^2 + 28x + 48$

75. $f(x) = x^{2/3}(x - 4)$

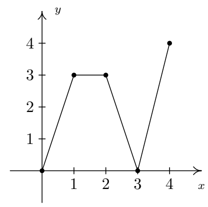
76. $f(x) = \sqrt{9 - x^2}$

77. $f(x) = x\sqrt{9 - x^2}$

In Exercises 78 – 85, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.



$y = f(x)$



$y = g(x)$

78. $(f + g)(0)$

79. $(f + g)(1)$

80. $(f - g)(1)$

81. $(g - f)(2)$

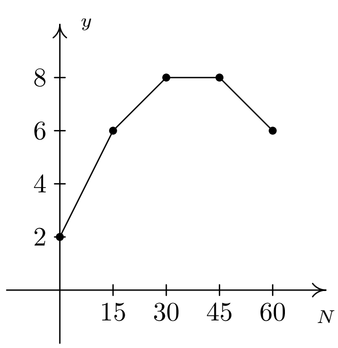
82. $(fg)(2)$

83. $(fg)(1)$

84. $\left(\dfrac{f}{g}\right)(4)$

85. $\left(\dfrac{g}{f}\right)(2)$

The graph below represents the height h of a Sasquatch (in feet) as a function of its age N in years. Use it to answer the questions in Exercises 86 – 90.



86. Find and interpret $h(0)$.

87. How tall is the Sasquatch when she is 15 years old?

88. Solve $h(N) = 6$ and interpret.

89. List the interval over which h is constant and interpret your answer.

90. List the interval over which h is decreasing and interpret your answer.

For Exercises 91 – 93, let $f(x) = \lfloor x \rfloor$ be the greatest integer function as defined in Exercise 76 in Section 2.3.

91. Graph $y = f(x)$. Be careful to correctly describe the behaviour of the graph near the integers.

92. Is f even, odd, or neither? Explain.

93. Discuss with your classmates which points on the graph are local minimums, local maximums or both. Is f ever increasing? Decreasing? Constant?

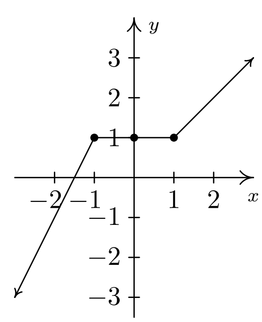
94. In Exercise 72 in Section 2.3, we saw that the population of Sasquatch in Portage County could be modeled by the function $P(t) = \dfrac{150t}{t + 15}$, where $t = 0$ represents the year 1803. Use your graphing calculator to analyze the general function behaviour of P. Will there ever be a time when 200 Sasquatch roam Portage County?

95. Suppose f and g are both even functions. What can be said about the functions $f + g$, $f - g$, fg and $\frac{f}{g}$? What if f and g are both odd? What if f is even but g is odd?

96. One of the most important aspects of the Cartesian Coordinate Plane is its ability to put Algebra into geometric terms and Geometry into algebraic terms. We've spent most of this chapter looking at this very phenomenon and now you should spend some time with your classmates reviewing what we've done. What major results do we have that tie Algebra and Geometry together? What concepts from Geometry have we not yet described algebraically? What topics from Intermediate Algebra have we not yet discussed geometrically?

It's now time to "thoroughly vet the pathologies induced" by the precise definitions of local maximum and local minimum. You and your classmates should carefully discuss Exercises 97 – 99. You will need to refer back to Definition 27 (Increasing, Decreasing and Constant) and Definition 28 (Maximum and Minimum) during the discussion.

97. Consider the graph of the function f given below.



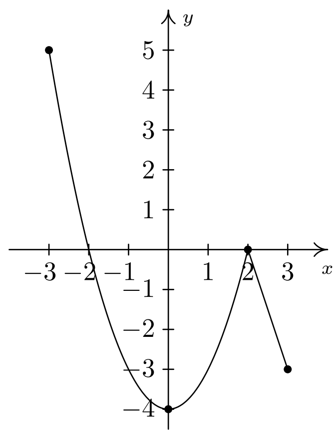
(a) Show that f has a local maximum but not a local minimum at the point $(-1, 1)$.

(b) Show that f has a local minimum but not a local maximum at the point $(1, 1)$.

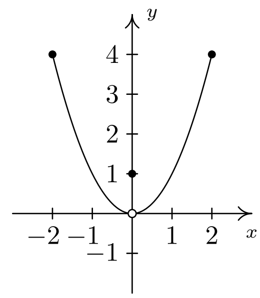
(c) Show that f has a local maximum AND a local minimum at the point $(0, 1)$.

(d) Show that f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

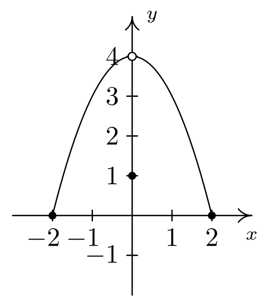
98. Using Example 33 as a guide, show that the function g whose graph is given below does <u>not</u> have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute. What's unique about the point $(0, -4)$ on this graph? Also find the intervals on which g is increasing and those on which g is decreasing.



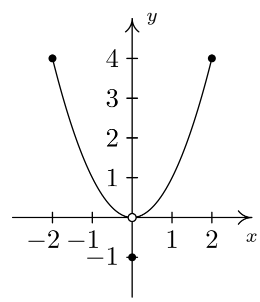
99. We said earlier in the section that it is not good enough to say local extrema exist where a function changes from increasing to decreasing or vice versa. As a previous exercise showed, we could have local extrema when a function is constant so now we need to examine some functions whose graphs do indeed change direction. Consider the functions graphed below. Notice that all four of them change direction at an open circle on the graph. Examine each for local extrema. What is the effect of placing the "dot" on the y-axis above or below the open circle? What could you say if no function value were assigned to $x = 0$?



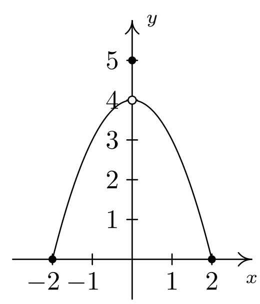
(a)



(b)



(c)



(d)

2.6 Transformations

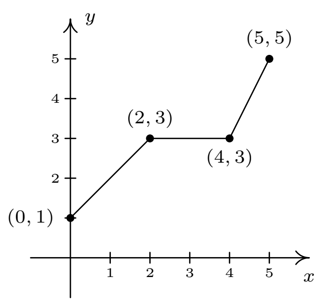


Figure 2.35: The graph of a function f

In this section, we study how the graphs of functions change, or **transform**, when certain specialized modifications are made to their formulas. The transformations we will study fall into three broad categories: shifts, reflections and scalings, and we will present them in that order. Suppose that Figure 2.35 the complete graph of a function f.

The Fundamental Graphing Principle for Functions says that for a point (a, b) to be on the graph, $f(a) = b$. In particular, we know $f(0) = 1$, $f(2) = 3$, $f(4) = 3$ and $f(5) = 5$. Suppose we wanted to graph the function defined by the formula $g(x) = f(x) + 2$. Let's take a minute to remind ourselves of what g is doing. We start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. In order to graph g, we need to graph the points $(x, g(x))$. How are we to find the values for $g(x)$ without a formula for $f(x)$? The answer is that we don't need a *formula* for $f(x)$, we just need the *values* of $f(x)$. The values of $f(x)$ are the y values on the graph of $y = f(x)$. For example, using the points indicated on the graph of f, we can make the following table.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	3	$(0, 3)$
2	$(2, 3)$	3	5	$(2, 5)$
4	$(4, 3)$	3	5	$(4, 5)$
5	$(5, 5)$	5	7	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$, so $g(a) = f(a) + 2 = b + 2$. Hence, $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f. Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is usually described as 'shifting the graph up 2 units'. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four points we moved in the same manner in which they were connected before: see Figure 2.34.

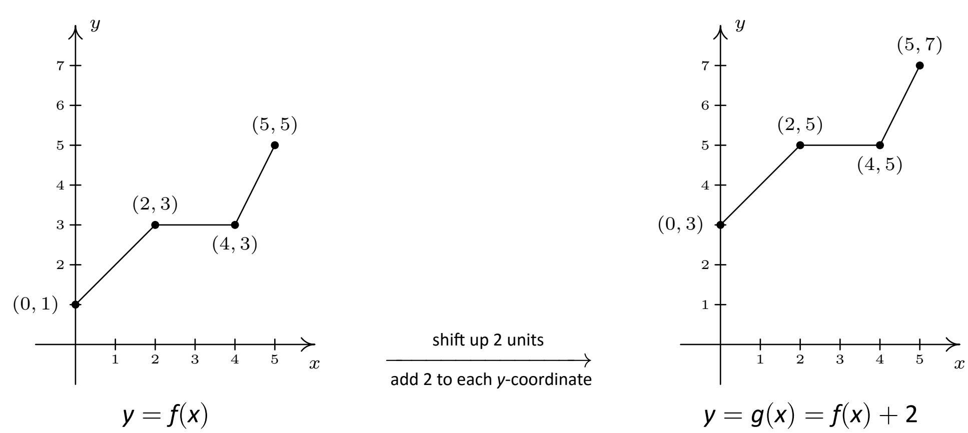


Figure 2.34: Shifting the graph of f up by 2 units

You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range. You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. What we have discussed is generalized in the following theorem.

Theorem 7 Vertical Shifts

Suppose f is a function and k is a positive number.

- To graph $y = f(x) + k$, shift the graph of $y = f(x)$ up k units by adding k to the y-coordinates of the points on the graph of f.
- To graph $y = f(x) - k$, shift the graph of $y = f(x)$ down k units by subtracting k from the y-coordinates of the points on the graph of f.

The key to understanding Theorem 7 and, indeed, all of the theorems in this section comes from an understanding of the Fundamental Graphing Principle for Functions. If (a, b) is on the graph of f, then $f(a) = b$. Substituting $x = a$ into the equation $y = f(x) + k$ gives $y = f(a) + k = b + k$. Hence, $(a, b + k)$ is on the graph of $y = f(x) + k$, and we have the result. In the language of 'inputs' and 'outputs', Theorem 7 can be paraphrased as "Adding to, or subtracting from, the *output* of a function causes the graph to shift up or down, respectively." So what happens if we add to or subtract from the *input* of the function?

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function.[1] Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0, 1)$	1	$f(0+2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$f(2+2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(4+2) = f(6) = ?$	
5	$(5, 5)$	5	$f(5+2) = f(7) = ?$	

When we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$. What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ is on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives

[1] We have spent a lot of time in this text showing you that $f(x + 2)$ and $f(x) + 2$ are, in general, wildly different algebraic animals. We will see momentarily that their geometry is also dramatically different.

$g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we get

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(5) = 5$	$(3, 5)$

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2, b)$ is on the graph of $y = g(x)$. The point $(a-2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.

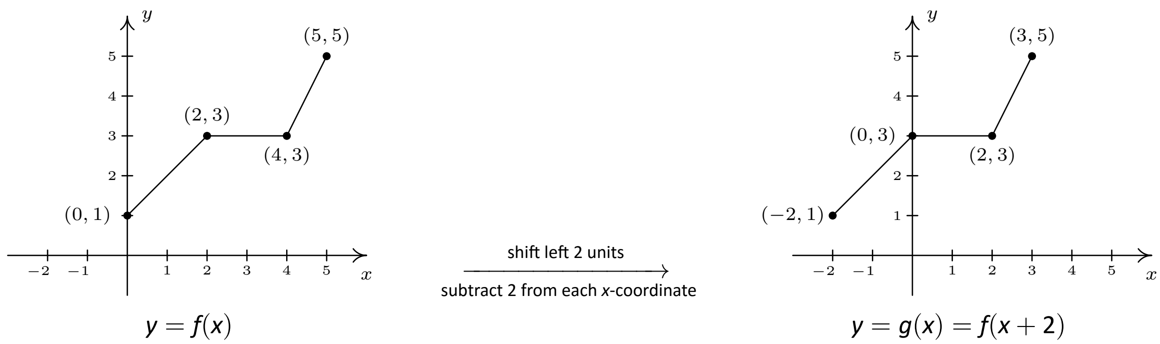


Figure 2.36: Shifting the graph of f left by 2 units

Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 8 Horizontal Shifts

Suppose f is a function and h is a positive number.

- To graph $y = f(x+h)$, shift the graph of $y = f(x)$ left h units by subtracting h from the x-coordinates of the points on the graph of f.
- To graph $y = f(x-h)$, shift the graph of $y = f(x)$ right h units by adding h to the x-coordinates of the points on the graph of f.

In other words, Theorem 8 says that adding to or subtracting from the *input* to a function amounts to shifting the graph left or right, respectively. Theorems 7 and 8 present a theme which will run common throughout the section: changes to the outputs from a function affect the y-coordinates of the graph, resulting in some kind of vertical change; changes to the inputs to a function affect the x-coordinates of the graph, resulting in some kind of horizontal change.

Example 36 **Transforming with vertical and horizontal shifts**

1. Graph $f(x) = \sqrt{x}$. Plot at least three points.
2. Use your graph in 1 to graph $g(x) = \sqrt{x} - 1$.
3. Use your graph in 1 to graph $j(x) = \sqrt{x-1}$.
4. Use your graph in 1 to graph $m(x) = \sqrt{x+3} - 2$.

Solution

1. Owing to the square root, the domain of f is $x \geq 0$, or $[0, \infty)$. We choose perfect squares to build our table and graph below. From the graph we verify the domain of f is $[0, \infty)$ and the range of f is also $[0, \infty)$. The original function is plotted in Figure 2.37

2. The domain of g is the same as the domain of f, since the only condition on both functions is that $x \geq 0$. If we compare the formula for $g(x)$ with $f(x)$, we see that $g(x) = f(x) - 1$. In other words, we have subtracted 1 from the output of the function f. By Theorem 7, we know that in order to graph g, we shift the graph of f down one unit by subtracting 1 from each of the y-coordinates of the points on the graph of f. Applying this to the three points we have specified on the graph, we move $(0,0)$ to $(0,-1)$, $(1,1)$ to $(1,0)$, and $(4,2)$ to $(4,1)$. The rest of the points follow suit, and we connect them with the same basic shape as before. We confirm the domain of g is $[0, \infty)$ and find the range of g to be $[-1, \infty)$. The graph of g is given in Figure 2.38.

3. Solving $x - 1 \geq 0$ gives $x \geq 1$, so the domain of j is $[1, \infty)$. To graph j, we note that $j(x) = f(x-1)$. In other words, we are subtracting 1 from the *input* of f. According to Theorem 8, this induces a shift to the right of the graph of f. We add 1 to the x-coordinates of the points on the graph of f and get the result below. The graph reaffirms that the domain of j is $[1, \infty)$ and tells us that the range of j is $[0, \infty)$.

4. To find the domain of m, we solve $x + 3 \geq 0$ and get $[-3, \infty)$. Comparing the formulas of $f(x)$ and $m(x)$, we have $m(x) = f(x+3) - 2$. We have 3 being added to an input, indicating a horizontal shift, and 2 being subtracted from an output, indicating a vertical shift. We leave it to the reader to verify that, in this particular case, the order in which we perform these transformations is immaterial; we will arrive at the same graph regardless as to which transformation we apply first. (We shall see in the next example that order is generally important when applying more than one transformation to a graph.) We follow the convention 'inputs first', and to that end we first tackle the horizontal shift. Letting $m_1(x) = f(x+3)$ denote this intermediate step, Theorem 8 tells us that the graph of $y = m_1(x)$ is the graph of f shifted to the left 3 units. Hence, we subtract 3 from each of the x-coordinates of the points on the graph of f.

   Since $m(x) = f(x+3) - 2$ and $f(x+3) = m_1(x)$, we have $m(x) = m_1(x) - 2$. We can apply Theorem 7 and obtain the graph of m by subtracting 2 from the y-coordinates of each of the points on the graph of $m_1(x)$. The graph verifies that the domain of m is $[-3, \infty)$ and we find the range of m to be $[-2, \infty)$.

x	$f(x)$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$

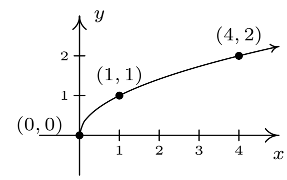


Figure 2.37: The graph $y = f(x) = \sqrt{x}$

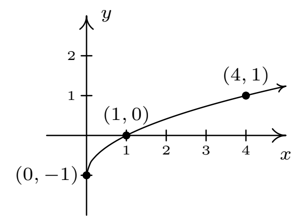


Figure 2.38: Graphing $g(x) = \sqrt{x} - 1$

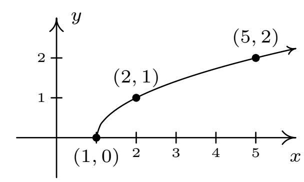


Figure 2.39: Graphing $j(x) = \sqrt{x} - 1$

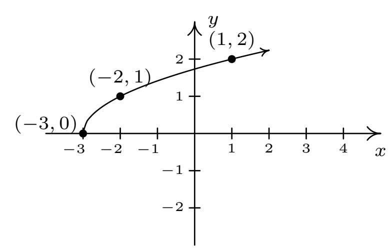


Figure 2.40: Graphing $m_1(x) = \sqrt{x+3}$

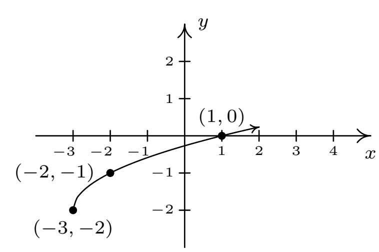


Figure 2.41: Graphing $m(x) = \sqrt{x+3} - 2$

Keep in mind that we can check our answer to any of these kinds of problems by showing that any of the points we've moved lie on the graph of our final answer. For example, we can check that $(-3, -2)$ is on the graph of m by computing $m(-3) = \sqrt{(-3)+3} - 2 = \sqrt{0} - 2 = -2$ ✓

We now turn our attention to reflections. We know from Section 1.3 that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of f reflected across the y-axis. Returning to the language of inputs and outputs, multiplying the output from a function by -1 reflects its graph across the x-axis, while multiplying the input to a function by -1 reflects the graph across the y-axis.[2]

Theorem 9 Reflections

Suppose f is a function.

- To graph $y = -f(x)$, reflect the graph of $y = f(x)$ across the x-axis by multiplying the y-coordinates of the points on the graph of f by -1.
- To graph $y = f(-x)$, reflect the graph of $y = f(x)$ across the y-axis by multiplying the x-coordinates of the points on the graph of f by -1.

Applying Theorem 9 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis

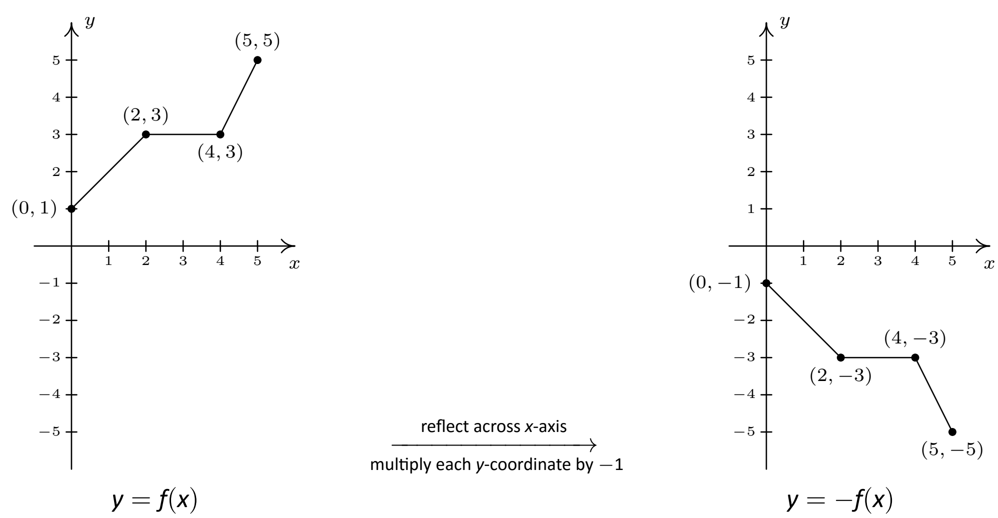


Figure 2.42: Reflecting the graph of f across the x-axis

By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

[2]The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section 2.5 to test if a function was even, odd or neither. The interested reader is invited to explore the role of reflections and symmetry of functions. What happens if you reflect an even function across the y-axis? What happens if you reflect an odd function across the y-axis? What about the x-axis?

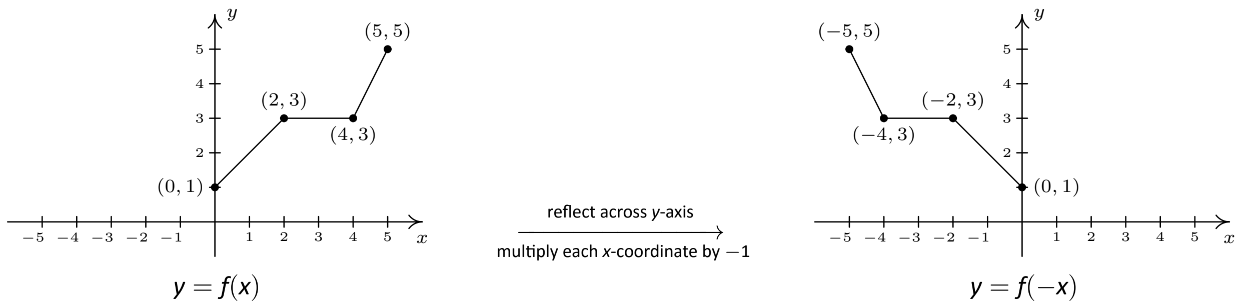


Figure 2.43: Reflecting the graph of f across the y-axis

With the addition of reflections, it is now more important than ever to consider the order of transformations, as the next example illustrates.

Example 37 Graphing reflections

Let $f(x) = \sqrt{x}$. Use the graph of f from Example 36 to graph the following functions. Also, state their domains and ranges.

1. $g(x) = \sqrt{-x}$
2. $j(x) = \sqrt{3-x}$
3. $m(x) = 3 - \sqrt{x}$

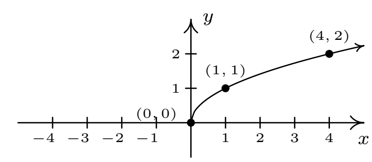


Figure 2.44: The graph $y = f(x)$ from Example 36

Solution

1. The mere sight of $\sqrt{-x}$ usually causes alarm, if not panic. When we discussed domains in Section 2.3, we clearly banished negatives from the radicands of even roots. However, we must remember that x is a variable, and as such, the quantity $-x$ isn't always negative. For example, if $x = -4$, $-x = 4$, thus $\sqrt{-x} = \sqrt{-(-4)} = 2$ is perfectly well-defined. To find the domain analytically, we set $-x \geq 0$ which gives $x \leq 0$, so that the domain of g is $(-\infty, 0]$. Since $g(x) = f(-x)$, Theorem 9 tells us that the graph of g is the reflection of the graph of f across the y-axis. We accomplish this by multiplying each x-coordinate on the graph of f by -1, so that the points $(0,0)$, $(1,1)$, and $(4,2)$ move to $(0,0)$, $(-1,1)$, and $(-4,2)$, respectively. Graphically, we see that the domain of g is $(-\infty, 0]$ and the range of g is the same as the range of f, namely $[0,\infty)$.

   If we had done the reflection first, then $j_1(x) = f(-x)$. Following this by a shift left would give us $j(x) = j_1(x+3) = f(-(x+3)) = f(-x-3) = \sqrt{-x-3}$ which isn't what we want. However, if we did the reflection first and followed it by a shift to the right 3 units, we would have arrived at the function $j(x)$. We leave it to the reader to verify the details.

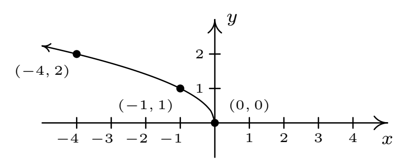


Figure 2.45: Reflecting $y = f(x)$ across the y-axis to obtain the graph of $g(x) = \sqrt{-x}$

2. To determine the domain of $j(x) = \sqrt{3-x}$, we solve $3 - x \geq 0$ and get $x \leq 3$, or $(-\infty, 3]$. To determine which transformations we need to apply to the graph of f to obtain the graph of j, we rewrite $j(x) = \sqrt{-x+3} = f(-x+3)$. Comparing this formula with $f(x) = \sqrt{x}$, we see that not only are we multiplying the input x by -1, which results in a reflection across the y-axis, but also we are adding 3, which indicates a horizontal shift to the left. Does it matter in which order we do the transformations? If so, which order is the correct order? Let's consider the point $(4,2)$ on the graph of f. We refer to the discussion leading up to Theorem 8. We know $f(4) = 2$ and wish to find the point on $y = j(x) = f(-x+3)$ which corresponds to $(4,2)$. We set $-x+3 = 4$ and solve. Our first step is to subtract 3 from both sides to get $-x = 1$. Subtracting 3 from the x-coordinate 4 is shifting the point $(4,2)$ to the left. From $-x = 1$, we then multiply[3]

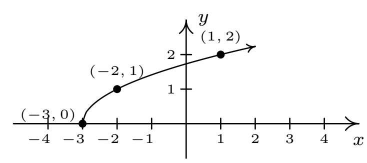


Figure 2.46: The intermediate function $j_1(x) = f(x+3)$

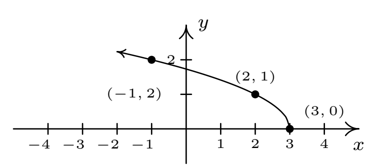


Figure 2.47: Reflecting $y = j_1(x)$ across the y-axis to obtain the graph of $j(x) = \sqrt{3-x}$

[3] Or divide - it amounts to the same thing.

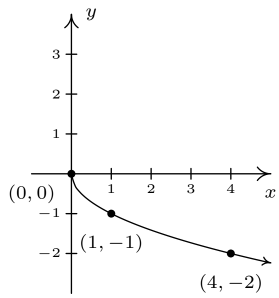


Figure 2.48: Reflecting $y = f(x)$ across the x-axis to obtain the graph of $m_1(x) = -\sqrt{x}$

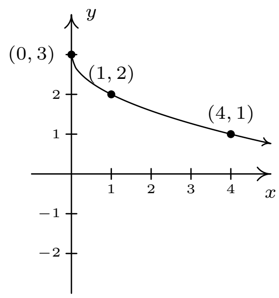
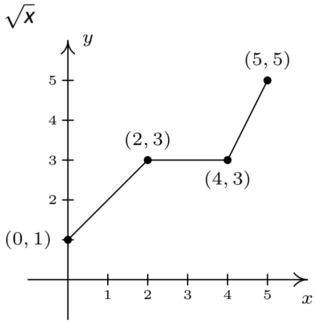
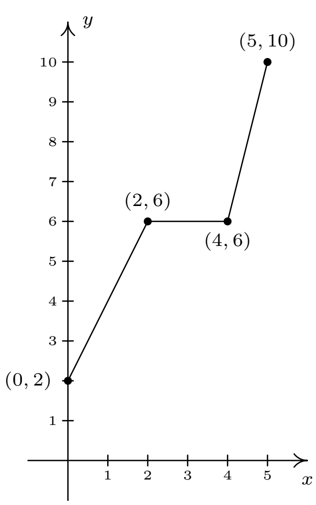


Figure 2.49: Shifting $y = m_1(x)$ up by three units to obtain the graph of $m(x) = 3 - \sqrt{x}$



$y = f(x)$



$y = 2f(x) = g(x)$

Figure 2.50: Graphing $g(x) = 2f(x)$

both sides by -1 to get $x = -1$. Multiplying the x-coordinate by -1 corresponds to reflecting the point about the y-axis. Hence, we perform the horizontal shift first, then follow it with the reflection about the y-axis. Starting with $f(x) = \sqrt{x}$, we let $j_1(x)$ be the intermediate function which shifts the graph of f 3 units to the left, $j_1(x) = f(x+3)$.

To obtain the function j, we reflect the graph of j_1 about y-axis. Theorem 9 tells us we have $j(x) = j_1(-x)$. Putting it all together, we have $j(x) = j_1(-x) = f(-x+3) = \sqrt{-x+3}$, which is what we want. From the graph, we confirm the domain of j is $(-\infty, 3]$ and we get that the range is $[0, \infty)$.

3. The domain of m works out to be the domain of f, $[0, \infty)$. Rewriting $m(x) = -\sqrt{x} + 3$, we see $m(x) = -f(x) + 3$. Since we are multiplying the output of f by -1 and then adding 3, we once again have two transformations to deal with: a reflection across the x-axis and a vertical shift. To determine the correct order in which to apply the transformations, we imagine trying to determine the point on the graph of m which corresponds to $(4, 2)$ on the graph of f. Since in the formula for $m(x)$, the input to f is just x, we substitute to find $m(4) = -f(4) + 3 = -2 + 3 = 1$. Hence, $(4, 1)$ is the corresponding point on the graph of m. If we closely examine the arithmetic, we see that we first multiply $f(4)$ by -1, which corresponds to the reflection across the x-axis, and then we add 3, which corresponds to the vertical shift. If we define an intermediate function $m_1(x) = -f(x)$ to take care of the reflection, we get the graph in Figure 2.48.

   To shift the graph of m_1 up 3 units, we set $m(x) = m_1(x) + 3$. Since $m_1(x) = -f(x)$, when we put it all together, we get $m(x) = m_1(x) + 3 = -f(x) + 3 = -\sqrt{x} + 3$. We see from the graph that the range of m is $(-\infty, 3]$.

We now turn our attention to our last class of transformations known as **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**. Simply put, they do not change the *shape* of the graph, only its position and orientation in the plane. If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be changing the shape of the graph. This type of transformation is called **non-rigid** for obvious reasons. Not only will it be important for us to differentiate between modifying inputs versus outputs, we must also pay close attention to the magnitude of the changes we make. As you will see shortly, the Mathematics turns out to be easier than the associated grammar.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given in Figure 2.35 the beginning of the section. From its graph, we can build a table of values for g as before:

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling (or 'vertical stretching', or 'vertical expansion' or 'vertical dilation') by a factor of 2', and the results are given in Figure 2.50

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling by a factor of $\frac{1}{2}$' (also called 'vertical shrinking', 'vertical compression' or 'vertical contraction' by a factor of 2) as seen in Figure 2.51

These results are generalized in the following theorem.

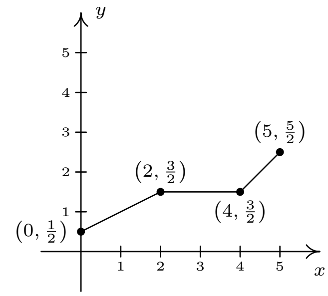


Figure 2.51: Vertical scaling by $\frac{1}{2}$

Theorem 10 Vertical Scalings

Suppose f is a function and $a > 0$. To graph $y = af(x)$, multiply all of the y-coordinates of the points on the graph of f by a. We say the graph of f has been vertically scaled by a factor of a.

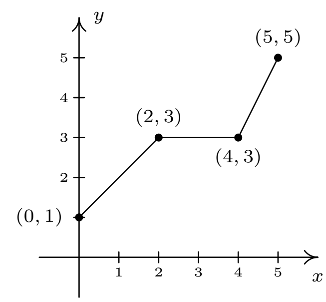
- If $a > 1$, we say the graph of f has undergone a vertical stretching (expansion, dilation) by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrinking (compression, contraction) by a factor of $\frac{1}{a}$.

A few remarks about Theorem 10 are in order. First, a note about the verbiage. To the authors, the words 'stretching', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrinking', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretching by a factor of a' and 'shrinking by a factor of $\frac{1}{a}$' in the statement of the theorem. Second, in terms of inputs and outputs, Theorem 10 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

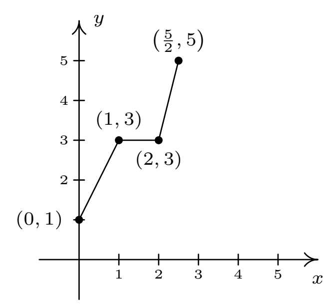
Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 8, as seen in the table below.

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

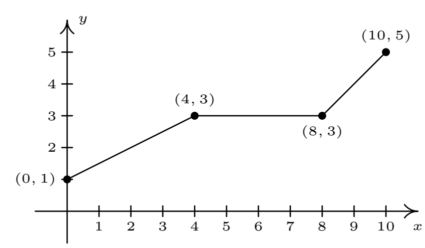
We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we can complete our table as follows:



The graph $y = f(x)$ from Figure 2.35



The graph $y = g(x) = f(2x)$



The graph $y = h(x) = f(\frac{1}{2}x)$

Figure 2.52: The effect of horizontal scaling on a graph

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling by a factor of $\frac{1}{2}$ (also called 'horizontal shrinking', 'horizontal compression' or 'horizontal contraction' by a factor of 2).

If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[4] by a factor of 2. The effect of both horizontal scalings is shown in Figure 2.52.

We have the following theorem.

Theorem 11 Horizontal Scalings.

Suppose f is a function and $b > 0$. To graph $y = f(bx)$, divide all of the x-coordinates of the points on the graph of f by b. We say the graph of f has been horizontally scaled by a factor of $\frac{1}{b}$.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretching (expansion, dilation) by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrinking (compression, contraction) by a factor of b.

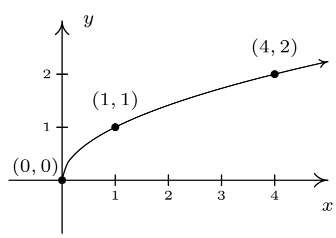


Figure 2.53: The graph $y = \sqrt{x}$

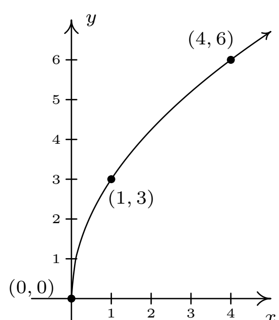


Figure 2.54: The graph $y = g(x) = 3\sqrt{x}$

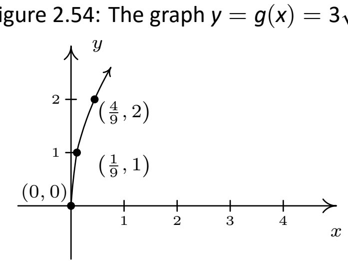


Figure 2.55: The graph $y = j(x) = \sqrt{9x}$

Theorem 11 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$ since the x-values are divided by b to produce corresponding points on the graph of $y = f(bx)$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 38 Applying vertical and horizontal scalings

Let $f(x) = \sqrt{x}$. Use the graph of f from Example 36 (see Figure 2.53) to graph the following functions. Also, state their domains and ranges.

1. $g(x) = 3\sqrt{x}$
2. $j(x) = \sqrt{9x}$
3. $m(x) = 1 - \sqrt{\frac{x+3}{2}}$

Solution

1. First we note that the domain of g is $[0, \infty)$ for the usual reason. Next, we have $g(x) = 3f(x)$ so by Theorem 10, we obtain the graph of g by multiplying all of the y-coordinates of the points on the graph of f by 3. The result is a vertical scaling of the graph of f by a factor of 3. We find the range of g is also $[0, \infty)$. The graph of g is given in Figure 2.54.

[4] Also called 'horizontal stretching', 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

2. To determine the domain of j, we solve $9x \geq 0$ to find $x \geq 0$. Our domain is once again $[0, \infty)$. We recognize $j(x) = f(9x)$ and by Theorem 11, we obtain the graph of j by dividing the x-coordinates of the points on the graph of f by 9. From the graph in Figure 2.55, we see the range of j is also $[0, \infty)$.

3. Solving $\frac{x+3}{2} \geq 0$ gives $x \geq -3$, so the domain of m is $[-3, \infty)$. To take advantage of what we know of transformations, we rewrite $m(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}} + 1$, or $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$. Focusing on the inputs first, we note that the input to f in the formula for $m(x)$ is $\frac{1}{2}x + \frac{3}{2}$. Multiplying the x by $\frac{1}{2}$ corresponds to a horizontal stretching by a factor of 2, and adding the $\frac{3}{2}$ corresponds to a shift to the left by $\frac{3}{2}$. As before, we resolve which to perform first by thinking about how we would find the point on m corresponding to a point on f, in this case, $(4, 2)$. To use $f(4) = 2$, we solve $\frac{1}{2}x + \frac{3}{2} = 4$. Our first step is to subtract the $\frac{3}{2}$ (the horizontal shift) to obtain $\frac{1}{2}x = \frac{5}{2}$. Next, we multiply by 2 (the horizontal stretching) and obtain $x = 5$. We define two intermediate functions to handle first the shift, then the stretching. In accordance with Theorem 8, $m_1(x) = f\left(x + \frac{3}{2}\right) = \sqrt{x + \frac{3}{2}}$ will shift the graph of f to the left $\frac{3}{2}$ units: see Figure 2.56

   Next, $m_2(x) = m_1\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$ will, according to Theorem 11, horizontally stretch the graph of m_1 by a factor of 2: see Figure 2.57

   We now examine what's happening to the outputs. From $m(x) = -f\left(\frac{1}{2}x + \frac{3}{2}\right) + 1$, we see that the output from f is being multiplied by -1 (a reflection about the x-axis) and then a 1 is added (a vertical shift up 1). As before, we can determine the correct order by looking at how the point $(4, 2)$ is moved. We already know that to make use of the equation $f(4) = 2$, we need to substitute $x = 5$. We get $m(5) = -f\left(\frac{1}{2}(5) + \frac{3}{2}\right) + 1 = -f(4) + 1 = -2 + 1 = -1$. We see that $f(4)$ (the output from f) is first multiplied by -1 then the 1 is added meaning we first reflect the graph about the x-axis then shift up 1. Theorem 9 tells us $m_3(x) = -m_2(x)$ will handle the reflection.

   Finally, to handle the vertical shift, Theorem 7 gives $m(x) = m_3(x) + 1$, and we see that the range of m is $(-\infty, 1]$. The graph of m is given in Figure 2.59.

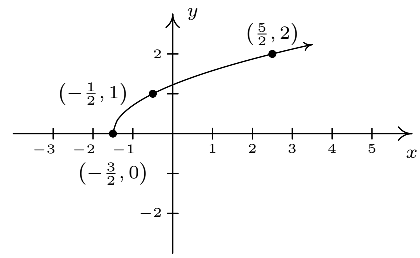


Figure 2.56: The graph $y = m_1(x) = f(x + \frac{3}{2}) = \sqrt{x + \frac{3}{2}}$

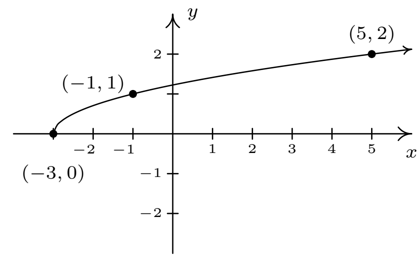


Figure 2.57: The graph $y = m_2(x) = m_1(\frac{1}{2}x) = \sqrt{\frac{1}{2}x + \frac{3}{2}}$

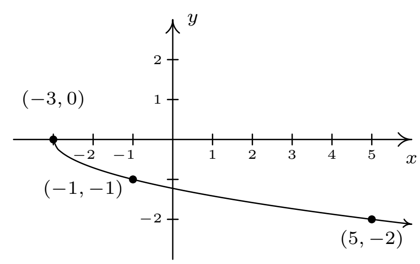


Figure 2.58: The graph $y = m_3(x) = -m_2(x) = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

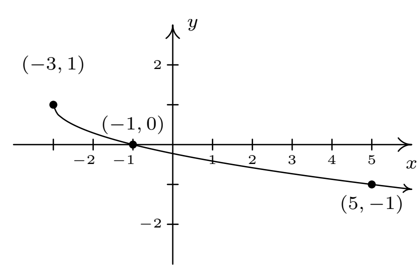


Figure 2.59: The graph $y = m(x) = m_3(x) + 1 = -\sqrt{\frac{1}{2}x + \frac{3}{2}}$

Some comments about Example 38 are in order. First, recalling the properties of radicals from Intermediate Algebra, we know that the functions g and j are the same, since j and g have the same domains and $j(x) = \sqrt{9x} = \sqrt{9}\sqrt{x} = 3\sqrt{x} = g(x)$. (We invite the reader to verify that all of the points we plotted on the graph of g lie on the graph of j and vice-versa.) Hence, for $f(x) = \sqrt{x}$, a vertical stretch by a factor of 3 and a horizontal shrinking by a factor of 9 result in the same transformation. While this kind of phenomenon is not universal, it happens commonly enough with some of the families of functions studied in College Algebra that it is worthy of note. Secondly, to graph the function m, we applied a series of four transformations. While it would have been easier on the authors to simply inform the reader of which steps to take, we have strived to explain why the order in which the transformations were applied made sense. We generalize the procedure in the theorem below.

Theorem 12 Transformations

Suppose f is a function. If $A \neq 0$ and $B \neq 0$, then to graph

$$g(x) = Af(Bx + H) + K$$

1. Subtract H from each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the left if $H > 0$ or right if $H < 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by B. This results in a horizontal scaling, but may also include a reflection about the y-axis if $B < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by A. This results in a vertical scaling, but may also include a reflection about the x-axis if $A < 0$.
4. Add K to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $K > 0$ or down if $K < 0$.

Theorem 12 can be established by generalizing the techniques developed in this section. Suppose (a, b) is on the graph of f. Then $f(a) = b$, and to make good use of this fact, we set $Bx + H = a$ and solve. We first subtract the H (causing the horizontal shift) and then divide by B. If B is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{B}$. If $B < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{a-H}{B}$ as the input to g which corresponds to the input $x = a$ to f. We now evaluate $g\left(\frac{a-H}{B}\right) = Af\left(B \cdot \frac{a-H}{B} + H\right) + K = Af(a) + K = Ab + K$. We notice that the output from f is first multiplied by A. As with the constant B, if $A > 0$, this induces only a vertical scaling. If $A < 0$, then the -1 induces a reflection across the x-axis. Finally, we add K to the result, which is our vertical shift. A less precise, but more intuitive way to paraphrase Theorem 12 is to think of the quantity $Bx + H$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation. What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations agreement: we first multiply by A then add K to find the corresponding y-coordinates on the graph of g.

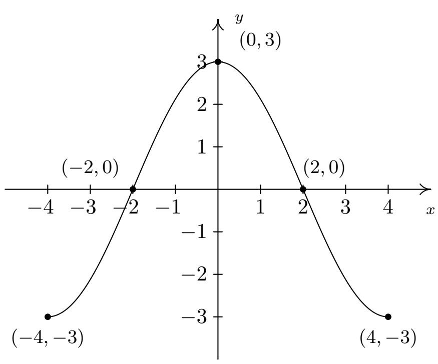


Figure 2.60: The graph $y = f(x)$ for Example 39

Example 39 Graphing a general transformation

The complete graph of $y = f(x)$ is shown in Figure 2.60. Use it to graph $g(x) = \frac{4-3f(1-2x)}{2}$.

Solution We use Theorem 12 to track the five 'key points' $(-4, -3)$, $(-2, 0)$, $(0, 3)$, $(2, 0)$ and $(4, -3)$ indicated on the graph of f to their new locations. We first rewrite $g(x)$ in the form presented in Theorem 12, $g(x) = -\frac{3}{2}f(-2x + 1) + 2$. We set $-2x + 1$ equal to the x-coordinates of the key points and solve. For example, solving $-2x + 1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process:

dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in the table in Figure 2.62

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table in Figure 2.63.

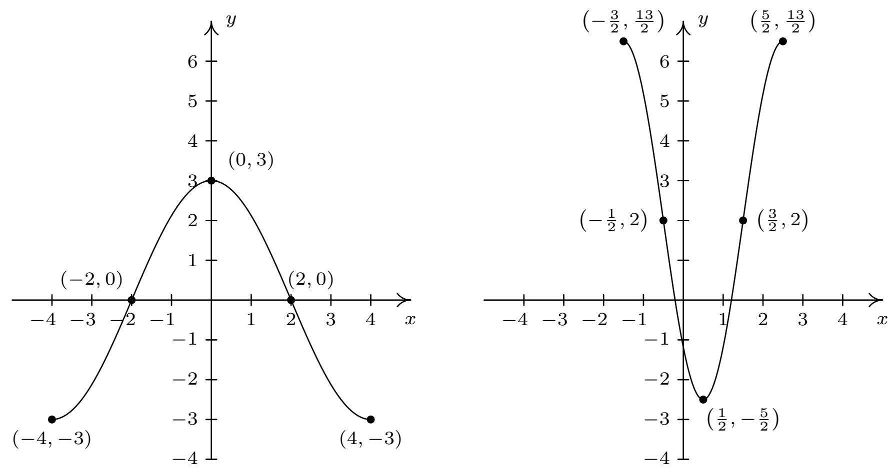
To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond. Plotting f and g side-by-side gives

$(a, f(a))$	$-2x+1=a$	x
$(-4,-3)$	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	$-2x+1=4$	$x=-\frac{3}{2}$

Figure 2.62: Tracking the x coordinates of transformed points

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

Figure 2.63: Getting the corresponding y coordinates



$y = f(x)$ $\quad$ $y = g(x) = g(x) = -\frac{3}{2}f(-2x+1)+2$

Figure 2.61: Determining the graph of $g(x) = -\frac{3}{2}f(-2x+1)+2$

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g. (You really should do this once in your life.) We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages.

Our last example turns the tables and asks for the formula of a function given a desired sequence of transformations. If nothing else, it is a good review of function notation.

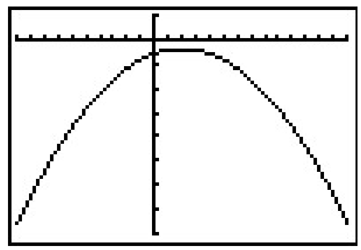
Example 40 Determining the formula for a transformed function

Let $f(x) = x^2$. Find and simplify the formula of the function $g(x)$ whose graph is the result of f undergoing the following sequence of transformations. Check your answer using a graphing calculator.

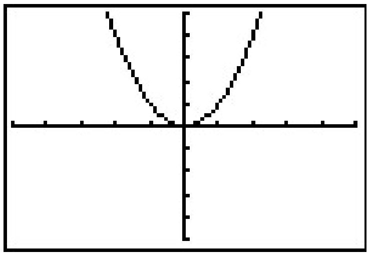
1. Vertical shift up 2 units
2. Reflection across the x-axis
3. Horizontal shift right 1 unit
4. Horizontal stretching by a factor of 2

Solution We build up to a formula for $g(x)$ using intermediate functions as we've seen in previous examples. We let g_1 take care of our first step. Theorem 7 tells us $g_1(x) = f(x) + 2 = x^2 + 2$. Next, we reflect the graph of g_1 about the x-axis using Theorem 9: $g_2(x) = -g_1(x) = -\left(x^2 + 2\right) = -x^2 - 2$. We shift the graph to the right 1 unit, according to Theorem 8, by setting $g_3(x) = g_2(x-1) = -(x-1)^2 - 2 = -x^2 + 2x - 3$. Finally, we induce a horizontal stretch by a factor of 2 using Theorem 11 to get $g(x) = g_3\left(\frac{1}{2}x\right) = -\left(\frac{1}{2}x\right)^2 + 2\left(\frac{1}{2}x\right) - 3$ which yields $g(x) = -\frac{1}{4}x^2 + x - 3$. We use the calculator to graph the stages below to confirm our result.

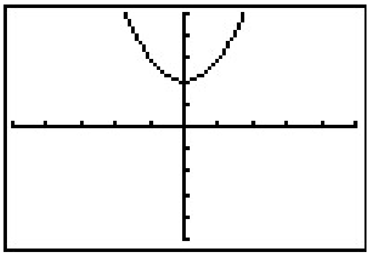
We have kept the viewing window the same in all of the graphs above. This had the undesirable consequence of making the last graph look 'incomplete' in that we cannot see the original shape of $f(x) = x^2$. Altering the viewing window results in a more complete graph of the transformed function shown below:



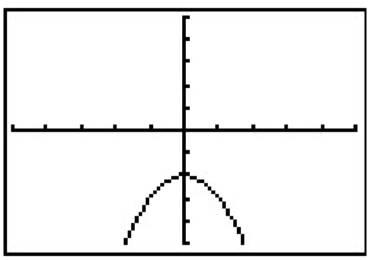
This example brings our first chapter to a close. In the chapters which lie ahead, be on the lookout for the concepts developed here to resurface as we study different families of functions.



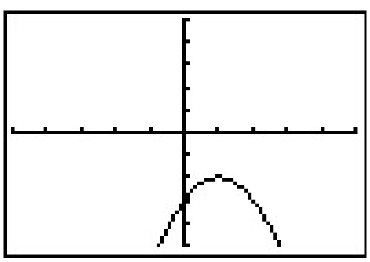
$y = f(x) = x^2$



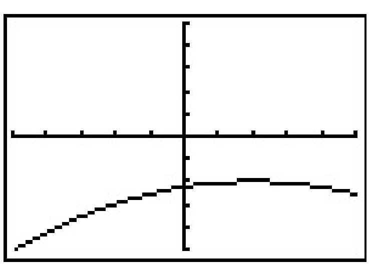
$y = g_1(x) = f(x) + 2 = x^2 + 2$
(Shift up by 2)



$y = g_2(x) = -g_1(x) = -x^2 - 2$
(Reflect across x-axis)



$y = g_3(x) = g_2(x-1) = -x^2 + 2x - 3$
(Shift right one unit)



$y = g(x) = g_3((\frac{1}{2}x) = -\frac{1}{4}x^2 + x - 3$
(Horizontal stretch by a factor of 2)

Figure 2.64: The sequence of transformations in Example 40

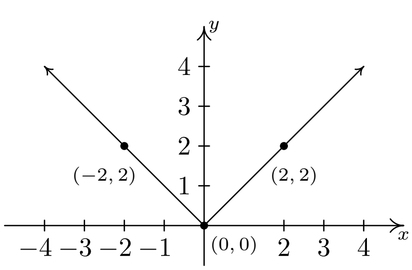
Exercises 2.6

Problems

Suppose $(2, -3)$ **is on the graph of** $y = f(x)$**. In Exercises 1 – 18, use Theorem 12 to find a point on the graph of the given transformed function.**

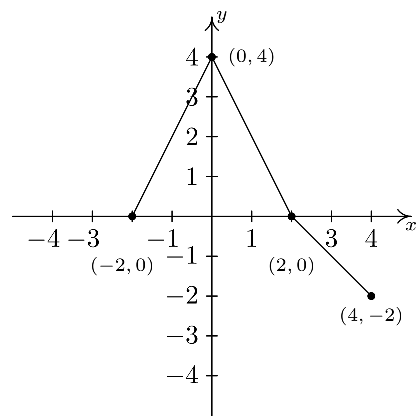
1. $y = f(x) + 3$
2. $y = f(x + 3)$
3. $y = f(x) - 1$
4. $y = f(x - 1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x - 3) + 1$
10. $y = 2f(x + 1)$
11. $y = 10 - f(x)$
12. $y = 3f(2x) - 1$
13. $y = \frac{1}{2}f(4 - x)$
14. $y = 5f(2x + 1) + 3$
15. $y = 2f(1 - x) - 1$
16. $y = f\left(\dfrac{7 - 2x}{4}\right)$
17. $y = \dfrac{f(3x) - 1}{2}$
18. $y = \dfrac{4 - f(3x - 1)}{7}$

The complete graph of $y = f(x)$ **is given below. In Exercises 19 – 27, use it and Theorem 12 to graph the given transformed function.**



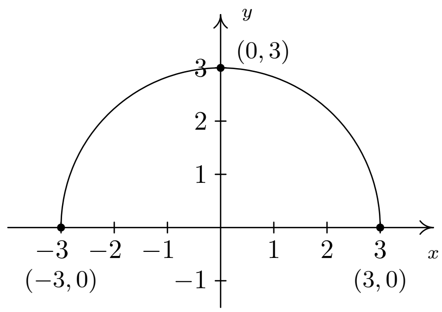
19. $y = f(x) + 1$
20. $y = f(x) - 2$
21. $y = f(x + 1)$
22. $y = f(x - 2)$
23. $y = 2f(x)$
24. $y = f(2x)$
25. $y = 2 - f(x)$
26. $y = f(2 - x)$
27. $y = 2 - f(2 - x)$

The complete graph of $y = f(x)$ **is given below. In Exercises 28 – 36, use it and Theorem 12 to graph the given transformed function.**



28. $y = f(x) - 1$
29. $y = f(x + 1)$
30. $y = \frac{1}{2}f(x)$
31. $y = f(2x)$
32. $y = -f(x)$
33. $y = f(-x)$
34. $y = f(x + 1) - 1$
35. $y = 1 - f(x)$
36. $y = \frac{1}{2}f(x + 1) - 1$

The complete graph of $y = f(x)$ **is given below. In Exercises 37 – 48, use it and Theorem 12 to graph the given transformed function.**



37. $g(x) = f(x) + 3$

38. $h(x) = f(x) - \frac{1}{2}$

39. $j(x) = f\left(x - \frac{2}{3}\right)$

40. $a(x) = f(x + 4)$

41. $b(x) = f(x + 1) - 1$

42. $c(x) = \frac{3}{5}f(x)$

43. $d(x) = -2f(x)$

44. $k(x) = f\left(\frac{2}{3}x\right)$

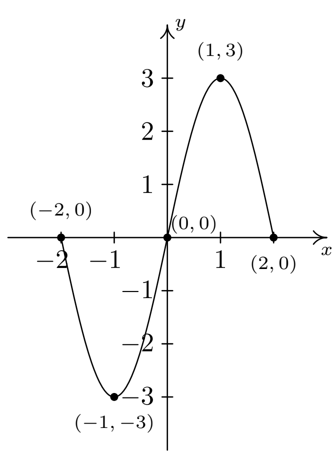
45. $m(x) = -\frac{1}{4}f(3x)$

46. $n(x) = 4f(x - 3) - 6$

47. $p(x) = 4 + f(1 - 2x)$

48. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(x)$ is given below. The purpose of Exercises 49 – 52 is to graph $y = \frac{1}{2}S(-x+1)+1$ by graphing each transformation, one step at a time.



49. $y = S_1(x) = S(x + 1)$

50. $y = S_2(x) = S_1(-x) = S(-x + 1)$

51. $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x + 1)$

52. $y = S_4(x) = S_3(x) + 1 = \frac{1}{2}S(-x + 1) + 1$

Let $f(x) = \sqrt{x}$. In Exercises 53 – 62, find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

53. (1) shift right 2 units; (2) shift down 3 units

54. (1) shift down 3 units; (2) shift right 2 units

55. (1) reflect across the x-axis; (2) shift up 1 unit

56. (1) shift up 1 unit; (2) reflect across the x-axis

57. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

58. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

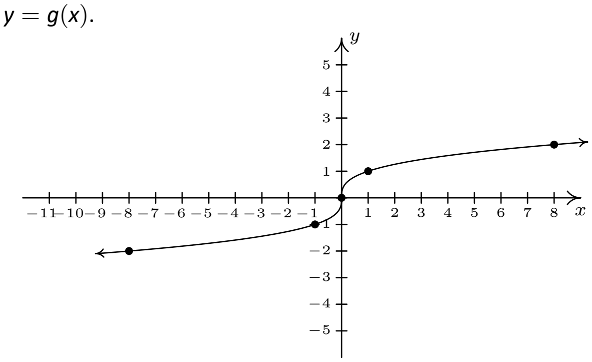
59. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

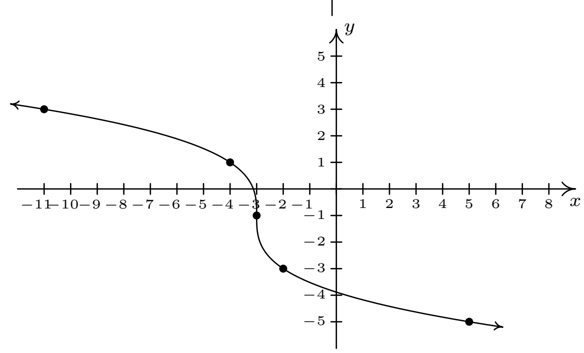
60. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

61. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

62. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

63. The graph of $y = f(x) = \sqrt[3]{x}$ is given immediately below, and the graph of $y = g(x)$ is given below that of $y = f(x)$. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.





64. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, we stated earlier that $\sqrt{9x} = 3\sqrt{x}$. With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$. What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

65. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

66. What happens if you reflect an even function across the y-axis?

67. What happens if you reflect an odd function across the y-axis?

68. What happens if you reflect an even function across the x-axis?

69. What happens if you reflect an odd function across the x-axis?

70. How would you describe symmetry about the origin in terms of reflections?

71. As we saw in Example 40, the viewing window on the graphing calculator affects how we see the transformations done to a graph. Using two different calculators, find viewing windows so that $f(x) = x^2$ on the one calculator looks like $g(x) = 3x^2$ on the other.

3: LINEAR AND QUADRATIC FUNCTIONS

3.1 Linear Functions

We now begin the study of families of functions. Our first family, linear functions, are old friends as we shall soon see. Recall from Geometry that two distinct points in the plane determine a unique line containing those points, as indicated in Figure 3.1.

To give a sense of the 'steepness' of the line, we recall that we can compute the **slope** of the line using the formula below.

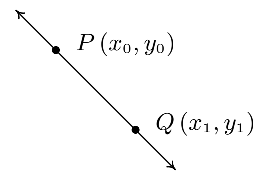


Figure 3.1: The line between two points P and Q

Definition 29 Slope

The **slope** m of the line containing the points $P(x_0, y_0)$ and $Q(x_1, y_1)$ is:

$$m = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_1 \neq x_0$.

A couple of notes about Definition 29 are in order. First, don't ask why we use the letter 'm' to represent slope. There are many explanations out there, but apparently no one really knows for sure. Secondly, the stipulation $x_1 \neq x_0$ ensures that we aren't trying to divide by zero. The reader is invited to pause to think about what is happening geometrically; the anxious reader can skip along to the next example.

See www.mathforum.org or www.mathworld.wolfram.com for discussions on the use of the letter m to indicate slope.

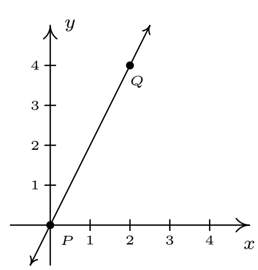
Example 41 Finding the slope of a line

Find the slope of the line containing the following pairs of points, if it exists. Plot each pair of points and the line containing them.

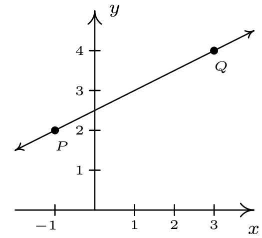
1. $P(0, 0)$, $Q(2, 4)$
2. $P(-1, 2)$, $Q(3, 4)$
3. $P(-2, 3)$, $Q(2, -3)$
4. $P(-3, 2)$, $Q(4, 2)$
5. $P(2, 3)$, $Q(2, -1)$
6. $P(2, 3)$, $Q(2.1, -1)$

SOLUTION In each of these examples, we apply the slope formula, from Definition 29.

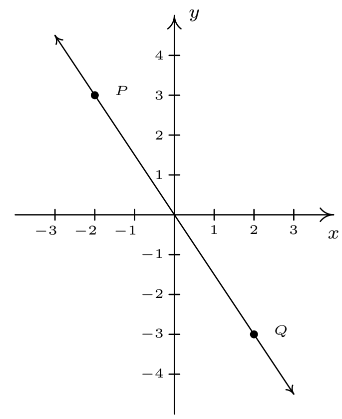
1. $m = \dfrac{4-0}{2-0} = \dfrac{4}{2} = 2$



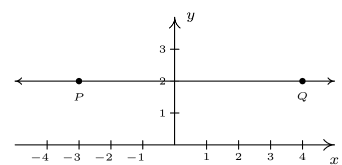
2. $m = \dfrac{4-2}{3-(-1)} = \dfrac{2}{4} = \dfrac{1}{2}$



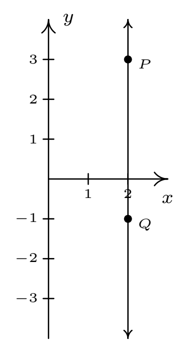
3. $m = \dfrac{-3-3}{2-(-2)} = \dfrac{-6}{4} = -\dfrac{3}{2}$



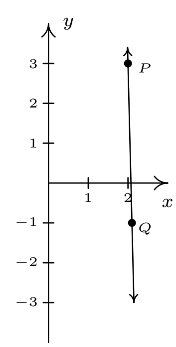
4. $m = \dfrac{2-2}{4-(-3)} = \dfrac{0}{7} = 0$



5. $m = \dfrac{-1-3}{2-2} = \dfrac{-4}{0}$, which is undefined



6. $m = \dfrac{-1-3}{2.1-2} = \dfrac{-4}{0.1} = -40$



A few comments about Example 41 are in order. First, for reasons which will be made clear soon, if the slope is positive then the resulting line is said to be increasing. If it is negative, we say the line is decreasing. A slope of 0 results in a horizontal line which we say is constant, and an undefined slope results in a vertical line. Second, the larger the slope is in absolute value, the steeper the line. You may recall from high school that slope can be described as the ratio '$\frac{\text{rise}}{\text{run}}$'. For example, in the second part of Example 41, we found the slope to be $\frac{1}{2}$. We can interpret this as a rise of 1 unit upward for every 2 units to the right we travel along the line, as shown in Figure 3.2.

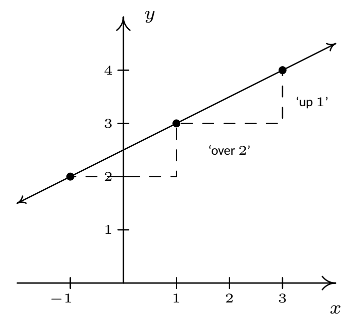


Figure 3.2: Slope as "rise over run"

Using more formal notation, given points (x_0, y_0) and (x_1, y_1), we use the Greek letter delta 'Δ' to write $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$. In most scientific circles, the symbol Δ means 'change in'.

Hence, we may write

$$m = \frac{\Delta y}{\Delta x},$$

which describes the slope as the **rate of change** of y with respect to x. Rates of change abound in the 'real world', as the next example illustrates.

Example 42 Temperature rate of change

Suppose that two separate temperature readings were taken at the ranger station on the top of Mt. Sasquatch: at 6 AM the temperature was 2°C and at 10 AM it was 8°C.

1. Find the slope of the line containing the points $(6, 2)$ and $(10, 8)$.
2. Interpret your answer to the first part in terms of temperature and time.
3. Predict the temperature at noon.

Solution

1. For the slope, we have $m = \frac{8-2}{10-6} = \frac{6}{4} = \frac{3}{2}$.
2. Since the values in the numerator correspond to the temperatures in °C, and the values in the denominator correspond to time in hours, we can interpret the slope as $\frac{3}{2} = \frac{3^\circ \text{ C}}{2 \text{ hour}}$, or 1.5°C per hour. Since the slope is positive, we know this corresponds to an increasing line. Hence, the temperature is increasing at a rate of 1.5°C per hour.
3. Noon is two hours after 10 AM. Assuming a temperature increase of 1.5°C per hour, in two hours the temperature should rise 3°C. Since the temperature at 10 AM is 82°C, we would expect the temperature at noon to be $8 + 3 = 11$°C.

Now it may well happen that in the previous scenario, at noon the temperature is only 10°C. This doesn't mean our calculations are incorrect, rather, it means that the temperature change throughout the day isn't a constant 1.5°C per hour. As discussed in Section 2.3.1, mathematical models are just that: models. The predictions we get out of the models may be mathematically accurate, but may not resemble what happens in the real world.

In Section 2.1, we discussed the equations of vertical and horizontal lines. Using the concept of slope, we can develop equations for the other varieties of lines. Suppose a line has a slope of m and contains the point (x_0, y_0). Suppose (x, y) is another point on the line, as indicated in Figure 3.3.

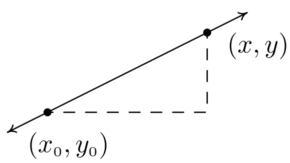


Figure 3.3: Deriving the point-slope formula

Definition 29 yields

$$\begin{aligned} m &= \frac{y - y_0}{x - x_0} \\ m(x - x_0) &= y - y_0 \\ y - y_0 &= m(x - x_0) \end{aligned}$$

We have just derived the **point-slope form** of a line.

Key Idea 15 The point-slope form of a line

The **point-slope form** of the equation of a line with slope m containing the point (x_0, y_0) is the equation $y - y_0 = m(x - x_0)$.

Example 43 Using the point-slope form

Write the equation of the line containing the points $(-1, 3)$ and $(2, 1)$.

Solution In order to use Key Idea 15 we need to find the slope of the line in question so we use Definition 29 to get $m = \frac{\Delta y}{\Delta x} = \frac{1-3}{2-(-1)} = -\frac{2}{3}$. We are spoiled for choice for a point (x_0, y_0). We'll use $(-1, 3)$ and leave it to the reader to check that using $(2, 1)$ results in the same equation. Substituting into the point-slope form of the line, we get

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= -\frac{2}{3}(x - (-1)) \\ y - 3 &= -\frac{2}{3}(x + 1) \\ y - 3 &= -\frac{2}{3}x - \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{7}{3}. \end{aligned}$$

We can check our answer by showing that both $(-1, 3)$ and $(2, 1)$ are on the graph of $y = -\frac{2}{3}x + \frac{7}{3}$ algebraically, as we did in Section 2.1.1.

In simplifying the equation of the line in the previous example, we produced another form of a line, the **slope-intercept form**. This is the familiar $y = mx + b$ form you have probably seen in high school. The 'intercept' in 'slope-intercept' comes from the fact that if we set $x = 0$, we get $y = b$. In other words, the y-intercept of the line $y = mx + b$ is $(0, b)$.

Key Idea 16 Slope intercept form of a line

The **slope-intercept form** of the line with slope m and y-intercept $(0, b)$ is the equation $y = mx + b$.

Note that if we have slope $m = 0$, we get the equation $y = b$ which matches our formula for a horizontal line given in Section 2.1. The formula given in Key Idea 16 can be used to describe all lines except vertical lines. All lines except vertical lines are functions (Why is this?) so we have finally reached a good point to introduce **linear functions**.

Definition 30 Linear function

A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

For the case $m = 0$, we get $f(x) = b$. These are given their own classification.

Definition 31 Constant function

A **constant function** is a function of the form

$$f(x) = b,$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Recall that to graph a function, f, we graph the equation $y = f(x)$. Hence, the graph of a linear function is a line with slope m and y-intercept $(0, b)$; the graph of a constant function is a horizontal line (a line with slope $m = 0$) and a y-intercept of $(0, b)$. Now think back to Section 2.5.1, specifically Definition 27 concerning increasing, decreasing and constant functions. A line with positive slope was called an increasing line because a linear function with $m > 0$ is an increasing function. Similarly, a line with a negative slope was called a decreasing line because a linear function with $m < 0$ is a decreasing function. And horizontal lines were called constant because, well, we hope you've already made the connection.

Example 44 Graphing linear functions

Graph the following functions. Identify the slope and y-intercept.

1. $f(x) = 3$
2. $f(x) = 3x - 1$
3. $f(x) = \dfrac{3 - 2x}{4}$
4. $f(x) = \dfrac{x^2 - 4}{x - 2}$

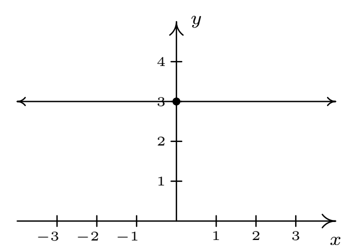


Figure 3.4: The graph of $f(x) = 3$

Solution

1. To graph $f(x) = 3$, we graph $y = 3$. This is a horizontal line ($m = 0$) through $(0, 3)$: see Figure 3.4.

2. The graph of $f(x) = 3x - 1$ is the graph of the line $y = 3x - 1$. Comparison of this equation with Equation 16 yields $m = 3$ and $b = -1$. Hence, our slope is 3 and our y-intercept is $(0, -1)$. To get another point on the line, we can plot $(1, f(1)) = (1, 2)$. Constructing the line through these points gives us Figure 3.5.

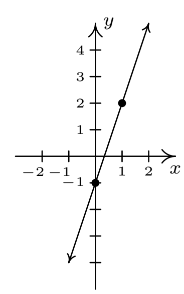


Figure 3.5: The graph of $f(x) = 3x - 1$

3. At first glance, the function $f(x) = \dfrac{3 - 2x}{4}$ does not fit the form in Definition 30 but after some rearranging we get $f(x) = \frac{3-2x}{4} = \frac{3}{4} - \frac{2x}{4} =$

$-\frac{1}{2}x+\frac{3}{4}$. We identify $m=-\frac{1}{2}$ and $b=\frac{3}{4}$. Hence, our graph is a line with a slope of $-\frac{1}{2}$ and a y-intercept of $\left(0,\frac{3}{4}\right)$. Plotting an additional point, we can choose $(1,f(1))$ to get $\left(1,\frac{1}{4}\right)$: see Figure 3.6.

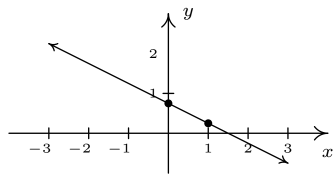


Figure 3.6: The graph of $f(x)=\dfrac{3-2x}{4}$

4. If we simplify the expression for f, we get

$$f(x)=\frac{x^2-4}{x-2}=\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}=x+2.$$

If we were to state $f(x)=x+2$, we would be committing a sin of omission. Remember, to find the domain of a function, we do so **before** we simplify! In this case, f has big problems when $x=2$, and as such, the domain of f is $(-\infty,2)\cup(2,\infty)$. To indicate this, we write $f(x)=x+2,\ x\neq 2$. So, except at $x=2$, we graph the line $y=x+2$. The slope $m=1$ and the y-intercept is $(0,2)$. A second point on the graph is $(1,f(1))=(1,3)$. Since our function f is not defined at $x=2$, we put an open circle at the point that would be on the line $y=x+2$ when $x=2$, namely $(2,4)$, as shown in Figure 3.7.

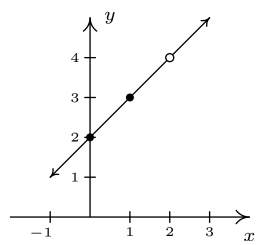


Figure 3.7: The graph of $f(x)=\dfrac{x^2-4}{x-2}$

The last two functions in the previous example showcase some of the difficulty in defining a linear function using the phrase 'of the form' as in Definition 30, since some algebraic manipulations may be needed to rewrite a given function to match 'the form'. Keep in mind that the domains of linear and constant functions are all real numbers $(-\infty,\infty)$, so while $f(x)=\frac{x^2-4}{x-2}$ simplified to a formula $f(x)=x+2$, f is not considered a linear function since its domain excludes $x=2$. However, we would consider

$$f(x)=\frac{2x^2+2}{x^2+1}$$

to be a constant function since its domain is all real numbers (Can you tell us why?) and

$$f(x)=\frac{2x^2+2}{x^2+1}=\frac{2\cancel{(x^2+1)}}{\cancel{(x^2+1)}}=2$$

The following example uses linear functions to model some basic economic relationships.

The similarity of this name to PortaJohn is deliberate.

Example 45 Pricing for a game system

The cost C, in dollars, to produce x PortaBoy game systems for a local retailer is given by $C(x)=80x+150$ for $x\geq 0$.

1. Find and interpret $C(10)$.
2. How many PortaBoys can be produced for \$15,000?
3. Explain the significance of the restriction on the domain, $x\geq 0$.
4. Find and interpret $C(0)$.
5. Find and interpret the slope of the graph of $y=C(x)$.

Solution

1. To find $C(10)$, we replace every occurrence of x with 10 in the formula for $C(x)$ to get $C(10) = 80(10) + 150 = 950$. Since x represents the number of PortaBoys produced, and $C(x)$ represents the cost, in dollars, $C(10) = 950$ means it costs $\$950$ to produce 10 PortaBoys for the local retailer.

2. To find how many PortaBoys can be produced for $\$15{,}000$, we solve $C(x) = 15000$, or $80x+150 = 15000$. Solving, we get $x = \frac{14850}{80} = 185.625$. Since we can only produce a whole number amount of PortaBoys, we can produce 185 PortaBoys for $\$15{,}000$.

3. The restriction $x \geq 0$ is the applied domain, as discussed in Section 2.3.1. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems.

4. We find $C(0) = 80(0) + 150 = 150$. This means it costs $\$150$ to produce 0 PortaBoys. As mentioned on page 73, this is the fixed, or start-up cost of this venture.

5. If we were to graph $y = C(x)$, we would be graphing the portion of the line $y = 80x + 150$ for $x \geq 0$. We recognize the slope, $m = 80$. Like any slope, we can interpret this as a rate of change. Here, $C(x)$ is the cost in dollars, while x measures the number of PortaBoys so

$$m = \frac{\Delta y}{\Delta x} = \frac{\Delta C}{\Delta x} = 80 = \frac{80}{1} = \frac{\$80}{1 \text{ PortaBoy}}.$$

In other words, the cost is increasing at a rate of $\$80$ per PortaBoy produced. This is often called the **variable cost** for this venture.

Actually, it makes no sense to produce a fractional part of a game system, either, as we saw in the previous part of this example. This absurdity, however, seems quite forgiveable in some textbooks but not to us.

The next example asks us to find a linear function to model a related economic problem.

Example 46 Modelling demand

The local retailer in Example 45 has determined that the number x of PortaBoy game systems sold in a week is related to the price p in dollars of each system. When the price was $\$220$, 20 game systems were sold in a week. When the systems went on sale the following week, 40 systems were sold at $\$190$ a piece.

1. Find a linear function which fits this data. Use the weekly sales x as the independent variable and the price p as the dependent variable.
2. Find a suitable applied domain.
3. Interpret the slope.
4. If the retailer wants to sell 150 PortaBoys next week, what should the price be?
5. What would the weekly sales be if the price were set at $\$150$ per system?

Solution

1. We recall from Section 2.3 the meaning of 'independent' and 'dependent' variable. Since x is to be the independent variable, and p the dependent variable, we treat x as the input variable and p as the output variable. Hence, we are looking for a function of the form $p(x) = mx + b$. To

determine m and b, we use the fact that 20 PortaBoys were sold during the week when the price was 220 dollars and 40 units were sold when the price was 190 dollars. Using function notation, these two facts can be translated as $p(20) = 220$ and $p(40) = 190$. Since m represents the rate of change of p with respect to x, we have

$$m = \frac{\Delta p}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5.$$

We now have determined $p(x) = -1.5x + b$. To determine b, we can use our given data again. Using $p(20) = 220$, we substitute $x = 20$ into $p(x) = 1.5x + b$ and set the result equal to 220: $-1.5(20) + b = 220$. Solving, we get $b = 250$. Hence, we get $p(x) = -1.5x + 250$. We can check our formula by computing $p(20)$ and $p(40)$ to see if we get 220 and 190, respectively. You may recall from page 73 that the function $p(x)$ is called the price-demand (or simply demand) function for this venture.

2. To determine the applied domain, we look at the physical constraints of the problem. Certainly, we can't sell a negative number of PortaBoys, so $x \geq 0$. However, we also note that the slope of this linear function is negative, and as such, the price is decreasing as more units are sold. Thus another constraint on the price is $p(x) \geq 0$. Solving $-1.5x + 250 \geq 0$ results in $-1.5x \geq -250$ or $x \leq \frac{500}{3} = 166.\overline{6}$. Since x represents the number of PortaBoys sold in a week, we round down to 166. As a result, a reasonable applied domain for p is $[0, 166]$.

3. The slope $m = -1.5$, once again, represents the rate of change of the price of a system with respect to weekly sales of PortaBoys. Since the slope is negative, we have that the price is decreasing at a rate of $\$1.50$ per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every $\$1.50$ drop in price.)

4. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be $\$25$.

5. If the price of a PortaBoy were set at $\$150$, we have $p(x) = 150$, or, $-1.5x + 250 = 150$. Solving, we get $-1.5x = -100$ or $x = 66.\overline{6}$. This means you would be able to sell 66 PortaBoys a week if the price were $\$150$ per system.

Not all real-world phenomena can be modelled using linear functions. Nevertheless, it is possible to use the concept of slope to help analyze non-linear functions using the following.

Definition 32 Average rate of change

Let f be a function defined on the interval $[a, b]$. The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta f}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, if we have the graph of $y = f(x)$, the average rate of change over $[a, b]$ is the slope of the line which connects $(a, f(a))$ and $(b, f(b))$. This is called the **secant line** through these points. For that reason, some textbooks use the notation m_{sec} for the average rate of change of a function. Note that for a linear function $m = m_{\text{sec}}$, or in other words, its rate of change over an interval is the same as its average rate of change.

The interested reader may question the adjective 'average' in the phrase 'average rate of change'. In the figure above, we can see that the function changes wildly on $[a, b]$, yet the slope of the secant line only captures a snapshot of the action at a and b. This situation is entirely analogous to the average speed on a trip. Suppose it takes you 2 hours to travel 100 kilometres. Your average speed is $\frac{100\,\text{km}}{2\,\text{h}} = 50\,\text{km/h}$. However, it is entirely possible that at the start of your journey, you travelled 25 kilometres per hour, then sped up to 65 kilometres per hour, and so forth. The average rate of change is akin to your average speed on the trip. Your speedometer measures your speed at any one instant along the trip, your **instantaneous rate of change**, and this is one of the central themes of Calculus.

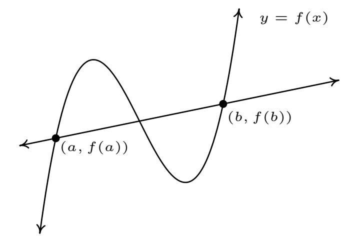


Figure 3.8: The graph of $y = f(x)$ and its secant line through $(a, f(a))$ and $(b, f(b))$

When interpreting rates of change, we interpret them the same way we did slopes. In the context of functions, it may be helpful to think of the average rate of change as:

$$\frac{\text{change in outputs}}{\text{change in inputs}}$$

Example 47 A non-linear revenue model

Recall from page 73, the revenue from selling x units at a price p per unit is given by the formula $R = xp$. Suppose we are in the scenario of Examples 45 and 46.

1. Find and simplify an expression for the weekly revenue $R(x)$ as a function of weekly sales x.
2. Find and interpret the average rate of change of $R(x)$ over the interval $[0, 50]$.
3. Find and interpret the average rate of change of $R(x)$ as x changes from 50 to 100 and compare that to your result in part 2.
4. Find and interpret the average rate of change of weekly revenue as weekly sales increase from 100 PortaBoys to 150 PortaBoys.

SOLUTION

1. Since $R = xp$, we substitute $p(x) = -1.5x + 250$ from Example 46 to get $R(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since we determined the price-demand function $p(x)$ is restricted to $0 \leq x \leq 166$, $R(x)$ is restricted to these values of x as well.
2. Using Definition 32, we get that the average rate of change is

$$\frac{\Delta R}{\Delta x} = \frac{R(50) - R(0)}{50 - 0} = \frac{8750 - 0}{50 - 0} = 175.$$

   Interpreting this slope as we have in similar situations, we conclude that for every additional PortaBoy sold during a given week, the weekly revenue increases $175.

3. The wording of this part is slightly different than that in Definition 32, but its meaning is to find the average rate of change of R over the interval $[50, 100]$. To find this rate of change, we compute

$$\frac{\Delta R}{\Delta x} = \frac{R(100) - R(50)}{100 - 50} = \frac{10000 - 8750}{50} = 25.$$

   In other words, for each additional PortaBoy sold, the revenue increases by $\$25$. Note that while the revenue is still increasing by selling more game systems, we aren't getting as much of an increase as we did in part 2 of this example. (Can you think of why this would happen?)

4. Translating the English to the mathematics, we are being asked to find the average rate of change of R over the interval $[100, 150]$. We find

$$\frac{\Delta R}{\Delta x} = \frac{R(150) - R(100)}{150 - 100} = \frac{3750 - 10000}{50} = -125.$$

   This means that we are losing $\$125$ dollars of weekly revenue for each additional PortaBoy sold. (Can you think why this is possible?)

We close this section with a new look at difference quotients which were first introduced in Section 2.3. If we wish to compute the average rate of change of a function f over the interval $[x, x + h]$, then we would have

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

As we have indicated, the rate of change of a function (average or otherwise) is of great importance in Calculus. (So we are not torturing you with these for nothing.) Also, we have the geometric interpretation of difference quotients which was promised to you back on page 72 – a difference quotient yields the slope of a secant line.

Exercises 3.1

Problems

In Exercises 1 – 10, find both the point-slope form and the slope-intercept form of the line with the given slope which passes through the given point.

1. $m = 3, \quad P(3, -1)$
2. $m = -2, \quad P(-5, 8)$
3. $m = -1, \quad P(-7, -1)$
4. $m = \frac{2}{3}, \quad P(-2, 1)$
5. $m = \frac{2}{3}, \quad P(-2, 1)$
6. $m = \frac{1}{7}, \quad P(-1, 4)$
7. $m = 0, \quad P(3, 117)$
8. $m = -\sqrt{2}, \quad P(0, -3)$
9. $m = -5, \quad P(\sqrt{3}, 2\sqrt{3})$
10. $m = 678, \quad P(-1, -12)$

In Exercises 11 – 20, find the slope-intercept form of the line which passes through the given points.

11. $P(0, 0),\ Q(-3, 5)$
12. $P(-1, -2),\ Q(3, -2)$
13. $P(5, 0),\ Q(0, -8)$
14. $P(3, -5),\ Q(7, 4)$
15. $P(-1, 5),\ Q(7, 5)$
16. $P(4, -8),\ Q(5, -8)$
17. $P\left(\frac{1}{2}, \frac{3}{4}\right),\ Q\left(\frac{5}{2}, -\frac{7}{4}\right)$
18. $P\left(\frac{2}{3}, \frac{7}{2}\right),\ Q\left(-\frac{1}{3}, \frac{3}{2}\right)$
19. $P\left(\sqrt{2}, -\sqrt{2}\right),\ Q\left(-\sqrt{2}, \sqrt{2}\right)$
20. $P\left(-\sqrt{3}, -1\right),\ Q\left(\sqrt{3}, 1\right)$

In Exercises 21 – 26, graph the function. Find the slope, *y*-intercept and *x*-intercept, if any exist.

21. $f(x) = 2x - 1$
22. $f(x) = 3 - x$
23. $f(x) = 3$
24. $f(x) = 0$
25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
26. $f(x) = \dfrac{1 - x}{2}$
27. Find all of the points on the line $y = 2x + 1$ which are 4 units from the point $(-1, 3)$.
28. Jeff can walk comfortably at 3 miles per hour. Find a linear function d that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.
29. Carl can stuff 6 envelopes per *minute*. Find a linear function E that represents the total number of envelopes Carl can stuff after *t hours*, assuming he doesn't take any breaks.
30. A landscaping company charges $\$45$ per cubic yard of mulch plus a delivery charge of $\$20$. Find a linear function which computes the total cost C (in dollars) to deliver x cubic yards of mulch.
31. A plumber charges $\$50$ for a service call plus $\$80$ per hour. If she spends no longer than 8 hours a day at any one site, find a linear function that represents her total daily charges C (in dollars) as a function of time t (in hours) spent at any one given location.
32. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find a linear function that represents her total weekly pay, W (in dollars) in terms of x. What must her weekly sales be in order for her to earn \$475.00 for the week?
33. An on-demand publisher charges $\$22.50$ to print a 600 page book and $\$15.50$ to print a 400 page book. Find a linear function which models the cost of a book C as a function of the number of pages p. Interpret the slope of the linear function and find and interpret $C(0)$.
34. The Topology Taxi Company charges $\$2.50$ for the first fifth of a mile and $\$0.45$ for each additional fifth of a mile. Find a linear function which models the taxi fare F as a function of the number of miles driven, m. Interpret the slope of the linear function and find and interpret $F(0)$.
35. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.
    (a) Find a linear function F that expresses temperature in the Fahrenheit scale in terms of degrees Celsius. Use this function to convert 20°C into Fahrenheit.
    (b) Find a linear function C that expresses temperature in the Celsius scale in terms of degrees Fahrenheit. Use this function to convert 110°F into Celsius.
    (c) Is there a temperature n such that $F(n) = C(n)$?

36. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is 40°*F* outside and only 5 times per hour if it's 70°*F*. Assuming that the number of howls per hour, *N*, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only 20°*F* outside. What is the applied domain of this function? Why?

37. Economic forces beyond anyone's control have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 45 with this new cost function.

38. In response to the economic forces in Exercise 37 above, the local retailer sets the selling price of a PortaBoy at $250. Remarkably, 30 units were sold each week. When the systems went on sale for $220, 40 units per week were sold. Rework Examples 46 and 47 with this new data. What difficulties do you encounter?

39. A local pizza store offers medium two-topping pizzas delivered for $6.00 per pizza plus a $1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are ordered, they are $5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost *C* (in dollars) of *p* medium two-topping pizzas delivered during a weekend.

40. A restaurant offers a buffet which costs $15 per person. For parties of 10 or more people, a group discount applies, and the cost is $12.50 per person. Write a piecewise-defined linear function which calculates the total bill *T* of a party of *n* people who all choose the buffet.

41. A mobile plan charges a base monthly rate of $10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost *C* (in dollars) for using *m* minutes of air time.

    HINT: You may want to revisit Exercise 75 in Section 2.3

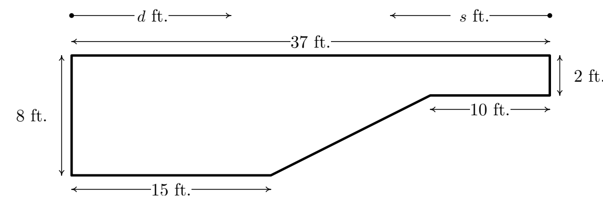
42. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price *P*, in dollars, of purchasing *c* crickets.

43. The cross-section of a swimming pool is below. Write a piecewise-defined linear function which describes the depth of the pool, *D* (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, *d*.

    (b) the distance (in feet) from the edge of the deep end of the pool, *s*.

    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.



In Exercises 44 – 49, compute the average rate of change of the function over the specified interval.

44. $f(x) = x^3,\ [-1, 2]$

45. $f(x) = \dfrac{1}{x},\ [1, 5]$

46. $f(x) = \sqrt{x},\ [0, 16]$

47. $f(x) = x^2,\ [-3, 3]$

48. $f(x) = \dfrac{x+4}{x-3},\ [5, 7]$

49. $f(x) = 3x^2 + 2x - 7,\ [-4, 2]$

In Exercises 50 – 53, compute the average rate of change of the given function over the interval $[x, x+h]$. Here we assume $[x, x+h]$ is in the domain of the function.

50. $f(x) = x^3$

51. $f(x) = \dfrac{1}{x}$

52. $f(x) = \dfrac{x+4}{x-3}$

53. $f(x) = 3x^2 + 2x - 7$

54. Using data from Bureau of Transportation Statistics, the average fuel economy *F* in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where *t* is the number of years since 1980. Find and interpret the average rate of change of *F* over the interval $[0, 28]$.

55. The temperature *T* in degrees Fahrenheit *t* hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

    (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.

    (b) Find and interpret the average rate of change of *T* over the interval $[4, 8]$.

    (c) Find and interpret the average rate of change of *T* from $t = 8$ to $t = 12$.

    (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

56. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

57. With the help of your classmates find several other "real-world" examples of rates of change that are used to describe non-linear phenomena.

58. With the help of your classmates find several other "real-world" examples of rates of change that are used to describe non-linear phenomena.

(Parallel Lines) Recall from high school that parallel lines have the same slope. (Please note that two vertical lines are also parallel to one another even though they have an undefined slope.) In Exercises 59 – 64, you are given a line and a point which is not on that line. Find the line parallel to the given line which passes through the given point.

59. $y = 3x + 2,\ P(0,0)$

60. $y = -6x + 5,\ P(3,2)$

61. $y = \frac{2}{3}x - 7,\ P(6,0)$

62. $y = \dfrac{4-x}{3},\ P(1,-1)$

63. $y = 6,\ P(3,-2)$

64. $x = 1,\ P(-5,0)$

(Perpendicular Lines) Recall from high school that two non-vertical lines are perpendicular if and only if they have negative reciprocal slopes. That is to say, if one line has slope m_1 and the other has slope m_2 then $m_1 \cdot m_2 = -1$. (You will be guided through a proof of this result in Exercise 71.) Please note that a horizontal line is perpendicular to a vertical line and vice versa, so we assume $m_1 \neq 0$ and $m_2 \neq 0$. In Exercises 65 – 70, you are given a line and a point which is not on that line. Find the line perpendicular to the given line which passes through the given point.

65. $y = \frac{1}{3}x + 2,\ P(0,0)$

66. $y = -6x + 5,\ P(3,2)$

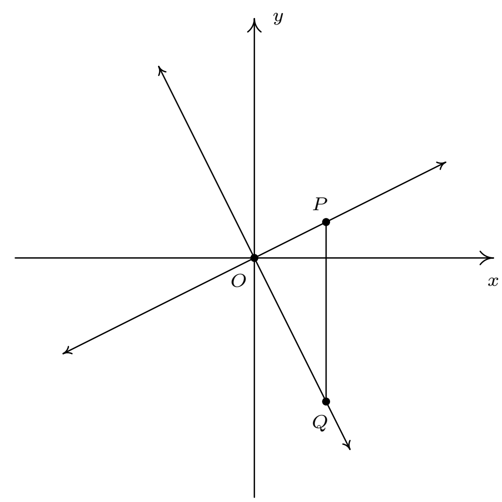
67. $y = \frac{2}{3}x - 7,\ P(6,0)$

68. $y = \dfrac{4-x}{3},\ P(1,-1)$

69. $y = 6,\ P(3,-2)$

70. $x = 1,\ P(-5,0)$

71. We shall now prove that $y = m_1x + b_1$ is perpendicular to $y = m_2x + b_2$ if and only if $m_1 \cdot m_2 = -1$. To make our lives easier we shall assume that $m_1 > 0$ and $m_2 < 0$. We can also "move" the lines so that their point of intersection is the origin without messing things up, so we'll assume $b_1 = b_2 = 0$. (Take a moment with your classmates to discuss why this is okay.) Graphing the lines and plotting the points $O(0,0)$, $P(1,m_1)$ and $Q(1,m_2)$ gives us the following set up.



The line $y = m_1x$ will be perpendicular to the line $y = m_2x$ if and only if $\triangle OPQ$ is a right triangle. Let d_1 be the distance from O to P, let d_2 be the distance from O to Q and let d_3 be the distance from P to Q. Use the Pythagorean Theorem to show that $\triangle OPQ$ is a right triangle if and only if $m_1 \cdot m_2 = -1$ by showing $d_1^2 + d_2^2 = d_3^2$ if and only if $m_1 \cdot m_2 = -1$.

72. Show that if $a \neq b$, the line containing the points (a,b) and (b,a) is perpendicular to the line $y = x$. (Coupled with the result from Example 11 on page 31, we have now shown that the line $y = x$ is a *perpendicular* bisector of the line segment connecting (a,b) and (b,a). This means the points (a,b) and (b,a) are symmetric about the line $y = x$. We will revisit this symmetry in section 6.2.)

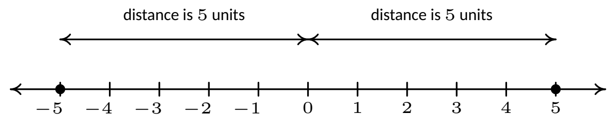
73. The function defined by $I(x) = x$ is called the Identity Function.

    (a) Discuss with your classmates why this name makes sense.

    (b) Show that the point-slope form of a line (Equation 15) can be obtained from I using a sequence of the transformations defined in Section 2.6.

3.2 Absolute Value Functions

There are a few ways to describe what is meant by the absolute value $|x|$ of a real number x. You may have been taught that $|x|$ is the distance from the real number x to 0 on the number line. So, for example, $|5| = 5$ and $|-5| = 5$, since each is 5 units from 0 on the number line.



Another way to define absolute value is by the equation $|x| = \sqrt{x^2}$. Using this definition, we have $|5| = \sqrt{(5)^2} = \sqrt{25} = 5$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$. The long and short of both of these procedures is that $|x|$ takes negative real numbers and assigns them to their positive counterparts while it leaves positive numbers alone. This last description is the one we shall adopt, and is summarized in the following definition.

Definition 33 Absolute value function

The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

In Definition 33, we define $|x|$ using a piecewise-defined function. (See page 64 in Section 2.3.) To check that this definition agrees with what we previously understood as absolute value, note that since $5 \geq 0$, to find $|5|$ we use the rule $|x| = x$, so $|5| = 5$. Similarly, since $-5 < 0$, we use the rule $|x| = -x$, so that $|-5| = -(-5) = 5$. This is one of the times when it's best to interpret the expression '$-x$' as 'the opposite of x' as opposed to 'negative x'. Before we begin studying absolute value functions, we remind ourselves of the properties of absolute value.

Theorem 13 Properties of Absolute Value

Let a, b and x be real numbers and let n be an integer. Then

- **Product Rule:** $|ab| = |a||b|$
- **Power Rule:** $|a^n| = |a|^n$ whenever a^n is defined
- **Quotient Rule:** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$, provided $b \neq 0$

Equality Properties:

- $|x| = 0$ if and only if $x = 0$.
- For $c > 0$, $|x| = c$ if and only if $x = c$ or $-x = c$.
- For $c < 0$, $|x| = c$ has no solution.

The proofs of the Product and Quotient Rules in Theorem 13 boil down to checking four cases: when both a and b are positive; when they are both negative; when one is positive and the other is negative; and when one or both are zero.

For example, suppose we wish to show that $|ab| = |a||b|$. We need to show that this equation is true for all real numbers a and b. If a and b are both positive, then so is ab. Hence, $|a| = a$, $|b| = b$ and $|ab| = ab$. Hence, the equation $|ab| = |a||b|$ is the same as $ab = ab$ which is true. If both a and b are negative, then ab is positive. Hence, $|a| = -a$, $|b| = -b$ and $|ab| = ab$. The equation $|ab| = |a||b|$ becomes $ab = (-a)(-b)$, which is true. Suppose a is positive and b is negative. Then ab is negative, and we have $|ab| = -ab$, $|a| = a$ and $|b| = -b$. The equation $|ab| = |a||b|$ reduces to $-ab = a(-b)$ which is true. A symmetric argument shows the equation $|ab| = |a||b|$ holds when a is negative and b is positive. Finally, if either a or b (or both) are zero, then both sides of $|ab| = |a||b|$ are zero, so the equation holds in this case, too. All of this rhetoric has shown that the equation $|ab| = |a||b|$ holds true in all cases.

The proof of the Quotient Rule is very similar, with the exception that $b \neq 0$. The Power Rule can be shown by repeated application of the Product Rule. The 'Equality Properties' can be proved using Definition 33 and by looking at the cases when $x \geq 0$, in which case $|x| = x$, or when $x < 0$, in which case $|x| = -x$. For example, if $c > 0$, and $|x| = c$, then if $x \geq 0$, we have $x = |x| = c$. If, on the other hand, $x < 0$, then $-x = |x| = c$, so $x = -c$. The remaining properties are proved similarly and are left for the Exercises. Our first example reviews how to solve basic equations involving absolute value using the properties listed in Theorem 13.

Example 48 Solving equations with absolute values

Solve each of the following equations.

1. $|3x - 1| = 6$
2. $3 - |x + 5| = 1$
3. $3|2x+1|-5 = 0$
4. $4 - |5x + 3| = 5$
5. $|x| = x^2 - 6$
6. $|x - 2| + 1 = x$

Solution

1. The equation $|3x-1| = 6$ is of the form $|x| = c$ for $c > 0$, so by the Equality Properties, $|3x-1| = 6$ is equivalent to $3x-1 = 6$ or $3x-1 = -6$. Solving the former, we arrive at $x = \frac{7}{3}$, and solving the latter, we get $x = -\frac{5}{3}$. We may check both of these solutions by substituting them into the original equation and showing that the arithmetic works out.

2. To use the Equality Properties to solve $3 - |x+5| = 1$, we first isolate the absolute value.

$$\begin{aligned} 3 - |x+5| &= 1 \\ -|x+5| &= -2 \qquad \text{subtract } 3 \\ |x+5| &= 2 \qquad \text{divide by } -1 \end{aligned}$$

From the Equality Properties, we have $x + 5 = 2$ or $x + 5 = -2$, and get our solutions to be $x = -3$ or $x = -7$. We leave it to the reader to check both answers in the original equation.

3. As in the previous example, we first isolate the absolute value in the equation $3|2x+1|-5=0$ and get $|2x+1|=\frac{5}{3}$. Using the Equality Properties, we have $2x+1=\frac{5}{3}$ or $2x+1=-\frac{5}{3}$. Solving the former gives $x=\frac{1}{3}$ and solving the latter gives $x=-\frac{4}{3}$. As usual, we may substitute both answers in the original equation to check.

4. Upon isolating the absolute value in the equation $4-|5x+3|=5$, we get $|5x+3|=-1$. At this point, we know there cannot be any real solution, since, by definition, the absolute value of *anything* is never negative. We are done.

5. The equation $|x|=x^2-6$ presents us with some difficulty, since x appears both inside and outside of the absolute value. Moreover, there are values of x for which x^2-6 is positive, negative and zero, so we cannot use the Equality Properties without the risk of introducing extraneous solutions, or worse, losing solutions. For this reason, we break equations like this into cases by rewriting the term in absolute values, $|x|$, using Definition 33. For $x<0$, $|x|=-x$, so for $x<0$, the equation $|x|=x^2-6$ is equivalent to $-x=x^2-6$. Rearranging this gives us $x^2+x-6=0$, or $(x+3)(x-2)=0$. We get $x=-3$ or $x=2$. Since only $x=-3$ satisfies $x<0$, this is the answer we keep. For $x\geq 0$, $|x|=x$, so the equation $|x|=x^2-6$ becomes $x=x^2-6$. From this, we get $x^2-x-6=0$ or $(x-3)(x+2)=0$. Our solutions are $x=3$ or $x=-2$, and since only $x=3$ satisfies $x\geq 0$, this is the one we keep. Hence, our two solutions to $|x|=x^2-6$ are $x=-3$ and $x=3$.

6. To solve $|x-2|+1=x$, we first isolate the absolute value and get $|x-2|=x-1$. Since we see x both inside and outside of the absolute value, we break the equation into cases. The term with absolute values here is $|x-2|$, so we replace 'x' with the quantity '$(x-2)$' in Definition 33 to get

$$|x-2|=\begin{cases} -(x-2), & \text{if} \quad (x-2)<0 \\ (x-2), & \text{if} \quad (x-2)\geq 0 \end{cases}$$

Simplifying yields

$$|x-2|=\begin{cases} -x+2, & \text{if} \quad x<2 \\ x-2, & \text{if} \quad x\geq 2 \end{cases}$$

So, for $x<2$, $|x-2|=-x+2$ and our equation $|x-2|=x-1$ becomes $-x+2=x-1$, which gives $x=\frac{3}{2}$. Since this solution satisfies $x<2$, we keep it. Next, for $x\geq 2$, $|x-2|=x-2$, so the equation $|x-2|=x-1$ becomes $x-2=x-1$. Here, the equation reduces to $-2=-1$, which signifies we have no solutions here. Hence, our only solution is $x=\frac{3}{2}$.

Next, we turn our attention to graphing absolute value functions. Our strategy in the next example is to make liberal use of Definition 33 along with what we know about graphing linear functions (from Section 3.1) and piecewise-defined functions (from Section 2.3).

Example 49 Graphing absolute value functions

Graph each of the following functions.

1. $f(x) = |x|$
2. $g(x) = |x - 3|$
3. $h(x) = |x| - 3$
4. $i(x) = 4 - 2|3x + 1|$

Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

Since functions can have at most one y-intercept (Do you know why?), as soon as we found $(0,0)$ as the x-intercept for $f(x)$ in Example 49, we knew this was also the y-intercept.

SOLUTION

1. To find the zeros of f, we set $f(x) = 0$. We get $|x| = 0$, which, by Theorem 13 gives us $x = 0$. Since the zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$, we get $(0,0)$ as our only x-intercept. To find the y-intercept, we set $x = 0$, and find $y = f(0) = 0$, so that $(0,0)$ is our y-intercept as well. Using Definition 33, we get

$$f(x) = |x| = \begin{cases} -x, & \text{if} \quad x < 0 \\ x, & \text{if} \quad x \geq 0 \end{cases}$$

   Hence, for $x < 0$, we are graphing the line $y = -x$; for $x \geq 0$, we have the line $y = x$. Proceeding as we did in Section 2.5, we get the first two graphs in Figure 3.9.

   Notice that we have an 'open circle' at $(0,0)$ in the graph when $x < 0$. As we have seen before, this is due to the fact that the points on $y = -x$ approach $(0,0)$ as the x-values approach 0. Since x is required to be strictly less than zero on this stretch, the open circle is drawn at the origin. However, notice that when $x \geq 0$, we get to fill in the point at $(0,0)$, which effectively 'plugs' the hole indicated by the open circle. Thus our final result is the graph at the bottom of Figure 3.9.

   By projecting the graph to the x-axis, we see that the domain is $(-\infty, \infty)$. Projecting to the y-axis gives us the range $[0, \infty)$. The function is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. The relative minimum value of f is the same as the absolute minimum, namely 0 which occurs at $(0,0)$. There is no relative maximum value of f. There is also no absolute maximum value of f, since the y values on the graph extend infinitely upwards.

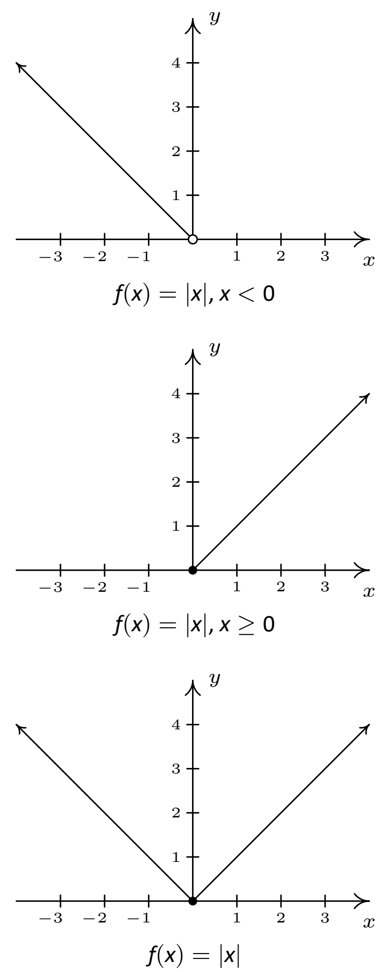


Figure 3.9: Constructing the graph of $f(x) = |x|$

2. To find the zeros of g, we set $g(x) = |x - 3| = 0$. By Theorem 13, we get $x - 3 = 0$ so that $x = 3$. Hence, the x-intercept is $(3,0)$. To find our y-intercept, we set $x = 0$ so that $y = g(0) = |0 - 3| = 3$, which yields $(0,3)$ as our y-intercept. To graph $g(x) = |x - 3|$, we use Definition 33 to rewrite g as

$$g(x) = |x - 3| = \begin{cases} -(x-3), & \text{if} \quad (x-3) < 0 \\ (x-3), & \text{if} \quad (x-3) \geq 0 \end{cases}$$

   Simplifying, we get

$$g(x) = \begin{cases} -x + 3, & \text{if} \quad x < 3 \\ x - 3, & \text{if} \quad x \geq 3 \end{cases}$$

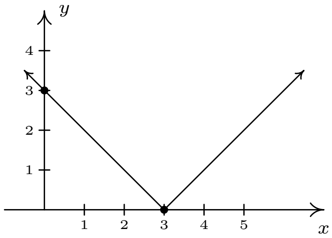


Figure 3.10: $g(x) = |x - 3|$

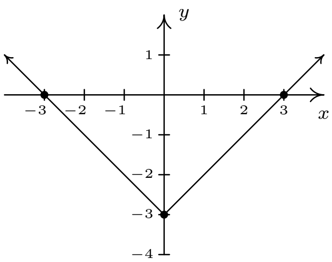


Figure 3.11: $h(x) = |x| - 3$

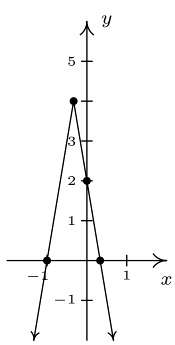


Figure 3.12: $i(x) = 4 - 2|3x + 1|$

As before, the open circle we introduce at $(3, 0)$ from the graph of $y = -x+3$ is filled by the point $(3, 0)$ from the line $y = x-3$. We determine the domain as $(-\infty, \infty)$ and the range as $[0, \infty)$. The function g is increasing on $[3, \infty)$ and decreasing on $(-\infty, 3]$. The relative and absolute minimum value of g is 0 which occurs at $(3, 0)$. As before, there is no relative or absolute maximum value of g.

3. Setting $h(x) = 0$ to look for zeros gives $|x| - 3 = 0$. As in Example 48, we isolate the absolute value to get $|x| = 3$ so that $x = 3$ or $x = -3$. As a result, we have a pair of x-intercepts: $(-3, 0)$ and $(3, 0)$. Setting $x = 0$ gives $y = h(0) = |0| - 3 = -3$, so our y-intercept is $(0, -3)$. As before, we rewrite the absolute value in h to get

$$h(x) = \begin{cases} -x - 3, & \text{if} \quad x < 0 \\ x - 3, & \text{if} \quad x \geq 0 \end{cases}$$

Once again, the open circle at $(0, -3)$ from one piece of the graph of h is filled by the point $(0, -3)$ from the other piece of h. From the graph, we determine the domain of h is $(-\infty, \infty)$ and the range is $[-3, \infty)$. On $[0, \infty)$, h is increasing; on $(-\infty, 0]$ it is decreasing. The relative minimum occurs at the point $(0, -3)$ on the graph, and we see -3 is both the relative and absolute minimum value of h. Also, h has no relative or absolute maximum value.

4. As before, we set $i(x) = 0$ to find the zeros of i and get $4 - 2|3x + 1| = 0$. Isolating the absolute value term gives $|3x + 1| = 2$, so either $3x + 1 = 2$ or $3x + 1 = -2$. We get $x = \frac{1}{3}$ or $x = -1$, so our x-intercepts are $\left(\frac{1}{3}, 0\right)$ and $(-1, 0)$. Substituting $x = 0$ gives $y = i(0) = 4 - 2|3(0) + 1| = 2$, for a y-intercept of $(0, 2)$. Rewriting the formula for $i(x)$ without absolute values gives

$$\begin{aligned} i(x) &= \begin{cases} 4 - 2(-(3x + 1)), & \text{if} \quad (3x + 1) < 0 \\ 4 - 2(3x + 1), & \text{if} \quad (3x + 1) \geq 0 \end{cases} \\ &= \begin{cases} 6x + 6, & \text{if} \quad x < -\frac{1}{3} \\ -6x + 2, & \text{if} \quad x \geq -\frac{1}{3} \end{cases} \end{aligned}$$

The usual analysis near the trouble spot $x = -\frac{1}{3}$ gives that the 'corner' of this graph is $\left(-\frac{1}{3}, 4\right)$, and we get the distinctive '$\vee$' shape: see Figure 3.12.

The domain of i is $(-\infty, \infty)$ while the range is $(-\infty, 4]$. The function i is increasing on $\left(-\infty, -\frac{1}{3}\right]$ and decreasing on $\left[-\frac{1}{3}, \infty\right)$. The relative maximum occurs at the point $\left(-\frac{1}{3}, 4\right)$ and the relative and absolute maximum value of i is 4. Since the graph of i extends downwards forever more, there is no absolute minimum value. As we can see from the graph, there is no relative minimum, either.

Note that all of the functions in the previous example bear the characteristic '$\vee$' shape of the graph of $y = |x|$. We could have graphed the functions g, h and i in Example 49 starting with the graph of $f(x) = |x|$ and applying transformations as in Section 2.6 as our next example illustrates.

Example 50 Graphing using transformations

Graph the following functions starting with the graph of $f(x) = |x|$ and using transformations.

1. $g(x) = |x - 3|$
2. $h(x) = |x| - 3$
3. $i(x) = 4 - 2|3x + 1|$

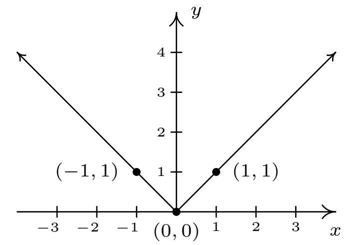


Figure 3.13: $f(x) = |x|$ with three labelled points

Solution We begin by graphing $f(x) = |x|$ and labelling three points, $(-1, 1)$, $(0, 0)$ and $(1, 1)$, as in Figure 3.13

1. Since $g(x) = |x - 3| = f(x - 3)$, Theorem 12 tells us to *add* 3 to each of the x-values of the points on the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. This shifts the graph of $y = f(x)$ to the *right* 3 units and moves the point $(-1, 1)$ to $(2, 1)$, $(0, 0)$ to $(3, 0)$ and $(1, 1)$ to $(4, 1)$. Connecting these points in the classic '$\vee$' fashion produces the graph of $y = g(x)$ in Figure 3.14.

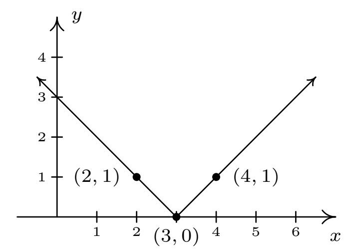


Figure 3.14: $g(x) = |x - 3| = f(x - 3)$

2. For $h(x) = |x| - 3 = f(x) - 3$, Theorem 12 tells us to *subtract* 3 from each of the y-values of the points on the graph of $y = f(x)$ to obtain the graph of $y = h(x)$. This shifts the graph of $y = f(x)$ *down* 3 units and moves $(-1, 1)$ to $(-1, -2)$, $(0, 0)$ to $(0, -3)$ and $(1, 1)$ to $(1, -2)$. Connecting these points with the '$\vee$' shape produces our graph of $y = h(x)$: see Figure 3.15.

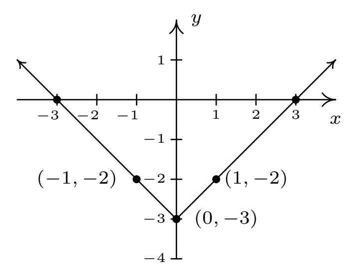


Figure 3.15: $h(x) = |x| - 3 = f(x) - 3$

3. We re-write $i(x) = 4 - 2|3x + 1| = 4 - 2f(3x + 1) = -2f(3x + 1) + 4$ and apply Theorem 12. First, we take care of the changes on the 'inside' of the absolute value. Instead of $|x|$, we have $|3x + 1|$, so, in accordance with Theorem 12, we first *subtract* 1 from each of the x-values of points on the graph of $y = f(x)$, then *divide* each of those new values by 3. This effects a horizontal shift *left* 1 unit followed by a horizontal *shrink* by a factor of 3. These transformations move $(-1, 1)$ to $\left(-\frac{2}{3}, 1\right)$, $(0, 0)$ to $\left(-\frac{1}{3}, 0\right)$ and $(1, 1)$ to $(0, 1)$. Next, we take care of what's happening 'outside of' the absolute value. Theorem 12 instructs us to first *multiply* each y-value of these new points by -2 then *add* 4. Geometrically, this corresponds to a vertical *stretch* by a factor of 2, a reflection across the x-axis and finally, a vertical shift *up* 4 units. These transformations move $\left(-\frac{2}{3}, 1\right)$ to $\left(-\frac{2}{3}, 2\right)$, $\left(-\frac{1}{3}, 0\right)$ to $\left(-\frac{1}{3}, 4\right)$, and $(0, 1)$ to $(0, 2)$. Connecting these points with the usual '$\vee$' shape produces our graph of $y = i(x)$.

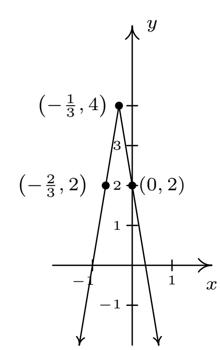


Figure 3.16: $i(x) = 4 - 2|3x + 1| = -2f(3x + 1) + 4$

While the methods in Section 2.6 can be used to graph an entire family of absolute value functions, not all functions involving absolute values posses the characteristic '$\vee$' shape. As the next example illustrates, often there is no substitute for appealing directly to the definition.

Example 51 A more complicated example

Graph each of the following functions. Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

1. $f(x) = \dfrac{|x|}{x}$

2. $g(x) = |x+2| - |x-3| + 1$

Solution

1. We first note that, due to the fraction in the formula of $f(x)$, $x \neq 0$. Thus the domain is $(-\infty, 0) \cup (0, \infty)$. To find the zeros of f, we set $f(x) = \frac{|x|}{x} = 0$. This last equation implies $|x| = 0$, which, from Theorem 13, implies $x = 0$. However, $x = 0$ is not in the domain of f, which means we have, in fact, no x-intercepts. We have no y-intercepts either, since $f(0)$ is undefined. Re-writing the absolute value in the function gives

$$f(x) = \begin{cases} \dfrac{-x}{x}, & \text{if } \quad x < 0 \\ \dfrac{x}{x}, & \text{if } \quad x > 0 \end{cases} = \begin{cases} -1, & \text{if } \quad x < 0 \\ 1, & \text{if } \quad x > 0 \end{cases}$$

To graph this function, we graph two horizontal lines: $y = -1$ for $x < 0$ and $y = 1$ for $x > 0$. We have open circles at $(0, -1)$ and $(0, 1)$ (Can you explain why?) so we get the graph in figure 3.17.

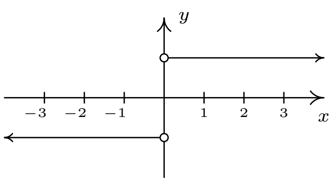


Figure 3.17: $f(x) = \dfrac{|x|}{x}$

As we found earlier, the domain is $(-\infty, 0) \cup (0, \infty)$. The range consists of just two y-values: $\{-1, 1\}$. The function f is constant on $(-\infty, 0)$ and $(0, \infty)$. The local minimum value of f is the absolute minimum value of f, namely -1; the local maximum and absolute maximum values for f also coincide – they both are 1. Every point on the graph of f is simultaneously a relative maximum and a relative minimum. (Can you remember why in light of Definition 28? This was explored in the Exercises in Section 2.5.)

2. To find the zeros of g, we set $g(x) = 0$. The result is $|x+2| - |x-3| + 1 = 0$. Attempting to isolate the absolute value term is complicated by the fact that there are **two** terms with absolute values. In this case, it easier to proceed using cases by re-writing the function g with two separate applications of Definition 33 to remove each instance of the absolute values, one at a time. In the first round we get

$$\begin{aligned} g(x) &= \begin{cases} -(x+2) - |x-3| + 1, & \text{if } \quad (x+2) < 0 \\ (x+2) - |x-3| + 1, & \text{if } \quad (x+2) \geq 0 \end{cases} \\ &= \begin{cases} -x - 1 - |x-3|, & \text{if } \quad x < -2 \\ x + 3 - |x-3|, & \text{if } \quad x \geq -2 \end{cases} \end{aligned}$$

Given that

$$|x-3| = \begin{cases} -(x-3), & \text{if } \quad (x-3) < 0 \\ x-3, & \text{if } \quad (x-3) \geq 0 \end{cases} = \begin{cases} -x+3, & \text{if } \quad x < 3 \\ x-3, & \text{if } \quad x \geq 3 \end{cases},$$

we need to break up the domain again at $x = 3$. Note that if $x < -2$, then $x < 3$, so we replace $|x-3|$ with $-x+3$ for that part of the domain, too. Our completed revision of the form of g yields

$$g(x) = \begin{cases} -x - 1 - (-x + 3), & \text{if} \quad x < -2 \\ x + 3 - (-x + 3), & \text{if} \quad x \geq -2 \text{ and } x < 3 \\ x + 3 - (x - 3), & \text{if} \quad x \geq 3 \end{cases}$$

$$= \begin{cases} -4, & \text{if} \quad x < -2 \\ 2x, & \text{if} \quad -2 \leq x < 3 \\ 6, & \text{if} \quad x \geq 3 \end{cases}$$

To solve $g(x) = 0$, we see that the only piece which contains a variable is $g(x) = 2x$ for $-2 \leq x < 3$. Solving $2x = 0$ gives $x = 0$. Since $x = 0$ is in the interval $[-2, 3)$, we keep this solution and have $(0, 0)$ as our only x-intercept. Accordingly, the y-intercept is also $(0, 0)$. To graph g, we start with $x < -2$ and graph the horizontal line $y = -4$ with an open circle at $(-2, -4)$. For $-2 \leq x < 3$, we graph the line $y = 2x$ and the point $(-2, -4)$ patches the hole left by the previous piece. An open circle at $(3, 6)$ completes the graph of this part. Finally, we graph the horizontal line $y = 6$ for $x \geq 3$, and the point $(3, 6)$ fills in the open circle left by the previous part of the graph. The finished graph is given in Figure 3.18

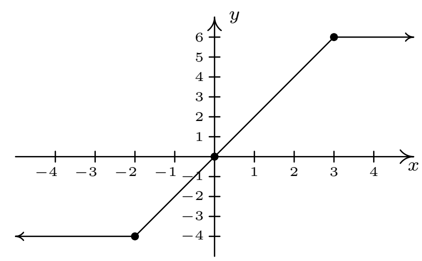


Figure 3.18: $g(x) = |x + 2| - |x - 3| + 1$

The domain of g is all real numbers, $(-\infty, \infty)$, and the range of g is all real numbers between -4 and 6 inclusive, $[-4, 6]$. The function is increasing on $[-2, 3]$ and constant on $(-\infty, -2]$ and $[3, \infty)$. The relative minimum value of f is -4 which matches the absolute minimum. The relative and absolute maximum values also coincide at 6. Every point on the graph of $y = g(x)$ for $x < -2$ and $x > 3$ yields both a relative minimum and relative maximum. The point $(-2, -4)$, however, gives only a relative minimum and the point $(3, 6)$ yields only a relative maximum. (Recall the Exercises in Section 2.5 which dealt with constant functions.)

Many of the applications that the authors are aware of involving absolute values also involve absolute value inequalities. For that reason, we save our discussion of applications for Section 3.4.

Exercises 3.2

Problems

In Exercises 1 – 15, solve the equation.

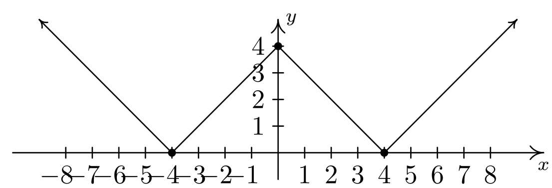
1. $|x| = 6$
2. $|3x - 1| = 10$
3. $|4 - x| = 7$
4. $4 - |x| = 3$
5. $2|5x + 1| - 3 = 0$
6. $|7x - 1| + 2 = 0$
7. $\dfrac{5 - |x|}{2} = 1$
8. $\frac{2}{3}|5 - 2x| - \frac{1}{2} = 5$
9. $|x| = x + 3$
10. $|2x - 1| = x + 1$
11. $4 - |x| = 2x + 1$
12. $|x - 4| = x - 5$
13. $|x| = x^2$
14. $|x| = 12 - x^2$
15. $|x^2 - 1| = 3$

Prove that if $|f(x)| = |g(x)|$ then either $f(x) = g(x)$ or $f(x) = -g(x)$. Use that result to solve the equations in Exercises 16 – 21.

16. $|3x - 2| = |2x + 7|$
17. $|3x + 1| = |4x|$
18. $|1 - 2x| = |x + 1|$
19. $|4 - x| - |x + 2| = 0$
20. $|2 - 5x| = 5|x + 1|$
21. $3|x - 1| = 2|x + 1|$

In Exercises 22 – 33, graph the function. Find the zeros of each function and the x- and y-intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, list the intervals on which the function is increasing, decreasing or constant, and find the relative and absolute extrema, if they exist.

22. $f(x) = |x + 4|$
23. $f(x) = |x| + 4$
24. $f(x) = |4x|$
25. $f(x) = -3|x|$
26. $f(x) = 3|x + 4| - 4$
27. $f(x) = \frac{1}{3}|2x - 1|$
28. $f(x) = \dfrac{|x + 4|}{x + 4}$
29. $f(x) = \dfrac{|2 - x|}{2 - x}$
30. $f(x) = x + |x| - 3$
31. $f(x) = |x + 2| - x$
32. $f(x) = |x + 2| - |x|$
33. $f(x) = |x + 4| + |x - 2|$
34. With the help of your classmates, find an absolute value function whose graph is given below.



35. With help from your classmates, prove the second, third and fifth parts of Theorem 13.
36. Prove **The Triangle Inequality:** For all real numbers a and b, $|a + b| \leq |a| + |b|$.

3.3 Quadratic Functions

You may recall studying quadratic equations in high school. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 34 Quadratic function

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The most basic quadratic function is $f(x) = x^2$, whose graph appears below. Its shape should look familiar from high school – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola. In this case, the vertex is a relative minimum and is also the where the absolute minimum value of f can be found.

Much like many of the absolute value functions in Section 3.2, knowing the graph of $f(x) = x^2$ enables us to graph an entire family of quadratic functions using transformations.

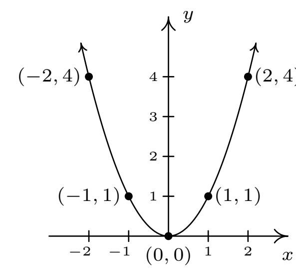


Figure 3.19: The graph of the basic quadratic function $f(x) = x^2$

Example 52 Graphics quadratic functions

Graph the following functions starting with the graph of $f(x) = x^2$ and using transformations. Find the vertex, state the range and find the x- and y-intercepts, if any exist.

1. $g(x) = (x + 2)^2 - 3$
2. $h(x) = -2(x - 3)^2 + 1$

Solution

1. Since $g(x) = (x + 2)^2 - 3 = f(x + 2) - 3$, Theorem 12 instructs us to first *subtract* 2 from each of the x-values of the points on $y = f(x)$. This shifts the graph of $y = f(x)$ to the *left* 2 units and moves $(-2, 4)$ to $(-4, 4)$, $(-1, 1)$ to $(-3, 1)$, $(0, 0)$ to $(-2, 0)$, $(1, 1)$ to $(-1, 1)$ and $(2, 4)$ to $(0, 4)$. Next, we *subtract* 3 from each of the y-values of these new points. This moves the graph *down* 3 units and moves $(-4, 4)$ to $(-4, 1)$, $(-3, 1)$ to $(-3, -2)$, $(-2, 0)$ to $(-2, 3)$, $(-1, 1)$ to $(-1, -2)$ and $(0, 4)$ to $(0, 1)$. We connect the dots in parabolic fashion to get the graph in Figure 3.21.

   From the graph, we see that the vertex has moved from $(0, 0)$ on the graph of $y = f(x)$ to $(-2, -3)$ on the graph of $y = g(x)$. This sets $[-3, \infty)$ as the range of g. We see that the graph of $y = g(x)$ crosses the x-axis twice, so we expect two x-intercepts. To find these, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x + 2)^2 - 3 = 0$, or $(x + 2)^2 = 3$. Extracting square roots gives $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2 - \sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2 + \sqrt{3}, 0) \approx (-0.27, 0)$. The y-intercept of the graph, $(0, 1)$ was one of the points we originally plotted, so we are done.

2. Following Theorem 12 once more, to graph $h(x) = -2(x - 3)^2 + 1 = -2f(x - 3) + 1$, we first start by *adding* 3 to each of the x-values of the

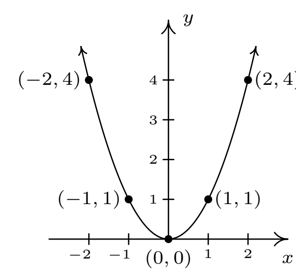


Figure 3.20: The graph $y = x^2$ with points labelled

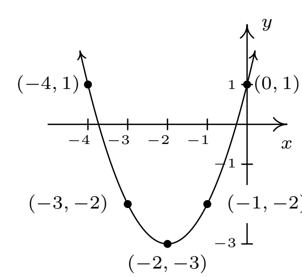


Figure 3.21: $g(x) = f(x + 2) - 3 = (x + 2)^2 - 3$

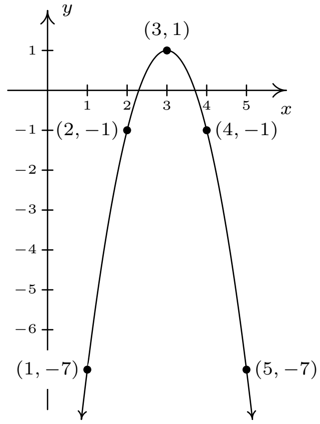


Figure 3.22: $h(x) = -2f(x-3)+1 = -2(x-3)^2+1$

points on the graph of $y = f(x)$. This effects a horizontal shift *right* 3 units and moves $(-2,4)$ to $(1,4)$, $(-1,1)$ to $(2,1)$, $(0,0)$ to $(3,0)$, $(1,1)$ to $(4,1)$ and $(2,4)$ to $(5,4)$. Next, we *multiply* each of our y-values first by -2 and then *add* 1 to that result. Geometrically, this is a vertical *stretch* by a factor of 2, followed by a reflection about the x-axis, followed by a vertical shift *up* 1 unit. This moves $(1,4)$ to $(1,-7)$, $(2,1)$ to $(2,-1)$, $(3,0)$ to $(3,1)$, $(4,1)$ to $(4,-1)$ and $(5,4)$ to $(5,-7)$, giving us the graph in Figure 3.22.

The vertex is $(3,1)$ which makes the range of h $(-\infty, 1]$. From our graph, we know that there are two x-intercepts, so we set $y = h(x) = 0$ and solve. We get $-2(x-3)^2+1 = 0$ which gives $(x-3)^2 = \frac{1}{2}$. Extracting square roots (and rationalizing denominators!) gives $x - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side, (and get common denominators!) we get $x = \frac{6\pm\sqrt{2}}{2}$. Hence, our x-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. Although our graph doesn't show it, there is a y-intercept which can be found by setting $x = 0$. With $h(0) = -2(0-3)^2+1 = -17$, we have that our y-intercept is $(0, -17)$.

A few remarks about Example 52 are in order. First note that neither the formula given for $g(x)$ nor the one given for $h(x)$ match the form given in Definition 34. We could, of course, convert both $g(x)$ and $h(x)$ into that form by expanding and collecting like terms. Doing so, we find $g(x) = (x+2)^2-3 = x^2+4x+1$ and $h(x) = -2(x-3)^2+1 = -2x^2+12x-17$. While these 'simplified' formulas for $g(x)$ and $h(x)$ satisfy Definition 34, they do not lend themselves to graphing easily. For that reason, the form of g and h presented in Example 53 is given a special name, which we list below, along with the form presented in Definition 34.

Definition 35 Standard and General Form of Quadratic Functions

Suppose f is a quadratic function.

- The **general form** of the quadratic function f is $f(x) = ax^2+bx+c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2+k$, where a, h and k are real numbers with $a \neq 0$.

It is important to note at this stage that we have no guarantees that *every* quadratic function can be written in standard form. This is actually true, and we prove this later in the exposition, but for now we celebrate the advantages of the standard form, starting with the following theorem.

Theorem 14 Vertex Formula for Quadratics in Standard Form

For the quadratic function $f(x) = a(x-h)^2+k$, where a, h and k are real numbers with $a \neq 0$, the vertex of the graph of $y = f(x)$ is (h, k).

We can readily verify the formula given Theorem 14 with the two functions given in Example 52. After a (slight) rewrite, $g(x) = (x+2)^2 - 3 = (x-(-2))^2 + (-3)$, and we identify $h = -2$ and $k = -3$. Sure enough, we found the vertex of the graph of $y = g(x)$ to be $(-2, -3)$. For $h(x) = -2(x-3)^2 + 1$, no rewrite is needed. We can directly identify $h = 3$ and $k = 1$ and, sure enough, we found the vertex of the graph of $y = h(x)$ to be $(3, 1)$.

To see why the formula in Theorem 14 produces the vertex, consider the graph of the equation $y = a(x-h)^2 + k$. When we substitute $x = h$, we get $y = k$, so (h, k) is on the graph. If $x \neq h$, then $x - h \neq 0$ so $(x-h)^2$ is a positive number. If $a > 0$, then $a(x-h)^2$ is positive, thus $y = a(x-h)^2 + k$ is always a number larger than k. This means that when $a > 0$, (h, k) is the lowest point on the graph and thus the parabola must open upwards, making (h, k) the vertex. A similar argument shows that if $a < 0$, (h, k) is the highest point on the graph, so the parabola opens downwards, and (h, k) is also the vertex in this case.

Alternatively, we can apply the machinery in Section 2.6. Since the vertex of $y = x^2$ is $(0, 0)$, we can determine the vertex of $y = a(x-h)^2 + k$ by determining the final destination of $(0, 0)$ as it is moved through each transformation. To obtain the formula $f(x) = a(x-h)^2 + k$, we start with $g(x) = x^2$ and first define $g_1(x) = ag(x) = ax^2$. This is results in a vertical scaling and/or reflection. (Just a scaling if $a > 0$. If $a < 0$, there is a reflection involved.) Since we multiply the output by a, we multiply the y-coordinates on the graph of g by a, so the point $(0, 0)$ remains $(0, 0)$ and remains the vertex. Next, we define $g_2(x) = g_1(x-h) = a(x-h)^2$. This induces a horizontal shift right or left h units (right if $h > 0$, left if $h < 0$.) moves the vertex, in either case, to $(h, 0)$. Finally, $f(x) = g_2(x) + k = a(x-h)^2 + k$ which effects a vertical shift up or down k units (up if $k > 0$, down if $k < 0$) resulting in the vertex moving from $(h, 0)$ to (h, k).

In addition to verifying Theorem 14, the arguments in the two preceding paragraphs have also shown us the role of the number a in the graphs of quadratic functions. The graph of $y = a(x-h)^2 + k$ is a parabola 'opening upwards' if $a > 0$, and 'opening downwards' if $a < 0$. Moreover, the symmetry enjoyed by the graph of $y = x^2$ about the y-axis is translated to a symmetry about the vertical line $x = h$ which is the vertical line through the vertex. (You should use transformations to verify this!) This line is called the **axis of symmetry** of the parabola and is shown as the dashed line in Figure 3.23

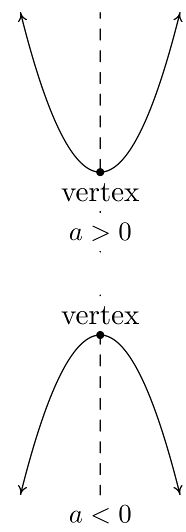


Figure 3.23: The axis of symmetry of a parabola

Without a doubt, the standard form of a quadratic function, coupled with the machinery in Section 2.6, allows us to list the attributes of the graphs of such functions quickly and elegantly. What remains to be shown, however, is the fact that every quadratic function *can be written* in standard form. To convert a quadratic function given in general form into standard form, we employ the ancient rite of 'Completing the Square'. We remind the reader how this is done in our next example.

Example 53 Converting from general to standard form

Convert the functions below from general form to standard form. Find the vertex, axis of symmetry and any x- or y-intercepts. Graph each function and determine its range.

1. $f(x) = x^2 - 4x + 3$.
2. $g(x) = 6 - x - x^2$

Solution

1. To convert from general form to standard form, we complete the square. First, we verify that the coefficient of x^2 is 1. Next, we find the coefficient

If you forget why we do what we do to complete the square, start with $a(x-h)^2+k$, multiply it out, step by step, and then reverse the process.

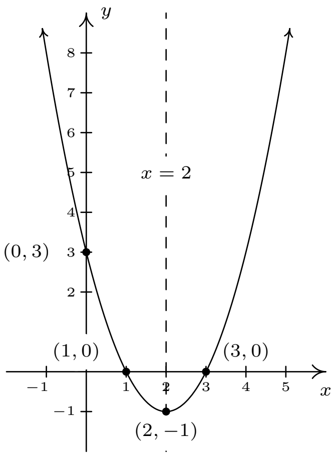


Figure 3.24: $f(x) = x^2 - 4x + 3$

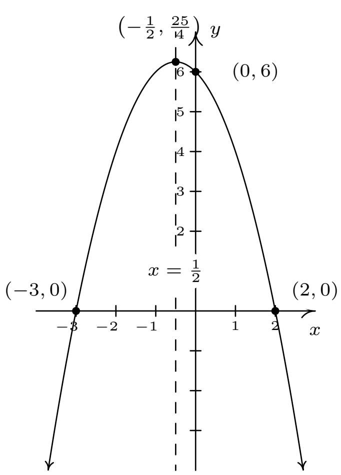


Figure 3.25: $g(x) = 6 - x - x^2$

of x, in this case -4, and take half of it to get $\frac{1}{2}(-4) = -2$. This tells us that our target perfect square quantity is $(x-2)^2$. To get an expression equivalent to $(x-2)^2$, we need to add $(-2)^2 = 4$ to the $x^2 - 4x$ to create a perfect square trinomial, but to keep the balance, we must also subtract it. We collect the terms which create the perfect square and gather the remaining constant terms. Putting it all together, we get

$$\begin{aligned}
f(x) &= x^2 - 4x + 3 && \text{(Compute } \tfrac{1}{2}(-4) = -2.\text{)}\\
&= \left(x^2 - 4x + \underline{4} - \underline{4}\right) + 3 && \text{(Add and subtract } (-2)^2 = 4 \text{ to } (x^2+4x).\text{)}\\
&= \left(x^2 - 4x + 4\right) - 4 + 3 && \text{(Group the perfect square trinomial.)}\\
&= (x-2)^2 - 1 && \text{(Factor the perfect square trinomial.)}
\end{aligned}$$

Of course, we can always check our answer by multiplying out $f(x) = (x-2)^2 - 1$ to see that it simplifies to $f(x) = x^2 - 4x - 1$. In the form $f(x) = (x-2)^2 - 1$, we readily find the vertex to be $(2,-1)$ which makes the axis of symmetry $x = 2$. To find the x-intercepts, we set $y = f(x) = 0$. We are spoiled for choice, since we have *two* formulas for $f(x)$. Since we recognize $f(x) = x^2 - 4x + 3$ to be easily factorable, (experience pays off, here!) we proceed to solve $x^2 - 4x + 3 = 0$. Factoring gives $(x-3)(x-1) = 0$ so that $x = 3$ or $x = 1$. The x-intercepts are then $(1,0)$ and $(3,0)$. To find the y-intercept, we set $x = 0$. Once again, the general form $f(x) = x^2 - 4x + 3$ is easiest to work with here, and we find $y = f(0) = 3$. Hence, the y-intercept is $(0,3)$. With the vertex, axis of symmetry and the intercepts, we get a pretty good graph without the need to plot additional points. We see that the range of f is $[-1,\infty)$ and we are done. The graph of f is given in Figure 3.24.

2. To get started, we rewrite $g(x) = 6 - x - x^2 = -x^2 - x + 6$ and note that the coefficient of x^2 is -1, not 1. This means our first step is to factor out the (-1) from both the x^2 and x terms. We then follow the completing the square recipe as above.

$$\begin{aligned}
g(x) &= -x^2 - x + 6\\
&= (-1)\left(x^2 + x\right) + 6 && \text{(Factor the coefficient of } x^2 \text{ from } x^2 \text{ and } x.\text{)}\\
&= (-1)\left(x^2 + x + \underline{\frac{1}{4}} - \underline{\frac{1}{4}}\right) + 6\\
&= (-1)\left(x^2 + x + \frac{1}{4}\right) + (-1)\left(-\frac{1}{4}\right) + 6\\
& && \text{(Group the perfect square trinomial.)}\\
&= -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}
\end{aligned}$$

From $g(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{25}{4}$, we get the vertex to be $\left(-\frac{1}{2}, \frac{25}{4}\right)$ and the axis of symmetry to be $x = -\frac{1}{2}$. To get the x-intercepts, we opt to set the given formula $g(x) = 6 - x - x^2 = 0$. Solving, we get $x = -3$ and $x = 2$, so the x-intercepts are $(-3,0)$ and $(2,0)$. Setting $x = 0$, we find $g(0) = 6$, so the y-intercept is $(0,6)$. Plotting these points gives us the graph in Figure 3.25. We see that the range of g is $\left(-\infty, \frac{25}{4}\right]$.

With Example 53 fresh in our minds, we are now in a position to show that every quadratic function can be written in standard form. We begin with $f(x) = ax^2 + bx + c$, assume $a \neq 0$, and complete the square in *complete* generality.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c && \text{(Factor out coefficient of } x^2 \text{ from } x^2 \text{ and } x.) \\ &= a\left(x^2 + \frac{b}{a}x + \underline{\frac{b^2}{4a^2}} - \underline{\frac{b^2}{4a^2}}\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - a\left(\frac{b^2}{4a^2}\right) + c \\ & && \text{(Group the perfect square trinomial.)} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Factor and get a common denominator.)} \end{aligned}$$

Comparing this last expression with the standard form, we identify $(x - h)$ with $\left(x + \frac{b}{2a}\right)$ so that $h = -\frac{b}{2a}$. Instead of memorizing the value $k = \frac{4ac-b^2}{4a}$, we see that $f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$. As such, we have derived a vertex formula for the general form. We summarize both vertex formulas in the box at the top of the next page.

Theorem 15 Vertex Formulas for Quadratic Functions

Suppose a, b, c, h and k are real numbers with $a \neq 0$.

- If $f(x) = a(x - h)^2 + k$, the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$, the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

There are two more results which can be gleaned from the completed-square form of the general form of a quadratic function,

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

We have seen that the number a in the standard form of a quadratic function determines whether the parabola opens upwards (if $a > 0$) or downwards (if $a < 0$). We see here that this number a is none other than the coefficient of x^2 in the general form of the quadratic function. In other words, it is the coefficient of x^2 alone which determines this behavior – a result that is generalized in Section 4.1. The second treasure is a re-discovery of the **quadratic formula**.

Theorem 16 The Quadratic Formula

If a, b and c are real numbers with $a \neq 0$, then the solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Assuming the conditions of Equation 16, the solutions to $ax^2 + bx + c = 0$ are precisely the zeros of $f(x) = ax^2 + bx + c$. Since

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

the equation $ax^2 + bx + c = 0$ is equivalent to

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Solving gives

$$\begin{aligned}
a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} &= 0 \\
a\left(x + \frac{b}{2a}\right)^2 &= -\frac{4ac - b^2}{4a} \\
\frac{1}{a}\left[a\left(x + \frac{b}{2a}\right)^2\right] &= \frac{1}{a}\left(\frac{b^2 - 4ac}{4a}\right) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{extract square roots} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

In our discussions of domain, we were warned against having negative numbers underneath the square root. Given that $\sqrt{b^2 - 4ac}$ is part of the Quadratic Formula, we will need to pay special attention to the radicand $b^2 - 4ac$. It turns out that the quantity $b^2 - 4ac$ plays a critical role in determining the nature of the solutions to a quadratic equation. It is given a special name.

Definition 36 Discriminant

If a, b and c are real numbers with $a \neq 0$, then the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$.

The discriminant 'discriminates' between the kinds of solutions we get from a quadratic equation. These cases, and their relation to the discriminant, are summarized below.

Theorem 17 Discriminant Trichotomy

Let a, b and c be real numbers with $a \neq 0$.

- If $b^2 - 4ac < 0$, the equation $ax^2 + bx + c = 0$ has no real solutions.
- If $b^2 - 4ac = 0$, the equation $ax^2 + bx + c = 0$ has exactly one real solution.
- If $b^2 - 4ac > 0$, the equation $ax^2 + bx + c = 0$ has exactly two real solutions.

The proof of Theorem 17 stems from the position of the discriminant in the quadratic equation, and is left as a good mental exercise for the reader. The next example exploits the fruits of all of our labor in this section thus far.

Example 54 Computing and maximizing profit

Recall that the profit (defined on page 73) for a product is defined by the equation Profit = Revenue − Cost, or $P(x) = R(x) - C(x)$. In Example 47 the weekly revenue, in dollars, made by selling x PortaBoy Game Systems was found to be $R(x) = -1.5x^2 + 250x$ with the restriction (carried over from the price-demand function) that $0 \leq x \leq 166$. The cost, in dollars, to produce x PortaBoy Game Systems is given in Example 45 as $C(x) = 80x + 150$ for $x \geq 0$.

1. Determine the weekly profit function $P(x)$.
2. Graph $y = P(x)$. Include the x- and y-intercepts as well as the vertex and axis of symmetry.
3. Interpret the zeros of P.
4. Interpret the vertex of the graph of $y = P(x)$.
5. Recall that the weekly price-demand equation for PortaBoys is $p(x) = -1.5x + 250$, where $p(x)$ is the price per PortaBoy, in dollars, and x is the weekly sales. What should the price per system be in order to maximize profit?

Solution

1. To find the profit function $P(x)$, we subtract

   $$P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150.$$

   Since the revenue function is valid when $0 \leq x \leq 166$, P is also restricted to these values.

2. To find the x-intercepts, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. The mere thought of trying to factor the left hand side of this equation could do serious psychological damage, so we resort to the quadratic formula, Equation 16. Identifying $a = -1.5$, $b = 170$, and $c = -150$, we obtain

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\
&= \frac{-170 \pm \sqrt{28000}}{-3} \\
&= \frac{170 \pm 20\sqrt{70}}{3}
\end{aligned}
$$

We get two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right)$. To find the y-intercept, we set $x = 0$ and find $y = P(0) = -150$ for a y-intercept of $(0, -150)$. To find the vertex, we use the fact that $P(x) = -1.5x^2 + 170x - 150$ is in the general form of a quadratic function and appeal to Equation 15. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3}$ and find that our vertex is $\left(\frac{170}{3}, \frac{14000}{3}\right)$. The axis of symmetry is the vertical line passing through the vertex so it is the line $x = \frac{170}{3}$. To sketch a reasonable graph, we approximate the x-intercepts, $(0.89, 0)$ and $(112.44, 0)$, and the vertex, $(56.67, 4666.67)$. (Note that in order to get the x-intercepts and the vertex to show up in the same picture, we had to scale the x-axis differently than the y-axis in Figure 3.26. This results in the left-hand x-intercept and the y-intercept being uncomfortably close to each other and to the origin in the picture.)

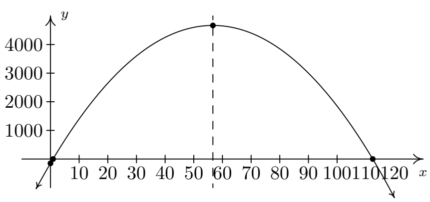


Figure 3.26: The graph of the profit function $P(x)$

3. The zeros of P are the solutions to $P(x) = 0$, which we have found to be approximately 0.89 and 112.44. As we saw in Example 29, these are the 'break-even' points of the profit function, where enough product is sold to recover the cost spent to make the product. More importantly, we see from the graph that as long as x is between 0.89 and 112.44, the graph $y = P(x)$ is above the x-axis, meaning $y = P(x) > 0$ there. This means that for these values of x, a profit is being made. Since x represents the weekly sales of PortaBoy Game Systems, we round the zeros to positive integers and have that as long as 1, but no more than 112 game systems are sold weekly, the retailer will make a profit.

4. From the graph, we see that the maximum value of P occurs at the vertex, which is approximately $(56.67, 4666.67)$. As above, x represents the weekly sales of PortaBoy systems, so we can't sell 56.67 game systems. Comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

5. In the previous part, we found that we need to sell 57 PortaBoys per week to maximize profit. To find the price per PortaBoy, we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. The price should be set at \$164.50.

Our next example is another classic application of quadratic functions.

Example 55 Optimizing pasture dimensions

Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives. The time is finally right for him to pursue his dream of farming alpaca. He wishes to build a rectangular pasture, and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a stream (so no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

SOLUTION It is always helpful to sketch the problem situation, so we do so in Figure 3.27.

We are tasked to find the dimensions of the pasture which would give a maximum area. We let w denote the width of the pasture and we let l denote the length of the pasture. Since the units given to us in the statement of the problem are feet, we assume w and l are measured in feet. The area of the pasture, which we'll call A, is related to w and l by the equation $A = wl$. Since w and l are both measured in feet, A has units of feet^2, or square feet. We are given the total amount of fencing available is 200 feet, which means $w + l + w = 200$, or, $l + 2w = 200$. We now have two equations, $A = wl$ and $l + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or l. This is where we use the equation $l + 2w = 200$. Solving for l, we find $l = 200 - 2w$, and we substitute this into our equation for A. We get $A = wl = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A(w) = 200w - 2w^2 = -2w^2 + 200w$.

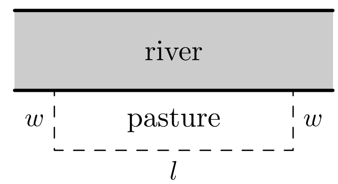


Figure 3.27: A diagram of pasture dimensions

Before we go any further, we need to find the applied domain of A so that we know what values of w make sense in this problem situation. (Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.) Since w represents the width of the pasture, $w > 0$. Likewise, l represents the length of the pasture, so $l = 200 - 2w > 0$. Solving this latter inequality, we find $w < 100$. Hence, the function we wish to maximize is $A(w) = -2w^2 + 200w$ for $0 < w < 100$. Since A is a quadratic function (of w), we know that the graph of $y = A(w)$ is a parabola. Since the coefficient of w^2 is -2, we know that this parabola opens downwards. This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $l = 200 - 2w$ and find $l = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca.

We conclude this section with the graph of a more complicated absolute value function.

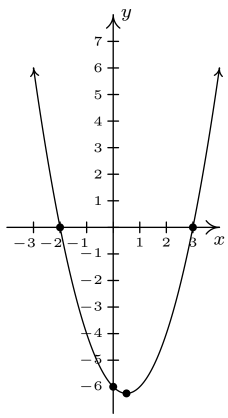
Example 56 Graphing the absolute value of a quadratic function

Graph $f(x) = |x^2 - x - 6|$.

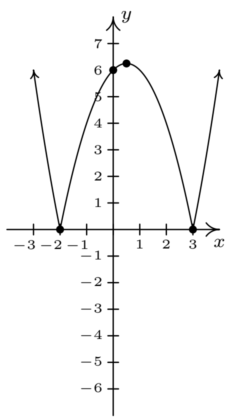
SOLUTION Using the definition of absolute value, Definition 33, we have

$$f(x) = \begin{cases} -\left(x^2 - x - 6\right), & \text{if } \; x^2 - x - 6 < 0 \\ x^2 - x - 6, & \text{if } \; x^2 - x - 6 \geq 0 \end{cases}$$

The trouble is that we have yet to develop any analytic techniques to solve nonlinear inequalities such as $x^2 - x - 6 < 0$. You won't have to wait long; this is one of the main topics of Section 3.4. Nevertheless, we can attack this problem graphically. To that end, we graph $y = g(x) = x^2 - x - 6$ using the intercepts and the vertex. To find the x-intercepts, we solve $x^2 - x - 6 = 0$. Factoring gives $(x-3)(x+2) = 0$ so $x = -2$ or $x = 3$. Hence, $(-2, 0)$ and $(3, 0)$ are x-intercepts. The y-intercept $(0, -6)$ is found by setting $x = 0$. To plot the vertex, we find $x = -\frac{b}{2a} = -\frac{-1}{2(1)} = \frac{1}{2}$, and $y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6 = -\frac{25}{4} = -6.25$. Plotting, we get the parabola seen below on the left. To obtain points on the graph of $y = f(x) = |x^2 - x - 6|$, we can take points on the graph of $g(x) = x^2 - x - 6$ and apply the absolute value to each of the y values on the parabola. We see from the graph of g that for $x \leq -2$ or $x \geq 3$, the y values on the parabola are greater than or equal to zero (since the graph is on or above the x-axis), so the absolute value leaves these portions of the graph alone. For x between -2 and 3, however, the y values on the parabola are negative. For example, the point $(0, -6)$ on $y = x^2 - x - 6$ would result in the point $(0, |-6|) = (0, -(-6)) = (0, 6)$ on the graph of $f(x) = |x^2 - x - 6|$. Proceeding in this manner for all points with x-coordinates between -2 and 3 results in the graph seen at the bottom of Figure 3.28.



$y = x^2 - x - 6$



$y = |x^2 - x - 6$

Figure 3.28: Obtaining the graph of $f(x) = |x^2 - x - 6|$

If we take a step back and look at the graphs of g and f in the last example, we notice that to obtain the graph of f from the graph of g, we reflect a *portion* of the graph of g about the x-axis. We can see this analytically by substituting $g(x) = x^2 - x - 6$ into the formula for $f(x)$ and calling to mind Theorem 9 from Section 2.6.

$$f(x) = \begin{cases} -g(x), & \text{if} \quad g(x) < 0 \\ g(x), & \text{if} \quad g(x) \geq 0 \end{cases}$$

The function f is defined so that when $g(x)$ is negative (i.e., when its graph is below the x-axis), the graph of f is its refection across the x-axis. This is a general template to graph functions of the form $f(x) = |g(x)|$. From this perspective, the graph of $f(x) = |x|$ can be obtained by reflecting the portion of the line $g(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.

Exercises 3.3

Problems

In Exercises 1 – 9, graph the quadratic function. Find the x- and y-intercepts of each graph, if any exist. If it is given in general form, convert it into standard form; if it is given in standard form, convert it into general form. Find the domain and range of the function and list the intervals on which the function is increasing or decreasing. Identify the vertex and the axis of symmetry and determine whether the vertex yields a relative and absolute maximum or minimum.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $f(x) = -2(x+1)^2 + 4$
5. $f(x) = 2x^2 - 4x - 1$
6. $f(x) = -3x^2 + 4x - 7$
7. $f(x) = x^2 + x + 1$
8. $f(x) = -3x^2 + 5x + 4$
9. $f(x) = x^2 - \frac{1}{100}x - 1$[1]

In Exercises 10 – 14, the cost and price-demand functions are given for different scenarios. For each scenario,

- **Find the profit function $P(x)$.**
- **Find the number of items which need to be sold in order to maximize profit.**
- **Find the maximum profit.**
- **Find the price to charge per item in order to maximize profit.**
- **Find and interpret break-even points.**

10. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.
11. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.
12. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, $0 \leq x \leq 30$.
13. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.
14. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, $0 \leq x \leq 70$.
15. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$. How many cookies should they bake in order to maximize their profit?
16. Using data from Bureau of Transportation Statistics, the average fuel economy F in miles per gallon for passenger cars in the US can be modelled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.
17. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

    What is the warmest temperature of the day? When does this happen?

18. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?
19. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?
20. In the situation of Example 55, Donnie has a nightmare that one of his alpaca herd fell into the river and drowned. To avoid this, he wants to move his rectangular pasture *away* from the river. This means that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?
21. What is the largest rectangular area one can enclose with 14 inches of string?

[1] We have already seen the graph of this function. It was used as an example in Section 2.5 to show how the graphing calculator can be misleading.

22. The height of an object dropped from the roof of an eight story building is modelled by $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, h is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

23. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

24. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height h in feet of the hammer above the ground t seconds after Jason lets it go is modeled by $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

25. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.
    (a) What is the applied domain of this function?
    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.
    (c) Come up with a scenario in which $s_0 < 0$.
    (d) Let's say a slingshot is used to shoot a marble straight up from the ground $(s_0 = 0)$ with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?
    (e) Now shoot the marble from the top of a tower which is 25 meters tall. When does it hit the ground?
    (f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

26. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[2] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

27. Graph $f(x) = |1 - x^2|$

28. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

29. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

30. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

In Exercises 31 – 36, solve the quadratic equation for the indicated variable.

31. $x^2 - 10y^2 = 0$ for x

32. $y^2 - 4y = x^2 - 4$ for x

33. $x^2 - mx = 1$ for x

34. $y^2 - 3y = 4x$ for y

35. $y^2 - 4y = x^2 - 4$ for y

36. $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)

[2] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line forms not a parabola, but a **catenary**, a curve that is defined using exponential functions.

3.4 Inequalities with Absolute Value and Quadratic Functions

In this section, not only do we develop techniques for solving various classes of inequalities analytically, we also look at them graphically. The first example motivates the core ideas.

Example 57 Inequalities with linear functions

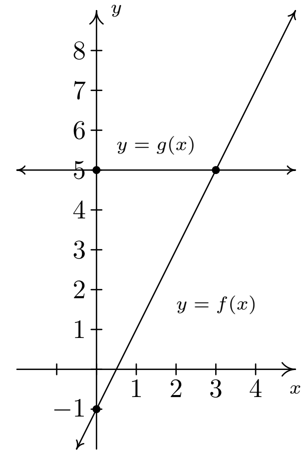
Let $f(x) = 2x - 1$ and $g(x) = 5$.

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) > g(x)$.
4. Graph $y = f(x)$ and $y = g(x)$ on the same set of axes and interpret your solutions to parts 1 through 3 above.

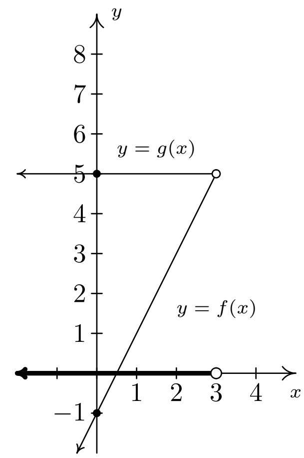
Solution

1. To solve $f(x) = g(x)$, we replace $f(x)$ with $2x - 1$ and $g(x)$ with 5 to get $2x - 1 = 5$. Solving for x, we get $x = 3$.
2. The inequality $f(x) < g(x)$ is equivalent to $2x - 1 < 5$. Solving gives $x < 3$ or $(-\infty, 3)$.
3. To find where $f(x) > g(x)$, we solve $2x - 1 > 5$. We get $x > 3$, or $(3, \infty)$.
4. To graph $y = f(x)$, we graph $y = 2x - 1$, which is a line with a y-intercept of $(0, -1)$ and a slope of 2. The graph of $y = g(x)$ is $y = 5$ which is a horizontal line through $(0, 5)$.

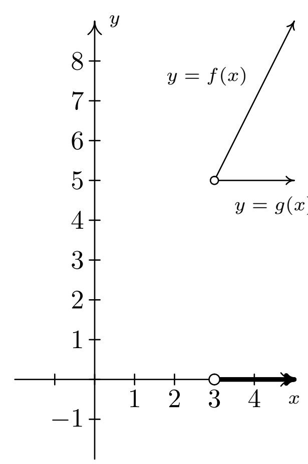
   To see the connection between the graph and the Algebra, we recall the Fundamental Graphing Principle for Functions in Section 2.5: the point (a, b) is on the graph of f if and only if $f(a) = b$. In other words, a generic point on the graph of $y = f(x)$ is $(x, f(x))$, and a generic point on the graph of $y = g(x)$ is $(x, g(x))$. When we seek solutions to $f(x) = g(x)$, we are looking for x values whose y values on the graphs of f and g are the same. In part 1, we found $x = 3$ is the solution to $f(x) = g(x)$. Sure enough, $f(3) = 5$ and $g(3) = 5$ so that the point $(3, 5)$ is on both graphs. In other words, the graphs of f and g *intersect* at $(3, 5)$. In part 2, we set $f(x) < g(x)$ and solved to find $x < 3$. For $x < 3$, the point $(x, f(x))$ is *below* $(x, g(x))$ since the y values on the graph of f are less than the y values on the graph of g there. Analogously, in part 3, we solved $f(x) > g(x)$ and found $x > 3$. For $x > 3$, note that the graph of f is *above* the graph of g, since the y values on the graph of f are greater than the y values on the graph of g for those values of x: see Figure 3.29.



Intersecting graphs $y = f(x)$ and $y = g(x)$



$f(x) < g(x)$ on $(-\infty, 3)$



$f(x) > g(x)$ on $(3, \infty)$

Figure 3.29: Graphical interpretation of Example 57

The preceding example demonstrates the following, which is a consequence of the Fundamental Graphing Principle for Functions.

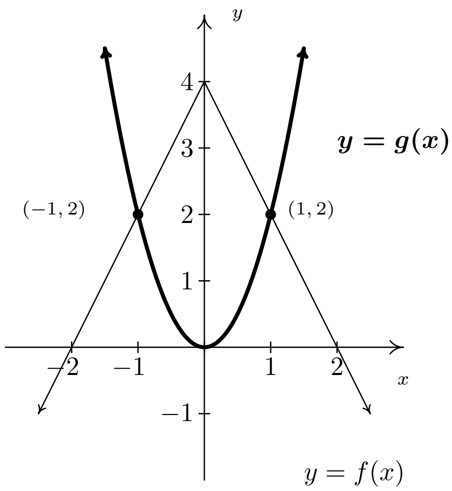


Figure 3.30: The graphs $y = f(x)$ and $y = g(x)$ for Example 58

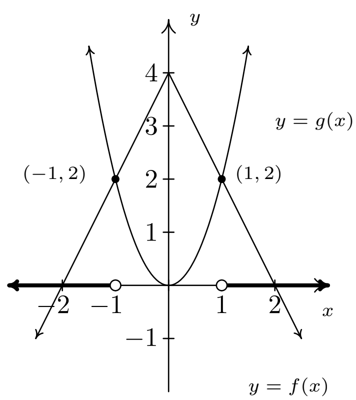


Figure 3.31: The solution to $f(x) < g(x)$

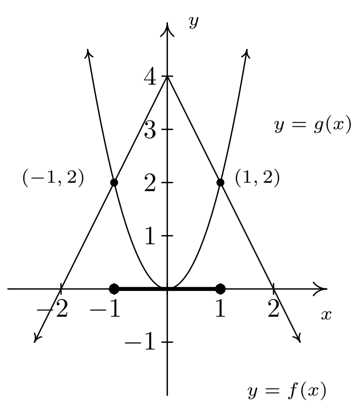


Figure 3.32: The solution to $f(x) \geq g(x)$

Key Idea 17 Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions.

- The solutions to $f(x) = g(x)$ are the x values where the graphs of $y = f(x)$ and $y = g(x)$ intersect.
- The solution to $f(x) < g(x)$ is the set of x values where the graph of $y = f(x)$ is *below* the graph of $y = g(x)$.
- The solution to $f(x) > g(x)$ is the set of x values where the graph of $y = f(x)$ *above* the graph of $y = g(x)$.

The next example turns the tables and furnishes the graphs of two functions and asks for solutions to equations and inequalities.

Example 58 Using graphs to solve equations and inequalities

The graphs of f and g are shown in Figure 3.30. (The graph of $y = g(x)$ is in bold.) Use these graphs to answer the following questions.

1. Solve $f(x) = g(x)$.
2. Solve $f(x) < g(x)$.
3. Solve $f(x) \geq g(x)$.

Solution

1. To solve $f(x) = g(x)$, we look for where the graphs of f and g intersect. These appear to be at the points $(-1, 2)$ and $(1, 2)$, so our solutions to $f(x) = g(x)$ are $x = -1$ and $x = 1$.
2. To solve $f(x) < g(x)$, we look for where the graph of f is below the graph of g. This appears to happen for the x values less than -1 and greater than 1. Our solution is $(-\infty, -1) \cup (1, \infty)$.
3. To solve $f(x) \geq g(x)$, we look for solutions to $f(x) = g(x)$ as well as $f(x) > g(x)$. We solved the former equation and found $x = \pm 1$. To solve $f(x) > g(x)$, we look for where the graph of f is above the graph of g. This appears to happen between $x = -1$ and $x = 1$, on the interval $(-1, 1)$. Hence, our solution to $f(x) \geq g(x)$ is $[-1, 1]$.

We now turn our attention to solving inequalities involving the absolute value. We have the following theorem to help us.

Theorem 18 Inequalities Involving the Absolute Value

Let c be a real number.

- For $c > 0$, $|x| < c$ is equivalent to $-c < x < c$.
- For $c > 0$, $|x| \leq c$ is equivalent to $-c \leq x \leq c$.
- For $c \leq 0$, $|x| < c$ has no solution, and for $c < 0$, $|x| \leq c$ has no solution.
- For $c \geq 0$, $|x| > c$ is equivalent to $x < -c$ or $x > c$.
- For $c \geq 0$, $|x| \geq c$ is equivalent to $x \leq -c$ or $x \geq c$.
- For $c < 0$, $|x| > c$ and $|x| \geq c$ are true for all real numbers.

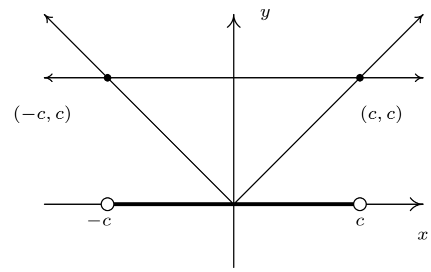


Figure 3.33: Solving $|x| < c$ graphically

As with Theorem 13 in Section 3.2, we could argue Theorem 18 using cases. However, in light of what we have developed in this section, we can understand these statements graphically. For instance, if $c > 0$, the graph of $y = c$ is a horizontal line which lies above the x-axis through $(0, c)$. To solve $|x| < c$, we are looking for the x values where the graph of $y = |x|$ is below the graph of $y = c$. We know that the graphs intersect when $|x| = c$, which, from Section 3.2, we know happens when $x = c$ or $x = -c$.

In Figure 3.33 we see that the graph of $y = |x|$ is below $y = c$ for x between $-c$ and c, and hence we get $|x| < c$ is equivalent to $-c < x < c$. The other properties in Theorem 18 can be shown similarly.

Example 59 Solving absolute value inequalities

Solve the following inequalities analytically; check your answers graphically.

1. $|x - 1| \geq 3$
2. $4 - 3|2x + 1| > -2$
3. $2 < |x - 1| \leq 5$
4. $|x + 1| \geq \dfrac{x + 4}{2}$

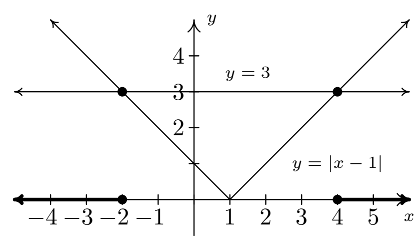


Figure 3.34: Solving $|x - 1| \geq 3$ in Example 59

Solution

1. From Theorem 18, $|x - 1| \geq 3$ is equivalent to $x - 1 \leq -3$ or $x - 1 \geq 3$. Solving, we get $x \leq -2$ or $x \geq 4$, which, in interval notation is $(-\infty, -2] \cup [4, \infty)$. Graphically, we have Figure 3.34.

   We see that the graph of $y = |x - 1|$ is above the horizontal line $y = 3$ for $x < -2$ and $x > 4$ hence this is where $|x - 1| > 3$. The two graphs intersect when $x = -2$ and $x = 4$, so we have graphical confirmation of our analytic solution.

2. To solve $4 - 3|2x + 1| > -2$ analytically, we first isolate the absolute value before applying Theorem 18. To that end, we get $-3|2x + 1| > -6$ or $|2x + 1| < 2$. Rewriting, we now have $-2 < 2x + 1 < 2$ so that $-\frac{3}{2} < x < \frac{1}{2}$. In interval notation, we write $\left(-\frac{3}{2}, \frac{1}{2}\right)$. Graphically we see in Figure 3.35 that the graph of $y = 4 - 3|2x + 1|$ is above $y = -2$ for x values between $-\frac{3}{2}$ and $\frac{1}{2}$.

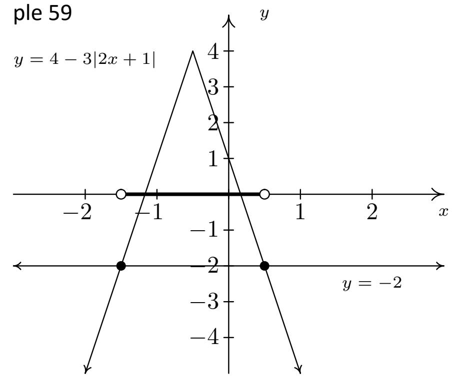


Figure 3.35: Solving $4 - 3|2x + 1| > -2$ in Example 59

3. Rewriting the compound inequality $2 < |x-1| \leq 5$ as '$2 < |x-1|$ and $|x-1| \leq 5$' allows us to solve each piece using Theorem 18. The first inequality, $2 < |x-1|$ can be re-written as $|x-1| > 2$ so $x-1 < -2$ or $x-1 > 2$. We get $x < -1$ or $x > 3$. Our solution to the first inequality is then $(-\infty,-1)\cup(3,\infty)$. For $|x-1| \leq 5$, we combine results in Theorems 13 and 18 to get $-5 \leq x-1 \leq 5$ so that $-4 \leq x \leq 6$, or $[-4,6]$. Our solution to $2 < |x-1| \leq 5$ is comprised of values of x which satisfy both parts of the inequality, so we take the intersection of $(-\infty,-1)\cup(3,\infty)$ and $[-4,6]$ to get $[-4,-1)\cup(3,6]$. (see Definition 4 in Section 1.1.1.) Graphically, we see that the graph of $y = |x-1|$ is 'between' the horizontal lines $y = 2$ and $y = 5$ for x values between -4 and -1 as well as those between 3 and 6. Including the x values where $y = |x-1|$ and $y = 5$ intersect, we get Figure 3.36.

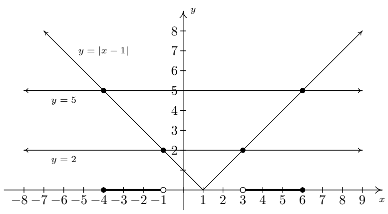


Figure 3.36: Solving $2 < |x-1| \leq 5$ in Example 59

4. We need to exercise some special caution when solving $|x+1| \geq \frac{x+4}{2}$. As we saw in Example 48 in Section 3.2, when variables are both inside and outside of the absolute value, it's usually best to refer to the definition of absolute value, Definition 33, to remove the absolute values and proceed from there. To that end, we have $|x+1| = -(x+1)$ if $x < -1$ and $|x+1| = x+1$ if $x \geq -1$. We break the inequality into cases, the first case being when $x < -1$. For these values of x, our inequality becomes $-(x+1) \geq \frac{x+4}{2}$. Solving, we get $-2x-2 \geq x+4$, so that $-3x \geq 6$, which means $x \leq -2$. Since all of these solutions fall into the category $x < -1$, we keep them all. For the second case, we assume $x \geq -1$. Our inequality becomes $x+1 \geq \frac{x+4}{2}$, which gives $2x+2 \geq x+4$ or $x \geq 2$. Since all of these values of x are greater than or equal to -1, we accept all of these solutions as well. Our final answer is $(-\infty,-2]\cup[2,\infty)$.

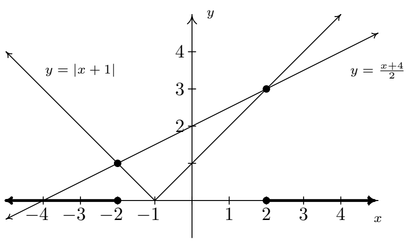


Figure 3.37: Solving $|x+1| \geq \frac{x+4}{2}$ in Example 59

We now turn our attention to quadratic inequalities. In the last example of Section 3.3, we needed to determine the solution to $x^2 - x - 6 < 0$. We will now re-visit this problem using some of the techniques developed in this section not only to reinforce our solution in Section 3.3, but to also help formulate a general analytic procedure for solving all quadratic inequalities. If we consider $f(x) = x^2 - x - 6$ and $g(x) = 0$, then solving $x^2 - x - 6 < 0$ corresponds graphically to finding the values of x for which the graph of $y = f(x) = x^2 - x - 6$ (the parabola) is below the graph of $y = g(x) = 0$ (the x-axis). See Figure 3.38 for reference.

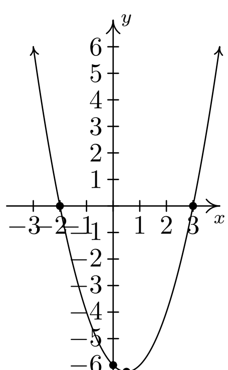
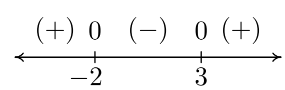


Figure 3.38: $y = x^2 - x - 6$

We can see that the graph of f does dip below the x-axis between its two x-intercepts. The zeros of f are $x = -2$ and $x = 3$ in this case and they divide the domain (the x-axis) into three intervals: $(-\infty,-2)$, $(-2,3)$ and $(3,\infty)$. For every number in $(-\infty,-2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty,-2)$. Similarly, $f(x) < 0$ for all x in $(-2,3)$, and $f(x) > 0$ for all x in $(3,\infty)$. We can schematically represent this with the **sign diagram** below.



Here, the $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x; the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. We see at once that the solution to $f(x) < 0$ is $(-2,3)$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. An important property of quadratic functions is that if the function is positive at one point and negative at another, the function must have at least one zero in between. Graphically, this means that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval. This gives us the following.

Key Idea 18 Steps for Solving a Quadratic Inequality

1. Rewrite the inequality, if necessary, as a quadratic function $f(x)$ on one side of the inequality and 0 on the other.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a real number, called a **test value**, in each of the intervals determined in step 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.
5. Choose the intervals which correspond to the correct sign to solve the inequality.

Example 60 Solving quadratic inequalities

Solve the following inequalities analytically using sign diagrams. Verify your answer graphically.

1. $2x^2 \leq 3 - x$
2. $x^2 - 2x > 1$
3. $x^2 + 1 \leq 2x$
4. $2x - x^2 \geq |x - 1| - 1$

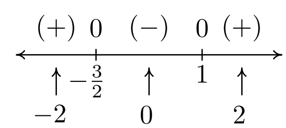


Figure 3.39: The sign diagram for $f(x) = 2x^2 + x - 3$

SOLUTION

1. To solve $2x^2 \leq 3 - x$, we first get 0 on one side of the inequality which yields $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$ for x. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$. Since we are solving $2x^2 + x - 3 \leq 0$, we look for solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$. Checking the sign diagram, we see this is $\left(-\frac{3}{2}, 1\right)$. We know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

   To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values

We have to choose a test value in each interval to construct the sign diagram. You'll get the same sign chart if you choose different test values than the ones chosen here.

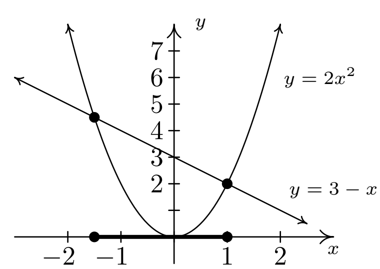


Figure 3.40: Verifying the solution to $2x^2 \leq 3 - x$ graphically

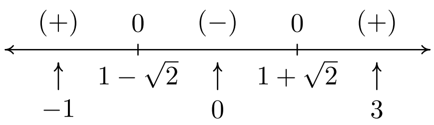


Figure 3.41: The sign diagram for $f(x) = x^2 - 2x - 1$

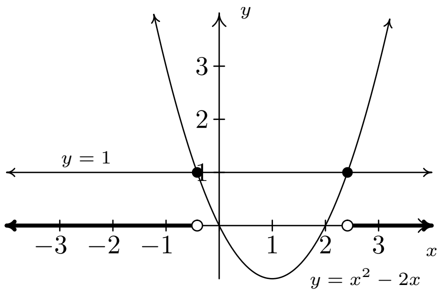


Figure 3.42: Verifying the solution to $x^2 - 2x > 1$ graphically

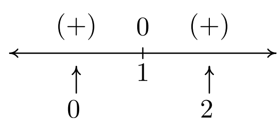


Figure 3.43: The sign diagram for $f(x) = x^2 - 2x + 1$

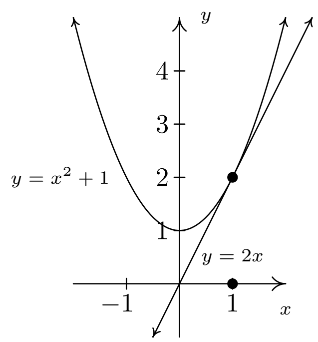


Figure 3.44: Verifying the solution to $x^2 + 1 \leq 2x$ graphically

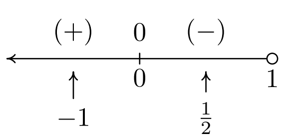


Figure 3.45: The sign diagram for $f(x) = x^2 - 3x$, where $x < 1$

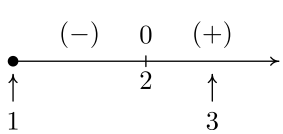


Figure 3.46: The sign diagram for $g(x) = x^2 - x - 2$, where $x \geq 1$

where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). See Figure 3.40.

2. Once again, we re-write $x^2 - 2x > 1$ as $x^2 - 2x - 1 > 0$ and we identify $f(x) = x^2 - 2x - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $x^2 - 2x - 1$ doesn't factor nicely. Hence, we resort to the quadratic formula to solve $x^2 - 2x - 1 = 0$, and arrive at $x = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $x = -1$, $x = 0$ and $x = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again. Our solution to $x^2 - 2x - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$. To check the inequality $x^2 - 2x > 1$ graphically, we set $g(x) = x^2 - 2x$ and $h(x) = 1$. We are looking for the x values where the graph of g is above the graph of h: see Figure 3.42.

3. To solve $x^2 + 1 \leq 2x$, as before, we solve $x^2 - 2x + 1 \leq 0$. Setting $f(x) = x^2 - 2x + 1 = 0$, we find the only one zero of f, $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$. Since we are looking for solutions to $x^2 - 2x + 1 \leq 0$, we are looking for x values where $x^2 - 2x + 1 < 0$ as well as where $x^2 - 2x + 1 = 0$. Looking at our sign diagram, there are no places where $x^2 - 2x + 1 < 0$ (there are no $(-)$), so our solution is only $x = 1$ (where $x^2 - 2x + 1 = 0$). We write this as $\{1\}$. Graphically, we solve $x^2 + 1 \leq 2x$ by graphing $g(x) = x^2 + 1$ and $h(x) = 2x$. We are looking for the x values where the graph of g is below the graph of h (for $x^2 + 1 < 2x$) and where the two graphs intersect ($x^2 + 1 = 2x$); see Figure 3.44. Notice that the line and the parabola touch at $(1, 2)$, but the parabola is always above the line otherwise.

   In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1, 2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

4. To solve our last inequality, $2x - x^2 \geq |x - 1| - 1$, we re-write the absolute value using cases. For $x < 1$, $|x - 1| = -(x - 1) = 1 - x$, so we get $2x - x^2 \geq 1 - x - 1$, or $x^2 - 3x \leq 0$. Finding the zeros of $f(x) = x^2 - 3x$, we get $x = 0$ and $x = 3$. However, we are only concerned with the portion of the number line where $x < 1$, so the only zero that we concern ourselves with is $x = 0$. This divides the interval $x < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $x = -1$ and $x = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$, giving us the signs in Figure 3.45. Hence, our solution to $x^2 - 3x \leq 0$ for $x < 1$ is $[0, 1)$. Next, we turn our attention to the case $x \geq 1$. Here, $|x - 1| = x - 1$, so our original inequality becomes $2x - x^2 \geq x - 1 - 1$, or $x^2 - x - 2 \leq 0$. Setting $g(x) = x^2 - x - 2$, we find the zeros of g to be $x = -1$ and $x = 2$. Of these, only $x = 2$ lies in the region $x \geq 1$, so we ignore $x = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $x = 1$ and $x = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$, yielding the sign diagram in Figure 3.46. Hence, our solution to $g(x) = x^2 - x - 2 \leq 0$, in this region is $[1, 2)$.

   Combining these into one sign diagram, we have that our solution is $[0, 2]$. Graphically, to check $2x - x^2 \geq |x - 1| - 1$, we set $h(x) = 2x - x^2$ and $i(x) = |x - 1| - 1$ and look for the x values where the graph of h is above the the graph of i (the solution of $h(x) > i(x)$) as well as the x-coordinates of the intersection points of both graphs (where $h(x) = i(x)$). The combined sign chart is given in Figure 3.47 and the graphs are plotted in Figure 3.48.

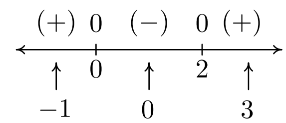


Figure 3.47: The overall sign diagram for Problem 4 in Example 60

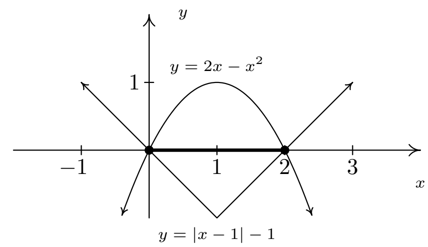


Figure 3.48: Verifying the inequality $2x - x^2 \geq |x-1| - 1$ graphically

One of the classic applications of inequalities is the notion of tolerances. Recall that for real numbers x and c, the quantity $|x - c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x - c| \leq d$ for $d \geq 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 61 Computing tolerance

The area A (in square inches) of a square piece of particle board which measures x inches on each side is $A(x) = x^2$. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution Mathematically, we express the desire for the area $A(x)$ to be within 0.25 square inches of 576 as $|A - 576| \leq 0.25$. Since $A(x) = x^2$, we get $|x^2 - 576| \leq 0.25$, which is equivalent to $-0.25 \leq x^2 - 576 \leq 0.25$. One way to proceed at this point is to solve the two inequalities $-0.25 \leq x^2 - 576$ and $x^2 - 576 \leq 0.25$ individually using sign diagrams and then taking the intersection of the solution sets. While this way will (eventually) lead to the correct answer, we take this opportunity to showcase the increasing property of the square root: if $0 \leq a \leq b$, then $\sqrt{a} \leq \sqrt{b}$. To use this property, we proceed as follows

$$\begin{array}{rcll}
-0.25 \leq & x^2 - 576 & \leq 0.25 & \\
575.75 \leq & x^2 & \leq 576.25 & \text{(add 576 across the inequalities.)} \\
\sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\
\sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|)
\end{array}$$

By Theorem 18, we find the solution to $\sqrt{575.75} \leq |x|$ to be

$$\left(-\infty, -\sqrt{575.75}\right] \cup \left[\sqrt{575.75}, \infty\right)$$

and the solution to $|x| \leq \sqrt{576.25}$ to be $\left[-\sqrt{576.25}, \sqrt{576.25}\right]$. To solve $\sqrt{575.75} \leq |x| \leq \sqrt{576.25}$, we intersect these two sets to get

$$[-\sqrt{576.25}, -\sqrt{575.75}] \cup [\sqrt{575.75}, \sqrt{576.25}].$$

Since x represents a length, we discard the negative answers and get the interval $[\sqrt{575.75}, \sqrt{576.25}]$. This means that the side of the piece of particle board must be cut between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the target length of 24 inches.

Our last example in the section demonstrates how inequalities can be used to describe regions in the plane, as we saw earlier in Section 2.1.

Example 62 Relations determined by inequalities

Sketch the following relations.

1. $R = \{(x, y) : y > |x|\}$

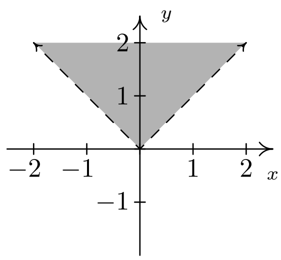


Figure 3.49: Graph of the relation *R* in Example 62

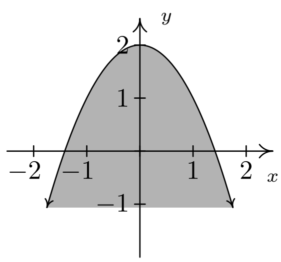


Figure 3.50: Graph of the relation *S* in Example 62

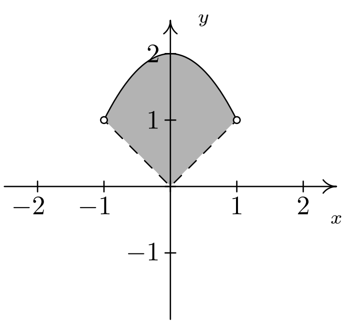


Figure 3.51: Graph of the relation *T* in Example 62

2. $S = \{(x, y) : y \leq 2 - x^2\}$
3. $T = \{(x, y) : |x| < y \leq 2 - x^2\}$

Solution

1. The relation R consists of all points (x, y) whose y-coordinate is greater than $|x|$. If we graph $y = |x|$, then we want all of the points in the plane *above* the points on the graph. Dotting the graph of $y = |x|$ as we have done before to indicate that the points on the graph itself are not in the relation, we get the shaded region in Figure 3.49.

2. For a point to be in S, its y-coordinate must be less than or equal to the y-coordinate on the parabola $y = 2 - x^2$. This is the set of all points *below* or *on* the parabola $y = 2 - x^2$: see Figure 3.50.

3. Finally, the relation T takes the points whose y-coordinates satisfy both the conditions given in R and those of S. Thus we shade the region between $y = |x|$ and $y = 2 - x^2$, keeping those points on the parabola, but not the points on $y = |x|$. To get an accurate graph, we need to find where these two graphs intersect, so we set $|x| = 2 - x^2$. Proceeding as before, breaking this equation into cases, we get $x = -1, 1$. Graphing yields Figure 3.51.

Exercises 3.4

Problems

In Exercises 1 – 32, solve the inequality. Write your answer using interval notation.

1. $|3x-5| \leq 4$
2. $|7x+2| > 10$
3. $|2x+1| - 5 < 0$
4. $|2-x| - 4 \geq -3$
5. $|3x+5| + 2 < 1$
6. $2|7-x| + 4 > 1$
7. $2 \leq |4-x| < 7$
8. $1 < |2x-9| \leq 3$
9. $|x+3| \geq |6x+9|$
10. $|x-3| - |2x+1| < 0$
11. $|1-2x| \geq x+5$
12. $x+5 < |x+5|$
13. $x \geq |x+1|$
14. $|2x+1| \leq 6-x$
15. $x + |2x-3| < 2$
16. $|3-x| \geq x-5$
17. $x^2 + 2x - 3 \geq 0$
18. $16x^2 + 8x + 1 > 0$
19. $x^2 + 9 < 6x$
20. $9x^2 + 16 \geq 24x$
21. $x^2 + 4 \leq 4x$
22. $x^2 + 1 < 0$
23. $3x^2 \leq 11x + 4$
24. $x > x^2$
25. $2x^2 - 4x - 1 > 0$
26. $5x + 4 \leq 3x^2$
27. $2 \leq |x^2 - 9| < 9$
28. $x^2 \leq |4x-3|$
29. $x^2 + x + 1 \geq 0$
30. $x^2 \geq |x|$
31. $x|x+5| \geq -6$
32. $x|x-3| < 2$
33. The profit, in dollars, made by selling x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is given by $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$. How many bottles of tonic must be sold to make at least $\$50$ in profit?
34. Suppose $C(x) = x^2 - 10x + 27$, $x \geq 0$ represents the costs, in *hundreds* of dollars, to produce x *thousand* pens. Find the number of pens which can be produced for no more than $\$1100$.
35. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 32$, for $0 \leq t \leq 12$. When is it warmer than 42° Fahrenheit?
36. The height h in feet of a model rocket above the ground t seconds after lift-off is given by $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When is the rocket at least 250 feet off the ground? Round your answer to two decimal places.
37. If a slingshot is used to shoot a marble straight up into the air from 2 meters above the ground with an initial velocity of 30 meters per second, for what values of time t will the marble be over 35 meters above the ground? (Refer to Exercise 25 in Section 3.3 for assistance if needed.) Round your answers to two decimal places.
38. What temperature values in degrees Celsius are equivalent to the temperature range $50^\circ F$ to $95^\circ F$? (Refer to Exercise 35 in Section 3.1 for assistance if needed.)

In Exercises 39 – 42, write and solve an inequality involving absolute values for the given statement.

39. Find all real numbers x so that x is within 4 units of 2.
40. Find all real numbers x so that $3x$ is within 2 units of -1.
41. Find all real numbers x so that x^2 is within 1 unit of 3.
42. Find all real numbers x so that x^2 is at least 7 units away from 4.
43. The surface area S of a cube with edge length x is given by $S(x) = 6x^2$ for $x > 0$. Suppose the cubes your company

manufactures are supposed to have a surface area of exactly 42 square centimetres, but the machines you own are old and cannot always make a cube with the precise surface area desired. Write an inequality using absolute value that says the surface area of a given cube is no more than 3 square centimetres away (high or low) from the target of 42 square centimetres. Solve the inequality and write your answer using interval notation.

44. Suppose f is a function, L is a real number and is a positive number. Discuss with your classmates what the inequality $|f(x) - L| <$ means algebraically and graphically.(Understanding this type of inequality is really important in Calculus.)

In Exercises 45 – 50, sketch the graph of the relation.

45. $R = \{(x, y) : y \leq x - 1\}$

46. $R = \left\{(x, y) : y > x^2 + 1\right\}$

47. $R = \{(x, y) : -1 < y \leq 2x + 1\}$

48. $R = \left\{(x, y) : x^2 \leq y < x + 2\right\}$

49. $R = \{(x, y) : |x| - 4 < y < 2 - x\}$

50. $R = \left\{(x, y) : x^2 < y \leq |4x - 3|\right\}$

4: Polynomial Functions

4.1 Graphs of Polynomial Functions

Three of the families of functions studied thus far – constant, linear and quadratic – belong to a much larger group of functions called **polynomials**. We begin our formal study of general polynomials with a definition and some examples.

Definition 37 Polynomial function

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, ..., a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

There are several things about Definition 37 that may be off-putting or downright frightening. The best thing to do is look at an example. Consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. Is this a polynomial function? We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. Comparing this with Definition 37, we identify $n = 5$, $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. In other words, a_5 is the coefficient of x^5, a_4 is the coefficient of x^4, and so forth; the subscript on the a's merely indicates to which power of x the coefficient belongs. The business of restricting n to be a natural number lets us focus on well-behaved algebraic animals. (Yes, there are examples of worse behaviour still to come!)

Example 63 Identifying polynomial functions

Determine if the following functions are polynomials. Explain your reasoning.

1. $g(x) = \dfrac{4 + x^3}{x}$
2. $p(x) = \dfrac{4x + x^3}{x}$
3. $q(x) = \dfrac{4x + x^3}{x^2 + 4}$
4. $f(x) = \sqrt[3]{x}$
5. $h(x) = |x|$
6. $z(x) = 0$

Solution

1. We note directly that the domain of $g(x) = \dfrac{x^3 + 4}{x}$ is $x \neq 0$. By definition, a polynomial has all real numbers as its domain. Hence, g can't be a polynomial.

2. Even though $p(x) = \dfrac{x^3 + 4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like the form given in Definition 37, the domain of p, which, as you may recall, we determine *before* we simplify, excludes 0. Alas, p is not a polynomial function for the same reason g isn't.

3. After what happened with p in the previous part, you may be a little shy about simplifying $q(x) = \dfrac{x^3 + 4x}{x^2 + 4}$ to $q(x) = x$, which certainly fits Definition 37. If we look at the domain of q before we simplified, we see that it is, indeed, all real numbers. A function which can be written in the form of Definition 37 whose domain is all real numbers is, in fact, a polynomial.

4. We can rewrite $f(x) = \sqrt[3]{x}$ as $f(x) = x^{\frac{1}{3}}$. Since $\frac{1}{3}$ is not a natural number, f is not a polynomial.

5. The function $h(x) = |x|$ isn't a polynomial, since it can't be written as a combination of powers of x even though it can be written as a piecewise function involving polynomials. As we shall see in this section, graphs of polynomials possess a quality that the graph of h does not.

6. There's nothing in Definition 37 which prevents all the coefficients a_n, etc., from being 0. Hence, $z(x) = 0$, is an honest-to-goodness polynomial.

Once we get to calculus, we'll see that the absolute value function is the classic example of a function which is continuous everywhere, but fails to have a derivative everywhere: the graph of $h(x) = |x|$ fails to be "smooth" at the origin.

Definition 38 Polynomial terminology

Suppose f is a polynomial function.

- Given $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term $a_n x^n$ is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.

The reader may well wonder why we have chosen to separate off constant functions from the other polynomials in Definition 38. Why not just lump them all together and, instead of forcing n to be a natural number, $n = 1, 2, \ldots$, allow n to be a whole number, $n = 0, 1, 2, \ldots$. We could unify all of the cases, since, after all, isn't $a_0 x^0 = a_0$? The answer is 'yes, as long as $x \neq 0$.' The function $f(x) = 3$ and $g(x) = 3x^0$ are different, because their domains are different. The number $f(0) = 3$ is defined, whereas $g(0) = 3(0)^0$ is not. Indeed, much of the theory we will develop in this chapter doesn't include the constant functions, so we might as well treat them as outsiders from the start. One good thing that comes from Definition 38 is that we can now think of linear functions as degree 1 (or 'first degree') polynomial functions and quadratic functions as degree 2 (or 'second degree') polynomial functions.

In the context of limits, results such as 0^0 are known as *indeterminant forms*. These are cases where the function fails to be defined, but the methods of calculus might still be able to extract information.

Example 64 Using polynomial terminiology

Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x + x^3$
3. $h(x) = \dfrac{4-x}{5}$
4. $p(x) = (2x-1)^3(x-2)(3x+2)$

Solution

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 38, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.
2. The form given in Definition 38 has the highest power of x first. To that end, we re-write $g(x) = 12x + x^3 = x^3 + 12x$, and see that the degree of g is 3, the leading term is x^3, the leading coefficient is 1 and the constant term is 0.
3. We need to rewrite the formula for h so that it resembles the form given in Definition 38: $h(x) = \frac{4-x}{5} = \frac{4}{5} - \frac{x}{5} = -\frac{1}{5}x + \frac{4}{5}$. The degree of h is 1, the leading term is $-\frac{1}{5}x$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.
4. It may seem that we have some work ahead of us to get p in the form of Definition 38. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2x-1)^3(x-2)(3x+2)$. The leading term of p will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $p(x)$ is the product of the leading terms of the factors of $p(x)$. Hence, the leading term of p is $(2x)^3(x)(3x) = 24x^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

Our next example shows how polynomials of higher degree arise 'naturally' in even the most basic geometric applications.

Example 65 Optimizing a box construction

A box with no top is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner: see Figure 4.1.

1. Find the volume V of the box as a function of x. Include an appropriate applied domain.
2. Use software or a graphing calculator to graph $y = V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

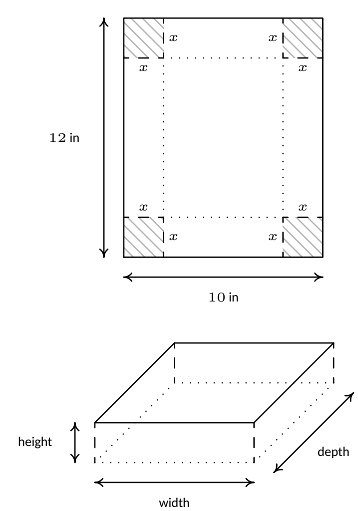


Figure 4.1: Constructing the box in Example 65

Solution

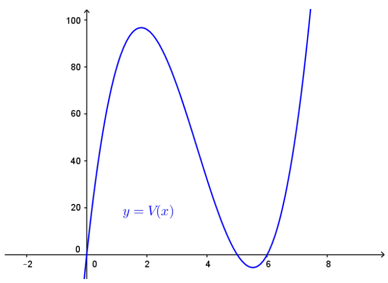
When we write $V(x)$, it is in the context of function notation, not the volume V times the quantity x. There's no harm in taking the time here to make sure that our definition of $V(x)$ makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.

1. From Geometry, we know that Volume $=$ width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10-2x$ inches for the width. As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12 - 2x$ inches remaining. As a function of x, the volume is

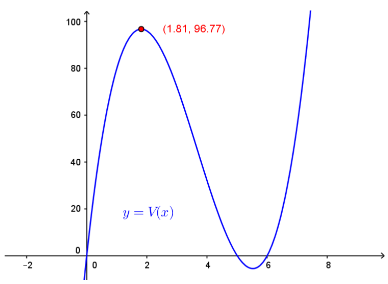
$$V(x) = x(10 - 2x)(12 - 2x) = 4x^3 - 44x^2 + 120x$$

   To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using GeoGebra to plot $V(x)$, we see that the graph of $y = V(x)$ has a relative maximum. The graph of V is shown in Figure 4.2; note that we had to rescale the y-axis significantly to get everything to fit on the screen. For $0 < x < 5$, this is also the absolute maximum. Using the 'Max' command, we get $x \approx 1.81$, $y \approx 96.77$. This yields a height of $x \approx 1.81$ inches, a width of $10 - 2x \approx 6.38$ inches, and a depth of $12 - 2x \approx 8.38$ inches. The y-coordinate is the maximum volume, which is approximately 96.77 cubic inches (also written in^3).



The graph $y = V(x)$



The graph $y = V(x)$ with maximum shown

Figure 4.2: Optimizing the volume of the box in Example 65

In order to solve Example 65, we made good use of the graph of the polynomial $y = V(x)$, so we ought to turn our attention to graphs of polynomials in general. In Figure 4.3 the graphs of $y = x^2$, $y = x^4$ and $y = x^6$, are shown. We have omitted the axes to allow you to see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand, (make sure you choose some x-values between -1 and 1.) you will see why.

All of these functions are even, (Do you remember how to show this?) and it is exactly because the exponent is even. (Herein lies one of the possible origins of the term 'even' when applied to functions.) This symmetry is important, but we want to explore a different yet equally important feature of these functions which we can be seen graphically – their **end behaviour**.

When $x \to \infty$ we think of x as moving far to the right of zero and becoming a very large *positive* number. When $x \to -\infty$ we think of x as becoming a very large (in the sense of its absolute value) *negative* number far to the left of zero.

The end behaviour of a function is a way to describe what is happening to the function values (the y-values) as the x-values approach the 'ends' of the x-axis. (Of course, there are no ends to the x-axis.) That is, what happens to y as x becomes small without bound (written $x \to -\infty$) and, on the flip side, as x becomes large without bound (written $x \to \infty$).

See Theorems 9 and 10 in Section 2.6 if you need a reminder on the effect of scalings and reflections on the graph of a function.

For example, given $f(x) = x^2$, as $x \to -\infty$, we imagine substituting $x = -100$, $x = -1000$, etc., into f to get $f(-100) = 10000$, $f(-1000) = 1000000$, and so on. Thus the function values are becoming larger and larger positive numbers (without bound). To describe this behaviour, we write: as $x \to -\infty$, $f(x) \to \infty$. If we study the behaviour of f as $x \to \infty$, we see that in this case, too, $f(x) \to \infty$. (We told you that the symmetry was important!) The same can be said for any function of the form $f(x) = x^n$ where n is an even natural number. If we generalize just a bit to include vertical scalings and reflections across the x-axis, we have

Key Idea 19 End behaviour of functions $f(x) = ax^n$, n even.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

This is illustrated graphically below:

$a > 0$ $a < 0$

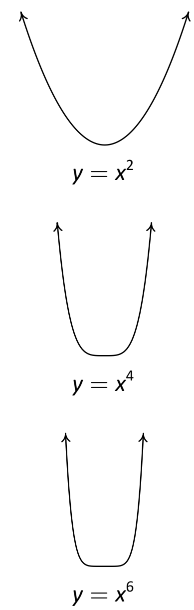


Figure 4.3: Graphing even powers of x

We now turn our attention to functions of the form $f(x) = x^n$ where $n \geq 3$ is an odd natural number. (We ignore the case when $n = 1$, since the graph of $f(x) = x$ is a line and doesn't fit the general pattern of higher-degree odd polynomials.) In Figure 4.4 we have graphed $y = x^3$, $y = x^5$, and $y = x^7$. The 'flattening' and 'steepening' that we saw with the even powers presents itself here as well, and, it should come as no surprise that all of these functions are odd. (And are, perhaps, the inspiration for the moniker 'odd function'.) The end behaviour of these functions is all the same, with $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

As with the even degreed functions we studied earlier, we can generalize their end behaviour.

Key Idea 20 End behaviour of functions $f(x) = ax^n$, n odd.

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behaviour of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$

This is illustrated graphically as follows:

$a > 0$ $a < 0$

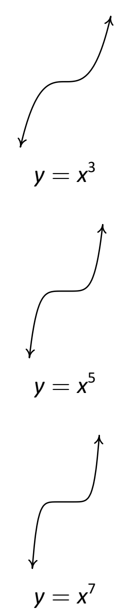


Figure 4.4: Graphing odd powers of x

Despite having different end behaviour, all functions of the form $f(x) = ax^n$ for natural numbers n share two properties which help distinguish them from other animals in the algebra zoo: they are **continuous** and **smooth**. While these concepts are formally defined using Calculus, informally, graphs of continuous functions have no 'breaks' or 'holes' in them, and the graphs of smooth functions

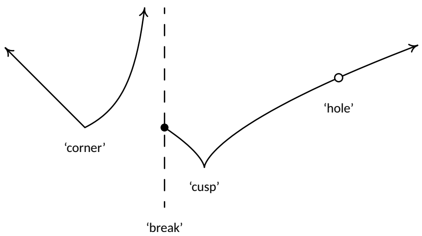


Figure 4.5: Pathologies not found on graphs of polynomials

In fact, when we get to Calculus, you'll find that smooth functions are automatically continuous, so that saying 'polynomials are continuous and smooth' is redundant.

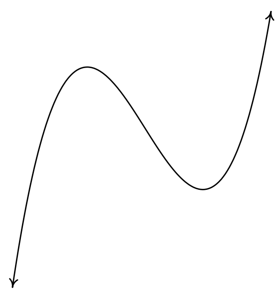


Figure 4.6: The graph of a polynomial

have no 'sharp turns'. It turns out that these traits are preserved when functions are added together, so general polynomial functions inherit these qualities. In Figure 4.5, we find the graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison. The function whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in Figure 4.6.

The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial. The notion of continuity is what allowed us to construct the sign diagram for quadratic inequalities as we did in Section 3.4. This last result is formalized in the following theorem.

Theorem 19 The Intermediate Value Theorem (Zero Version)

Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most often used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense since it says, geometrically, that the graph of a polynomial function cannot be above the x-axis at one point and below the x-axis at another point without crossing the x-axis somewhere in between. We'll return to the Intermediate Value Theorem later in the Calculus portion of the course, when we study continuity in general. The following example uses the Intermediate Value Theorem to establish a fact that that most students take for granted. Many students, and sadly some instructors, will find it silly.

Example 66 Existence of $\sqrt{2}$

Use the Intermediate Value Theorem to establish that $\sqrt{2}$ is a real number.

The validity of the result in Example 66 of course relies on having a rigorous proof of Theorem 19. Although intuitive, its proof is one of the most difficult in single variable calculus. At most universities, you don't see a proof until a first course in Analysis, like Math 3500.

Solution Consider the polynomial function $f(x) = x^2 - 2$. Then $f(1) = -1$ and $f(3) = 7$. Since $f(1)$ and $f(3)$ have different signs, the Intermediate Value Theorem guarantees us a real number c between 1 and 3 with $f(c) = 0$. If $c^2 - 2 = 0$ then $c = \pm\sqrt{2}$. Since c is between 1 and 3, c is positive, so $c = \sqrt{2}$.

Our primary use of the Intermediate Value Theorem is in the construction of sign diagrams, as in Section 3.4, since it guarantees us that polynomial functions are always positive $(+)$ or always negative $(-)$ on intervals which do not contain any of its zeros. The general algorithm for polynomials is given below.

Key Idea 21 Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine the sign of $f(x)$ for each test value in step 2, and write that sign above the corresponding interval.

Example 67 Using a sign diagram to sketch a polynomial

Construct a sign diagram for $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$. Use it to give a rough sketch of the graph of $y = f(x)$.

SOLUTION First, we find the zeros of f by solving $x^3(x-3)^2(x+2)\left(x^2+1\right) = 0$. We get $x = 0$, $x = 3$ and $x = -2$. (The equation $x^2 + 1 = 0$ produces no real solutions.) These three points divide the real number line into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. We select the test values $x = -3$, $x = -1$, $x = 1$ and $x = 4$. We find $f(-3)$ is $(+)$, $f(-1)$ is $(-)$ and $f(1)$ is $(+)$ as is $f(4)$. Wherever f is $(+)$, its graph is above the x-axis; wherever f is $(-)$, its graph is below the x-axis. The x-intercepts of the graph of f are $(-2, 0)$, $(0, 0)$ and $(3, 0)$. Knowing f is smooth and continuous allows us to sketch its graph in Figure 4.8.

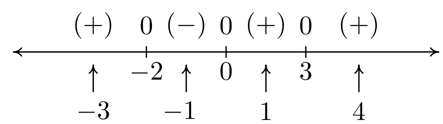


Figure 4.7: The sign diagram of f in Example 67

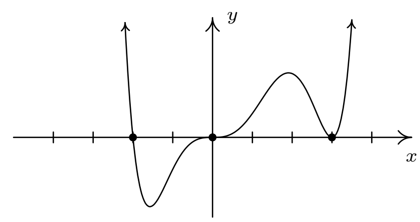


Figure 4.8: The graph $y = f(x)$ for Example 67

A couple of notes about the Example 67 are in order. First, note that we purposefully did not label the y-axis in the sketch of the graph of $y = f(x)$. This is because the sign diagram gives us the zeros and the relative position of the graph - it doesn't give us any information as to how high or low the graph strays from the x-axis. Furthermore, as we have mentioned earlier in the text, without Calculus, the values of the relative maximum and minimum can only be found approximately using a calculator. If we took the time to find the leading term of f, we would find it to be x^8. Looking at the end behaviour of f, we notice that it matches the end behaviour of $y = x^8$. This is no accident, as we find out in the next theorem.

Theorem 20 End behaviour for Polynomial Functions

The end behaviour of a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behaviour of $y = a_nx^n$.

To see why Theorem 20 is true, let's first look at a specific example. Consider $f(x) = 4x^3 - x + 5$. If we wish to examine end behaviour, we look to see the behaviour of f as $x \to \pm\infty$. Since we're concerned with x's far down the x-axis, we are far away from $x = 0$ so can rewrite $f(x)$ for these values of x as

$$f(x) = 4x^3\left(1 - \frac{1}{4x^2} + \frac{5}{4x^3}\right)$$

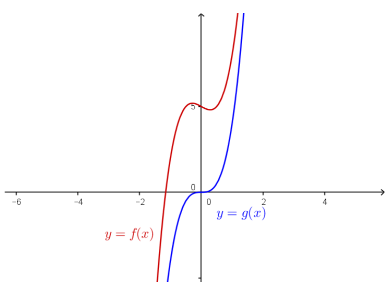
As x becomes unbounded (in either direction), the terms $\frac{1}{4x^2}$ and $\frac{5}{4x^3}$ become closer and closer to 0, as the table below indicates.

x	$\frac{1}{4x^2}$	$\frac{5}{4x^3}$
-1000	0.00000025	-0.00000000125
-100	0.000025	-0.00000125
-10	0.0025	-0.00125
10	0.0025	0.00125
100	0.000025	0.00000125
1000	0.00000025	0.00000000125

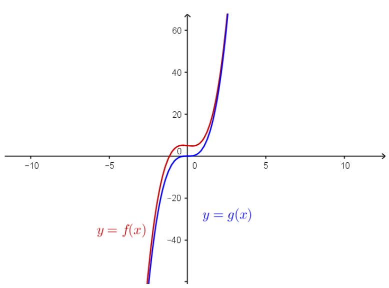
In other words, as $x \to \pm\infty$, $f(x) \approx 4x^3(1 - 0 + 0) = 4x^3$, which is the leading term of f. The formal proof of Theorem 20 works in much the same way. Factoring out the leading term leaves

$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \ldots + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

As $x \to \pm\infty$, any term with an x in the denominator becomes closer and closer to 0, and we have $f(x) \approx a_n x^n$. Geometrically, Theorem 20 says that if we graph $y = f(x)$ using a graphing calculator, and continue to 'zoom out', the graph of it and its leading term become indistinguishable. In Figure 4.9 the graphs of $y = 4x^3 - x + 5$ and $y = 4x^3$) in two different windows.



A view close to the origin



A 'zoomed out' view

Figure 4.9: Two views of the polynomials $f(x)$ and $g(x)$

Let's return to the function in Example 67, $f(x) = x^3(x-3)^2(x+2)\left(x^2+1\right)$, whose sign diagram and graph are given in Figures 4.7 and 4.8. Theorem 20 tells us that the end behaviour is the same as that of its leading term x^8. This tells us that the graph of $y = f(x)$ starts and ends above the x-axis. In other words, $f(x)$ is $(+)$ as $x \to \pm\infty$, and as a result, we no longer need to evaluate f at the test values $x = -3$ and $x = 4$. Is there a way to eliminate the need to evaluate f at the other test values? What we would really need to know is how the function behaves near its zeros - does it cross through the x-axis at these points, as it does at $x = -2$ and $x = 0$, or does it simply touch and rebound like it does at $x = 3$. From the sign diagram, the graph of f will cross the x-axis whenever the signs on either side of the zero switch (like they do at $x = -2$ and $x = 0$); it will touch when the signs are the same on either side of the zero (as is the case with $x = 3$). What we need to determine is the reason behind whether or not the sign change occurs.

Fortunately, f was given to us in factored form: $f(x) = x^3(x-3)^2(x+2)$. When we attempt to determine the sign of $f(-4)$, we are attempting to find the sign of the number $(-4)^3(-7)^2(-2)$, which works out to be $(-)(+)(-)$ which is $(+)$. If we move to the other side of $x = -2$, and find the sign of $f(-1)$, we are determining the sign of $(-1)^3(-4)^2(+1)$, which is $(-)(+)(+)$ which gives us the $(-)$. Notice that signs of the first two factors in both expressions are the same in $f(-4)$ and $f(-1)$. The only factor which switches sign is the third factor, $(x+2)$, precisely the factor which gave us the zero $x = -2$. If we move to the other side of 0 and look closely at $f(1)$, we get the sign pattern $(+1)^3(-2)^2(+3)$ or $(+)(+)(+)$ and we note that, once again, going from $f(-1)$ to $f(1)$, the only factor which changed sign was the first factor, x^3, which corresponds to the zero $x = 0$. Finally, to find $f(4)$, we substitute to get $(+4)^3(+2)^2(+5)$ which is $(+)(+)(+)$ or $(+)$. The sign didn't change for the middle factor $(x-3)^2$. Even though this is the factor which corresponds to the zero $x = 3$, the fact that the quantity is *squared* kept the sign of the middle factor the same on either side of 3. If we look back at the exponents on the factors $(x+2)$ and x^3, we see that they are both odd, so as we substitute values to the left and right of the corresponding zeros, the signs of the corresponding factors change which results in the sign of the function value changing. This is the key to the behaviour of the function near the zeros. We need a definition and then a theorem.

Definition 39 Multiplicity of a zero

Suppose f is a polynomial function and m is a natural number. If $(x-c)^m$ is a factor of $f(x)$ but $(x-c)^{m+1}$ is not, then we say $x=c$ is a zero of **multiplicity** m.

Hence, rewriting $f(x)=x^3(x-3)^2(x+2)$ as $f(x)=(x-0)^3(x-3)^2(x-(-2))^1$, we see that $x=0$ is a zero of multiplicity 3, $x=3$ is a zero of multiplicity 2 and $x=-2$ is a zero of multiplicity 1.

Theorem 21 The Role of Multiplicity

Suppose f is a polynomial function and $x=c$ is a zero of multiplicity m.

- If m is even, the graph of $y=f(x)$ touches and rebounds from the x-axis at $(c,0)$.
- If m is odd, the graph of $y=f(x)$ crosses through the x-axis at $(c,0)$.

Our last example shows how end behaviour and multiplicity allow us to sketch a decent graph without appealing to a sign diagram.

Example 68 Using end behaviour and multiplicity

Sketch the graph of $f(x)=-3(2x-1)(x+1)^2$ using end behaviour and the multiplicity of its zeros.

Solution The end behaviour of the graph of f will match that of its leading term. To find the leading term, we multiply by the leading terms of each factor to get $(-3)(2x)(x)^2=-6x^3$. This tells us that the graph will start above the x-axis, in Quadrant II, and finish below the x-axis, in Quadrant IV. Next, we find the zeros of f. Fortunately for us, f is factored. (Obtaining the factored form of a polynomial is the main focus of the next few sections.) Setting each factor equal to zero gives is $x=\frac{1}{2}$ and $x=-1$ as zeros. To find the multiplicity of $x=\frac{1}{2}$ we note that it corresponds to the factor $(2x-1)$. This isn't strictly in the form required in Definition 39. If we factor out the 2, however, we get $(2x-1)=2\left(x-\frac{1}{2}\right)$, and we see that the multiplicity of $x=\frac{1}{2}$ is 1. Since 1 is an odd number, we know from Theorem 21 that the graph of f will cross through the x-axis at $\left(\frac{1}{2},0\right)$. Since the zero $x=-1$ corresponds to the factor $(x+1)^2=(x-(-1))^2$, we find its multiplicity to be 2 which is an even number. As such, the graph of f will touch and rebound from the x-axis at $(-1,0)$. Though we're not asked to, we can find the y-intercept by finding $f(0)=-3(2(0)-1)(0+1)^2=3$. Thus $(0,3)$ is an additional point on the graph. Putting this together gives us the graph in Figure 4.10.

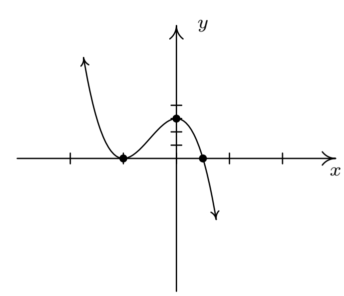


Figure 4.10: The graph $y=f(x)$ for Example 68

Exercises 4.1

Problems

In Exercises 1 – 10, solve the inequality. Write your answer using interval notation.

1. $f(x) = 4 - x - 3x^2$
2. $g(x) = 3x^5 - 2x^2 + x + 1$
3. $q(r) = 1 - 16r^4$
4. $Z(b) = 42b - b^3$
5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
6. $s(t) = -4.9t^2 + v_0 t + s_0$
7. $P(x) = (x-1)(x-2)(x-3)(x-4)$
8. $p(t) = -t^2(3-5t)(t^2+t+4)$
9. $f(x) = -2x^3(x+1)(x+2)^2$
10. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$

In Exercises 11 – 20, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with a sign chart to provide a rough sketch of the graph of the polynomial. Compare your answer with the result from a graphing utility.

11. $a(x) = x(x+2)^2$
12. $g(x) = x(x+2)^3$
13. $f(x) = -2(x-2)^2(x+1)$
14. $g(x) = (2x+1)^2(x-3)$
15. $F(x) = x^3(x+2)^2$
16. $P(x) = (x-1)(x-2)(x-3)(x-4)$
17. $Q(x) = (x+5)^2(x-3)^4$
18. $h(x) = x^2(x-2)^2(x+2)^2$
19. $H(t) = (3-t)(t^2+1)$
20. $Z(b) = b(42-b^2)$

In Exercises 21 – 26, given the pair of functions f and g, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice through the transformations. State the domain and range of g.

21. $f(x) = x^3$, $g(x) = (x+2)^3 + 1$
22. $f(x) = x^4$, $g(x) = (x+2)^4 + 1$
23. $f(x) = x^4$, $g(x) = 2 - 3(x-1)^4$
24. $f(x) = x^5$, $g(x) = -x^5 - 3$
25. $f(x) = x^5$, $g(x) = (x+1)^5 + 10$
26. $f(x) = x^6$, $g(x) = 8 - x^6$
27. Use the Intermediate Value Theorem to prove that $f(x) = x^3 - 9x + 5$ has a real zero in each of the following intervals: $[-4, -3]$, $[0, 1]$ and $[2, 3]$.
28. Rework Example 65 assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[1]

In Exercises 29 – 31, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \le x \le 10.07$.

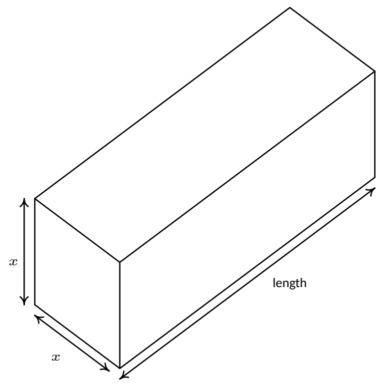
29. Use a graphing utility to graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?
30. Assume that the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by $C(x) = 200x + 25$ for $x \ge 0$. Find and simplify an expression for the profit function $P(x)$. (Remember: Profit = Revenue - Cost.)
31. Use a graphing utility to graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?
32. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 45) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \ge 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.
33. According to US Postal regulations, a rectangular shipping box must satisfy the inequality "Length + Girth $\le$ 130 inches" for Parcel Post and "Length + Girth $\le$ 108 inches" for other services. Let's assume we have a closed rectangular box with a square face of side length x as drawn below. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

[1] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

(a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth $= 130$ inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

(b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

(c) Repeat parts 33a and 33b if the box is shipped using "other services".



34. Show that the end behaviour of a linear function $f(x) = mx + b$ is as it should be according to the results we've established in the section for polynomials of odd degree.[2] (That is, show that the graph of a linear function is "up on one side and down on the other" just like the graph of $y = a_n x^n$ for odd numbers n.)

35. There is one subtlety about the role of multiplicity that we need to discuss further; specifically we need to see 'how' the graph crosses the x-axis at a zero of odd multiplicity. In the section, we deliberately excluded the function $f(x) = x$ from the discussion of the end behaviour of $f(x) = x^n$ for odd numbers n and we said at the time that it was due to the fact that $f(x) = x$ didn't fit the pattern we were trying to establish. You just showed in the previous exercise that the end behaviour of a linear function behaves like every other polynomial of odd degree, so what doesn't $f(x) = x$ do that $g(x) = x^3$ does? It's the 'flattening' for values of x near zero. It is this local behaviour that will distinguish between a zero of multiplicity 1 and one of higher odd multiplicity. Look again closely at the graphs of $a(x) = x(x+2)^2$ and $F(x) = x^3(x+2)^2$ from Exercise 21. Discuss with your classmates how the graphs are fundamentally different at the origin. It might help to use a graphing calculator to zoom in on the origin to see the different crossing behaviour. Also compare the behaviour of $a(x) = x(x+2)^2$ to that of $g(x) = x(x+2)^3$ near the point $(-2, 0)$. What do you predict will happen at the zeros of $f(x) = (x-1)(x-2)^2(x-3)^3(x-4)^4(x-5)^5$?

36. Here are a few other questions for you to discuss with your classmates.

(a) How many local extrema could a polynomial of degree n have? How few local extrema can it have?

(b) Could a polynomial have two local maxima but no local minima?

(c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

(d) Can a polynomial have local extrema without having any real zeros?

(e) Why must every polynomial of odd degree have at least one real zero?

(f) Can a polynomial have two distinct real zeros and no local extrema?

(g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?

(h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

[2] Remember, to be a linear function, $m \neq 0$.

4.2 The Factor Theorem and the Remainder Theorem

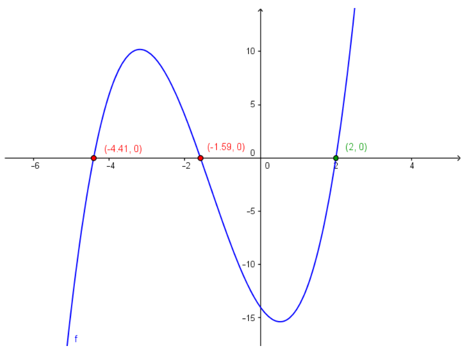


Figure 4.11: The graph $y = x^3 + 4x^2 - 5x - 14$

Suppose we wish to find the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned (and probably forgot) in high school, this equation foils us at every turn. If we graph f using GeoGebra, we get the result in Figure 4.11.

The graph suggests that the function has three zeros, one of which is $x = 2$. It's easy to show that $f(2) = 0$, but the other two zeros seem to be less friendly. Asking GeoGebra to intersect the graph with the x-axis gives us the decimal approximations shown in the figure, but we seek a method to find the remaining zeros *exactly*. Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\, q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. Dividing $x^3 + 4x^2 - 5x - 14$ by $x - 2$ gives

$$
\begin{array}{rrrrr}
 & x^2 & +\ 6x & +\ 7 & \\
\cline{2-5}
x-2 \big) & x^3 & +\ 4x^2 & -\ 5x & -\ 14 \\
 & -\left(x^3 - 2x^2\right) & & & \\
\cline{2-2}
 & & 6x^2 & -\ 5x & \\
 & & -\left(6x^2 - 12x\right) & & \\
\cline{3-3}
 & & & 7x & -\ 14 \\
 & & & -\left(7x - 14\right) & \\
\cline{4-5}
 & & & & 0
\end{array}
$$

As you may recall, this means $x^3 + 4x^2 - 5x - 14 = (x - 2)\left(x^2 + 6x + 7\right)$, so to find the zeros of f, we now solve $(x - 2)\left(x^2 + 6x + 7\right) = 0$. We get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2 + 6x + 7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

Theorem 22 Polynomial Division

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that $p(x) = d(x)\, q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 22 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The proof of Theorem 22 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 23 The Remainder Theorem

Suppose p is a polynomial of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$.

The proof of Theorem 23 is a direct consequence of Theorem 22. When a polynomial is divided by $x - c$, the remainder is either 0 or has degree less than the degree of $x - c$. Since $x - c$ is degree 1, the degree of the remainder must be 0, which means the remainder is a constant. Hence, in either case, $p(x) = (x - c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one more piece of 'low hanging fruit' to collect, which we present below.

Theorem 24 The Factor Theorem

Suppose p is a nonzero polynomial. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

The proof of The Factor Theorem is a consequence of what we already know. If $(x - c)$ is a factor of $p(x)$, this means $p(x) = (x - c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c - c)\,q(c) = 0$, so c is a zero of p. Conversely, if c is a zero of p, then $p(c) = 0$. In this case, The Remainder Theorem tells us the remainder when $p(x)$ is divided by $(x - c)$, namely $p(c)$, is 0, which means $(x - c)$ is a factor of p. What we have established is the fundamental connection between zeros of polynomials and factors of polynomials.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomials by quantities of the form $x - c$. Fortunately, people like Ruffini and Horner have already blazed this trail. Let's take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$
\begin{array}{r|rrrr}
 & & x^2 & +6x & +7 \\ \hline
x-2 & x^3 & +4x^2 & -5x & -14 \\
 & -x^3 & +2x^2 & & \\ \cline{2-3}
 & & 6x^2 & -5x & \\
 & & -6x^2 & +12x & \\ \cline{3-4}
 & & & 7x & -14 \\
 & & & -7x & +14 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +\,6x & +\,7 \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & & \\ \cline{2-3}
 & & 6x^2 & & \\
 & & & 12x & \\ \cline{3-4}
 & & & 7x & \\
 & & & & 14 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +\,6x & +\,7 \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
\multicolumn{1}{r}{} & x^3 & 6x^2 & 7x & 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|rrrr}
-2 & x^3 & +4x^2 & -\,5x & -14 \\
 & & 2x^2 & 12x & 14 \\ \cline{2-5}
\multicolumn{1}{r}{} & x^3 & 6x^2 & 7x & 0
\end{array}
$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$
\begin{array}{r|rrrr}
2 & 1 & 4 & -5 & -14 \\
 & & 2 & 12 & 14 \\ \cline{2-5}
\multicolumn{1}{r}{} & 1 & 6 & 7 & 0
\end{array}
$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$
\begin{array}{r|rrrr}
2 & 1 & 4 & -5 & -14 \\
 & & & & \\ \cline{2-5}
\multicolumn{1}{r}{} & & & &
\end{array}
\qquad
\begin{array}{r|rrrr}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & & & \\ \cline{2-5}
\multicolumn{1}{r}{} & 1 & & &
\end{array}
$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Caution: It is important to note that it works *only* for divisors of the form $x - a$, where a is a constant. For divisors of the form $ax+b$, you need to either first factor out the a, or use long division. For divisors of higher degree (such as x^2+1), you have no other option but to use long division.

Example 69 Using synthetic division

Use synthetic division to perform the following polynomial divisions. Find the quotient and the remainder polynomials, then write the dividend, quotient and remainder in the form given in Theorem 22.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(x^3 + 8\right) \div (x + 2)$
3. $\dfrac{4 - 8x - 12x^2}{2x - 3}$

Solution

1. When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. According to Theorem 22, we have $5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$.

2. For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

We get the quotient $q(x) = x^2 - 2x + 4$ and the remainder $r(x) = 0$. Relating the dividend, quotient and remainder gives $x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$.

3. To divide $4 - 8x - 12x^2$ by $2x - 3$, two things must be done. First, we write the dividend in descending powers of x as $-12x^2 - 8x + 4$. Second, since synthetic division works only for factors of the form $x - c$, we factor $2x - 3$ as $2\left(x - \frac{3}{2}\right)$. Our strategy is to first divide $-12x^2 - 8x + 4$ by 2, to get $-6x^2 - 4x + 2$. Next, we divide by $\left(x - \frac{3}{2}\right)$. The tableau becomes

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

From this, we get $-6x^2 - 4x + 2 = \left(x - \frac{3}{2}\right)(-6x - 13) - \frac{35}{2}$. Multiplying both sides by 2 and distributing gives $-12x^2 - 8x + 4 = (2x - 3)(-6x - 13) - 35$. At this stage, we have written $-12x^2 - 8x + 4$ in the **form** $(2x - 3)q(x) + r(x)$, but how can we be sure the quotient polynomial is $-6x - 13$ and the remainder is -35? The answer is the word 'unique' in Theorem 22. The theorem states that there is only one way to decompose $-12x^2 - 8x + 4$ into a multiple of $(2x - 3)$ plus a constant term. Since we have found such a way, we can be sure it is the only way.

The next example pulls together all of the concepts discussed in this section.

Example 70 **Factoring a cubic polynomial**

Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.

2. Use the fact that $x = 1$ is a zero of p to factor $p(x)$ and then find all of the real zeros of p.

Solution

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x - (-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0, and proceed as above.

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. The Factor Theorem tells us that since $x = 1$ is a zero of p, $x - 1$ is a factor of $p(x)$. To factor $p(x)$, we divide by $x - 1$, giving us

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

We get a remainder of 0 which verifies that, indeed, $p(1) = 0$. Our quotient polynomial is a second degree polynomial with coefficients 2, 2, and -3. So $q(x) = 2x^2 + 2x - 3$. Theorem 22 tells us $p(x) = (x - 1)\left(2x^2 + 2x - 3\right)$. To find the remaining real zeros of p, we need to solve $2x^2 + 2x - 3 = 0$ for x. Since this doesn't factor nicely, we use the quadratic formula to find that the remaining zeros are $x = \frac{-1 \pm \sqrt{7}}{2}$.

In Section 4.1, we discussed the notion of the multiplicity of a zero. Roughly speaking, a zero with multiplicity 2 can be divided twice into a polynomial; multiplicity 3, three times and so on. This is illustrated in the next example.

Example 71 Factoring out a zero of multiplicity two

Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Given that $x = \frac{1}{2}$ is a zero of multiplicity 2, find all of the real zeros of p.

Solution We set up for synthetic division. Since we are told the multiplicity of $\frac{1}{2}$ is two, we continue our tableau and divide $\frac{1}{2}$ into the quotient polynomial

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\ & \downarrow & 2 & -1 & -6 & 3 \\ \hline \frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\ & \downarrow & 2 & 0 & -6 & \\ \hline & 4 & 0 & -12 & \boxed{0} & \end{array}$$

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division tells us $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$. Combining these results, we have $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 - 12\right)$. To find the remaining zeros of p, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$.

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common sight in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula. Secondly, we found $x = \pm\sqrt{3}$ are zeros of p. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of p. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
\sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\
 & \downarrow & 4\sqrt{3} & 12 & & \\
\hline
-\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\
 & \downarrow & -4\sqrt{3} & & & \\
\hline
 & 4 & \boxed{0} & & &
\end{array}
$$

This gives us $4x^4-4x^3-11x^2+12x-3 = \left(x-\frac{1}{2}\right)^2\left(x-\sqrt{3}\right)\left(x-\left(-\sqrt{3}\right)\right)(4)$, or, when written with the constant in front

$$p(x) = 4\left(x-\frac{1}{2}\right)^2\left(x-\sqrt{3}\right)\left(x-\left(-\sqrt{3}\right)\right)$$

We have shown that p is a product of its leading coefficient times linear factors of the form $(x-c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization; however, some of the zeros may be **complex** numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

Theorem 25 Number of zeros is bounded above by degree

Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Theorem 25 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x-c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

Key Idea 22 Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x-c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

Exercises 4.2

Problems

In Exercises 1 – 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

In Exercises 7 – 20, use synthetic division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

7. $\left(3x^2 - 2x + 1\right) \div (x - 1)$
8. $\left(x^2 - 5\right) \div (x - 5)$
9. $\left(3 - 4x - 2x^2\right) \div (x + 1)$
10. $\left(4x^2 - 5x + 3\right) \div (x + 3)$
11. $\left(x^3 + 8\right) \div (x + 2)$
12. $\left(4x^3 + 2x - 3\right) \div (x - 3)$
13. $\left(18x^2 - 15x - 25\right) \div \left(x - \frac{5}{3}\right)$
14. $\left(4x^2 - 1\right) \div \left(x - \frac{1}{2}\right)$
15. $\left(2x^3 + x^2 + 2x + 1\right) \div \left(x + \frac{1}{2}\right)$
16. $\left(3x^3 - x + 4\right) \div \left(x - \frac{2}{3}\right)$
17. $\left(2x^3 - 3x + 1\right) \div \left(x - \frac{1}{2}\right)$
18. $\left(4x^4 - 12x^3 + 13x^2 - 12x + 9\right) \div \left(x - \frac{3}{2}\right)$
19. $\left(x^4 - 6x^2 + 9\right) \div \left(x - \sqrt{3}\right)$
20. $\left(x^6 - 6x^4 + 12x^2 - 8\right) \div \left(x + \sqrt{2}\right)$

In Exercises 21 – 30, determine $p(c)$ using the Remainder Theorem for the given polynomial functions and value of c. If $p(c) = 0$, factor $p(x) = (x - c)q(x)$.

21. $p(x) = 2x^2 - x + 1, c = 4$
22. $p(x) = 4x^2 - 33x - 180, c = 12$
23. $p(x) = 2x^3 - x + 6, c = -3$
24. $p(x) = x^3 + 2x^2 + 3x + 4, c = -1$
25. $p(x) = 3x^3 - 6x^2 + 4x - 8, c = 2$
26. $p(x) = 8x^3 + 12x^2 + 6x + 1, c = -\frac{1}{2}$
27. $p(x) = x^4 - 2x^2 + 4, c = \frac{3}{2}$
28. $p(x) = 6x^4 - x^2 + 2, c = -\frac{2}{3}$
29. $p(x) = x^4 + x^3 - 6x^2 - 7x - 7, c = -\sqrt{7}$
30. $p(x) = x^2 - 4x + 1, c = 2 - \sqrt{3}$

In Exercises 31 – 40, you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

31. $x^3 - 6x^2 + 11x - 6, \quad c = 1$
32. $x^3 - 24x^2 + 192x - 512, \quad c = 8$
33. $3x^3 + 4x^2 - x - 2, \quad c = \frac{2}{3}$
34. $2x^3 - 3x^2 - 11x + 6, \quad c = \frac{1}{2}$
35. $x^3 + 2x^2 - 3x - 6, \quad c = -2$
36. $2x^3 - x^2 - 10x + 5, \quad c = \frac{1}{2}$
37. $4x^4 - 28x^3 + 61x^2 - 42x + 9, c = \frac{1}{2}$ is a zero of multiplicity 2
38. $x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12, c = -1$ is a zero of multiplicity 3
39. $125x^5 - 275x^4 - 2265x^3 - 3213x^2 - 1728x - 324, c = -\frac{3}{5}$ is a zero of multiplicity 3
40. $x^2 - 2x - 2, \quad c = 1 - \sqrt{3}$

In Exercises 41 – 45, create a polynomial p which has the desired characteristics. You may leave the polynomial in factored form.

41.
   - The zeros of p are $c = \pm 2$ and $c = \pm 1$
   - The leading term of $p(x)$ is $117x^4$.
42.
   - The zeros of p are $c = 1$ and $c = 3$
   - $c = 3$ is a zero of multiplicity 2.
   - The leading term of $p(x)$ is $-5x^3$
43.
   - The solutions to $p(x) = 0$ are $x = \pm 3$ and $x = 6$
   - The leading term of $p(x)$ is $7x^4$

- The point $(-3, 0)$ is a local minimum on the graph of $y = p(x)$.

44. - The solutions to $p(x) = 0$ are $x = \pm 3$, $x = -2$, and $x = 4$.
    - The leading term of $p(x)$ is $-x^5$.
    - The point $(-2, 0)$ is a local maximum on the graph of $y = p(x)$.

45. - p is degree 4.
    - as $x \to \infty$, $p(x) \to -\infty$
    - p has exactly three x-intercepts: $(-6, 0)$, $(1, 0)$ and $(117, 0)$
    - The graph of $y = p(x)$ crosses through the x-axis at $(1, 0)$.

46. Find a quadratic polynomial with integer coefficients which has $x = \frac{3}{5} \pm \frac{\sqrt{29}}{5}$ as its real zeros.

4.3 Real Zeros of Polynomials

In Section 4.2, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division. Our ability to find zeros of a polynomial depends on a number of factors, including the degree of the polynomial (by now, you should know exactly what to do if handed a linear or quadratic polynomial!) and whether or not we have access to technology. Whatever approach we are using, when searching for zeros, it helps to know where to look. The following theorem by the famous mathematician Augustin Cauchy, gives us an interval on which all of the real zeros of a polynomial can be found.

Theorem 26 Cauchy's Bound

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}$, $\frac{|a_1|}{|a_n|}$, ..., $\frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

The proof of Theorem 26 is not easily explained within the confines of this text. This paper contains the result and gives references to its proof.

Like many of the results in this section, Cauchy's Bound is best understood with an example.

Example 72 Using Cauchy's Bound

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all of the real zeros of f.

Solution To find the M stated in Cauchy's Bound, we take the absolute value of the leading coefficient, in this case $|2| = 2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6| = 6$. This yields $M = 3$ so it is guaranteed that all of the real zeros of f lie in the interval $[-4, 4]$.

Whereas the previous result tells us *where* we can find the real zeros of a polynomial, the next theorem gives us a list of *possible* real zeros.

Theorem 27 Rational Zeros Theorem

Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and a_0, a_1, ...a_n are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$$

Rearranging this equation, we get

$$a_n p^n = -a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1} - a_0 q^n$$

Now, the left hand side is an integer multiple of p, and the right hand side is an integer multiple of q. (Can you see why?) This means $a_n p^n$ is both a multiple of p and a multiple of q. Since p and q have no common factors, a_n must be a multiple of q. If we rearrange the equation

$$a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1}p^{n-1}q - \ldots - a_1 pq^{n-1}$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result.

Example 73 Finding rational zeros

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

SOLUTION To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\}$ or $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$.

At this stage, we know not only the interval in which all of the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ are located, but we also know some potential candidates. If we have access to software or a graphing calculator, we can use it to help us determine all of the real zeros of f, as illustrated in the next example.

Example 74 Using technology to find the zeros of a polynomial

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Graph $y = f(x)$ on the calculator or computer using the interval obtained in Example 72 as a guide.
2. Use the graph to shorten the list of possible rational zeros obtained in Example 73.
3. Use synthetic division to find the real zeros of f, and state their multiplicities.

SOLUTION

1. In Example 72, we determined all of the real zeros of f lie in the interval $[-4, 4]$. We plot $f(x)$ using GeoGebra, and zoom in to show the portion of the graph where $-4 \leq x \leq 4$: see Figure 4.12.
2. In Example 73, we learned that any rational zero of f must be in the list $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$. From the graph, it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the x-axis at a

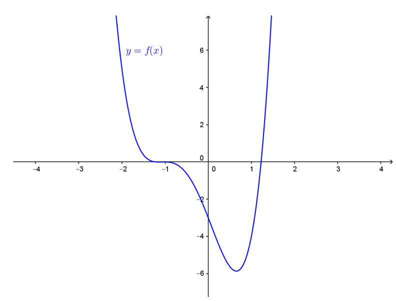


Figure 4.12: The graph $y = f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

value just a little greater than 1. On the negative side, -1 looks good, so we try that for our synthetic division.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

We have a winner! Remembering that f was a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have knocked the quotient down to a quadratic, and, if all else fails, we can use the quadratic formula to find the last two zeros. Since there seems to be no other rational zeros to try, we continue with -1. Also, the shape of the crossing at $x = -1$ leads us to wonder if the zero $x = -1$ has multiplicity 3.

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Success! Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, or $x^2 = \frac{3}{2}$, which gives us $x = \pm\frac{\sqrt{6}}{2}$. Concerning multiplicities, based on our division, we have that -1 has a multiplicity of at least 2. The Factor Theorem tells us our remaining zeros, $\pm\frac{\sqrt{6}}{2}$, each have multiplicity at least 1. However, Theorem 25 tells us f can have at most 4 real zeros, counting multiplicity, and so we conclude that -1 is of multiplicity exactly 2 and $\pm\frac{\sqrt{6}}{2}$ each has multiplicity 1. (Thus, we were wrong to think that -1 had multiplicity 3.)

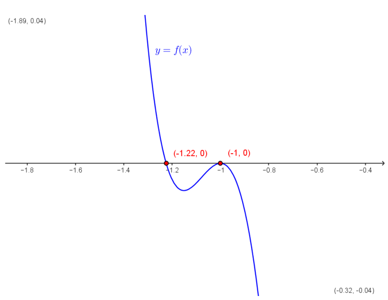


Figure 4.13: Zooming in on the repeated zero in Example 74

The y-axis isn't visible in Figure 4.13, so it's worth pointing out that in order to get a good view of the two local extrema, we had to shrink the y scale significantly: the y-value of the local minimum at $x = -\sqrt{6}/2$ is just shy of -0.01.

It is interesting to note that we could greatly improve on the graph of $y = f(x)$ in the previous example given to us by GeoGebra. For instance, from our determination of the zeros of f and their multiplicities, we know the graph crosses at $x = -\frac{\sqrt{6}}{2} \approx -1.22$ then turns back upwards to touch the $x-$axis at $x = -1$. This tells us that, despite what the software showed us the first time, there is a relative maximum occurring at $x = -1$ and not a 'flattened crossing' as we originally believed. After zooming in and rescaling the coordinate axes, we see not only the relative maximum but also a relative minimum (this is an example of what is called 'hidden behaviour.') just to the left of $x = -1$ which shows us, once again, that Mathematics enhances the technology, instead of vice-versa: see Figure 4.13.

Our next example shows how even a mild-mannered polynomial can cause problems.

Example 75 Factoring using a u-substitution

Let $f(x) = x^4 + x^2 - 12$.

1. Use Cauchy's Bound to determine an interval in which all of the real zeros of f lie.
2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.
3. Graph $y = f(x)$ using your graphing calculator.

4. Find all of the real zeros of f and their multiplicities.

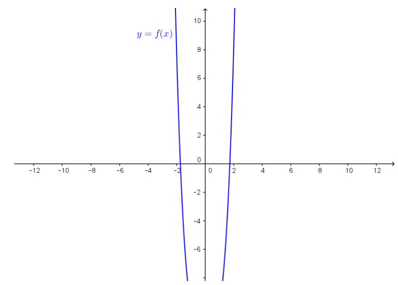
Solution

1. Applying Cauchy's Bound, we find $M = 12$, so all of the real zeros lie in the interval $[-13, 13]$.

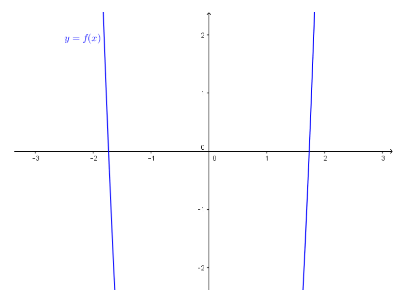
2. Applying the Rational Zeros Theorem with constant term $a_0 = -12$ and leading coefficient $a_4 = 1$, we get the list $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$.

3. Graphing $y = f(x)$ on the interval $[-13, 13]$ produces the graph in Figure 4.14 (a). Zooming in a bit gives the graph (b). Based on the graph, none of our rational zeros will work. (Do you see why not?)

4. From the graph, we know f has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of f by setting $f(x) = x^4 + x^2 - 12 = 0$ and solving. If we stare at this equation long enough, we may recognize it as a 'quadratic in disguise' or 'quadratic in form'. In other words, we have three terms: x^4, x^2 and 12, and the exponent on the first term, x^4, is exactly twice that of the second term, x^2. We may rewrite this as $\left(x^2\right)^2 + \left(x^2\right) - 12 = 0$. To better see the forest for the trees, we momentarily replace x^2 with the variable u. In terms of u, our equation becomes $u^2 + u - 12 = 0$, which we can readily factor as $(u + 4)(u - 3) = 0$. In terms of x, this means $x^4 + x^2 - 12 = \left(x^2 - 3\right)\left(x^2 + 4\right) = 0$. We get $x^2 = 3$, which gives us $x = \pm\sqrt{3}$, or $x^2 = -4$, which admits no real solutions. Since $\sqrt{3} \approx 1.73$, the two zeros match what we expected from the graph. In terms of multiplicity, the Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$ are factors of $f(x)$. Since $f(x)$ can be factored as $f(x) = \left(x^2 - 3\right)\left(x^2 + 4\right)$, and $x^2 + 4$ has no real zeros, the quantities $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$ must both be factors of $x^2 - 3$. According to Theorem 25, $x^2 - 3$ can have at most 2 zeros, counting multiplicity, hence each of $\pm\sqrt{3}$ is a zero of f of multiplicity 1.



(a)



(b)

Figure 4.14: Two views of the graph $y = f(x) = x^4 + x^2 - 12$ in Example 75

The technique used to factor $f(x)$ in Example 75 is called u-**substitution**. In general, substitution can help us identify a 'quadratic in disguise' provided that there are exactly three terms and the exponent of the first term is exactly twice that of the second. It is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate, which in this subsection means we use the 'Zero' command on the graphing calculator. (In GeoGebra, there is a 'root' command available, or you can simply use the Intersect tool to plot the points where the graph intersects the x-axis.)

Cauchy's Bound gives us a general bound on the zeros of a polynomial function. Our next result helps us determine bounds on the real zeros of a polynomial as we synthetically divide which are often sharper (that is, better, or more accurate) bounds than Cauchy's Bound.

Theorem 28 Upper and Lower Bounds

Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

  NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

The Upper and Lower Bounds Theorem works because of Theorem 22. For the upper bound part of the theorem, suppose $c > 0$ is divided into f and the resulting line in the division tableau contains, for example, all nonnegative numbers. This means $f(x) = (x - c)q(x) + r$, where the coefficients of the quotient polynomial and the remainder are nonnegative. (Note that the leading coefficient of q is the same as f so $q(x)$ is not the zero polynomial.) If $b > c$, then $f(b) = (b - c)q(b) + r$, where $(b - c)$ and $q(b)$ are both positive and $r \geq 0$. Hence $f(b) > 0$ which shows b cannot be a zero of f. Thus no real number $b > c$ can be a zero of f, as required. A similar argument proves $f(b) < 0$ if all of the numbers in the final line of the synthetic division tableau are non-positive. To prove the lower bound part of the theorem, we note that a lower bound for the negative real zeros of $f(x)$ is an upper bound for the positive real zeros of $f(-x)$. Applying the upper bound portion to $f(-x)$ gives the result. (Do you see where the alternating signs come in?) With the additional mathematical machinery of Descartes' Rule of Signs and the Upper and Lower Bounds Theorem, we can find the real zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ without the use of technology.

Let us now return to the function $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ from Example 74, and attempt to find its zeros without the aid of technology.

Example 76 Finding real zeros by hand

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the real zeros of f and their multiplicities.
2. Sketch the graph of $y = f(x)$.

Solution

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. We try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, we know $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f. We continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph

of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero, and we also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Although we did not find any positive rational zeros, we can conclude that there must be a positive irrational zero: we found that $f(1) = -4 < 0$ and $f\left(\frac{3}{2}\right) = \frac{75}{8} > 0$, so the Intermediate Value Theorem, Theorem 19, tells us the zero lies between 1 and $\frac{3}{2}$.

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array}$$

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a zero. Since we're only aware of one positive real zero and f has degree 4, we may have as many as three negative real zeros, counting multiplicity, so we try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\ \hline & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}} \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. We now have two zeros of multiplicity one yielding factors $\left(x - \frac{\sqrt{6}}{2}\right)$ and $\left(x + \frac{\sqrt{6}}{2}\right)$, respectively and one zero of multiplicity two, which yields the factor $(x+1)^2$. Since multiplying the corresponding factors together produces a polynomial of degree 4, we know that we have found all possible zeros of f. (If there were another zero, we would have another factor, and multiplying by this factor would produce a polynomial of degree 5 or more.)

2. We know the end behaviour of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of odd multiplicity, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and

rebounds off the x-axis at $(-1, 0)$. Putting this together, we get the graph in Figure 4.15.

You can see why the 'no calculator' approach is not very popular these days. It requires more computation and more theorems than the alternative. (This is apparently a bad thing.) In general, no matter how many theorems you throw at a polynomial, it may well be impossible to find their zeros exactly. The polynomial $f(x) = x^5 - x - 1$ is one such beast. The Rational Zeros Test gives us ± 1 as rational zeros to try but neither of these work since $f(1) = f(-1) = -1$. If we try the substitution technique we used in Example 75, we find $f(x)$ has three terms, but the exponent on the x^5 isn't exactly twice the exponent on x. How could we go about approximating the positive zero without resorting to the 'Zero' command of a graphing calculator? We use the **Bisection Method**. The first step in the Bisection Method is to find an interval on which f changes sign. We know $f(1) = -1$ and we find $f(2) = 29$. By the Intermediate Value Theorem, we know that the zero of f lies in the interval $[1, 2]$. Next, we 'bisect' this interval and find the midpoint is 1.5. We have that $f(1.5) \approx 5.09$. This means that our zero is between 1 and 1.5, since f changes sign on this interval. Now, we 'bisect' the interval $[1, 1.5]$ and find $f(1.25) \approx 0.80$, so now we have the zero between 1 and 1.25. Bisecting $[1, 1.25]$, we find $f(1.125) \approx -0.32$, which means the zero of f is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers which differ by no more than a desired accuracy. You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and checking the sign of f using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for approximating a real zero.

We don't use the word "impossible" lightly; it can be proven that the zeros of some polynomials cannot be expressed using the usual algebraic symbols. See this page, for example.

Our next example reminds us of the role finding zeros plays in solving equations and inequalities.

Example 77 **Solving a polynomial equation and inequality**

1. Find all of the real solutions to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.
2. Solve the inequality $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$.
3. Interpret your answer to part 2 graphically, and verify using a graphing calculator.

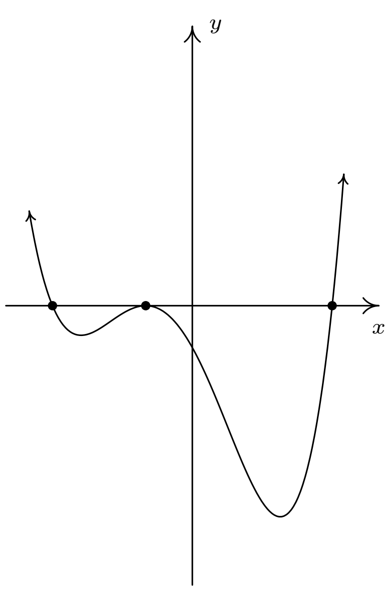


Figure 4.15: The graph $y = 2x^4 + 4x^3 - x^2 - 6x - 3$

Solution

1. Finding the real solutions to $2x^5+6x^3+3 = 3x^4+8x^2$ is the same as finding the real solutions to $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$. In other words, we are looking for the real zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr}
1 & 2 & -3 & 6 & -8 & 0 & 3 \\
 & \downarrow & 2 & -1 & 5 & -3 & -3 \\ \hline
1 & 2 & -1 & 5 & -3 & -3 & \boxed{0} \\
 & \downarrow & 2 & 1 & 6 & 3 & \\ \hline
-\frac{1}{2} & 2 & 1 & 6 & 3 & \boxed{0} & \\
 & \downarrow & -1 & 0 & -3 & & \\ \hline
 & 2 & 0 & 6 & \boxed{0} & &
\end{array}$$

The quotient polynomial is $2x^2 + 6$ which has no real zeros so we get $x = -\frac{1}{2}$ and $x = 1$.

2. To solve this nonlinear inequality, we follow the same guidelines set forth in Section 3.4: we get 0 on one side of the inequality and construct a sign diagram. Our original inequality can be rewritten as $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \leq 0$. We found the zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ in part 1 to be $x = -\frac{1}{2}$ and $x = 1$. We construct our sign diagram as before, giving us Figure 4.16.

   The solution to $p(x) < 0$ is $\left(-\infty, -\frac{1}{2}\right)$, and we know $p(x) = 0$ at $x = -\frac{1}{2}$ and $x = 1$. Hence, the solution to $p(x) \leq 0$ is $\left(-\infty, -\frac{1}{2}\right] \cup \{1\}$.

3. To interpret this solution graphically, we set $f(x) = 2x^5 + 6x^3 + 3$ and $g(x) = 3x^4 + 8x^2$. We recall that the solution to $f(x) \leq g(x)$ is the set of x values for which the graph of f is below the graph of g (where $f(x) < g(x)$) along with the x values where the two graphs intersect ($f(x) = g(x)$). Graphing f and g using GeoGebra produces Figure 4.17(a). (The end behaviour should tell you which is which.) We see that the graph of f is below the graph of g on $\left(-\infty, -\frac{1}{2}\right)$. However, it is difficult to see what is happening near $x = 1$. Zooming in (and making the graph of g thicker), we see in Figure 4.17(b) that the graphs of f and g do intersect at $x = 1$, but the graph of g remains below the graph of f on either side of $x = 1$.

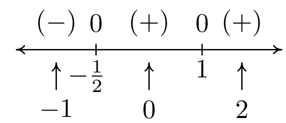
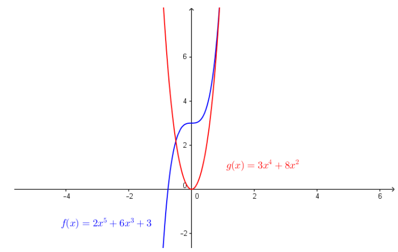
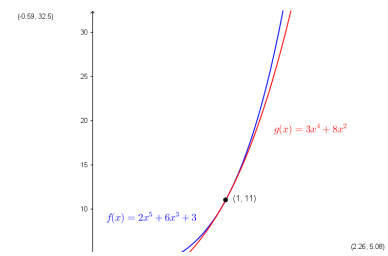


Figure 4.16: The sign diagram for $p(x)$ in Example 77



(a)



(b)

Figure 4.17: The polynomials $f(x)$ and $g(x)$ from Example 77, part 3

Our last example revisits an application from page 162 in the Exercises of Section 4.1.

Example 78 Calculating sales profits

Suppose the profit P, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$. How many TVs should be produced to make a profit? Check your answer using a graphing utility.

Solution To 'make a profit' means to solve $P(x) = -5x^3+35x^2-45x-25 > 0$, which we do analytically using a sign diagram. To simplify things, we first factor out the -5 common to all the coefficients to get $-5\left(x^3 - 7x^2 + 9x - 5\right) > 0$, so we can just focus on finding the zeros of $f(x) = x^3 - 7x^2 + 9x + 5$. The possible rational zeros of f are ± 1 and ± 5, and going through the usual computations, we find $x = 5$ is the only rational zero. Using this, we factor $f(x) = x^3 - 7x^2 + 9x + 5 = (x - 5)\left(x^2 - 2x - 1\right)$, and we find the remaining zeros by applying the Quadratic Formula to $x^2 - 2x - 1 = 0$. We find three real zeros, $x = 1 - \sqrt{2} = -0.414\ldots$, $x = 1 + \sqrt{2} = 2.414\ldots$, and $x = 5$, of which only the last two fall in the applied domain of $[0, 10.07]$. We choose $x = 0$, $x = 3$ and $x = 10.07$ as our test values and plug them into the function $P(x) = -5x^3 + 35x^2 - 45x - 25$ (not $f(x) = x^3 - 7x^2 + 9x - 5$) to get the sign diagram in Figure 4.18.

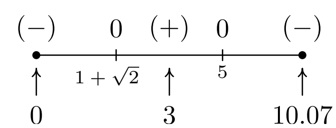


Figure 4.18: The sign diagram for $P(x)$ in Example 78

We see immediately that $P(x) > 0$ on $(1 + \sqrt{2}, 5)$. Since x measures the number of TVs in *hundreds*, $x = 1 + \sqrt{2}$ corresponds to $241.4\ldots$ TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that $P(2.41) < 0$ but $P(2.42) > 0$ so, in this case we take the next *larger* integer value and set the minimum production to 242 TVs. At the other end of the interval, we have $x = 5$ which corresponds to 500 TVs. Here, we take the next *smaller* integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. To check our answer using

GeoGebra, we graph $y = P(x)$ and use the Intersect tool to see where $y = P(x)$ intersects the x-axis. We see in Figure 4.19 that the software approximations bear out our analysis.

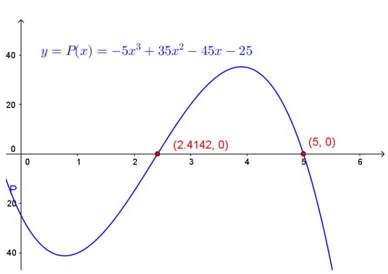


Figure 4.19: Plotting the profit function $P(x)$ in Example 78

Exercises 4.3

Problems

In Exercises 1 – 10, for the given polynomial:

- **Use Cauchy's Bound to find an interval containing all of the real zeros.**
- **Use the Rational Zeros Theorem to make a list of possible rational zeros.**

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $f(x) = x^4 - 9x^2 - 4x + 12$
4. $f(x) = x^3 + 4x^2 - 11x + 6$
5. $f(x) = x^3 - 7x^2 + x - 7$
6. $f(x) = -2x^3 + 19x^2 - 49x + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = 3x^3 + 3x^2 - 11x - 10$
10. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

In Exercises 11 – 30, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^3 - 7x^2 + x - 7$
16. $f(x) = -2x^3 + 19x^2 - 49x + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $f(x) = 3x^3 + 3x^2 - 11x - 10$
20. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
21. $f(x) = 9x^3 - 5x^2 - x$
22. $f(x) = 6x^4 - 5x^3 - 9x^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $f(x) = 3x^4 - 14x^2 - 5$
26. $f(x) = 2x^4 - 7x^2 + 6$
27. $f(x) = x^6 - 3x^3 - 10$
28. $f(x) = 2x^6 - 9x^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 – 33, use software or a graphing calculator[3] to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
34. Find the real zeros of $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$ by first finding a polynomial $q(x)$ with integer coefficients such that $q(x) = N \cdot f(x)$ for some integer N. (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that f and q have the same real zeros.

In Exercises 35 – 44, find the real solutions of the polynomial equation. (See Example 77.)

35. $9x^3 = 5x^2 + x$
36. $9x^2 + 5x^3 = 6x^4$
37. $x^3 + 6 = 2x^2 + 5x$
38. $x^4 + 2x^3 = 12x^2 + 40x + 32$
39. $x^3 - 7x^2 = 7 - x$
40. $2x^3 = 19x^2 - 49x + 20$
41. $x^3 + x^2 = \dfrac{11x + 10}{3}$
42. $x^4 + 2x^2 = 15$
43. $14x^2 + 5 = 3x^4$
44. $2x^5 + 3x^4 = 18x + 27$

[3] You *can* do theseby hand, but it may test your mettle!

In Exercises 45 – 54, solve the polynomial inequality and state your answer using interval notation.

45. $-2x^3 + 19x^2 - 49x + 20 > 0$

46. $x^4 - 9x^2 \leq 4x - 12$

47. $(x - 1)^2 \geq 4$

48. $4x^3 \geq 3x + 1$

49. $x^4 \leq 16 + 4x - x^3$

50. $3x^2 + 2x < x^4$

51. $\dfrac{x^3 + 2x^2}{2} < x + 2$

52. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$

53. $2x^4 > 5x^2 + 3$

54. $2x^4 > 5x^2 + 3$

55. In Example 65 in Section 4.1, a box with no top is constructed from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by $V(x) = 4x^3 - 44x^2 + 120x$, where x denotes the length of the side of the square which is removed from each corner (in inches), $0 < x < 5$. Solve the inequality $V(x) \geq 80$ analytically and interpret your answer in the context of that example.

56. From Exercise 32 in Section 4.1, $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ models the cost, in dollars, to produce x PortaBoy game systems. If the production budget is $\$5000$, find the number of game systems which can be produced and still remain under budget.

57. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the x- and y- intercepts of the graph of f. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of f, using more than one picture if necessary to show all of the important features of the graph.

58. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

4.4 Complex Zeros of Polynomials

In Section 4.3, we were focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well. Consider the polynomial $p(x) = x^2 + 1$. The zeros of p are the solutions to $x^2 + 1 = 0$, or $x^2 = -1$. This equation has no real solutions, but you may recall Section 1.4 that we can formally extract the square roots of both sides to get $x = \pm\sqrt{-1}$. You may want to review the basics of complex numbers in Section 1.4 before proceeding.

Suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic doesn't factor nicely, so we resort to the Quadratic Formula, Equation 16 and obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with real coefficients, the zeros will be a complex conjugate pair. (Do you see why?) Next, we note that in Example 12, part 6, we found $(x - [1 + 2i])(x - [1 - 2i]) = x^2 - 2x + 5$. This demonstrates that the factor theorem holds even for non-real zeros, i.e, $x = 1 + 2i$ is a zero of f, and, sure enough, $(x - [1 + 2i])$ is a factor of $f(x)$. It turns out that polynomial division works the same way for all complex numbers, real and non-real alike, so the Factor and Remainder Theorems hold as well. But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

Theorem 29 The Fundamental Theorem of Algebra

Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

The Fundamental Theorem of Algebra is an example of an 'existence' theorem in Mathematics. Like the Intermediate Value Theorem, Theorem 19, the Fundamental Theorem of Algebra guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. In fact, as we mentioned in Section 4.3, there are polynomials whose real zeros, though they exist, cannot be expressed using the 'usual' combinations of arithmetic symbols, and must be approximated. The authors are fully aware that the full impact and profound nature of the Fundamental Theorem of Algebra is lost on most students studying College Algebra, and that's fine. It took mathematicians literally hundreds of years to prove the theorem in its full generality, and some of that history is recorded in this Wikipedia article. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients, but to those with complex number coefficients as well.

The Fundamental Theorem of Algebra has since been proved many times, using many different methods, by many mathematicians. There are probably very few, if any, results in mathematics with the variety of proofs this result has. Unfortunately, none of the proofs can be understood within the realm of this text, but if the reader is sufficiently interested, a collection of proofs can be found at this website.

Suppose f is a polynomial of degree $n \geq 1$. The Fundamental Theorem of Algebra guarantees us at least one complex zero, z_1, and as such, the Factor Theorem guarantees that $f(x)$ factors as $f(x) = (x - z_1)\,q_1(x)$ for a polynomial function q_1, of degree exactly $n-1$. If $n-1 \geq 1$, then the Fundamental Theorem of Algebra guarantees a complex zero of q_1 as well, say z_2, so then the Factor Theorem gives us $q_1(x) = (x - z_2)\, q_2(x)$, and hence $f(x) = (x - z_1)(x - z_2)\, q_2(x)$. We can continue this process exactly n times, at which point our quotient polynomial q_n has degree 0 so it's a constant. This argument gives us the following factorization theorem.

Theorem 30 Complex Factorization Theorem

Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If z_1, z_2, …, z_k are the distinct zeros of f, with multiplicities m_1, m_2, …, m_k, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Note that the value a in Theorem 30 is the leading coefficient of $f(x)$ (Can you see why?) and as such, we see that a polynomial is completely determined by its zeros, their multiplicities, and its leading coefficient. We put this theorem to good use in the next example.

Example 79 Factoring using complex numbers

Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

1. Find all of the complex zeros of f and state their multiplicities.
2. Factor $f(x)$ using Theorem 30

Solution

1. Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. Using the techniques developed in Section 4.3, we get

$\frac{1}{2}$	12	-20	19	-6	-2	1
	$\downarrow$	6	-7	6	0	-1
$\frac{1}{2}$	12	-14	12	0	-2	$\boxed{0}$
	$\downarrow$	6	-4	4	2	
$-\frac{1}{3}$	12	-8	8	4	$\boxed{0}$	
	$\downarrow$	-4	4	-4		
	12	-12	12	$\boxed{0}$		

Our quotient is $12x^2 - 12x + 12$, whose zeros we find to be $\frac{1 \pm i\sqrt{3}}{2}$. From Theorem 30, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

2. Applying Theorem 30, we are guaranteed that f factors as

$$f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)\left(x - \left[\frac{1 + i\sqrt{3}}{2}\right]\right)\left(x - \left[\frac{1 - i\sqrt{3}}{2}\right]\right)$$

A true test of Theorem 30 (and a student's mettle!) would be to take the factored form of $f(x)$ in the previous example and multiply it out to see that it really does reduce to the original formula $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$. (You really should do this once in your life to convince yourself that all of the theory actually does work!) When factoring a polynomial using Theorem 30, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the **real numbers** then we would have stopped short of finding the nonreal zeros of f and factored f using our work from the synthetic division to write $f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(12x^2 - 12x + 12\right)$, or $f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x^2 - x + 1\right)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using *real* numbers.

The last two results of the section show us that, at least in theory, if we have a polynomial function with real coefficients, we can always factor it down enough so that any nonreal zeros come from irreducible quadratics.

Theorem 31 Conjugate Pairs Theorem

If f is a polynomial function with real number coefficients and z is a zero of f, then so is $\bar{z}$.

To prove the theorem, suppose f is a polynomial with real number coefficients. Specifically, let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$. If z is a zero of f, then $f(z) = 0$, which means $a_nz^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0 = 0$. Next, we consider $f(\bar{z})$ and apply Theorem 4 below.

$$
\begin{aligned}
f(\bar{z}) &= a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \ldots + a_2(\bar{z})^2 + a_1\bar{z} + a_0 \\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\bar{z} + a_0 && \text{since } (\bar{z})^n = \overline{z^n} \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \ldots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\bar{z} + \overline{a_0} && \text{since the coefficients are real} \\
&= \overline{a_nz^n} + \overline{a_{n-1}z^{n-1}} + \ldots + \overline{a_2z^2} + \overline{a_1z} + \overline{a_0} && \text{since } \bar{z}\,\overline{w} = \overline{zw} \\
&= \overline{a_nz^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0} && \text{since } \bar{z} + \overline{w} = \overline{z + w} \\
&= \overline{f(z)} \\
&= \overline{0} \\
&= 0
\end{aligned}
$$

This shows that $\bar{z}$ is a zero of f. So, if f is a polynomial function with real number coefficients, Theorem 31 tells us that if $a + bi$ is a nonreal zero of f, then so is $a - bi$. In other words, nonreal zeros of f come in conjugate pairs. The Factor Theorem kicks in to give us both $(x - [a + bi])$ and $(x - [a - bi])$ as factors of $f(x)$ which means $(x - [a + bi])(x - [a - bi]) = x^2 + 2ax + \left(a^2 + b^2\right)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of the section.

Theorem 32 Real Factorization Theorem

Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

We now present an example which pulls together all of the major ideas of this section.

Example 80 Factoring over the complex numbers

Let $f(x) = x^4 + 64$.

1. Use synthetic division to show that $x = 2 + 2i$ is a zero of f.
2. Find the remaining complex zeros of f.
3. Completely factor $f(x)$ over the complex numbers.
4. Completely factor $f(x)$ over the real numbers.

Solution

1. Remembering to insert the 0's in the synthetic division tableau we have

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline & 1 & 2+2i & 8i & -16+16i & \boxed{0} \end{array}$$

2. Since f is a fourth degree polynomial, we need to make two successful divisions to get a quadratic quotient. Since $2 + 2i$ is a zero, we know from Theorem 31 that $2 - 2i$ is also a zero. We continue our synthetic division tableau.

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline 2-2i & 1 & 2+2i & 8i & -16+16i & \boxed{0} \\ & \downarrow & 2-2i & 8-8i & 16-16i & \\ \hline & 1 & 4 & 8 & \boxed{0} & \end{array}$$

   Our quotient polynomial is $x^2 + 4x + 8$. Using the quadratic formula, we obtain the remaining zeros $-2 + 2i$ and $-2 - 2i$.

3. Using Theorem 30, we get $f(x) = (x - [2 - 2i])(x - [2 + 2i])(x - [-2 + 2i])(x - [-2 - 2i])$.

4. We multiply the linear factors of $f(x)$ which correspond to complex conjugate pairs. We find $(x - [2 - 2i])(x - [2 + 2i]) = x^2 - 4x + 8$, and $(x - [-2 + 2i])(x - [-2 - 2i]) = x^2 + 4x + 8$. Our final answer is $f(x) = \left(x^2 - 4x + 8\right)\left(x^2 + 4x + 8\right)$.

Our last example turns the tables and asks us to manufacture a polynomial with certain properties of its graph and zeros.

Example 81 Constructing a polynomial

Find a polynomial p of lowest degree that has integer coefficients and satisfies all of the following criteria:

- the graph of $y = p(x)$ touches (but doesn't cross) the x-axis at $\left(\frac{1}{3}, 0\right)$
- $x = 3i$ is a zero of p.
- as $x \to -\infty$, $p(x) \to -\infty$
- as $x \to \infty$, $p(x) \to -\infty$

Solution To solve this problem, we will need a good understanding of the relationship between the x-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behaviour of polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of p touches the x-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2. The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x - 3i)$ and $(x + 3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x-3i)(x+3i)$ for some real number a. Expanding this, we get $p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$. In order to obtain integer coefficients, we know a must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is ax^4, we need $a < 0$ to get $p(x) \to -\infty$ as $x \to \pm\infty$. Hence, if we choose $x = -9$, we get $p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$. We can verify our handiwork using the techniques developed in this chapter.

This example concludes our study of polynomial functions. (With the exception of the Exercises on the next page, of course.) The last few sections have contained what is considered by many to be 'heavy' Mathematics. Like a heavy meal, heavy Mathematics takes time to digest. Don't be overly concerned if it doesn't seem to sink in all at once, and pace yourself in the Exercises or you're liable to get mental cramps. But before we get to the Exercises, we'd like to offer a bit of an epilogue.

Our main goal in presenting the material on the complex zeros of a polynomial was to give the chapter a sense of completeness. Given that it can be shown that some polynomials have real zeros which cannot be expressed using the usual algebraic operations, and still others have no real zeros at all, it was nice to discover that every polynomial of degree $n \geq 1$ has n complex zeros. So like we said, it gives us a sense of closure. But the observant reader will note that we did not give any examples of applications which involve complex numbers. Students often wonder when complex numbers will be used in 'real-world' applications. After all, didn't we call i the imaginary unit? How can imaginary things be used in reality? It turns out that complex numbers are very useful in many applied fields such as fluid dynamics, electromagnetism and quantum mechanics, but most of the applications require Mathematics well beyond College Algebra to fully understand them. That does not mean you'll never be be able to understand them; in fact, it is the authors' sincere hope that all of you will reach a point in your studies when the glory, awe and splendour of complex numbers are revealed to you. For now, however, the really good stuff is beyond the scope of this text. We invite you and your classmates to find a few examples of complex number applications and see what you can make of them. A simple Internet search with the phrase 'complex numbers in real life' should get you started. Basic electronics classes are another place to look, but remember, they might use the letter *j* where we have used *i*.

For the remainder of the text we will restrict our attention to real numbers. We do this primarily because the calculus in the later chapters of this text involves only functions of real variables. Also, lots of really cool scientific things don't require any deep understanding of complex numbers to study them, but they do need more Mathematics like exponential, logarithmic and trigonometric functions. We believe it makes more sense pedagogically for you to learn about those functions now then take a course in Complex Function Theory in your junior or senior year once you've completed the Calculus sequence. It is in that course that the true power of the complex numbers is released. But for now, in order to fully prepare you for life immediately after College Algebra, we will say that functions like $f(x) = \frac{1}{x^2+1}$ have a domain of all real numbers, even though we know $x^2 + 1 = 0$ has two complex solutions, namely $x = \pm i$. Because $x^2 + 1 > 0$ for all *real* numbers x, the fraction $\frac{1}{x^2+1}$ is never undefined in the real variable setting.

Exercises 4.4

Problems

In Exercises 1 – 22, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

1. $f(x) = x^2 - 4x + 13$
2. $f(x) = x^2 - 2x + 5$
3. $f(x) = 3x^2 + 2x + 10$
4. $f(x) = x^3 - 2x^2 + 9x - 18$
5. $f(x) = x^3 + 6x^2 + 6x + 5$
6. $f(x) = 3x^3 - 13x^2 + 43x - 13$
7. $f(x) = x^3 + 3x^2 + 4x + 12$
8. $f(x) = 4x^3 - 6x^2 - 8x + 15$
9. $f(x) = x^3 + 7x^2 + 9x - 2$
10. $f(x) = 9x^3 + 2x + 1$
11. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3$
12. $f(x) = 2x^4 - 7x^3 + 14x^2 - 15x + 6$
13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$
14. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$
15. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5$
16. $f(x) = 8x^4 + 50x^3 + 43x^2 + 2x - 4$
17. $f(x) = x^4 + 9x^2 + 20$
18. $f(x) = x^4 + 5x^2 - 24$
19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
20. $f(x) = x^6 - 64$
21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
22. $f(x) = 2x^4 + 5x^3 + 13x^2 + 7x + 5$ (Hint: $x = -1 + 2i$ is a zero.)

In Exercises 23 – 27, create a polynomial *f* with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

23.
   - The zeros of f are $c = \pm 1$ and $c = \pm i$
   - The leading term of $f(x)$ is $42x^4$

24.
   - $c = 2i$ is a zero.
   - the point $(-1, 0)$ is a local minimum on the graph of $y = f(x)$
   - the leading term of $f(x)$ is $117x^4$

25.
   - The solutions to $f(x) = 0$ are $x = \pm 2$ and $x = \pm 7i$
   - The leading term of $f(x)$ is $-3x^5$
   - The point $(2, 0)$ is a local maximum on the graph of $y = f(x)$.

26.
   - f is degree 5.
   - $x = 6$, $x = i$ and $x = 1 - 3i$ are zeros of f
   - as $x \to -\infty$, $f(x) \to \infty$

27.
   - The leading term of $f(x)$ is $-2x^3$
   - $c = 2i$ is a zero
   - $f(0) = -16$

28. Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

5: Rational Functions

5.1 Introduction to Rational Functions

If we add, subtract or multiply polynomial functions according to the function arithmetic rules defined in Section 2.4, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. In this chapter we study **rational functions** - functions which are ratios of polynomials.

Definition 40 Rational Function

A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.

According to Definition 40, all polynomial functions are also rational functions, since we can take $q(x) = 1$.

As we recall from Section 2.3, we have domain issues any time the denominator of a fraction is zero. In the example below, we review this concept as well as some of the arithmetic of rational expressions.

Example 82 Domain of rational functions

Find the domain of the following rational functions. Write them in the form $\frac{p(x)}{q(x)}$ for polynomial functions p and q and simplify.

1. $f(x) = \dfrac{2x-1}{x+1}$
2. $g(x) = 2 - \dfrac{3}{x+1}$
3. $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$
4. $r(x) = \dfrac{2x^2-1}{x^2-1} \div \dfrac{3x-2}{x^2-1}$

Solution

1. To find the domain of f, we proceed as we did in Section 2.3: we find the zeros of the denominator and exclude them from the domain. Setting $x+1=0$ results in $x=-1$. Hence, our domain is $(-\infty,-1)\cup(-1,\infty)$. The expression $f(x)$ is already in the form requested and when we check for common factors among the numerator and denominator we find none, so we are done.

2. Proceeding as before, we determine the domain of g by solving $x+1=0$. As before, we find the domain of g is $(-\infty,-1)\cup(-1,\infty)$. To write $g(x)$ in the form requested, we need to get a common denominator

$$\begin{aligned} g(x) &= 2 - \frac{3}{x+1} = \frac{2}{1} - \frac{3}{x+1} = \frac{(2)(x+1)}{(1)(x+1)} - \frac{3}{x+1} \\ &= \frac{(2x+2)-3}{x+1} = \frac{2x-1}{x+1} \end{aligned}$$

This formula is now completely simplified.

3. The denominators in the formula for $h(x)$ are both $x^2 - 1$ whose zeros are $x = \pm 1$. As a result, the domain of h is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. We now proceed to simplify $h(x)$. Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned} h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} = \frac{\left(2x^2-1\right)-(3x-2)}{x^2-1} \\ &= \frac{2x^2-1-3x+2}{x^2-1} = \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} = \frac{(2x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

4. To find the domain of r, it may help to temporarily rewrite $r(x)$ as

$$r(x) = \frac{\dfrac{2x^2-1}{x^2-1}}{\dfrac{3x-2}{x^2-1}}$$

We need to set all of the denominators equal to zero which means we need to solve not only $x^2 - 1 = 0$, but also $\dfrac{3x-2}{x^2-1} = 0$. We find $x = \pm 1$ for the former and $x = \frac{2}{3}$ for the latter. Our domain is $(-\infty, -1) \cup \left(-1, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$. We simplify $r(x)$ by rewriting the division as multiplication by the reciprocal and then by cancelling the common factor

$$\begin{aligned} r(x) &= \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1} = \frac{2x^2-1}{x^2-1} \cdot \frac{x^2-1}{3x-2} = \frac{\left(2x^2-1\right)\left(x^2-1\right)}{\left(x^2-1\right)(3x-2)} \\ &= \frac{\left(2x^2-1\right)\cancel{\left(x^2-1\right)}}{\cancel{\left(x^2-1\right)}(3x-2)} = \frac{2x^2-1}{3x-2} \end{aligned}$$

A few remarks about Example 82 are in order. Note that the expressions for $f(x)$, $g(x)$ and $h(x)$ work out to be the same. However, only two of these functions are actually equal. Recall that functions are ultimately sets of ordered pairs (you should review Sections 2.1 and 2.2 if this statement caught you off guard), so for two functions to be equal, they need, among other things, to have the same domain. Since $f(x) = g(x)$ and f and g have the same domain, they are equal functions. Even though the formula $h(x)$ is the same as $f(x)$, the domain of h is different than the domain of f, and thus they are different functions.

We now turn our attention to the graphs of rational functions. Consider the function $f(x) = \dfrac{2x-1}{x+1}$ from Example 82. Using GeoGebra calculator, we obtain the graph in Figure 5.1

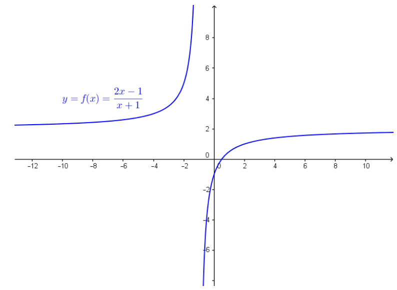


Figure 5.1: The graph of $f(x) = \dfrac{2x-1}{x+1}$

Two behaviours of the graph are worthy of further discussion. First, note that the graph appears to 'break' at $x = -1$. We know from our last example that $x = -1$ is not in the domain of f which means $f(-1)$ is undefined. When we make a table of values to study the behaviour of f *near* $x = -1$ we see that we can get 'near' $x = -1$ from two directions. We can choose values a little less than -1, for example $x = -1.1$, $x = -1.01$, $x = -1.001$, and so on. These values are said to 'approach -1 from the *left*.' Similarly, the values $x = -0.9$, $x = -0.99$, $x = -0.999$, etc., are said to 'approach -1 from the *right*.' If we make the two tables in Figure 5.2, we find that the numerical results confirm what we see graphically.

x	$f(x)$	$(x, f(x))$
-1.1	32	$(-1.1, 32)$
-1.01	302	$(-1.01, 302)$
-1.001	3002	$(-1.001, 3002)$
-1.0001	30002	$(-1.001, 30002)$

x	$f(x)$	$(x, f(x))$
-0.9	-28	$(-0.9, -28)$
-0.99	-298	$(-0.99, -298)$
-0.999	-2998	$(-0.999, -2998)$
-0.9999	-29998	$(-0.9999, -29998)$

Figure 5.2: Values of $f(x) = \frac{2x-1}{x+1}$ near $x = -1$

As the x values approach -1 from the left, the function values become larger and larger positive numbers. (We would need Calculus to confirm this analytically.) We express this symbolically by stating as $x \to -1^-$, $f(x) \to \infty$. Similarly, using analogous notation, we conclude from the table that as $x \to -1^+$, $f(x) \to -\infty$. For this type of unbounded behaviour, we say the graph of $y = f(x)$ has a **vertical asymptote** of $x = -1$. Roughly speaking, this means that near $x = -1$, the graph looks very much like the vertical line $x = -1$.

The other feature worthy of note about the graph of $y = f(x)$ is that it seems to 'level off' on the left and right hand sides of the screen. This is a statement about the end behaviour of the function. As we discussed in Section 4.1, the end behaviour of a function is its behaviour as x attains larger and larger negative values without bound (here, the word 'larger' means larger in absolute value), $x \to -\infty$, and as x becomes large without bound, $x \to \infty$.

From the tables in Figure 5.3, we see that as $x \to -\infty$, $f(x) \to 2^+$ and as $x \to \infty$, $f(x) \to 2^-$. Here the '$+$' means 'from above' and the '$-$' means 'from below'. In this case, we say the graph of $y = f(x)$ has a **horizontal asymptote** of $y = 2$. This means that the end behaviour of f resembles the horizontal line $y = 2$, which explains the 'levelling off' behaviour we see in Figure 5.1. We formalize the concepts of vertical and horizontal asymptotes in the following definitions.

x	$f(x) \approx$	$(x, f(x)) \approx$
-10	2.3333	$(-10, 2.3333)$
-100	2.0303	$(-100, 2.0303)$
-1000	2.0030	$(-1000, 2.0030)$
-10000	2.0003	$(-10000, 2.0003)$

x	$f(x) \approx$	$(x, f(x)) \approx$
10	1.7273	$(10, 1.7273)$
100	1.9703	$(100, 1.9703)$
1000	1.9970	$(1000, 1.9970)$
10000	1.9997	$(10000, 1.9997)$

Figure 5.3: Values of $f(x) = \dfrac{2x-1}{x+1}$ for large negative and positive values of x

Definition 41 Vertical Asymptote

The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Definition 42 Horizontal Asymptote

The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

As we shall see in the next section, the graphs of rational functions may, in fact, *cross* their horizontal asymptotes. If this happens, however, it does so only a *finite* number of times, and so for each choice of $x \to -\infty$ and $x \to \infty$, $f(x)$ will approach c from either below (in the case $f(x) \to c^-$) or above (in the case $f(x) \to c^+$.) We leave $f(x) \to c$ generic in our definition, however, to allow this concept to apply to less tame specimens in the Precalculus zoo, such as Exercise 50 in Section 8.5.

Note that in Definition 42, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so.

In our discussion following Example 82, we determined that, despite the fact that the formula for $h(x)$ reduced to the same formula as $f(x)$, the functions f and

x	$h(x) \approx$	$(x, h(x)) \approx$
0.9	0.4210	$(0.9, 0.4210)$
0.99	0.4925	$(0.99, 0.4925)$
0.999	0.4992	$(0.999, 0.4992)$
0.9999	0.4999	$(0.9999, 0.4999)$

x	$h(x) \approx$	$(x, h(x)) \approx$
1.1	0.5714	$(1.1, 0.5714)$
1.01	0.5075	$(1.01, 0.5075)$
1.001	0.5007	$(1.001, 0.5007)$
1.0001	0.5001	$(1.0001, 0.5001)$

Figure 5.4: Values of $h(x) = \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1}$ near $x = 1$

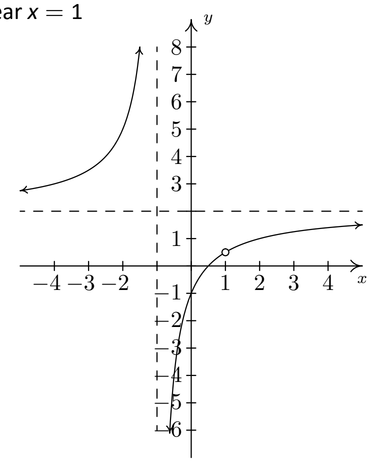


Figure 5.5: The graph $y = h(x)$ showing asymptotes and the 'hole'

In Calculus, we will see how these 'holes' in graphs can be 'plugged' once we've made a more advanced study of continuity.

In English, Theorem 33 says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$. In other words, Theorem 33 tells us 'How to tell your asymptote from a hole in the graph.'

h are different, since $x = 1$ is in the domain of f, but $x = 1$ is not in the domain of h. If we graph $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$ using a graphing calculator, we are surprised to find that the graph looks identical to the graph of $y = f(x)$. There is a vertical asymptote at $x = -1$, but near $x = 1$, everything seem fine. Tables of values provide numerical evidence which supports the graphical observation: see Figure 5.4.

We see that as $x \to 1^-$, $h(x) \to 0.5^-$ and as $x \to 1^+$, $h(x) \to 0.5^+$. In other words, the points on the graph of $y = h(x)$ are approaching $(1, 0.5)$, but since $x = 1$ is not in the domain of h, it would be inaccurate to fill in a point at $(1, 0.5)$. As we've done in past sections when something like this occurs (for instance, graphing piecewise defined functions in Section 2.5), we put an open circle (also called a **hole** in this case) at $(1, 0.5)$. Figure 5.5 is a detailed graph of $y = h(x)$, with the vertical and horizontal asymptotes as dashed lines.

Neither $x = -1$ nor $x = 1$ are in the domain of h, yet the behaviour of the graph of $y = h(x)$ is drastically different near these x-values. The reason for this lies in the second to last step when we simplified the formula for $h(x)$ in Example 82, where we had $h(x) = \dfrac{(2x-1)(x-1)}{(x+1)(x-1)}$. The reason $x = -1$ is not in the domain of h is because the factor $(x+1)$ appears in the denominator of $h(x)$; similarly, $x = 1$ is not in the domain of h because of the factor $(x-1)$ in the denominator of $h(x)$. The major difference between these two factors is that $(x-1)$ cancels with a factor in the numerator whereas $(x+1)$ does not. Loosely speaking, the trouble caused by $(x-1)$ in the denominator is cancelled away while the factor $(x+1)$ remains to cause mischief. This is why the graph of $y = h(x)$ has a vertical asymptote at $x = -1$ but only a hole at $x = 1$. These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

Theorem 33 Location of Vertical Asymptotes and Holes

Suppose r is a rational function which can be written as $r(x) = \dfrac{p(x)}{q(x)}$ where p and q have no common zeros (in other words, $r(x)$ is in lowest terms). Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \dfrac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

Example 83 Finding vertical asymptotes

Find the vertical asymptotes of, and/or holes in, the graphs of the following rational functions. Verify your answers using software or a graphing calculator, and describe the behaviour of the graph near them using proper notation.

1. $f(x) = \dfrac{2x}{x^2-3}$
2. $g(x) = \dfrac{x^2-x-6}{x^2-9}$
3. $h(x) = \dfrac{x^2-x-6}{x^2+9}$
4. $r(x) = \dfrac{x^2-x-6}{x^2+4x+4}$

Solution

1. To use Theorem 33, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms, there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. Plotting the function in GeoGebra verifies this claim, and from the graph in Figure 5.6, we see that as $x \to -\sqrt{3}^-$, $f(x) \to -\infty$, as $x \to -\sqrt{3}^+$, $f(x) \to \infty$, as $x \to \sqrt{3}^-$, $f(x) \to -\infty$, and finally as $x \to \sqrt{3}^+$, $f(x) \to \infty$.

2. Solving $x^2 - 9 = 0$ gives $x = \pm 3$. In lowest terms $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9} = \dfrac{(x-3)(x+2)}{(x-3)(x+3)} = \dfrac{x+2}{x+3}$. Since $x = -3$ continues to make trouble in the denominator, we know the line $x = -3$ is a vertical asymptote of the graph of $y = g(x)$. Since $x = 3$ no longer produces a 0 in the denominator, we have a hole at $x = 3$. To find the y-coordinate of the hole, we substitute $x = 3$ into $\dfrac{x+2}{x+3}$ and find the hole is at $\left(3, \frac{5}{6}\right)$. When we graph $y = g(x)$ using GeoGebra, we clearly see the vertical asymptote at $x = -3$, but everything seems calm near $x = 3$: see Figure 5.7. Hence, as $x \to -3^-$, $g(x) \to \infty$, as $x \to -3^+$, $g(x) \to -\infty$, as $x \to 3^-$, $g(x) \to \frac{5}{6}^-$, and as $x \to 3^+$, $g(x) \to \frac{5}{6}^+$.

3. The domain of h is all real numbers, since $x^2 + 9 = 0$ has no real solutions. Accordingly, the graph of $y = h(x)$ is devoid of both vertical asymptotes and holes, as see in Figure 5.8.

4. Setting $x^2 + 4x + 4 = 0$ gives us $x = -2$ as the only real number of concern. Simplifying, we see $r(x) = \dfrac{x^2 - x - 6}{x^2 + 4x + 4} = \dfrac{(x-3)(x+2)}{(x+2)^2} = \dfrac{x-3}{x+2}$. Since $x = -2$ continues to produce a 0 in the denominator of the reduced function, we know $x = -2$ is a vertical asymptote to the graph. The graph in Figure 5.9 bears this out, and, moreover, we see that as $x \to -2^-$, $r(x) \to \infty$ and as $x \to -2^+$, $r(x) \to -\infty$.

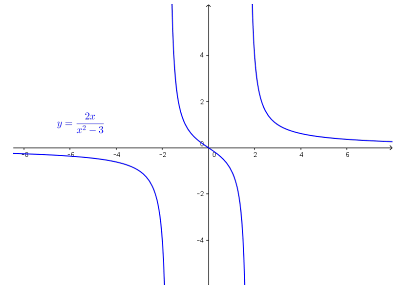


Figure 5.6: The graph $y = f(x)$ in Example 83

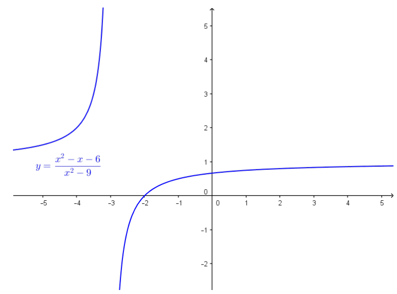


Figure 5.7: The graph $y = g(x)$ in Example 83

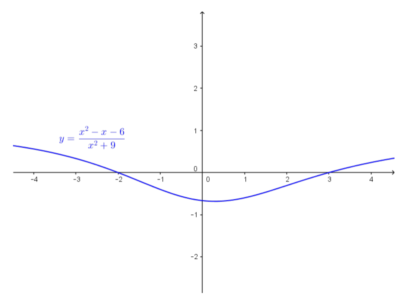


Figure 5.8: The graph $y = g(x)$ in Example 83

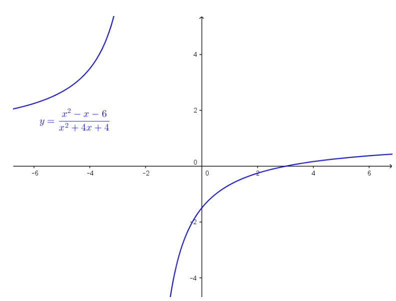


Figure 5.9: The graph $y = r(x)$ in Example 83

Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations. (These functions arise in Differential Equations. The unfortunate name will make sense shortly.)

Example 84 Doomsday population model

A mathematical model for the population P, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \dfrac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.

2. When will the population reach 100,000?

3. Determine the behaviour of P as $t \to 5^-$. Interpret this result graphically and within the context of the problem.

t	$P(t)$
4.9	10000
4.99	1000000
4.999	100000000
4.9999	10000000000

Figure 5.10: The behaviour of P as $t \to 5^-$

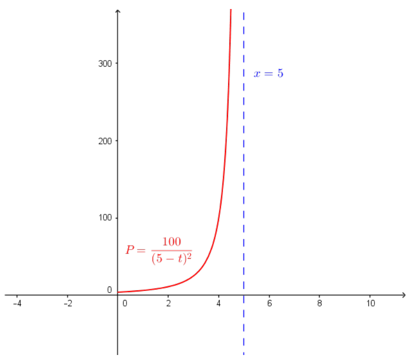


Figure 5.11: The graph of $P(t)$, for $0 \leq t < 5$

Solution

1. Substituting $t = 0$ gives $P(0) = \dfrac{100}{(5-0)^2} = 4$, which means 4000 bacteria are initially introduced into the environment.

2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in *thousands*. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Substituting for $P(t)$ gives the equation $\dfrac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

3. To determine the behaviour of P as $t \to 5^-$, we make the table in Figure 5.10.

   In other words, as $t \to 5^-$, $P(t) \to \infty$. Graphically, the line $t = 5$ is a vertical asymptote of the graph of $y = P(t)$: see Figure 5.11. Physically, this means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse.

Now that we have thoroughly investigated vertical asymptotes, we can turn our attention to horizontal asymptotes. The next theorem tells us when to expect horizontal asymptotes.

Theorem 34 Location of Horizontal Asymptotes

Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

Like Theorem 33, Theorem 34 is proved using Calculus. Nevertheless, we can understand the idea behind it using our example $f(x) = \dfrac{2x-1}{x+1}$. If we interpret $f(x)$ as a division problem, $(2x-1) \div (x+1)$, we find that the quotient is 2 with a remainder of -3. Using what we know about polynomial division, specifically Theorem 22, we get $2x - 1 = 2(x+1) - 3$. Dividing both sides by $(x+1)$ gives $\dfrac{2x-1}{x+1} = 2 - \dfrac{3}{x+1}$. (You may remember this as the formula for $g(x)$ in Example 82.) As x becomes unbounded in either direction, the quantity $\dfrac{3}{x+1}$

gets closer and closer to 0 so that the values of $f(x)$ become closer and closer (as seen in the tables in Figure 5.3) to 2. In symbols, as $x \to \pm\infty$, $f(x) \to 2$, and we have the result.

Alternatively, we can use what we know about end behaviour of polynomials to help us understand this theorem. From Theorem 20, we know the end behaviour of a polynomial is determined by its leading term. Applying this to the numerator and denominator of $f(x)$, we get that as $x \to \pm\infty$, $f(x) = \frac{2x-1}{x+1} \approx \frac{2x}{x} = 2$. This last approach is useful in Calculus, and, indeed, is made rigorous there. (Keep this in mind for the remainder of this paragraph.) Applying this reasoning to the general case, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively. If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case. If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{a}{bx^{m-n}} \to 0$ as $x \to \pm\infty$. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^{n-m}}{b}$, which becomes unbounded as $x \to \pm\infty$. As we said before, if a rational function has a horizontal asymptote, then it will have only one. (This is not true for other types of functions we shall see in later chapters.)

More specifically, as $x \to -\infty$, $f(x) \to 2^+$, and as $x \to \infty$, $f(x) \to 2^-$. Notice that the graph gets close to the same y value as $x \to -\infty$ or $x \to \infty$. This means that the graph can have only one horizontal asymptote if it is going to have one at all. Thus we were justified in using 'the' in the previous theorem.

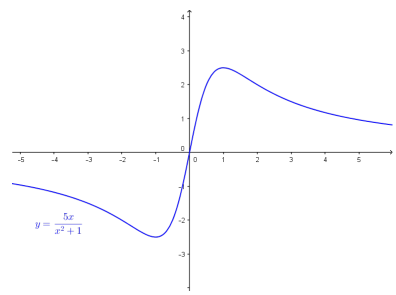
Example 85 Finding horizontal asymptotes

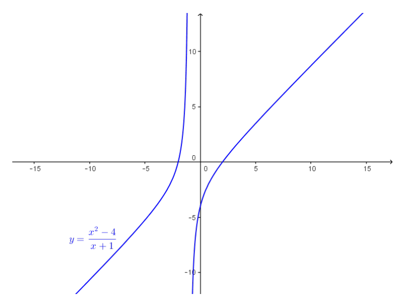
List the horizontal asymptotes, if any, of the graphs of the following functions. Verify your answers using a graphing calculator, and describe the behaviour of the graph near them using proper notation.

1. $f(x) = \frac{5x}{x^2+1}$
2. $g(x) = \frac{x^2-4}{x+1}$
3. $h(x) = \frac{6x^3-3x+1}{5-2x^3}$

Solution

1. The numerator of $f(x)$ is $5x$, which has degree 1. The denominator of $f(x)$ is x^2+1, which has degree 2. Applying Theorem 34, $y = 0$ is the horizontal asymptote. Sure enough, we see from the graph that as $x \to -\infty$, $f(x) \to 0^-$ and as $x \to \infty$, $f(x) \to 0^+$.
2. The numerator of $g(x)$, $x^2 - 4$, has degree 2, but the degree of the denominator, $x + 1$, has degree 1. By Theorem 34, there is no horizontal asymptote. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote. (Sit tight! We'll revisit this function and its end behaviour shortly.)
3. The degrees of the numerator and denominator of $h(x)$ are both three, so Theorem 34 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the calculator's graph that as $x \to -\infty$, $h(x) \to -3^+$, and as $x \to \infty$, $h(x) \to -3^-$.


$y = f(x)$


$y = g(x)$

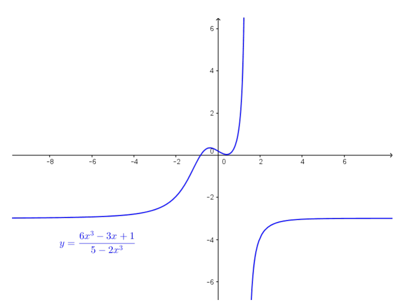

$y = h(x)$

Figure 5.12: Graphs of the three functions in Example 85

Our next example of the section gives us a real-world application of a horizontal asymptote.

Though the population in Example 86 is more accurately modelled with the functions in Chapter 7, we can approximate it (using Calculus, of course!) using a rational function.

Example 86 **Spread of the flu virus**

The number of students N at local college who have had the flu t months after the semester begins can be modelled by the formula $N(t) = 500 - \dfrac{450}{1+3t}$ for $t \geq 0$.

1. Find and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Determine the behaviour of N as $t \to \infty$. Interpret this result graphically and within the context of the problem.

t	$N(t)$
10	≈ 485.48
100	≈ 498.50
1000	≈ 499.85
10000	≈ 499.98

Figure 5.13: The long-term behaviour of $N(t)$

SOLUTION

1. $N(0) = 500 - \dfrac{450}{1+3(0)} = 50$. This means that at the beginning of the semester, 50 students have had the flu.
2. We set $N(t) = 300$ to get $500 - \dfrac{450}{1+3t} = 300$ and solve. Isolating the fraction gives $\dfrac{450}{1+3t} = 200$. Clearing denominators gives $450 = 200(1+3t)$. Finally, we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.
3. To determine the behaviour of N as $t \to \infty$, we can use the table in Figure 5.13.

   The table suggests that as $t \to \infty$, $N(t) \to 500$. (More specifically, 500^-.) This means as time goes by, only a total of 500 students will have ever had the flu.

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

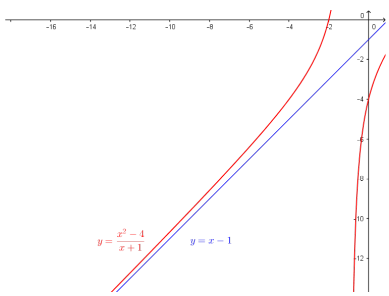


Figure 5.14: The graph $y = g(x)$ as $x \to -\infty$

We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \dfrac{x^2-4}{x+1}$ in Example 85. Performing long division, (see the remarks following Theorem 34) we get $g(x) = \dfrac{x^2-4}{x+1} = x - 1 - \dfrac{3}{x+1}$. Since the term $\dfrac{3}{x+1} \to 0$ as $x \to \pm\infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \dfrac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to \pm\infty$. We see this play out both numerically and graphically in Figures 5.14 and 5.15.

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

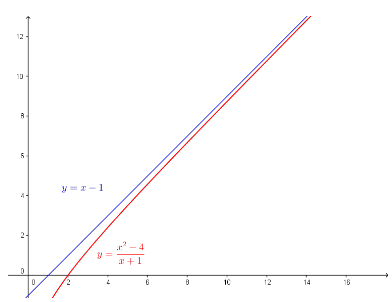


Figure 5.15: The graph $y = g(x)$ as $x \to +\infty$

The way we symbolize the relationship between the end behaviour of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to \pm\infty$, $g(x) \to x - 1$.' In this case, we say the line $y = x - 1$ is a **slant asymptote** (or 'oblique' asymptote) to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to \infty$ or as $x \to -\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as follows.

Definition 43 Slant Asymptote

The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 43 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$f(x) - (mx + b) \to 0$.' In other words, the graph of $y = f(x)$ has the *slant* asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a *horizontal* asymptote $y = 0$.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \dfrac{x^2 - 4}{x + 1}$, the degree of the numerator $x^2 - 4$ is 2, which is *exactly one more* than the degree if its denominator $x + 1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem. (Once again, this theorem is brought to you courtesy of Theorem 22 and Calculus.)

Theorem 35 Determination of Slant Asymptotes

Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where the degree of p is *exactly* one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 34 gives us an easy way to see if the graph of a rational function $r(x) = \dfrac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 35 gives us an easy way to check for slant asymptotes. Unlike Theorem 34, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 35 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it. (That's OK, though. In the next section, we'll use long division to analyze end behaviour and it's worth the effort!)

Example 87 Finding slant asymptotes

Find the slant asymptotes of the graphs of the following functions if they exist. Verify your answers using software or a graphing calculator and describe the behaviour of the graph near them using proper notation.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$

2. $g(x) = \dfrac{x^2 - 4}{x - 2}$

3. $h(x) = \dfrac{x^3+1}{x^2-4}$

Solution

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 35 guarantees us a slant asymptote. To find it, we divide $1-x=-x+1$ into x^2-4x+2 and get a quotient of $-x+3$, so our slant asymptote is $y=-x+3$. We confirm this graphically in Figure 5.16, and we see that as $x\to-\infty$, the graph of $y=f(x)$ approaches the asymptote from below, and as $x\to\infty$, the graph of $y=f(x)$ approaches the asymptote from above.

2. As with the previous example, the degree of the numerator $g(x)=\dfrac{x^2-4}{x-2}$ is 2 and the degree of the denominator is 1, so Theorem 35 applies. In this case,

$$g(x)=\frac{x^2-4}{x-2}=\frac{(x+2)(x-2)}{(x-2)}=\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}_{\,1}}=x+2,\quad x\neq 2$$

so we have that the slant asymptote $y=x+2$ is identical to the graph of $y=g(x)$ except at $x=2$ (where the latter has a 'hole' at $(2,4)$.) The graph (using GeoGebra) in Figure 5.17 supports this claim.

3. For $h(x)=\dfrac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $(x^3+1)\div(x^2-4)$ gives a quotient of just x, so our slant asymptote is the line $y=x$. The graph confirms this, and we find that as $x\to-\infty$, the graph of $y=h(x)$ approaches the asymptote from below, and as $x\to\infty$, the graph of $y=h(x)$ approaches the asymptote from above: see Figure 5.18.

The reader may be a bit disappointed with the authors at this point owing to the fact that in Examples 83, 85, and 87, we used the *calculator* to determine function behaviour near asymptotes. We rectify that in the next section where we, in excruciating detail, demonstrate the usefulness of 'number sense' to reveal this behaviour analytically.

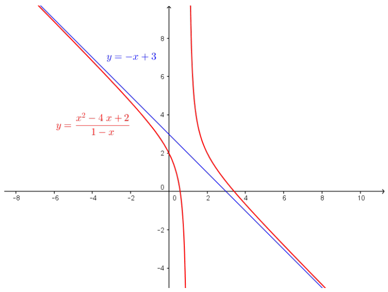


Figure 5.16: The graph $y=f(x)$ in Example 87

Note that we are purposefully avoiding notation like 'as $x\to\infty$, $f(x)\to(-x+3)^+$'. While it is possible to define these notions formally with Calculus, it is not standard to do so. Besides, with the introduction of the symbol '‽' in the next section, the authors feel we are in enough trouble already.

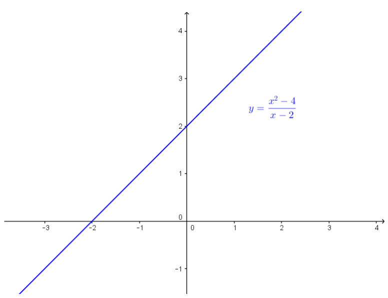


Figure 5.17: The graph $y=g(x)$ in Example 87

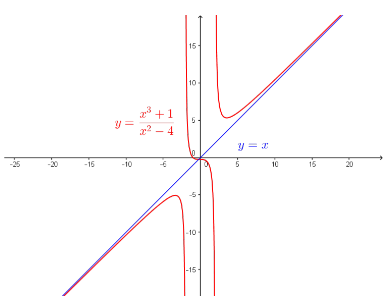


Figure 5.18: The graph $y=h(x)$ in Example 87

While the word 'asymptote' has the connotation of 'approaching but not equalling,' Definitions 42 and 43 invite the same kind of pathologies we saw with Definitions 28 in Section 2.5.

Exercises 5.1

Problems

In Exercises 1 – 18, for the given rational function f:

- **Find the domain of f.**
- **Identify any vertical asymptotes of the graph of $y = f(x)$.**
- **Identify any holes in the graph.**
- **Find the horizontal asymptote, if it exists.**
- **Find the slant asymptote, if it exists.**
- **Graph the function using a graphing utility and describe the behaviour near the asymptotes.**

1. $f(x) = \dfrac{x}{3x-6}$

2. $f(x) = \dfrac{3+7x}{5-2x}$

3. $f(x) = \dfrac{x}{x^2+x-12}$

4. $f(x) = \dfrac{x}{x^2+1}$

5. $f(x) = \dfrac{x+7}{(x+3)^2}$

6. $f(x) = \dfrac{x^3+1}{x^2-1}$

7. $f(x) = \dfrac{4x}{x^2+4}$

8. $f(x) = \dfrac{4x}{x^2-4}$

9. $f(x) = \dfrac{x^2-x-12}{x^2+x-6}$

10. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

11. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

12. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

13. $f(x) = \dfrac{2x^2+5x-3}{3x+2}$

14. $f(x) = \dfrac{-x^3+4x}{x^2-9}$

15. $f(x) = \dfrac{-5x^4-3x^3+x^2-10}{x^3-3x^2+3x-1}$

16. $f(x) = \dfrac{x^3}{1-x}$

17. $f(x) = \dfrac{18-2x^2}{x^2-9}$

18. $f(x) = \dfrac{x^3-4x^2-4x-5}{x^2+x+1}$

19. The cost C in dollars to remove $p\%$ of the invasive species of Ippizuti fish from Sasquatch Pond is given by

$$C(p) = \frac{1770p}{100-p}, \quad 0 \leq p < 100$$

    (a) Find and interpret $C(25)$ and $C(95)$.
    (b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?
    (c) What percentage of the Ippizuti fish can you remove for \$40000?

20. In Exercise 72 in Section 2.3, the population of Sasquatch in Portage County was modeled by the function

$$P(t) = \frac{150t}{t+15},$$

where $t = 0$ represents the year 1803. Find the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

21. Recall from Example 29 that the cost C (in dollars) to make x dOpi media players is $C(x) = 100x + 2000$, $x \geq 0$.

    (a) Find a formula for the average cost $\overline{C}(x)$. Recall: $\overline{C}(x) = \frac{C(x)}{x}$.
    (b) Find and interpret $\overline{C}(1)$ and $\overline{C}(100)$.
    (c) How many dOpis need to be produced so that the average cost per dOpi is \$200?
    (d) Interpret the behaviour of $\overline{C}(x)$ as $x \to 0^+$. (HINT: You may want to find the fixed cost $C(0)$ to help in your interpretation.)
    (e) Interpret the behaviour of $\overline{C}(x)$ as $x \to \infty$. (HINT: You may want to find the variable cost (defined in Example 45 in Section 3.1) to help in your interpretation.)

5.2 Graphs of Rational Functions

In this section, we take a closer look at graphing rational functions. In Section 5.1, we learned that the graphs of rational functions may have holes in them and could have vertical, horizontal and slant asymptotes. Theorems 33, 34 and 35 tell us exactly when and where these behaviours will occur, and if we combine these results with what we already know about graphing functions, we will quickly be able to generate reasonable graphs of rational functions.

Recall that at this stage (prior to discussing calculus), continuity of a function means that its graph is devoid of any breaks, jumps or holes. We'll define continuity more carefully once we've introduced limits.

One of the standard tools we will use is the sign diagram which was first introduced in Section 3.4, and then revisited in Section 4.1. In those sections, we operated under the belief that a function couldn't change its sign without its graph crossing through the *x*-axis. The major theorem we used to justify this belief was the Intermediate Value Theorem, Theorem 19. It turns out the Intermediate Value Theorem applies to all *continuous* functions, not just polynomials. Although rational functions are continuous on their domains, (another result from Calculus) Theorem 33 tells us that vertical asymptotes and holes occur at the values excluded from their domains. In other words, rational functions aren't continuous at these excluded values which leaves open the possibility that the function could change sign *without* crossing through the *x*-axis. Consider the graph of $y = h(x)$ from Example 82, reproduced in Figure 5.19 for convenience. We have added its *x*-intercept at $\left(\frac{1}{2}, 0\right)$ for the discussion that follows. Suppose we wish to construct a sign diagram for $h(x)$. Recall that the intervals where $h(x) > 0$, or $(+)$, correspond to the *x*-values where the graph of $y = h(x)$ is *above* the *x*-axis; the intervals on which $h(x) < 0$, or $(-)$ correspond to where the graph is *below* the *x*-axis.

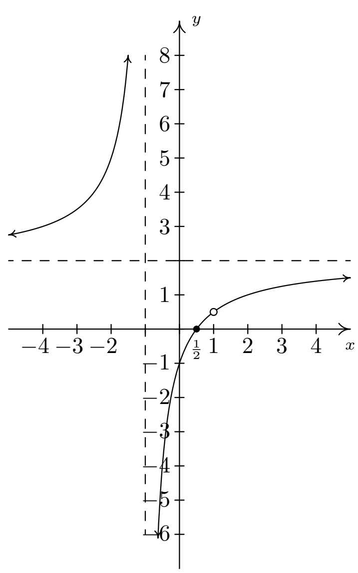


Figure 5.19: The graph $y = h(x)$ from Example 82

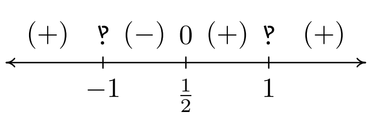


Figure 5.20: The sign diagram for the function $h(x)$ from Example 82

As we examine the graph of $y = h(x)$, reading from left to right, we note that from $(-\infty, -1)$, the graph is above the *x*-axis, so $h(x)$ is $(+)$ there. At $x = -1$, we have a vertical asymptote, at which point the graph 'jumps' across the *x*-axis. On the interval $\left(-1, \frac{1}{2}\right)$, the graph is below the *x*-axis, so $h(x)$ is $(-)$ there. The graph crosses through the *x*-axis at $\left(\frac{1}{2}, 0\right)$ and remains above the *x*-axis until $x = 1$, where we have a 'hole' in the graph. Since $h(1)$ is undefined, there is no sign here. So we have $h(x)$ as $(+)$ on the interval $\left(\frac{1}{2}, 1\right)$. Continuing, we see that on $(1, \infty)$, the graph of $y = h(x)$ is above the *x*-axis, so we mark $(+)$ there. To construct a sign diagram from this information, we not only need to denote the zero of *h*, but also the places not in the domain of *h*. As is our custom, we write '0' above $\frac{1}{2}$ on the sign diagram to remind us that it is a zero of *h*. We need a different notation for -1 and 1, and we have chosen to use '‽' - a nonstandard symbol called the interrobang. We use this symbol to convey a sense of surprise, caution and wonderment - an appropriate attitude to take when approaching these points. The moral of the story is that when constructing sign diagrams for rational functions, we include the zeros as well as the values excluded from the domain. The final result is shown in Figure 5.20.

Key Idea 23 Steps for Constructing a Sign Diagram for a Rational Function

Suppose r is a rational function.

1. Place any values excluded from the domain of r on the number line with an '‽' above them.
2. Find the zeros of r and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $r(x)$ for each test value in step 3, and write that sign above the corresponding interval.

We now present our procedure for graphing rational functions and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

Key Idea 24 Steps for Graphing Rational Functions

Suppose r is a rational function. To plot the graph $y = r(x)$, we use the following steps:

1. Find the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable.
3. Find the x- and y-intercepts of the graph of $y = r(x)$, if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behaviour of r on either side of the vertical asymptotes, if applicable.
5. Analyze the end behaviour of r. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = r(x)$.

Example 88 Graphing a rational function

Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2 - 4}$.

SOLUTION We follow the six step procedure outlined in Key Idea 24.

1. As usual, we set the denominator equal to zero to get $x^2 - 4 = 0$. We find $x = \pm 2$, so our domain is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
2. To reduce $f(x)$ to lowest terms, we factor the numerator and denominator

As we mentioned at least once earlier, since functions can have at most one y-intercept, once we find that $(0,0)$ is on the graph, we know it is the y-intercept as well as an x-intercept.

It's worth going through the analysis below to make sure we understand what's going on near the vertical asymptotes, but it's not 100% necessary. The sign diagram we construct in step 6 is the easiest way to determine the behaviour near the vertical asymptotes: once we know the value of $f(x)$ is going to be infinite, it only remains to determine if it will be $+\infty$ or $-\infty$. Keep this in mind — it will come in handy once we reach the discussion of infinite limits in our Calculus material.

which yields $f(x) = \dfrac{3x}{(x-2)(x+2)}$. There are no common factors which means $f(x)$ is already in lowest terms.

3. To find the x-intercepts of the graph of $y = f(x)$, we set $y = f(x) = 0$. Solving $\dfrac{3x}{(x-2)(x+2)} = 0$ results in $x = 0$. Since $x = 0$ is in our domain, $(0,0)$ is the x-intercept. To find the y-intercept, we set $x = 0$ and find $y = f(0) = 0$, so that $(0,0)$ is our y-intercept as well.

4. The two numbers excluded from the domain of f are $x = -2$ and $x = 2$. Since $f(x)$ didn't reduce at all, both of these values of x still cause trouble in the denominator. Thus by Theorem 33, $x = -2$ and $x = 2$ are vertical asymptotes of the graph. We can actually go a step further at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary, it is good practice for when we reach calculus. For the discussion that follows, it is best to use the factored form of $f(x) = \dfrac{3x}{(x-2)(x+2)}$.

   - *The behaviour of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use x-values a little less than -2, say -2.1, -2.01 and -2.001. While there is no harm in actually building a table like we did in Section 5.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to -6, the quantity $(x-2)$ would be very close to -4, and the factor $(x+2)$ would be very close to 0. More specifically, $(x+2)$ would be a little less than 0, in this case, -0.000001. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value. So, mentally, as $x \to -2^-$, we estimate

$$f(x) = \frac{3x}{(x-2)(x+2)} \approx \frac{-6}{(-4)\,(\text{very small }(-))} = \frac{3}{2\,(\text{very small }(-))}$$

   Now, the closer x gets to -2, the smaller $(x+2)$ will become, so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

   The term 'very big $(-)$' means a number with a large absolute value which is negative. (The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.) What all of this means is that as $x \to -2^-$, $f(x) \to -\infty$. Now suppose we wanted to determine the behaviour of $f(x)$ as $x \to -2^+$. If we imagine substituting something a little larger than -2 in for x, say -1.999999, we mentally estimate

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small }(+))} = \frac{3}{2\,(\text{very small }(+))} \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$$

We conclude that as $x \to -2^+$, $f(x) \to \infty$.

- *The behaviour of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

$$f(x) \approx \frac{6}{(\text{very small }(-))\,(4)} = \frac{3}{2\,(\text{very small }(-))}$$
$$\approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

We conclude that as $x \to 2^-$, $f(x) \to -\infty$. Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$ to get $f(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big (+). So as $x \to 2^+, f(x) \to \infty$.

The appearance of the graph $y = f(x)$ near $x = -2$ and $x = 2$ is shown in Figure 5.21.

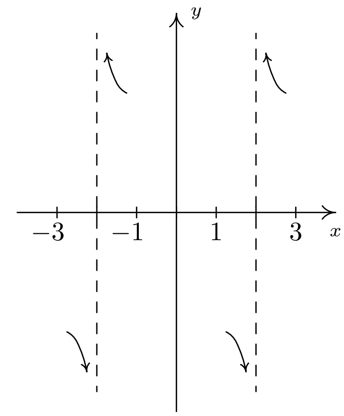


Figure 5.21: The graph $y = \frac{3x}{x^2-4}$ near its vertical asymptotes

5. Next, we determine the end behaviour of the graph of $y = f(x)$. Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 34 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \dfrac{3x}{x^2-4}$.

   - *The behaviour of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behaviour of f as $x \to -\infty$, we would substitute very 'large' negative numbers in for x, say for example, $x = -1$ billion. The numerator $3x$ would then be -3 billion, whereas the denominator $x^2 - 4$ would be $(-1\text{ billion})^2 - 4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

$$f(-1\text{ billion}) \approx \frac{-3\text{ billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}} \approx \text{very small }(-)$$

   Notice that if we substituted in $x = -1$ trillion, essentially the same kind of cancellation would happen, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit *below* the x-axis as we move to the far left.

   - *The behaviour of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for x and looking at the behaviour of $f(x)$. For example, let $x = 1$ billion. Proceeding as before, we get

$$f(1\text{ billion}) \approx \frac{3\text{ billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}} \approx \text{very small }(+)$$

   The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit *above* the x-axis as we look toward the far right. See Figure 5.22

We have deliberately left off the labels on the y-axis because we know only the behaviour near $x = \pm 2$, not the actual function values.

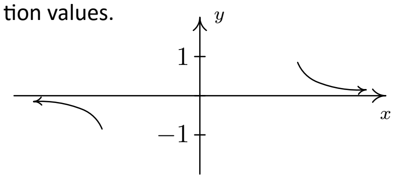


Figure 5.22: The end behavhiour of the graph $y = \frac{3x}{x^2-4}$

As with the vertical asymptotes in the previous step, we know only the behaviour of the graph as $x \to \pm\infty$. For that reason, we provide no x-axis labels.

6. Lastly, we construct a sign diagram for $f(x)$. The x-values excluded from the domain of f are $x = \pm 2$, and the only zero of f is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and

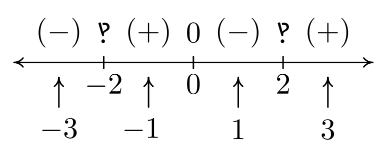


Figure 5.23: The sign diagram for $f(x) = \frac{3x}{x^2-4}$

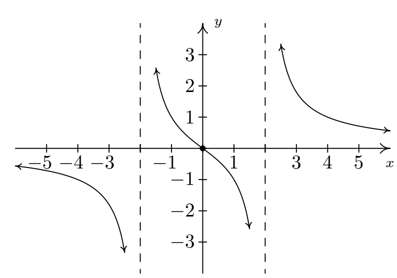


Figure 5.24: The complete graph $y = \frac{3x}{x^2-4}$ for Example 88

we choose the test values $x = -3$, $x = -1$, $x = 1$ and $x = 3$. We find $f(-3)$ is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$ and $f(3)$ is $(+)$. Combining this with our previous work, we get the graph of $y = f(x)$ in Figure 5.24.

A couple of notes are in order. First, the graph of $y = f(x)$ certainly seems to possess symmetry with respect to the origin. In fact, we can check $f(-x) = -f(x)$ to see that f is an odd function. In some textbooks, checking for symmetry is part of the standard procedure for graphing rational functions; but since it happens comparatively rarely we'll just point it out when we see it. Also note that while $y = 0$ is the horizontal asymptote, the graph of f actually crosses the x-axis at $(0, 0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false, (that's why we called it a MYTH!) as we shall see again in our next example.

Example 89 **Graphing a rational function**

Sketch a detailed graph of $g(x) = \dfrac{2x^2 - 3x - 5}{x^2 - x - 6}$.

Solution

1. Setting $x^2 - x - 6 = 0$ gives $x = -2$ and $x = 3$. Our domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

2. Factoring $g(x)$ gives $g(x) = \dfrac{(2x-5)(x+1)}{(x-3)(x+2)}$. There is no cancellation, so $g(x)$ is in lowest terms.

3. To find the x-intercept we set $y = g(x) = 0$. Using the factored form of $g(x)$ above, we find the zeros to be the solutions of $(2x - 5)(x + 1) = 0$. We obtain $x = \frac{5}{2}$ and $x = -1$. Since both of these numbers are in the domain of g, we have two x-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we set $x = 0$ and find $y = g(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

4. Since $g(x)$ was given to us in lowest terms, we have, once again by Theorem 33 vertical asymptotes $x = -2$ and $x = 3$. Keeping in mind $g(x) = \dfrac{(2x-5)(x+1)}{(x-3)(x+2)}$, we proceed to our analysis near each of these values.

   - *The behaviour of $y = g(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have

     $$g(x) \approx \frac{(-9)(-1)}{(-5)(\text{very small }(-))} \approx \frac{9}{\text{very small }(+)} \approx \text{very big (+)}$$

     so as $x \to -2^-$, $g(x) \to \infty$. On the flip side, as $x \to -2^+$, we get

     $$g(x) \approx \frac{9}{\text{very small }(-)} \approx \text{very big }(-)$$

     so $g(x) \to -\infty$.

   - *The behaviour of $y = g(x)$ as $x \to 3$:* As $x \to 3^-$, we imagine plugging in a number just shy of 3. We have

     $$g(x) \approx \frac{(1)(4)}{(\text{ very small }(-))(5)} \approx \frac{4}{\text{very small }(-)} \approx \text{very big }(-)$$

Hence, as $x \to 3^-$, $g(x) \to -\infty$. As $x \to 3^+$, we get

$$g(x) \approx \frac{4}{\text{very small }(+)} \approx \text{very big }(+)$$

so $g(x) \to \infty$.

Our results are given graphically (again, without labels on the y-axis) in Figure 5.25.

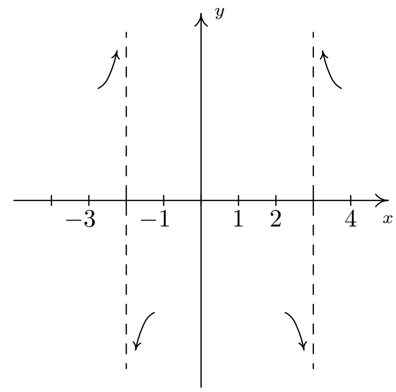


Figure 5.25: The graph $y = \frac{2x^2-3x-5}{x^2-x-6}$ near the vertical asymptotes

5. Since the degrees of the numerator and denominator of $g(x)$ are the same, we know from Theorem 34 that we can find the horizontal asymptote of the graph of g by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behaviour which would be lost otherwise.(That is, if you use a calculator to graph. Once again, Calculus is the ultimate graphing power tool.) As in the discussion following Theorem 34, we use the result of the long division $\left(2x^2-3x-5\right) \div \left(x^2-x-6\right)$ to rewrite $g(x) = \dfrac{2x^2-3x-5}{x^2-x-6}$ as $g(x) = 2 - \dfrac{x-7}{x^2-x-6}$. We focus our attention on the term $\dfrac{x-7}{x^2-x-6}$.

   - *The behaviour of $y = g(x)$ as $x \to -\infty$:* If imagine substituting $x = -1$ billion into $\dfrac{x-7}{x^2-x-6}$, we estimate $\dfrac{x-7}{x^2-x-6} \approx \dfrac{-1\text{ billion}}{1\text{billion}^2} \approx$ very small $(-)$. Hence,

   $$g(x) = 2 - \frac{x-7}{x^2-x-6} \approx 2 - \text{very small }(-) = 2 + \text{very small }(+)$$

   In other words, as $x \to -\infty$, the graph of $y = g(x)$ is a little bit *above* the line $y = 2$.

   - *The behaviour of $y = g(x)$ as $x \to \infty$.* To consider $\dfrac{x-7}{x^2-x-6}$ as $x \to \infty$, we imagine substituting $x = 1$ billion and, going through the usual mental routine, find

   $$\frac{x-7}{x^2-x-6} \approx \text{very small }(+)$$

   Hence, $g(x) \approx 2 -$ very small $(+)$, in other words, the graph of $y = g(x)$ is just *below* the line $y = 2$ as $x \to \infty$.

In the denominator for $g(-1000000000)$, we would have $(1\text{billion})^2 - 1\text{billion} - 6$. It's easy to see why the 6 is insignificant, but to ignore the 1 billion seems criminal. However, compared to (1 billion)^2, it's on the insignificant side; it's 10^{18} versus 10^9. We are once again using the fact that for polynomials, end behaviour is determined by the leading term, so in the denominator, the x^2 term wins out over the x term.

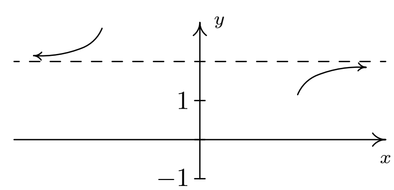


Figure 5.26: The end behaviour of $y = \frac{2x^2-3x-5}{x^2-x-6}$

Our end behaviour (again, without labels on the x-axis) is given in Figure 5.26.

6. Finally we construct our sign diagram. We place an '‽' above $x = -2$ and $x = 3$, and a '0' above $x = \frac{5}{2}$ and $x = -1$. Choosing test values in the test intervals gives us $f(x)$ is $(+)$ on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$ and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$, giving us the sign diagram in Figure 5.27. As we piece together all of the information, we note that the graph must cross the horizontal asymptote at some point after $x = 3$ in order for it to approach $y = 2$ from underneath. This is the subtlety that we would have missed had we skipped the long division and subsequent end behaviour analysis. We can, in fact, find exactly when the graph crosses $y = 2$. As a result of the long division, we have $g(x) = 2 - \dfrac{x-7}{x^2-x-6}$. For $g(x) = 2$, we would need $\dfrac{x-7}{x^2-x-6} = 0$. This gives

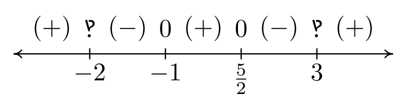


Figure 5.27: The sign diagram for $g(x) = \frac{2x^2-3x-5}{x^2-x-6}$

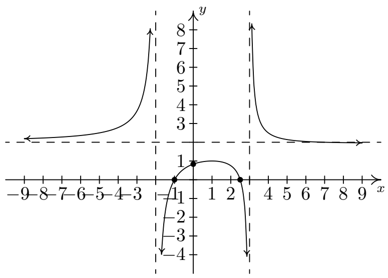


Figure 5.28: The complete graph $y = \frac{2x^2-3x-5}{x^2-x-6}$ for Example 89

$x-7=0$, or $x=7$. Note that $x-7$ is the remainder when $2x^2-3x-5$ is divided by x^2-x-6, so it makes sense that for $g(x)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7)=2$. Moreover, it stands to reason that g must attain a relative minimum at some point past $x=7$. Calculus verifies that at $x=13$, we have such a minimum at exactly $(13, 1.96)$.

Our next example gives us an opportunity to more thoroughly analyze a slant asymptote.

Example 90 A graph with a slant asymptote

Sketch a detailed graph of $h(x) = \dfrac{2x^3+5x^2+4x+1}{x^2+3x+2}$.

Solution

1. For domain, you know the drill. Solving $x^2+3x+2=0$ gives $x=-2$ and $x=-1$. Our answer is $(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$.

2. To reduce $h(x)$, we need to factor the numerator and denominator. To factor the numerator, we use the techniques set forth in Section 4.3 and we get

$$\begin{aligned} h(x) &= \frac{2x^3+5x^2+4x+1}{x^2+3x+2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)} \\ &= \frac{(2x+1)(x+1)^{\cancel{2}\,1}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2} \end{aligned}$$

   We will use this reduced formula for $h(x)$ as long as we're not substituting $x=-1$. To make this exclusion specific, we write $h(x) = \dfrac{(2x+1)(x+1)}{x+2}$, $x\neq -1$.

3. To find the x-intercepts, as usual, we set $h(x)=0$ and solve. Solving $\dfrac{(2x+1)(x+1)}{x+2} = 0$ yields $x=-\frac{1}{2}$ and $x=-1$. The latter isn't in the domain of h, so we exclude it. Our only x-intercept is $\left(-\frac{1}{2},0\right)$. To find the y-intercept, we set $x=0$. Since $0\neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0)=\frac{1}{2}$ for a y-intercept of $\left(0,\frac{1}{2}\right)$.

4. From Theorem 33, we know that since $x=-2$ still poses a threat in the denominator of the reduced function, we have a vertical asymptote there. As for $x=-1$, the factor $(x+1)$ was cancelled from the denominator when we reduced $h(x)$, so it no longer causes trouble there. This means that we get a hole when $x=-1$. To find the y-coordinate of the hole, we substitute $x=-1$ into $\dfrac{(2x+1)(x+1)}{x+2}$, per Theorem 33 and get 0. Hence, we have a hole on the x-axis at $(-1,0)$. It should make you uncomfortable plugging $x=-1$ into the reduced formula for $h(x)$, especially since we've made such a big deal concerning the stipulation about not letting $x=-1$ for that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near $x=-1$ which we will show below. Speaking of which, for the discussion that follows, we will use the formula $h(x) = \dfrac{(2x+1)(x+1)}{x+2}$, $x\neq -1$.

- *The behaviour of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small }(-))} \approx \frac{3}{(\text{very small }(-))} \approx$ very big $(-)$ thus as $x \to -2^-$, $h(x) \to -\infty$. On the other side of -2, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big $(+)$, so $h(x) \to \infty$.
- *The behaviour of $y = h(x)$ as $x \to -1$.* As $x \to -1^-$, we imagine plugging in a number a bit less than $x = -1$. We have $h(x) \approx \frac{(-1)(\text{very small }(-))}{1} =$ very small $(+)$ Hence, as $x \to -1^-$, $h(x) \to 0^+$. This means that as $x \to -1^-$, the graph is a bit above the point $(-1, 0)$. As $x \to -1^+$, we get $h(x) \approx \frac{(-1)(\text{very small }(+))}{1} =$ very small $(-)$. This gives us that as $x \to -1^+$, $h(x) \to 0^-$, so the graph is a little bit lower than $(-1, 0)$ here. Our results are shown graphically in Figure 5.29.

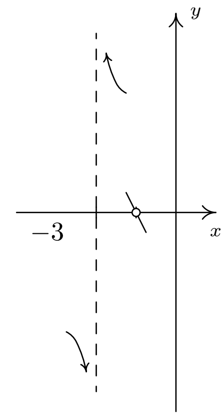


Figure 5.29: The behaviour of $y = h(x)$ near the hole and vertical asymptote

5. For end behaviour, we note that the degree of the numerator of $h(x)$, $2x^3 + 5x^2 + 4x + 1$, is 3 and the degree of the denominator, $x^2 + 3x + 2$, is 2 so by Theorem 35, the graph of $y = h(x)$ has a slant asymptote. For $x \to \pm\infty$, we are far enough away from $x = -1$ to use the reduced formula, $h(x) = \dfrac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \dfrac{2x^2 + 3x + 1}{x+2}$, $x \neq -1$, and rewrite $h(x) = 2x - 1 + \dfrac{3}{x+2}$, $x \neq -1$. By Theorem 35, the slant asymptote is $y = 2x - 1$, and to better see *how* the graph approaches the asymptote, we focus our attention on the term generated from the remainder, $\dfrac{3}{x+2}$.

   - *The behaviour of $y = h(x)$ as $x \to -\infty$:* Substituting $x = -1$ billion into $\dfrac{3}{x+2}$, we get the estimate $\frac{3}{-1\text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x - 1 + \dfrac{3}{x+2} \approx 2x - 1 +$ very small $(-)$. This means the graph of $y = h(x)$ is a little bit *below* the line $y = 2x - 1$ as $x \to -\infty$.
   - *The behaviour of $y = h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\dfrac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x - 1 +$ very small $(+)$, or that the graph of $y = h(x)$ is a little bit *above* the line $y = 2x - 1$ as $x \to \infty$. The end behaviour is shown in Figure 5.30

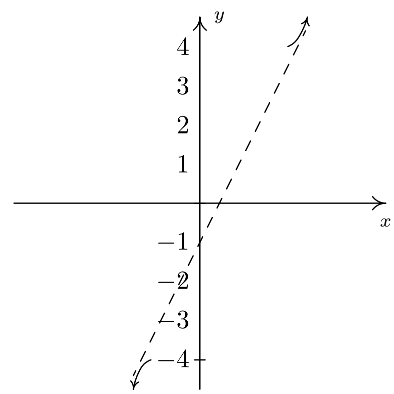


Figure 5.30: End behaviour for $y = h(x)$

6. To make our sign diagram, we place an '‽' above $x = -2$ and $x = -1$ and a '0' above $x = -\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2, -1)$ and $\left(-\frac{1}{2}, \infty\right)$ and $h(x)$ is $(-)$ on $(-\infty, -2)$ and $\left(-1, -\frac{1}{2}\right)$, giving us the sign diagram in Figure 5.31. Putting all of our work together yields the graph in Figure 5.32.

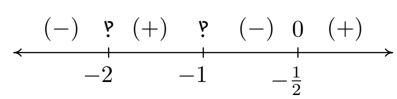


Figure 5.31: The sign diagram for $h(x) = \frac{2x^3+5x^2+4x+1}{x^2+3x+2}$

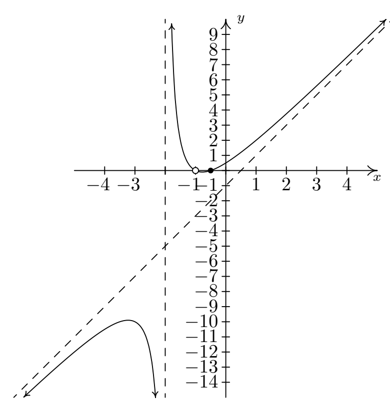


Figure 5.32: The graph $y = h(x)$ for Example 90

We could ask whether the graph of $y = h(x)$ crosses its slant asymptote. From the formula $h(x) = 2x - 1 + \dfrac{3}{x+2}$, $x \neq -1$, we see that if $h(x) = 2x - 1$, we would have $\dfrac{3}{x+2} = 0$. Since this will never happen, we conclude the graph never crosses its slant asymptote. (But rest assured, some graphs do!)

We end this section with an example that shows it's not all pathological weirdness when it comes to rational functions and technology still has a role to play in studying their graphs at this level (that is, prior to introducing the techniques of Calculus).

Example 91 **A graph requiring calculus for the details**

Sketch the graph of $r(x) = \dfrac{x^4+1}{x^2+1}$.

Solution

1. The denominator x^2+1 is never zero so the domain is $(-\infty, \infty)$.

2. With no real zeros in the denominator, x^2+1 is an irreducible quadratic. Our only hope of reducing $r(x)$ is if x^2+1 is a factor of x^4+1. Performing long division gives us

$$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$$

The remainder is not zero so $r(x)$ is already reduced.

3. To find the x-intercept, we'd set $r(x) = 0$. Since there are no real solutions to $\dfrac{x^4+1}{x^2+1} = 0$, we have no x-intercepts. Since $r(0) = 1$, we get $(0, 1)$ as the y-intercept.

4. This step doesn't apply to r, since its domain is all real numbers.

5. For end behaviour, we note that since the degree of the numerator is exactly *two* more than the degree of the denominator, neither Theorems 34 nor 35 apply. We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \dfrac{2}{x^2+1}$, so we focus our attention on the term corresponding to the remainder, $\dfrac{2}{x^2+1}$. It should be clear that as $x \to \pm\infty$, $\dfrac{2}{x^2+1} \approx$ very small $(+)$, which means $r(x) \approx x^2 - 1 +$ very small $(+)$. So the graph $y = r(x)$ is a little bit *above* the graph of the parabola $y = x^2 - 1$ as $x \to \pm\infty$. Graphically, we have Figure 5.33.

6. There isn't much work to do for a sign diagram for $r(x)$, since its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty, \infty)$, and since we know $r(0) = 1$, we conclude $r(x)$ is $(+)$ for all real numbers. At this point, we don't have much to go on for a graph. We leave it to the reader to show $r(-x) = r(x)$ so r is even, and, hence, its graph is symmetric about the y-axis. Figure 5.34 shows a comparison of what we have determined analytically versus what the computer shows us. We have no way to detect the relative extrema analytically (without appealing to Calculus, of course) apart from brute force plotting of points, which is done more efficiently by the computer.

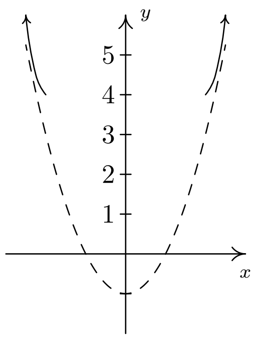
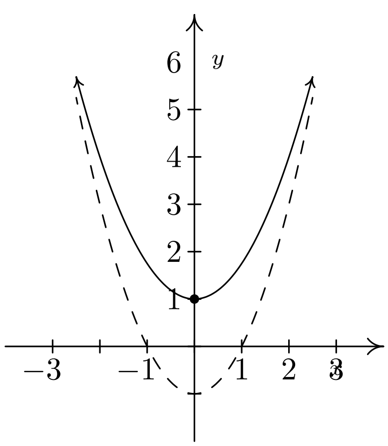
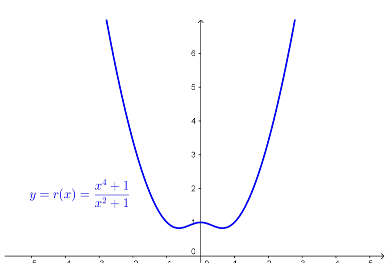


Figure 5.33: Comparing $y = r(x)$ to $y = x^2 - 1$



Plotting by hand (without calculus)



Plotting with GeoGebra

Figure 5.34: The limitations of our Precalculus methods

As usual, the authors offer no apologies for what may be construed as 'pedantry' in this section. We feel that the detail presented in this section is necessary to obtain a firm grasp of the concepts presented here and it also serves as an introduction to the methods employed in Calculus. As we have said many times in the past, your instructor will decide how much, if any, of the kinds of details presented here are 'mission critical' to your understanding of Precalculus. Without further delay, we present you with this section's Exercises.

Exercises 5.2

Problems

In Exercises 1 – 16, use the six-step procedure from Key Idea 24 to graph the rational function. Be sure to draw any asymptotes as dashed lines.

1. $f(x) = \dfrac{4}{x+2}$

2. $f(x) = \dfrac{5x}{6-2x}$

3. $f(x) = \dfrac{1}{x^2}$

4. $f(x) = \dfrac{1}{x^2+x-12}$

5. $f(x) = \dfrac{2x-1}{-2x^2-5x+3}$

6. $f(x) = \dfrac{x}{x^2+x-12}$

7. $f(x) = \dfrac{4x}{x^2+4}$

8. $f(x) = \dfrac{4x}{x^2-4}$

9. $f(x) = \dfrac{x^2-x-12}{x^2+x-6}$

10. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

11. $f(x) = \dfrac{x^2-x-6}{x+1}$

12. $f(x) = \dfrac{x^2-x}{3-x}$

13. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

14. $f(x) = \dfrac{-x^3+4x}{x^2-9}$

15. $f(x) = \dfrac{x^3-2x^2+3x}{2x^2+2}$

16. [1] $f(x) = \dfrac{x^2-2x+1}{x^3+x^2-2x}$

In Exercises 17 – 20, graph the rational function by applying transformations to the graph of $y = \dfrac{1}{x}$.

17. $f(x) = \dfrac{1}{x-2}$

18. $g(x) = 1 - \dfrac{3}{x}$

19. $h(x) = \dfrac{-2x+1}{x}$ (Hint: Divide)

20. $j(x) = \dfrac{3x-7}{x-2}$ (Hint: Divide)

21. Discuss with your classmates how you would graph $f(x) = \dfrac{ax+b}{cx+d}$. What restrictions must be placed on a, b, c and d so that the graph is indeed a transformation of $y = \dfrac{1}{x}$?

22. In Example 63 in Section 4.1 we showed that $p(x) = \frac{4x+x^3}{x}$ is not a polynomial even though its formula reduced to $4+x^2$ for $x \neq 0$. However, it is a rational function similar to those studied in the section. With the help of your classmates, graph $p(x)$.

23. Let $g(x) = \dfrac{x^4-8x^3+24x^2-72x+135}{x^3-9x^2+15x-7}$. With the help of your classmates, find the x- and y- intercepts of the graph of g. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Find all of the asymptotes of the graph of g and any holes in the graph, if they exist. Be sure to show all of your work including any polynomial or synthetic division. Sketch the graph of g, using more than one picture if necessary to show all of the important features of the graph.

Example 91 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function. Without Calculus, we need to use technology to reveal the hidden mysteries of rational function behaviour. Working with your classmates, use a computer or graphing calculator to examine the graphs of the rational functions given in Exercises 24 – 27. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

24. $f(x) = \dfrac{1}{x^2+1}$

25. $f(x) = \dfrac{x}{x^2+1}$

26. $f(x) = \dfrac{x^2}{x^2+1}$

27. $f(x) = \dfrac{x^3}{x^2+1}$

[1] Once you've done the six-step procedure, use a computer or graphing calculator to graph this function on the viewing window $[0, 12] \times [0, 0.25]$. What do you see?

5.3 Rational Inequalities and Applications

In this section, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving a rational equation and a rational inequality.

Example 92 **Rational equation and inequality**

1. Solve $\dfrac{x^3 - 2x + 1}{x - 1} = \dfrac{1}{2}x - 1$.
2. Solve $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1$.
3. Use your computer or calculator to graphically check your answers to 1 and 2.

Solution

1. To solve the equation, we clear denominators

$$\begin{aligned}
\frac{x^3 - 2x + 1}{x - 1} &= \frac{1}{2}x - 1 \\
\left(\frac{x^3 - 2x + 1}{x - 1}\right) \cdot 2(x - 1) &= \left(\frac{1}{2}x - 1\right) \cdot 2(x - 1) \\
2x^3 - 4x + 2 &= x^2 - 3x + 2 && \text{expand} \\
2x^3 - x^2 - x &= 0 \\
x(2x + 1)(x - 1) &= 0 && \text{factor} \\
x &= -\tfrac{1}{2},\ 0,\ 1
\end{aligned}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x = 1$ does not satisfy the original equation and must be discarded. Our solutions are $x = -\frac{1}{2}$ and $x = 0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation — namely by multiplying both sides by the quantity $(x - 1)$. The problem is that, depending on x, $(x - 1)$ may be positive (which doesn't affect the inequality) or $(x - 1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 205 in Section 5.2.

$$\begin{aligned}
\frac{x^3 - 2x + 1}{x - 1} &\geq \frac{1}{2}x - 1 \\
\frac{x^3 - 2x + 1}{x - 1} - \frac{1}{2}x + 1 &\geq 0 \\
\frac{2\left(x^3 - 2x + 1\right) - x(x - 1) + 1(2(x - 1))}{2(x - 1)} &\geq 0 \\
&\text{get a common denominator} \\
\frac{2x^3 - x^2 - x}{2x - 2} &\geq 0 && \text{expand}
\end{aligned}$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of r are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we obtain the sign diagram in Figure 5.35.

We are interested in where $r(x) \geq 0$. We find $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0, 1)$ and $(1, \infty)$. We add to these intervals the zeros of r, $-\frac{1}{2}$ and 0, to get our final solution: $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. Geometrically, if we set $f(x) = \dfrac{x^3 - 2x + 1}{x - 1}$ and $g(x) = \frac{1}{2}x - 1$, the solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above ($>$) the graph of $g(x)$. Entering these two functions into GeoGebra gives us Figure 5.36.

   Zooming in and using the Intersect tool, we see in Figure 5.37 that the graphs cross when $x = -\frac{1}{2}$ and $x = 0$. It is clear from the calculator that the graph of $y = f(x)$ is above the graph of $y = g(x)$ on $\left(-\infty, -\frac{1}{2}\right)$ as well as on $(0, \infty)$. According to the calculator, our solution is then $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ which *almost* matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, and, even though it isn't shown on the calculator, there is a hole in the graph of $y = f(x)$ when $x = 1$ which is why $x = 1$ is not part of our final answer. (There is no asymptote at $x = 1$ since the graph is well behaved near $x = 1$. According to Theorem 33, there must be a hole there.)

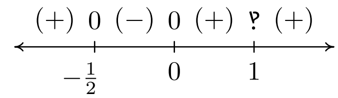


Figure 5.35: The sign diagram for the inequality in Example 92

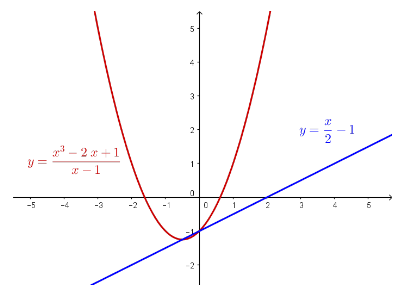


Figure 5.36: The initial plot of $f(x)$ and $g(x)$

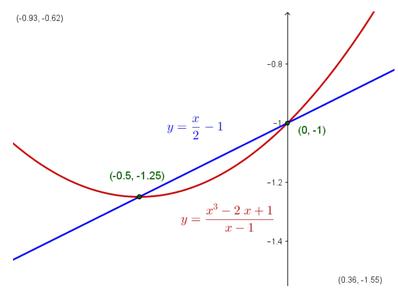


Figure 5.37: Zooming in to find the intersection points

Next, we explore how rational equations can be used to solve some classic problems involving rates.

Example 93 Calculating the speed of a river

Carl decides to explore the Meander River, the location of several recent Sasquatch sightings. From camp, he canoes downstream five miles to check out a purported Sasquatch nest. Finding nothing, he immediately turns around, retraces his route (this time travelling upstream), and returns to camp 3 hours after he left. If Carl canoes at a rate of 6 miles per hour in still water, how fast was the Meander River flowing on that day?

Solution We are given information about distances, rates (speeds) and times. The basic principle relating these quantities is:

$$\text{distance} = \text{rate} \cdot \text{time}$$

The first observation to make, however, is that the distance, rate and time given to us aren't 'compatible': the distance given is the distance for only *part* of the trip, the rate given is the speed Carl can canoe in still water, not in a flowing river, and the time given is the duration of the *entire* trip. Ultimately, we are after the speed of the river, so let's call that R measured in miles per hour to be consistent with the other rate given to us. To get started, let's divide the trip into its two parts: the initial trip downstream and the return trip upstream. For the downstream trip, all we know is that the distance travelled is 5 miles.

$$\begin{aligned} \text{distance downstream} &= \text{rate travelling downstream} \cdot \text{time travelling downstream} \\ 5 \text{ miles} &= \text{rate travelling downstream} \cdot \text{time travelling downstream} \end{aligned}$$

Since the return trip upstream followed the same route as the trip downstream, we know that the distance travelled upstream is also 5 miles.

$$\begin{aligned} \text{distance upstream} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5 \text{ miles} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \end{aligned}$$

We are told Carl can canoe at a rate of 6 miles per hour in still water. How does this figure into the rates travelling upstream and downstream? The speed the canoe travels in the river is a combination of the speed at which Carl can propel the canoe in still water, 6 miles per hour, and the speed of the river, which we're calling R. When travelling downstream, the river is helping Carl along, so we *add* these two speeds:

$$\begin{aligned} \text{rate traveling downstream} &= \text{rate Carl propels the canoe} + \text{speed of the river} \\ &= 6\,\tfrac{\text{miles}}{\text{hour}} + R\,\tfrac{\text{miles}}{\text{hour}} \end{aligned}$$

So our downstream speed is $(6+R)\,\frac{\text{miles}}{\text{hour}}$. Substituting this into our 'distance-rate-time' equation for the downstream part of the trip, we get:

$$\begin{aligned} 5 \text{ miles} &= \text{rate traveling downstream} \cdot \text{time traveling downstream} \\ 5 \text{ miles} &= (6+R)\,\tfrac{\text{miles}}{\text{hour}} \cdot \text{time traveling downstream} \end{aligned}$$

When travelling upstream, Carl works against the current. Since the canoe manages to travel upstream, the speed Carl can canoe in still water is greater than the river's speed, so we *subtract* the river's speed *from* Carl's canoeing speed to get:

$$\begin{aligned} \text{rate traveling upstream} &= \text{rate Carl propels the canoe} - \text{river speed} \\ &= 6\,\tfrac{\text{miles}}{\text{hour}} - R\,\tfrac{\text{miles}}{\text{hour}} \end{aligned}$$

Proceeding as before, we get

$$\begin{aligned} 5 \text{ miles} &= \text{rate traveling upstream} \cdot \text{time traveling upstream} \\ 5 \text{ miles} &= (6-R)\,\tfrac{\text{miles}}{\text{hour}} \cdot \text{time traveling upstream} \end{aligned}$$

The last piece of information given to us is that the total trip lasted 3 hours. If we let t_{down} denote the time of the downstream trip and t_{up} the time of the upstream trip, we have: $t_{\text{down}} + t_{\text{up}} = 3$ hours. Substituting t_{down} and t_{up} into the 'distance-rate-time' equations, we get (suppressing the units) *three* equations in *three* unknowns:

$$\left\{ \begin{array}{llll} E1 & (6+R)\,t_{\text{down}} &=& 5 \\ E2 & (6-R)\,t_{\text{up}} &=& 5 \\ E3 & t_{\text{down}} + t_{\text{up}} &=& 3 \end{array} \right.$$

This is an example of a *system* of equations. If you didn't encounter such creatures in high school, don't worry: you won't need to solve any systems in this course. If you're wondering if there's a general procedure for tackling such problems, you might want to check out Math 1410.

Although we usually discourage dividing both sides of an equation by a variable expression, we know $(6+R) \neq 0$ since otherwise we couldn't possibly multiply it by t_{down} and get 5.

Since we are ultimately after R, we need to use these three equations to get at least one equation involving only R. To that end, we solve $E1$ for t_{down} by dividing both sides by the quantity $(6+R)$ to get $t_{\text{down}} = \dfrac{5}{6+R}$. Similarly, we solve $E2$ for t_{up} and get $t_{\text{up}} = \dfrac{5}{6-R}$. Substituting these into $E3$, we get:

$$\frac{5}{6+R} + \frac{5}{6-R} = 3.$$

(The reader is encouraged to verify that the units in this equation are the same on both sides. To get you started, the units on the '3' is 'hours.') Clearing denominators, we get $5(6-R)+5(6+R)=3(6+R)(6-R)$ which reduces to $R^2=16$. We find $R=\pm 4$, and since R represents the speed of the river, we choose $R=4$. On the day in question, the Meander River is flowing at a rate of 4 miles per hour.

One of the important lessons to learn from Example 93 is that speeds, and more generally, rates, are additive. As we see in our next example, the concept of rate and its associated principles can be applied to a wide variety of problems - not just 'distance-rate-time' scenarios.

Example 94 Calculating work rates

Working alone, Taylor can weed the garden in 4 hours. If Carl helps, they can weed the garden in 3 hours. How long would it take for Carl to weed the garden on his own?

Solution The key relationship between work and time which we use in this problem is:

$$\text{amount of work done} = \text{rate of work} \cdot \text{time spent working}$$

We are told that, working alone, Taylor can weed the garden in 4 hours. In Taylor's case then:

$$\begin{aligned} \text{work done by Taylor} &= \text{rate of Taylor working} \cdot \text{time Taylor spent working} \\ 1 \text{ garden} &= (\text{rate of Taylor working}) \cdot (4 \text{ hours}) \end{aligned}$$

So we have that the rate Taylor works is $\frac{1 \text{ garden}}{4 \text{ hours}} = \frac{1}{4} \frac{\text{garden}}{\text{hour}}$. We are also told that when working together, Taylor and Carl can weed the garden in just 3 hours. We have:

$$\begin{aligned} \text{work done together} &= \text{rate of working together} \cdot \text{time working together} \\ 1 \text{ garden} &= (\text{rate of working together}) \cdot (3 \text{ hours}) \end{aligned}$$

From this, we find that the rate of Taylor and Carl working together is $\frac{1 \text{ garden}}{3 \text{ hours}} = \frac{1}{3} \frac{\text{garden}}{\text{hour}}$. We are asked to find out how long it would take for Carl to weed the garden on his own. Let us call this unknown t, measured in hours to be consistent with the other times given to us in the problem. Then:

$$\begin{aligned} \text{work done by Carl} &= \text{rate of Carl working} \cdot \text{time Carl spent working} \\ 1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours}) \end{aligned}$$

In order to find t, we need to find the rate of Carl working, so let's call this quantity R, with units $\frac{\text{garden}}{\text{hour}}$. Using the fact that rates are additive, we have:

$$\begin{aligned} \text{rate working together} &= \text{rate of Taylor working} + \text{rate of Carl working} \\ \frac{1}{3} \frac{\text{garden}}{\text{hour}} &= \frac{1}{4} \frac{\text{garden}}{\text{hour}} + R \frac{\text{garden}}{\text{hour}} \end{aligned}$$

so that $R = \frac{1}{12} \frac{\text{garden}}{\text{hour}}$. Substituting this into our 'work-rate-time' equation for Carl, we get:

$$\begin{aligned} 1 \text{ garden} &= (\text{rate of Carl working}) \cdot (t \text{ hours}) \\ 1 \text{ garden} &= \left(\frac{1}{12} \frac{\text{garden}}{\text{hour}}\right) \cdot (t \text{ hours}) \end{aligned}$$

Solving $1 = \frac{1}{12}t$, we get $t = 12$, so it takes Carl 12 hours to weed the garden on his own. (Carl would much rather spend his time writing open-source Mathematics texts than gardening anyway.)

As is common with 'word problems' like Examples 93 and 94, there is no short-cut to the answer. We encourage the reader to carefully think through and apply the basic principles of rate to each (potentially different!) situation. It is time well spent. We also encourage the tracking of units, especially in the early stages of the problem. Not only does this promote uniformity in the units, it also serves as a quick means to check if an equation makes sense. (In other words, make sure you don't try to add apples to oranges!)

Our next example deals with the average cost function, first introduced on page 73, as applied to PortaBoy Game systems from Example 45 in Section 3.1.

Example 95 A rational cost function

Given a cost function $C(x)$, which returns the total cost of producing x items, recall that the average cost function, $\overline{C}(x) = \frac{C(x)}{x}$ computes the cost per item when x items are produced. Suppose the cost C, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function $\overline{C}(x)$.
2. Solve $\overline{C}(x) < 100$ and interpret.
3. Determine the behaviour of $\overline{C}(x)$ as $x \to \infty$ and interpret.

Solution

1. From $\overline{C}(x) = \dfrac{C(x)}{x}$, we obtain $\overline{C}(x) = \dfrac{80x + 150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solving $\overline{C}(x) < 100$ means we solve $\dfrac{80x + 150}{x} < 100$. We proceed as in the previous example.

$$\begin{array}{rcll} \dfrac{80x + 150}{x} & < & 100 & \\ \dfrac{80x + 150}{x} - 100 & < & 0 & \\ \dfrac{80x + 150 - 100x}{x} & < & 0 & \text{common denominator} \\ \dfrac{150 - 20x}{x} & < & 0 & \end{array}$$

If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$, giving us the sign diagram in Figure 5.38.

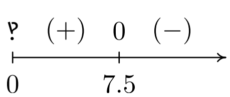


Figure 5.38: The sign digram for $r(x)$

In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than $\$100$ per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. When we apply Theorem 34 to $\overline{C}(x)$ we find that $y = 80$ is a horizontal asymptote to the graph of $y = \overline{C}(x)$. To more precisely determine the behaviour of $\overline{C}(x)$ as $x \to \infty$, we first use long division and rewrite $\overline{C}(x) = 80 + \frac{150}{x}$. (In this case, long division amounts to term-by-term division.) As $x \to \infty$, $\frac{150}{x} \to 0^+$, which means $\overline{C}(x) \approx 80 +$ very small $(+)$. Thus the average cost per system is getting closer to $\$80$ per system. If we set $\overline{C}(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than $\$80$ per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example 45, we realize $\$80$ is the variable cost per system $-$ the cost per system above and beyond the fixed initial cost of $\$150$. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively 'zero out' the fixed cost.

Our next example is another classic 'box with no top' problem.

Example 96 **Minimizing surface area**

A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimetres. Let x denote the width of the box, in centimetres as seen in Figure 5.39.

1. Express the height h in centimetres as a function of the width x and state the applied domain.
2. Solve $h(x) \geq x$ and interpret.
3. Find and interpret the behaviour of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.
4. Express the surface area S of the box as a function of x and state the applied domain.
5. Use a calculator to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

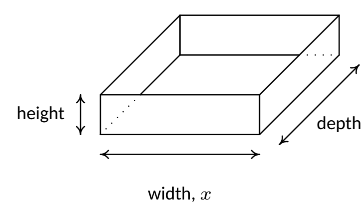


Figure 5.39: The box in Example 96

Solution

1. We are told that the volume of the box is 1000 cubic centimetres and that x represents the width, in centimetres. From geometry, we know Volume $=$ width $\times$ height $\times$ depth. Since the base of the box is a square, the width and the depth are both x centimetres. Using h for the height, we have $1000 = x^2 h$, so that $h = \frac{1000}{x^2}$. Using function notation, (that is, $h(x)$ means 'h of x', not 'h times x' here) $h(x) = \frac{1000}{x^2}$. As for the applied domain, in order for there to be a box at all, $x > 0$, and since every such choice of x will return a positive number for the height h we have no other restrictions and conclude our domain is $(0, \infty)$.

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \quad \text{common denominator} \end{aligned}$$

We consider the left hand side of the inequality as our rational function $r(x)$. We see r is undefined at $x = 0$, but, as in the previous example, the applied domain of the problem is $x > 0$, so we are considering only the behaviour of r on $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, which is $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the diagram in Figure 5.40.

We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, h represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimetres for this to happen.

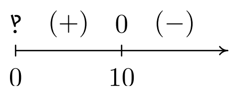


Figure 5.40: The sign digram for $h(x)$

3. As $x \to 0^+$, $h(x) = \frac{1000}{x^2} \to \infty$. This means that the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimetres. As $x \to \infty$, we find $h(x) \to 0^+$, which means that in order to maintain a volume of 1000 cubic centimetres, the width and depth must get bigger as the height becomes smaller.

4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by h. We get the surface area, $S = x^2 + 4xh$. To get S as a function of x, we substitute $h = \frac{1000}{x^2}$ to obtain $S = x^2 + 4x\left(\frac{1000}{x^2}\right)$. Hence, as a function of x, $S(x) = x^2 + \frac{4000}{x}$. The domain of S is the same as h, namely $(0, \infty)$, for the same reasons as above.

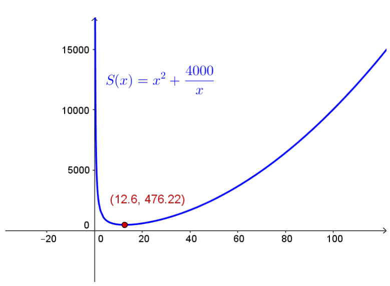


Figure 5.41: Minimizing the surface area in Example 96

5. A first attempt at the graph of $y = S(x)$ on the calculator or computer may lead to frustration. On the calculator, chances are good that the first window chosen to view the graph will suggest $y = S(x)$ has the x-axis as a horizontal asymptote. (On GeoGebra, you'll probably have to zoom out a long way before you can even see the graph!) From the formula $S(x) = x^2 + \frac{4000}{x}$, however, we get $S(x) \approx x^2$ as $x \to \infty$, so $S(x) \to \infty$. Readjusting the window, we find S does possess a relative minimum at $x \approx 12.60$. As far as we can tell, (without Calculus, that is) this is the only relative extremum, so it is the absolute minimum as well. This means that the width and depth of the box should each measure approximately 12.60 centimetres. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimetres.

5.3.1 Variation

In many instances in the sciences, rational functions are encountered as a result of fundamental natural laws which are typically a result of assuming certain basic relationships between variables. These basic relationships are summarized in the definition below.

Definition 44 Variation

Suppose x, y and z are variable quantities. We say

- y **varies directly with** (or is **directly proportional to**) x if there is a constant k such that $y = kx$.
- y **varies inversely with** (or is **inversely proportional to**) x if there is a constant k such that $y = \frac{k}{x}$.
- z **varies jointly with** (or is **jointly proportional to**) x and y if there is a constant k such that $z = kxy$.

The constant k in the above definitions is called the **constant of proportionality**.

Example 97 Some famous variational relationships

Translate the following into mathematical equations using Definition 44.

1. Hooke's Law: The force F exerted on a spring is directly proportional the extension x of the spring.
2. Boyle's Law: At a constant temperature, the pressure P of an ideal gas is inversely proportional to its volume V.
3. The volume V of a right circular cone varies jointly with the height h of the cone and the square of the radius r of the base.
4. Ohm's Law: The current I through a conductor between two points is directly proportional to the voltage V between the two points and inversely proportional to the resistance R between the two points.
5. Newton's Law of Universal Gravitation: Suppose two objects, one of mass m and one of mass M, are positioned so that the distance between their centers of mass is r. The gravitational force F exerted on the two objects varies directly with the product of the two masses and inversely with the square of the distance between their centers of mass.

Solution

1. Applying the definition of direct variation, we get $F = kx$ for some constant k.
2. Since P and V are inversely proportional, we write $P = \frac{k}{V}$.

3. There is a bit of ambiguity here. It's clear that the volume and the height of the cone are represented by the quantities V and h, respectively, but does r represent the radius of the base or the square of the radius of the base? It is the former. Usually, if an algebraic operation is specified (like squaring), it is meant to be expressed in the formula. We apply Definition 44 to get $V = khr^2$.

4. Even though the problem doesn't use the phrase 'varies jointly', it is implied by the fact that the current I is related to two different quantities. Since I varies directly with V but inversely with R, we write $I = \frac{kV}{R}$.

5. We write the product of the masses mM and the square of the distance as r^2. We have that F varies directly with mM and inversely with r^2, so $F = \frac{kmM}{r^2}$.

Exercises 5.3

Problems

In Exercises 1 – 6, solve the rational equation. Be sure to check for extraneous solutions.

1. $\dfrac{x}{5x+4} = 3$

2. $\dfrac{3x-1}{x^2+1} = 1$

3. $\dfrac{1}{x+3} + \dfrac{1}{x-3} = \dfrac{x^2-3}{x^2-9}$

4. $\dfrac{2x+17}{x+1} = x+5$

5. $\dfrac{x^2-2x+1}{x^3+x^2-2x} = 1$

6. $\dfrac{-x^3+4x}{x^2-9} = 4x$

In Exercises 7 – 20, solve the rational inequality. Express your answer using interval notation.

7. $\dfrac{1}{x+2} \geq 0$

8. $\dfrac{x-3}{x+2} \leq 0$

9. $\dfrac{x}{x^2-1} > 0$

10. $\dfrac{4x}{x^2+4} \geq 0$

11. $\dfrac{x^2-x-12}{x^2+x-6} > 0$

12. $\dfrac{3x^2-5x-2}{x^2-9} < 0$

13. $\dfrac{x^3+2x^2+x}{x^2-x-2} \geq 0$

14. $\dfrac{x^2+5x+6}{x^2-1} > 0$

15. $\dfrac{3x-1}{x^2+1} \leq 1$

16. $\dfrac{2x+17}{x+1} > x+5$

17. $\dfrac{-x^3+4x}{x^2-9} \geq 4x$

18. $\dfrac{1}{x^2+1} < 0$

19. $\dfrac{x^4-4x^3+x^2-2x-15}{x^3-4x^2} \geq x$

20. $\dfrac{5x^3-12x^2+9x+10}{x^2-1} \geq 3x-1$

21. Carl and Mike start a 3 mile race at the same time. If Mike ran the race at 6 miles per hour and finishes the race 10 minutes before Carl, how fast does Carl run?

22. One day, Donnie observes that the wind is blowing at 6 miles per hour. A unladen swallow nesting near Donnie's house flies three quarters of a mile down the road (in the direction of the wind), turns around, and returns exactly 4 minutes later. What is the airspeed of the unladen swallow? (Here, 'airspeed' is the speed that the swallow can fly in still air.)

23. In order to remove water from a flooded basement, two pumps, each rated at 40 gallons per minute, are used. After half an hour, the one pump burns out, and the second pump finishes removing the water half an hour later. How many gallons of water were removed from the basement?

24. A faucet can fill a sink in 5 minutes while a drain will empty the same sink in 8 minutes. If the faucet is turned on and the drain is left open, how long will it take to fill the sink?

25. Working together, Daniel and Donnie can clean the llama pen in 45 minutes. On his own, Daniel can clean the pen in an hour. How long does it take Donnie to clean the llama pen on his own?

26. In Exercise 32, the function $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ was used to model the cost (in dollars) to produce x PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost $\overline{C}$. Round your answer to the nearest number of systems.

27. Suppose we are in the same situation as Example 96. If the volume of the box is to be 500 cubic centimetres, use your calculator or computer to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

28. The box for the new Sasquatch-themed cereal, 'Crypt-Os', is to have a volume of 140 cubic inches. For aesthetic reasons, the height of the box needs to be 1.62 times the width of the base of the box.[2] Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.

[2] 1.62 is a crude approximation of the so-called 'Golden Ratio' $\phi = \frac{1+\sqrt{5}}{2}$.

29. Sally is Skippy's neighbour from Exercise 19 in Section 3.3. Sally also wants to plant a vegetable garden along the side of her home. She doesn't have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn't need fencing along that side.)

30. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[3]

    (a) Find an expression for the volume V of the can in terms of the height h and the base radius r.

    (b) Find an expression for the surface area S of the can in terms of the height h and the base radius r. (Hint: The top and bottom of the can are circles of radius r and the side of the can is really just a rectangle that has been bent into a cylinder.)

    (c) Using the fact that $V = 33.6$, write S as a function of r and state its applied domain.

    (d) Use your graphing calculator to find the dimensions of the can which has minimal surface area.

31. A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

32. In Exercise 72 in Section 2.3, the population of Sasquatch in Portage County was modeled by the function $P(t) = \frac{150t}{t+15}$, where $t = 0$ represents the year 1803. When were there fewer than 100 Sasquatch in Portage County?

In Exercises 33 – 38, translate the following into mathematical equations.

33. At a constant pressure, the temperature T of an ideal gas is directly proportional to its volume V. (This is Charles's Law)

34. The frequency of a wave f is inversely proportional to the wavelength of the wave[4] λ.

35. The density d of a material is directly proportional to the mass of the object m and inversely proportional to its volume V.

36. The square of the orbital period of a planet P is directly proportional to the cube of the semi-major axis of its orbit a. (This is Kepler's Third Law of Planetary Motion)

37. The drag of an object travelling through a fluid D varies jointly with the density of the fluid[5] ρ and the square of the velocity of the object ν.

38. Suppose two electric point charges, one with charge q and one with charge Q, are positioned r units apart. The electrostatic force F exerted on the charges varies directly with the product of the two charges and inversely with the square of the distance between the charges.[6] (This is Coulomb's Law)

39. According to this webpage, the frequency f of a vibrating string is given by $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ where T is the tension, μ is the linear mass[7] of the string and L is the length of the vibrating part of the string. Express this relationship using the language of variation.

40. According to the Centers for Disease Control and Prevention www.cdc.gov, a person's Body Mass Index B is directly proportional to his weight W in pounds and inversely proportional to the square of his height h in inches.

    (a) Express this relationship as a mathematical equation.

    (b) If a person who was 5 feet, 10 inches tall weighed 235 pounds had a Body Mass Index of 33.7, what is the value of the constant of proportionality?

    (c) Rewrite the mathematical equation found in part 40a to include the value of the constant found in part 40b and then find your Body Mass Index.

41. We know that the circumference of a circle varies directly with its radius with 2π as the constant of proportionality. (That is, we know $C = 2\pi r$.) With the help of your classmates, compile a list of other basic geometric relationships which can be seen as variations.

[3] According to www.dictionary.com, there are different values given for this conversion. We will stick with 33.6in^3 for this problem.
[4] The character λ is the lower case Greek letter 'lambda.'
[5] The characters ρ and ν are the lower case Greek letters 'rho' and 'nu,' respectively.
[6] Note the similarity to this formula and Newton's Law of Universal Gravitation as discussed in Example 5.
[7] Also known as the linear density. It is simply a measure of mass per unit length.

6: Function Composition and Inverses

6.1 Function Composition

Before we embark upon any further adventures with functions, we need to take some time to gather our thoughts and gain some perspective. Chapter 2 first introduced us to functions in Section 2.2. At that time, functions were specific kinds of relations - sets of points in the plane which passed the Vertical Line Test, Theorem 6. In Section 2.3, we developed the idea that functions are processes - rules which match inputs to outputs - and this gave rise to the concepts of domain and range. We spoke about how functions could be combined in Section 2.4 using the four basic arithmetic operations, took a more detailed look at their graphs in Section 2.5 and studied how their graphs behaved under certain classes of transformations in Section 2.6. In Chapter 3, we took a closer look at three families of functions: linear functions (Section 3.1), absolute value functions (which were introduced, as you may recall, as piecewise-defined linear functions in Section 3.2), and quadratic functions (Section 3.3). Linear and quadratic functions were special cases of polynomial functions, which we studied in generality in Chapter 4. One can prove (using complex numbers!) that all polynomial functions with real coefficients can be factored as products of linear and quadratic functions. Our next step was to enlarge our field (this is a really bad math pun) of study to rational functions in Chapter 5. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Chapters 3, 4, and 5 can be thought of as an exhaustive study of linear and quadratic functions and their arithmetic combinations as described in Section 2.4. We now wish to study other algebraic functions, such as $f(x) = \sqrt{x}$ and $g(x) = x^{2/3}$, and the purpose of the first two sections of this chapter is to see how these kinds of functions arise from polynomial and rational functions. To that end, we first study a new way to combine functions as defined below.

If we broaden our concept of functions to allow for complex valued coefficients, then every polynomial can be completely factored, so that every function we have studied thus far is in fact a combination of linear functions.

Definition 45 Composition of Functions

Suppose f and g are two functions. The **composite** of g with f, denoted $g \circ f$, is defined by the formula $(g \circ f)(x) = g(f(x))$, provided x is an element of the domain of f and $f(x)$ is an element of the domain of g.

The quantity $g \circ f$ is also read 'g composed with f' or, more simply 'g of f.' At its most basic level, Definition 45 tells us to obtain the formula for $(g \circ f)(x)$, we replace every occurrence of x in the formula for $g(x)$ with the formula we have for $f(x)$. If we take a step back and look at this from a procedural, 'inputs and outputs' perspective, Defintion 45 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. The result, $g(f(x))$, is the output from $g \circ f$. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. This is diagrammed abstractly in Figure 6.1.

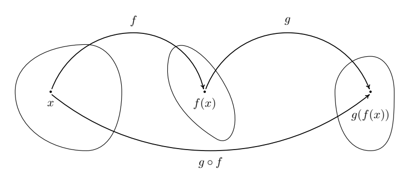


Figure 6.1: Composition of functions

In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. There are two ways to go about evaluating composite functions - 'inside out' and 'outside in' - depending on which function we replace with its formula first. Both ways are demonstrated in the following example.

Example 98 Evaluating composite functions

Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$.

Find the indicated function value for each of the following:

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

SOLUTION

1. Using Definition 45, $(g \circ f)(1) = g(f(1))$. We find $f(1) = -3$, so
$$(g \circ f)(1) = g(f(1)) = g(-3) = 2$$
2. As before, we use Definition 45 to write $(f \circ g)(1) = f(g(1))$. We find $g(1) = 0$, so
$$(f \circ g)(1) = f(g(1)) = f(0) = 0$$
3. Once more, Definition 45 tells us $(g \circ g)(6) = g(g(6))$. That is, we evaluate g at 6, then plug that result back into g. Since $g(6) = -1$,
$$(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$$

Example 99 Composing in different orders

With $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$ as in Example 98 find and simplify the composite functions $(g \circ f)(x)$ and $(f \circ g)(x)$. State the domain of each function.

SOLUTION By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

- *inside out*: We insert the expression $f(x)$ into g first to get
$$\begin{aligned}(g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) &= 2 - \sqrt{(x^2 - 4x) + 3}\\ &= 2 - \sqrt{x^2 - 4x + 3}\end{aligned}$$
Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.
- *outside in*: We use the formula for g first to get
$$\begin{aligned}(g \circ f)(x) = g(f(x)) = 2 - \sqrt{f(x) + 3} &= 2 - \sqrt{(x^2 - 4x) + 3}\\ &= 2 - \sqrt{x^2 - 4x + 3}\end{aligned}$$
We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. We accomplish this by following the rule set forth in Section 2.3, that is, we find the domain *before* we simplify. To that end, we examine $(g \circ f)(x) = 2 - \sqrt{(x^2 - 4x) + 3}$. To keep the square root happy, we solve the inequality $x^2 - 4x + 3 \geq 0$ by creating a sign diagram. If we let $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$. We obtain the sign diagram in Figure 6.2.

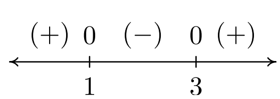


Figure 6.2: The sign diagram of $r(x) = x^2 - 4x + 3$

Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

To find $(f \circ g)(x)$, we find $f(g(x))$.

- *inside out*: We insert the expression $g(x)$ into f first to get

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) = f\left(2-\sqrt{x+3}\right)\\ &= \left(2-\sqrt{x+3}\right)^2 - 4\left(2-\sqrt{x+3}\right)\\ &= 4-4\sqrt{x+3}+\left(\sqrt{x+3}\right)^2 - 8 + 4\sqrt{x+3}\\ &= 4+x+3-8\\ &= x-1\end{aligned}$$

- *outside in*: We use the formula for $f(x)$ first to get

$$\begin{aligned}(f\circ g)(x) &= f(g(x)) = (g(x))^2 - 4\,(g(x))\\ &= \left(2-\sqrt{x+3}\right)^2 - 4\left(2-\sqrt{x+3}\right)\\ &= x-1 && \text{same algebra as before}\end{aligned}$$

Thus we get $(f\circ g)(x) = x-1$. To find the domain of $(f\circ g)$, we look to the step before we did any simplification and find $(f\circ g)(x) = \left(2-\sqrt{x+3}\right)^2 - 4\left(2-\sqrt{x+3}\right)$. To keep the square root happy, we set $x+3\geq 0$ and find our domain to be $[-3,\infty)$.

Notice that in Example 99, we found $(g\circ f)(x) \neq (f\circ g)(x)$. In Example 100 we add evidence that this is the rule, rather than the exception.

Example 100 Comparing order of composition

Find and simplify the functions $(g\circ h)(x)$ and $(h\circ g)(x)$, where we again take $g(x) = 2-\sqrt{x+3}$ and $h(x) = \dfrac{2x}{x+1}$. State the domain of each function.

Solution To find $(g\circ h)(x)$, we compute $g(h(x))$.

- *inside out*: We insert the expression $h(x)$ into g first to get

$$\begin{aligned}(g\circ h)(x) &= g(h(x)) = g\left(\frac{2x}{x+1}\right)\\ &= 2-\sqrt{\left(\frac{2x}{x+1}\right)+3}\\ &= 2-\sqrt{\frac{2x}{x+1}+\frac{3(x+1)}{x+1}} && \text{get common denominators}\\ &= 2-\sqrt{\frac{5x+3}{x+1}}\end{aligned}$$

- *outside in*: We use the formula for $g(x)$ first to get

$$\begin{aligned}(g\circ h)(x) &= g(h(x)) = 2-\sqrt{h(x)+3}\\ &= 2-\sqrt{\left(\frac{2x}{x+1}\right)+3}\\ &= 2-\sqrt{\frac{5x+3}{x+1}} && \text{get common denominators}\end{aligned}$$

To find the domain of $(g \circ h)$, we look to the step before we began to simplify:

$$(g \circ h)(x) = 2 - \sqrt{\left(\frac{2x}{x+1}\right) + 3}$$

To avoid division by zero, we need $x \neq -1$. To keep the radical happy, we need to solve

$$\frac{2x}{x+1} + 3 = \frac{5x+3}{x+1} \geq 0$$

Defining $r(x) = \dfrac{5x+3}{x+1}$, we see r is undefined at $x = -1$ and $r(x) = 0$ at $x = -\frac{3}{5}$. Our sign diagram is given in Figure 6.3.

Our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

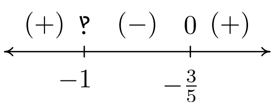


Figure 6.3: The sign diagram of $r(x) = \dfrac{5x+3}{x+1}$

Next, we find $(h \circ g)(x)$ by finding $h(g(x))$.

- *inside out*: We insert the expression $g(x)$ into h first to get

$$\begin{aligned}(h \circ g)(x) = h(g(x)) &= h\left(2 - \sqrt{x+3}\right)\\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1}\\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}\end{aligned}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned}(h \circ g)(x) = h(g(x)) &= \frac{2\left(g(x)\right)}{\left(g(x)\right) + 1}\\ &= \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1}\\ &= \frac{4 - 2\sqrt{x+3}}{3 - \sqrt{x+3}}\end{aligned}$$

To find the domain of $h \circ g$, we look to the step before any simplification:

$$(h \circ g)(x) = \frac{2\left(2 - \sqrt{x+3}\right)}{\left(2 - \sqrt{x+3}\right) + 1}$$

To keep the square root happy, we require $x + 3 \geq 0$ or $x \geq -3$. Setting the denominator equal to zero gives $\left(2 - \sqrt{x+3}\right) + 1 = 0$ or $\sqrt{x+3} = 3$. Squaring both sides gives us $x + 3 = 9$, or $x = 6$. Since $x = 6$ checks in the original equation, $\left(2 - \sqrt{x+3}\right) + 1 = 0$, we know $x = 6$ is the only zero of the denominator. Hence, the domain of $h \circ g$ is $[-3, 6) \cup (6, \infty)$.

Example 101 Composing a function with itself

Using the function $h(x) = \dfrac{2x}{x+1}$ from our previous examples, compute the function $(h \circ h)(x)$, and state its domain.

Solution To find $(h \circ h)(x)$, we substitute the function h into itself, $h(h(x))$.

- *inside out*: We insert the expression $h(x)$ into h to get

$$\begin{aligned}
(h \circ h)(x) = h(h(x)) &= h\left(\frac{2x}{x+1}\right) \\
&= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1} \\
&= \frac{\frac{4x}{x+1}}{\frac{2x}{x+1}+1} \cdot \frac{(x+1)}{(x+1)} \\
&= \frac{\frac{4x}{x+1} \cdot (x+1)}{\left(\frac{2x}{x+1}\right) \cdot (x+1) + 1 \cdot (x+1)} \\
&= \frac{\frac{4x}{\cancel{(x+1)}} \cdot \cancel{(x+1)}}{\frac{2x}{\cancel{(x+1)}} \cdot \cancel{(x+1)} + x + 1} \\
&= \frac{4x}{3x+1}
\end{aligned}$$

- *outside in*: This approach yields

$$\begin{aligned}
(h \circ h)(x) = h(h(x)) &= \frac{2(h(x))}{h(x)+1} \\
&= \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1} \\
&= \frac{4x}{3x+1} && \text{same algebra as before}
\end{aligned}$$

To find the domain of $h \circ h$, we analyze

$$(h \circ h)(x) = \frac{2\left(\frac{2x}{x+1}\right)}{\left(\frac{2x}{x+1}\right)+1}$$

To keep the denominator $x+1$ happy, we need $x \neq -1$. Setting the denominator

$$\frac{2x}{x+1} + 1 = 0$$

gives $x = -\frac{1}{3}$. Our domain is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

For our last example, we stick with the same three functions as above, but we consider two different compositions involving all three functions.

Example 102 Composing three functions

Let $f(x) = x^2 - 4x$, $g(x) = 2 - \sqrt{x+3}$, and $h(x) = \dfrac{2x}{x+1}$. Find and simplify the functions $(h \circ (g \circ f))(x)$ and $((h \circ g) \circ f)(x)$. State the domain of each function.

Solution The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from Example 99 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$. We now proceed as usual.

- *inside out*: We insert the expression $(g \circ f)(x)$ into h first to get

$$\begin{aligned} h \circ (g \circ f))(x) = h((g \circ f)(x)) &= h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

- *outside in*: We use the formula for $h(x)$ first to get

$$\begin{aligned} (h \circ (g \circ f))(x) = h((g \circ f)(x)) &= \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\ &= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

To find the domain of $(h \circ (g \circ f))$, we look at the step before we began to simplify,

$$(h \circ (g \circ f))(x) = \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1}$$

For the square root, we need $x^2 - 4x + 3 \geq 0$, which we determined in number 1 to be $(-\infty, 1] \cup [3, \infty)$. Next, we set the denominator to zero and solve: $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. To solve $x^2 - 4x - 6 = 0$, we use the quadratic formula and get $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1 = 0$. Hence we must exclude these numbers from the domain of $h \circ (g \circ f)$. Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From Example 100, we have

$$(h \circ g)(x) = \frac{4 - 2\sqrt{x + 3}}{3 - \sqrt{x + 3}}$$

We now proceed as before.

- *inside out*: We insert the expression $f(x)$ into $h \circ g$ first to get

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)\left(x^2 - 4x\right) \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}} \end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(x)$ first to get

$$\begin{aligned}((h \circ g) \circ f)(x) = (h \circ g)(f(x)) &= \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}} \\ &= \frac{4 - 2\sqrt{(x^2 - 4x) + 3}}{3 - \sqrt{(x^2 - 4x) + 3}} \\ &= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}\end{aligned}$$

We note that the formula for $((h \circ g) \circ f)(x)$ before simplification is identical to that of $(h \circ (g \circ f))(x)$ before we simplified it. Hence, the two functions have the same domain, $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

It should be clear from Examples 99 and 100 that, in general, when you compose two functions, such as f and g above, the order matters. We found that the functions $f \circ g$ and $g \circ f$ were different as were $g \circ h$ and $h \circ g$. Thinking of functions as processes, this isn't all that surprising. If we think of one process as putting on our socks, and the other as putting on our shoes, the order in which we do these two tasks does matter. Also note the importance of finding the domain of the composite function *before* simplifying. For instance, the domain of $f \circ g$ is much different than its simplified formula would indicate. Composing a function with itself, as in the case of finding $(g \circ g)(6)$ and $(h \circ h)(x)$, may seem odd. Looking at this from a procedural perspective, however, this merely indicates performing a task h and then doing it again - like setting the washing machine to do a 'double rinse'. Composing a function with itself is called 'iterating' the function, and we could easily spend an entire course on just that. The last two problems in Example 99 serve to demonstrate the **associative** property of functions. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. This property as well as another important property are listed in the theorem below.

This shows us function composition isn't **commutative**. An example of an operation we perform on two functions which is commutative is function addition, which we defined in Section 2.4. In other words, the functions $f + g$ and $g + f$ are always equal. Which of the remaining operations on functions we have discussed are commutative?

Theorem 36 Properties of Function Composition

Suppose f, g, and h are functions.

- $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- If I is defined as $I(x) = x$ for all real numbers x, then $I \circ f = f \circ I = f$.

By repeated applications of Definition 45, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 45. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise 73 in Section 3.1. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(x)$. This establishes that we have an identity for function composition much in the same way the real number 1 is an identity for real number multiplication. That is, just as for any real number x, $1 \cdot x = x \cdot 1 = x$, we have for any function f, $I \circ f = f \circ I = f$. We shall see the concept of an identity take on great significance in the next section. Out in the wild, function composition is often used to relate two quantities which may not be directly related, but have a variable in common, as illustrated in our next example.

Example 103 Inflating a sphere

The surface area S of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi\left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'middle man' r.

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates.

Example 104 Decomposing functions

Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

1. $F(x) = |3x - 1|$
2. $G(x) = \dfrac{2}{x^2 + 1}$
3. $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$

When we get to Calculus, we'll see that being able to decompose a complicated function into simpler pieces is a necessary skill for applying the Chain Rule for derivatives.

Solution There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 45, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x - 1|$ from an 'inside versus outside' perspective, we can think of $3x - 1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x - 1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x - 1| = F(x)$, so we are done.
2. We attack deconstructing G from an operational approach. Given an input x, the first step is to square x, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of three functions: f, g and h. Our first function, f, is the function that squares its input, $f(x) = x^2$. The next function is the function that adds 1

to its input, $g(x) = x + 1$. Our last function takes its input and divides it into 2, $h(x) = \frac{2}{x}$. The claim is that $G = h \circ g \circ f$. We find

$$(h \circ g \circ f)(x) = h(g(f(x))) = h(g\left(x^2\right)) = h\left(x^2 + 1\right) = \frac{2}{x^2 + 1} = G(x),$$

so we are done.

3. If we look $H(x) = \dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{x}$ is a simple piece of the larger function. If we define $f(x) = \sqrt{x}$, we have $H(x) = \frac{f(x)+1}{f(x)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(x)$ by looking at what is being done to $f(x)$. We take $g(x) = \frac{x+1}{x-1}$, so

$$(g \circ f)(x) = g(f(x)) = \frac{f(x) + 1}{f(x) - 1} = \frac{\sqrt{x} + 1}{\sqrt{x} - 1} = H(x),$$

as required.

Exercises 6.1

Problems

In Exercises 1 – 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(g \circ f)(-3)$
- $(f \circ g)(-1)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(2)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(x) = 2x + 1$
2. $f(x) = 4 - x$, $g(x) = 1 - x^2$
3. $f(x) = 4 - 3x$, $g(x) = |x|$
4. $f(x) = |x - 1|$, $g(x) = x^2 - 5$
5. $f(x) = 4x + 5$, $g(x) = \sqrt{x}$
6. $f(x) = \sqrt{3 - x}$, $g(x) = x^2 + 1$
7. $f(x) = 6 - x - x^2$, $g(x) = x\sqrt{x + 10}$
8. $f(x) = \sqrt[3]{x + 1}$, $g(x) = 4x^2 - x$
9. $f(x) = \dfrac{3}{1 - x}$, $g(x) = \dfrac{4x}{x^2 + 1}$
10. $f(x) = \dfrac{x}{x + 5}$, $g(x) = \dfrac{2}{7 - x^2}$
11. $f(x) = \dfrac{2x}{5 - x^2}$, $g(x) = \sqrt{4x + 1}$
12. $f(x) = \sqrt{2x + 5}$, $g(x) = \dfrac{10x}{x^2 + 1}$

In Exercises 13 – 24, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(x)$
- $(f \circ f)(x)$

13. $f(x) = 2x + 3$, $g(x) = x^2 - 9$
14. $f(x) = x^2 - x + 1$, $g(x) = 3x - 5$
15. $f(x) = x^2 - 4$, $g(x) = |x|$
16. $f(x) = 3x - 5$, $g(x) = \sqrt{x}$
17. $f(x) = |x + 1|$, $g(x) = \sqrt{x}$
18. $f(x) = 3 - x^2$, $g(x) = \sqrt{x + 1}$
19. $f(x) = |x|$, $g(x) = \sqrt{4 - x}$
20. $f(x) = x^2 - x - 1$, $g(x) = \sqrt{x - 5}$
21. $f(x) = 3x - 1$, $g(x) = \dfrac{1}{x + 3}$
22. $f(x) = \dfrac{3x}{x - 1}$, $g(x) = \dfrac{x}{x - 3}$
23. $f(x) = \dfrac{x}{2x + 1}$, $g(x) = \dfrac{2x + 1}{x}$
24. $f(x) = \dfrac{2x}{x^2 - 4}$, $g(x) = \sqrt{1 - x}$

In Exercises 25 – 31, use $f(x) = -2x$, $g(x) = \sqrt{x}$ and $h(x) = |x|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

25. $(h \circ g \circ f)(x)$
26. $(h \circ f \circ g)(x)$
27. $(g \circ f \circ h)(x)$
28. $(g \circ h \circ f)(x)$
29. $(f \circ h \circ g)(x)$
30. $(f \circ g \circ h)(x)$
31. $f(x) = |x|$, $g(x) = \sqrt{4 - x}$

In Exercises 32 – 41, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

32. $p(x) = (2x + 3)^3$
33. $P(x) = \left(x^2 - x + 1\right)^5$
34. $h(x) = \sqrt{2x - 1}$
35. $H(x) = |7 - 3x|$
36. $r(x) = \dfrac{2}{5x + 1}$
37. $R(x) = \dfrac{7}{x^2 - 1}$
38. $q(x) = \dfrac{|x| + 1}{|x| - 1}$
39. $Q(x) = \dfrac{2x^3 + 1}{x^3 - 1}$
40. $v(x) = \dfrac{2x + 1}{3 - 4x}$

41. $w(x) = \dfrac{x^2}{x^4+1}$

42. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

43. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2} - 4$?

44. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3 + 1$? What is the proper order of composition?

In Exercises 45 – 56, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Find the value if it exists.

45. $(f \circ g)(3)$

46. $f(g(-1))$

47. $(f \circ f)(0)$

48. $(f \circ g)(-3)$

49. $(g \circ f)(3)$

50. $g(f(-3))$

51. $(g \circ g)(-2)$

52. $(g \circ f)(-2)$

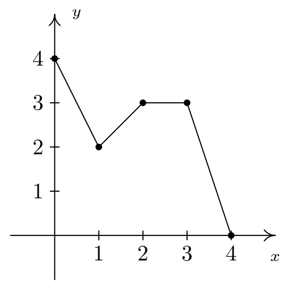
53. $g(f(g(0)))$

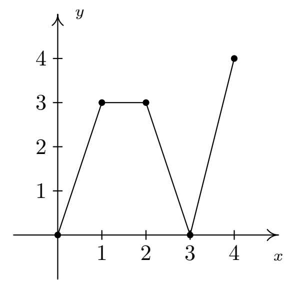
54. $f(f(f(-1)))$

55. $f(f(f(f(f(1)))))$

56. $(\underbrace{g \circ g \circ \cdots \circ g}_{n \text{ times}})(0)$

In Exercises 57 – 62, use the graphs of $y = f(x)$ and $y = g(x)$ below to find the function value.





57. $(g \circ f)(1)$

58. $(f \circ g)(3)$

59. $(g \circ f)(2)$

60. $(f \circ g)(0)$

61. $(f \circ f)(1)$

62. $(g \circ g)(1)$

63. The volume V of a cube is a function of its side length x. Let's assume that $x = t + 1$ is also a function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

64. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.

    (b) Find and simplify $(R \circ x)(t)$. What does this represent?

    (c) What is the revenue per hour at noon?

65. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

6.2 Inverse Functions

Thinking of a function as a process like we did in Section 2.3, in this section we seek another function which might reverse that process. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like cooking a steak) are not. We start by discussing a very basic function which is reversible, $f(x) = 3x + 4$. Thinking of f as a process, we start with an input x and apply two steps, as we saw in Section 2.3

1. multiply by 3
2. add 4

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the real-world reversible two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes, and then we take off the socks. In much the same way, the function g should undo the second step of f first. That is, the function g should

1. *subtract* 4
2. *divide* by 3

Following this procedure, we get $g(x) = \frac{x-4}{3}$. Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3. Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and put that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 6.1, the statements $g(f(x)) = x$ and $f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively. Abstractly, we can visualize the relationship between f and g in Figure 6.4.

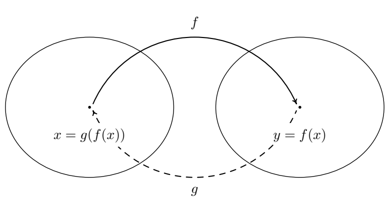


Figure 6.4: The relationship between a function and its inverse

The main idea to get from Figure 6.4 is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section.

Definition 46 Inverse of a function

Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f **and**
2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

We now formalize the concept that inverse functions exchange inputs and outputs.

Theorem 37 Properties of Inverse Functions

Suppose f and g are inverse functions.

- The range (recall this is the set of all outputs of a function) of f is the domain of g and the domain of f is the range of g
- $f(a) = b$ if and only if $g(b) = a$
- (a, b) is on the graph of f if and only if (b, a) is on the graph of g

Theorem 37 is a consequence of Definition 46 and the Fundamental Graphing Principle for Functions. We note the third property in Theorem 37 tells us that the graphs of inverse functions are reflections about the line $y = x$. For a proof of this, see Example 11 in Section 1.3 and Exercise 72 in Section 3.1. For example, we plot the inverse functions $f(x) = 3x + 4$ and $g(x) = \dfrac{x-4}{3}$ in Figure 6.5.

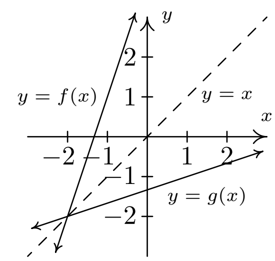


Figure 6.5: Reflecting $y = f(x)$ across $y = x$ to obtain $y = g(x)$

If we abstract one step further, we can express the sentiment in Definition 46 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted to the domain of f and I_2 is the identity function restricted to the domain of g. In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 36, we can show that function inverses are unique. (In other words, invertible functions have exactly one inverse.) Suppose g and h are both inverses of a function f. By Theorem 37, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required. (It is an excellent exercise to explain each step in this string of equalities.) We summarize the discussion of the last two paragraphs in the following theorem.

The identity function I, which was introduced in Section 3.1 and mentioned in Theorem 36, has a domain of all real numbers. Since the domains of f and g may not be all real numbers, we need the restrictions listed here.

Theorem 38 Uniqueness of Inverse Functions and Their Graphs

Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read f-inverse)
- The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses.

The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely *not* the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$. Why does this confusing notation persist? As we mentioned in Section 6.1, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same

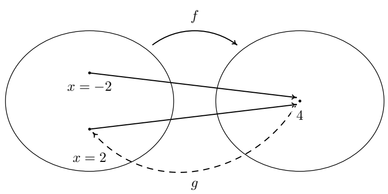
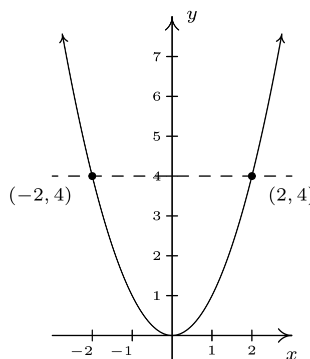
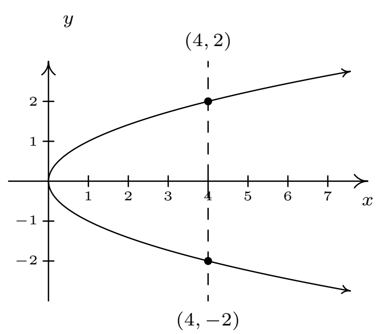


Figure 6.6: The function $f(x) = x^2$ is not invertible



(a) $y = f(x) = x^2$



(b) $y = f^{-1}(x)$?

Figure 6.7: Reflecting $y = x^2$ across the line $y = x$ does not produce a function

way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Let's turn our attention to the function $f(x) = x^2$. Is f invertible? A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. Checking the composition yields $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x for all x in the domain $(-\infty, \infty)$. For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$, which means g failed to return the input -2 from its output 4. What g did, however, is match the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. This issue is presented schematically in Figure 6.6.

We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$. (By definition, a function matches a real number with exactly one other real number.) From a graphical standpoint, we know that if $y = f^{-1}(x)$ exists, its graph can be obtained by reflecting $y = x^2$ about the line $y = x$, in accordance with Theorem 38. Doing so takes the graph in Figure 6.7 (a) to the one in Figure 6.7 (b).

We see that the line $x = 4$ intersects the graph of the supposed inverse twice - meaning the graph fails the Vertical Line Test, Theorem 6, and as such, does not represent y as a function of x. The vertical line $x = 4$ on the graph on the right corresponds to the *horizontal line* $y = 4$ on the graph of $y = f(x)$. The fact that the horizontal line $y = 4$ intersects the graph of f twice means two *different* inputs, namely $x = -2$ and $x = 2$, are matched with the *same* output, 4, which is the cause of all of the trouble. In general, for a function to have an inverse, *different* inputs must go to *different* outputs, or else we will run into the same problem we did with $f(x) = x^2$. We give this property a name.

Definition 47 One-to-one function

A function f is said to be **one-to-one** if f matches different inputs to different outputs. Equivalently, f is one-to-one if and only if whenever $f(c) = f(d)$, then $c = d$.

Graphically, we detect one-to-one functions using the test below.

Theorem 39 The Horizontal Line Test

A function f is one-to-one if and only if no horizontal line intersects the graph of f more than once.

We say that the graph of a function **passes** the Horizontal Line Test if no horizontal line intersects the graph more than once; otherwise, we say the graph of the function **fails** the Horizontal Line Test. We have argued that if f is invertible, then f must be one-to-one, otherwise the graph given by reflecting the graph of $y = f(x)$ about the line $y = x$ will fail the Vertical Line Test. It turns out that being one-to-one is also enough to guarantee invertibility. To see this, we think of f as the set of ordered pairs which constitute its graph. If switching the x- and y-coordinates of the points results in a function, then f is invertible and we have found f^{-1}. This is precisely what the Horizontal Line Test does for us: it checks to see whether or not a set of points describes x as a function of y. We summarize these results below.

Theorem 40 Equivalent Conditions for Invertibility

Suppose f is a function. The following statements are equivalent.

- f is invertible
- f is one-to-one
- The graph of f passes the Horizontal Line Test

We put this result to work in the next example.

Example 105 Finding one-to-one functions

Determine if the following functions are one-to-one in two ways: (a) analytically using Definition 47 and (b) graphically using the Horizontal Line Test.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$
3. $h(x) = x^2 - 2x + 4$
4. $F = \{(-1,1), (0,2), (2,1)\}$

Solution

1. (a) To determine if f is one-to-one analytically, we assume $f(c) = f(d)$ and attempt to deduce that $c = d$.

$$\begin{aligned} f(c) &= f(d) \\ \frac{1-2c}{5} &= \frac{1-2d}{5} \\ 1-2c &= 1-2d \\ -2c &= -2d \\ c &= d \checkmark \end{aligned}$$

Hence, f is one-to-one.

(b) To check if f is one-to-one graphically, we look to see if the graph of $y = f(x)$ passes the Horizontal Line Test. We have that f is a non-constant linear function, which means its graph is a non-horizontal line. Thus the graph of f passes the Horizontal Line Test: see Figure 6.8.

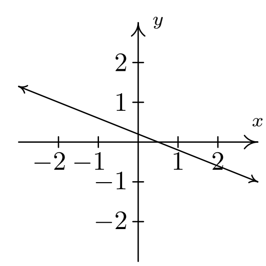


Figure 6.8: The function f is one-to-one

2. (a) We begin with the assumption that $g(c) = g(d)$ and try to show $c = d$.

$$\begin{aligned} g(c) &= g(d) \\ \frac{2c}{1-c} &= \frac{2d}{1-d} \\ 2c(1-d) &= 2d(1-c) \\ 2c - 2cd &= 2d - 2dc \\ 2c &= 2d \\ c &= d \checkmark \end{aligned}$$

We have shown that g is one-to-one.

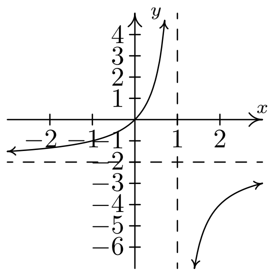


Figure 6.9: The function g is one-to-one

(b) We can graph g using the six step procedure outlined in Section 5.2. We get the sole intercept at $(0,0)$, a vertical asymptote $x = 1$ and a horizontal asymptote (which the graph never crosses) $y = -2$. We see from that the graph of g in Figure 6.9 that g passes the Horizontal Line Test.

3. (a) We begin with $h(c) = h(d)$. As we work our way through the problem, we encounter a nonlinear equation. We move the non-zero terms to the left, leave a 0 on the right and factor accordingly.

$$\begin{aligned} h(c) &= h(d) \\ c^2 - 2c + 4 &= d^2 - 2d + 4 \\ c^2 - 2c &= d^2 - 2d \\ c^2 - d^2 - 2c + 2d &= 0 \\ (c+d)(c-d) - 2(c-d) &= 0 \\ (c-d)((c+d)-2) &= 0 \quad \text{factor by grouping} \\ c - d = 0 \quad &\text{or} \quad c + d - 2 = 0 \\ c = d \quad &\text{or} \quad c = 2 - d \end{aligned}$$

We get $c = d$ as one possibility, but we also get the possibility that $c = 2 - d$. This suggests that f may not be one-to-one. Taking $d = 0$, we get $c = 0$ or $c = 2$. With $h(0) = 4$ and $h(2) = 4$, we have produced two different inputs with the same output meaning h is not one-to-one.

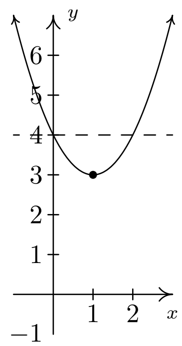


Figure 6.10: The function h is not one-to-one

(b) We note that h is a quadratic function and we graph $y = h(x)$ using the techniques presented in Section 3.3. The vertex is $(1,3)$ and the parabola opens upwards. We see immediately from the graph in Figure 6.10 that h is not one-to-one, since there are several horizontal lines which cross the graph more than once.

4. (a) The function F is given to us as a set of ordered pairs. The condition $F(c) = F(d)$ means the outputs from the function (the y-coordinates of the ordered pairs) are the same. We see that the points $(-1,1)$ and $(2,1)$ are both elements of F with $F(-1) = 1$ and $F(2) = 1$. Since $-1 \neq 2$, we have established that F is *not* one-to-one.

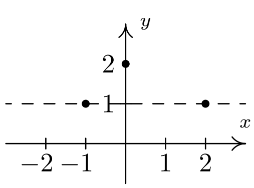


Figure 6.11: The function F is not one-to-one

(b) Graphically, we see in Figure 6.11 that the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.

We have shown that the functions f and g in Example 105 are one-to-one. This means they are invertible, so it is natural to wonder what $f^{-1}(x)$ and $g^{-1}(x)$ would be. For $f(x) = \frac{1-2x}{5}$, we can think our way through the inverse since there is only one occurrence of x. We can track step-by-step what is done to x and reverse those steps as we did at the beginning of the chapter. The function $g(x) = \frac{2x}{1-x}$ is a bit trickier since x occurs in two places. When one evaluates $g(x)$ for a specific value of x, which is first, the $2x$ or the $1 - x$? We can imagine functions more complicated than these so we need to develop a general methodology to attack this problem. Theorem 37 tells us equation $y = f^{-1}(x)$ is equivalent to $f(y) = x$ and this is the basis of our algorithm.

Key Idea 25 Steps for finding the Inverse of a One-to-one Function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

Note that we could have simply written 'Solve $x = f(y)$ for y' and be done with it. The act of interchanging the x and y is there to remind us that we are finding the inverse function by switching the inputs and outputs.

Example 106 Computing inverse functions

Find the inverse of the following one-to-one functions. Check your answers analytically using function composition and graphically.

1. $f(x) = \dfrac{1-2x}{5}$
2. $g(x) = \dfrac{2x}{1-x}$

Solution

1. As we mentioned earlier, it is possible to think our way through the inverse of f by recording the steps we apply to x and the order in which we apply them and then reversing those steps in the reverse order. We encourage the reader to do this. We, on the other hand, will practice the algorithm. We write $y = f(x)$ and proceed to switch x and y

$$\begin{aligned} y &= f(x) \\ y &= \frac{1-2x}{5} \\ x &= \frac{1-2y}{5} && \text{switch } x \text{ and } y \\ 5x &= 1-2y \\ 5x-1 &= -2y \\ \frac{5x-1}{-2} &= y \\ y &= -\frac{5}{2}x + \frac{1}{2} \end{aligned}$$

We have $f^{-1}(x) = -\frac{5}{2}x + \frac{1}{2}$. To check this answer analytically, we first check that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f, which is all real numbers.

$$\begin{aligned} \left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= -\frac{5}{2}f(x) + \frac{1}{2} \\ &= -\frac{5}{2}\left(\frac{1-2x}{5}\right) + \frac{1}{2} \\ &= -\frac{1}{2}(1-2x) + \frac{1}{2} \\ &= -\frac{1}{2} + x + \frac{1}{2} \\ &= x \checkmark \end{aligned}$$

We now check that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the range of f which is also all real numbers. (Recall that the domain of f^{-1}) is the range of f.)

$$\begin{aligned}\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x)) &= \frac{1 - 2f^{-1}(x)}{5} \\ &= \frac{1 - 2\left(-\frac{5}{2}x + \frac{1}{2}\right)}{5} = \frac{1 + 5x - 1}{5} \\ &= \frac{5x}{5} = x \checkmark\end{aligned}$$

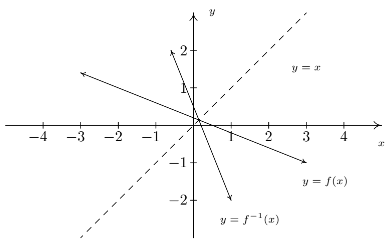


Figure 6.12: The graphs of f and f^{-1} from Example 106

To check our answer graphically, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes in Figure 6.12. They appear to be reflections across the line $y = x$.

2. To find $g^{-1}(x)$, we start with $y = g(x)$. We note that the domain of g is $(-\infty, 1) \cup (1, \infty)$.

$$\begin{aligned} y &= g(x)\frac{2x}{1-x} \\ x &= \frac{2y}{1-y} && \text{switch } x \text{ and } y \\ x(1-y) &= 2y \\ x - xy &= 2y \\ x &= xy + 2y = y(x+2) && \text{factor} \\ y &= \frac{x}{x+2}\end{aligned}$$

We obtain $g^{-1}(x) = \dfrac{x}{x+2}$. To check this analytically, we first check $\left(g^{-1} \circ g\right)(x) = x$ for all x in the domain of g, that is, for all $x \neq 1$.

$$\begin{aligned}\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) &= g^{-1}\left(\frac{2x}{1-x}\right) \\ &= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right) + 2} \\ &= \frac{\left(\dfrac{2x}{1-x}\right)}{\left(\dfrac{2x}{1-x}\right) + 2} \cdot \frac{(1-x)}{(1-x)} && \text{clear denominators} \\ &= \frac{2x}{2x + 2(1-x)} = \frac{2x}{2x + 2 - 2x} \\ &= \frac{2x}{2} = x \checkmark\end{aligned}$$

Next, we check $g\left(g^{-1}(x)\right) = x$ for all x in the range of g. From the graph of g in Example 105, we have that the range of g is $(-\infty, -2) \cup (-2, \infty)$.

This matches the domain we get from the formula $g^{-1}(x) = \frac{x}{x+2}$, as it should.

$$\begin{aligned}
\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\frac{x}{x+2}\right) & \\
= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} & \\
= \frac{2\left(\frac{x}{x+2}\right)}{1-\left(\frac{x}{x+2}\right)} \cdot \frac{(x+2)}{(x+2)} & \qquad \text{clear denominators} \\
= \frac{2x}{(x+2)-x} = \frac{2x}{2} & \\
= x \checkmark &
\end{aligned}$$

Graphing $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes is busy, but we can see the symmetric relationship if we thicken the curve for $y = g^{-1}(x)$. Note that the vertical asymptote $x = 1$ of the graph of g corresponds to the horizontal asymptote $y = 1$ of the graph of g^{-1}, as it should since x and y are switched. Similarly, the horizontal asymptote $y = -2$ of the graph of g corresponds to the vertical asymptote $x = -2$ of the graph of g^{-1}. See Figure 6.13

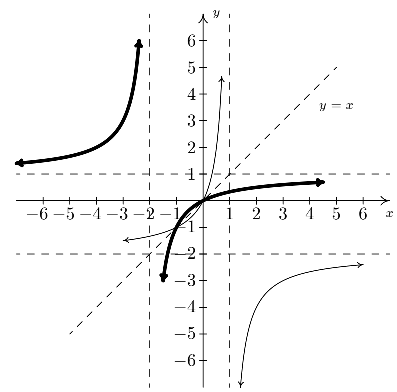


Figure 6.13: The graphs of g and g^{-1} from Example 106

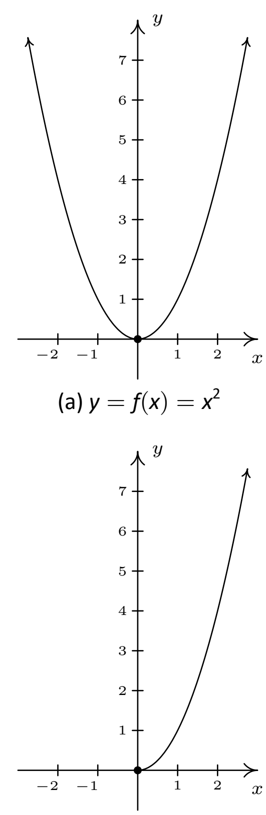
We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible. However, if we restrict the domain of f, we can produce a new function g which is one-to-one. If we define $g(x) = x^2, x \geq 0$, then we have the graph in Figure 6.14 (b).

The graph of g passes the Horizontal Line Test. To find an inverse of g, we proceed as usual

$$\begin{aligned}
y &= g(x) \\
y &= x^2, \;\; x \geq 0 \\
x &= y^2, \;\; y \geq 0 \quad \text{switch } x \text{ and } y \\
y &= \pm\sqrt{x} \\
y &= \sqrt{x} \qquad\qquad \text{since } y \geq 0
\end{aligned}$$

We get $g^{-1}(x) = \sqrt{x}$. At first it looks like we'll run into the same trouble as before, but when we check the composition, the domain restriction on g saves the day. We get $\left(g^{-1} \circ g\right)(x) = g^{-1}(g(x)) = g^{-1}\left(x^2\right) = \sqrt{x^2} = |x| = x$, since $x \geq 0$. Checking $\left(g \circ g^{-1}\right)(x) = g\left(g^{-1}(x)\right) = g\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^2 = x$. Graphing g and g^{-1} on the same set of axes in Figure 6.15 shows that they are reflections about the line $y = x$.

Our next example continues the theme of domain restriction.



(a) $y = f(x) = x^2$

(b) $y = g(x) = x^2, x \geq 0$

Figure 6.14: Restricting the domain of $f(x) = x^2$

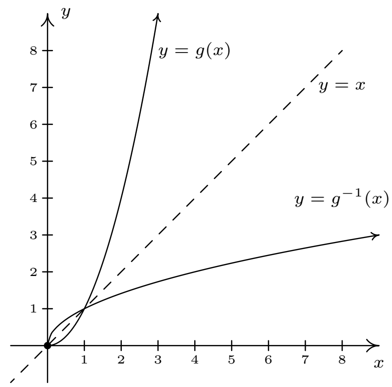


Figure 6.15: The restricted function g and its inverse

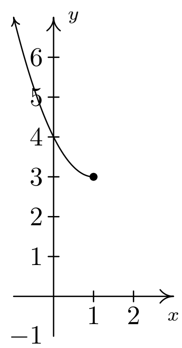


Figure 6.16: $y = x^2 - 2x + 4$, for $x \leq 1$

Example 107 Inverting restricted functions

Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(x) = \sqrt{x+2} - 1$

Solution

1. The function j is a restriction of the function h from Example 105. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. We see in Figure 6.16 that the graph of j passes the Horizontal Line Test and thus j is invertible.

   We now use our algorithm to find $j^{-1}(x)$. (Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.)

$$\begin{array}{rcll} y & = & j(x) & \\ y & = & x^2 - 2x + 4, \;\; x \leq 1 & \\ x & = & y^2 - 2y + 4, \;\; y \leq 1 & \text{switch } x \text{ and } y \\ 0 & = & y^2 - 2y + 4 - x & \\ y & = & \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} & \text{quadratic formula, } c = 4 - x \\ y & = & \dfrac{2 \pm \sqrt{4x - 12}}{2} & \\ y & = & \dfrac{2 \pm \sqrt{4(x-3)}}{2} & \\ y & = & \dfrac{2 \pm 2\sqrt{x-3}}{2} & \\ y & = & \dfrac{2\left(1 \pm \sqrt{x-3}\right)}{2} & \\ y & = & 1 \pm \sqrt{x-3} & \\ y & = & 1 - \sqrt{x-3} & \text{since } y \leq 1. \end{array}$$

   We have $j^{-1}(x) = 1 - \sqrt{x-3}$. When we simplify $\left(j^{-1} \circ j\right)(x)$, we need to remember that the domain of j is $x \leq 1$.

$$\begin{array}{rcll} \left(j^{-1} \circ j\right)(x) & = & j^{-1}(j(x)) & \\ & = & j^{-1}\left(x^2 - 2x + 4\right), \;\; x \leq 1 & \\ & = & 1 - \sqrt{(x^2 - 2x + 4) - 3} & \\ & = & 1 - \sqrt{x^2 - 2x + 1} & \\ & = & 1 - \sqrt{(x-1)^2} & \\ & = & 1 - |x - 1| & \\ & = & 1 - (-(x-1)) & \text{since } x \leq 1 \\ & = & x \checkmark & \end{array}$$

   Checking $j \circ j^{-1}$, we get

$$\begin{aligned}\left(j\circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) = j\left(1-\sqrt{x-3}\right)\\ &= \left(1-\sqrt{x-3}\right)^2 - 2\left(1-\sqrt{x-3}\right)+4\\ &= 1-2\sqrt{x-3}+\left(\sqrt{x-3}\right)^2-2+2\sqrt{x-3}+4\\ &= 3+x-3 = x\ \checkmark\end{aligned}$$

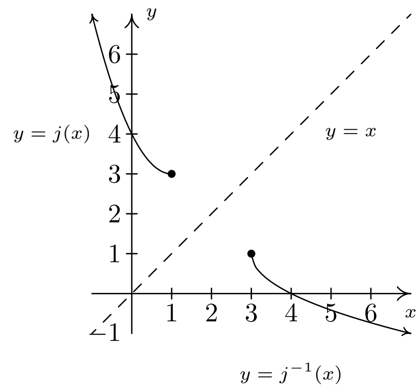


Figure 6.17: The graphs of j and j^{-1} from Example 107

Using what we know from Section 2.6, we graph $y = j^{-1}(x)$ and $y = j(x)$ in Figure 6.17.

2. We graph $y = k(x) = \sqrt{x+2}-1$ in Figure 6.18 using what we learned in Section 2.6 and see k is one-to-one.

   We now try to find k^{-1}.

$$\begin{aligned} y &= k(x)\\ y &= \sqrt{x+2}-1\\ x &= \sqrt{y+2}-1 \quad \text{switch } x \text{ and } y\\ x+1 &= \sqrt{y+2}\\ (x+1)^2 &= \left(\sqrt{y+2}\right)^2\\ x^2+2x+1 &= y+2\\ y &= x^2+2x-1\end{aligned}$$

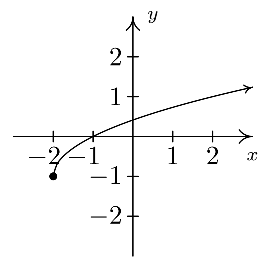


Figure 6.18: $y = \sqrt{x+2}-1$

We have $k^{-1}(x) = x^2+2x-1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted. Theorem 37 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1,\infty)$, which means we restrict the domain of k^{-1} to $x \geq -1$. We now check that this works in our compositions.

$$\begin{aligned}\left(k^{-1}\circ k\right)(x) &= k^{-1}(k(x))\\ &= k^{-1}\left(\sqrt{x+2}-1\right),\ \ x\geq -2\\ &= \left(\sqrt{x+2}-1\right)^2+2\left(\sqrt{x+2}-1\right)-1\\ &= \left(\sqrt{x+2}\right)^2-2\sqrt{x+2}+1+2\sqrt{x+2}-2-1\\ &= x+2-2\\ &= x\ \checkmark\end{aligned}$$

and

$$\begin{aligned}\left(k\circ k^{-1}\right)(x) &= k\left(x^2+2x-1\right)\ \ x\geq -1\\ &= \sqrt{(x^2+2x-1)+2}-1\\ &= \sqrt{x^2+2x+1}-1\\ &= \sqrt{(x+1)^2}-1\\ &= |x+1|-1\\ &= x+1-1 \qquad \text{since } x\geq -1\\ &= x\ \checkmark\end{aligned}$$

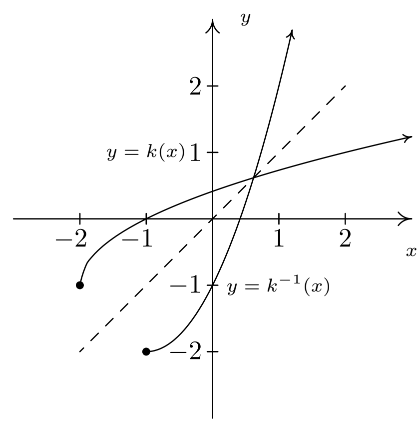


Figure 6.19: The graphs of k and k^{-1} from Example 107

Graphically, everything checks out as well in Figure 6.19, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola.

Our last example of the section gives an application of inverse functions.

Example 108 Inverting a price function

Recall from Section 3.1 that the price-demand equation for the PortaBoy game system is $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$, where x represents the number of systems sold weekly and p is the price per system in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section 3.3 that the weekly profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example 54.

Solution

1. We leave to the reader to show the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.
2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the weekly sales and furnished the price per system as the output, p^{-1} takes the price per system and returns the weekly sales as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in a week if the price is set at $\$220$ per system.
3. We compute $\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right) = -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$. After a hefty amount of Elementary Algebra, (it is good review to actually do this!) we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$. To understand what this means, recall that the original profit function P gave us the weekly profit as a function of the weekly sales. The function p^{-1} gives us the weekly sales as a function of the price. Hence, $P \circ p^{-1}$ takes as its input a price. The function p^{-1} returns the weekly sales, which in turn is fed into P to return the weekly profit. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the weekly profit (in dollars) as a function of the price per system, x, using the weekly sales $p^{-1}(x)$ as the 'middle man'.
4. We know from Section 3.3 that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation 15, $x = -\frac{b}{2a} = 165$. Hence, weekly profit is maximized if we set the price at $\$165$ per system. Comparing this with our answer from Example 54, there is a slight discrepancy to the tune of $\$0.50$. We leave it to the reader to balance the books appropriately.

Exercises 6.2

Problems

In Exercises 1 – 20, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify that the range of f is the domain of f^{-1} and vice-versa.

1. $f(x) = 6x - 2$
2. $f(x) = 42 - x$
3. $f(x) = \dfrac{x-2}{3} + 4$
4. $f(x) = 1 - \dfrac{4+3x}{5}$
5. $f(x) = \sqrt{3x-1} + 5$
6. $f(x) = 2 - \sqrt{x-5}$
7. $f(x) = 3\sqrt{x-1} - 4$
8. $f(x) = 1 - 2\sqrt{2x+5}$
9. $f(x) = \sqrt[5]{3x-1}$
10. $f(x) = 3 - \sqrt[3]{x-2}$
11. $f(x) = x^2 - 10x,\ x \geq 5$
12. $f(x) = 3(x+4)^2 - 5,\ x \leq -4$
13. $f(x) = x^2 - 6x + 5,\ x \leq 3$
14. $f(x) = 4x^2 + 4x + 1,\ x < -1$
15. $f(x) = \dfrac{3}{4-x}$
16. $f(x) = \dfrac{x}{1-3x}$
17. $f(x) = \dfrac{2x-1}{3x+4}$
18. $f(x) = \dfrac{4x+2}{3x-6}$
19. $f(x) = \dfrac{-3x-2}{x+3}$
20. $f(x) = \dfrac{x-2}{2x-1}$

With help from your classmates, find the inverses of the functions in Exercises 21 – 24.

21. $f(x) = ax + b,\ a \neq 0$
22. $f(x) = a\sqrt{x-h} + k,\ a \neq 0, x \geq h$
23. $f(x) = ax^2 + bx + c$ where $a \neq 0,\ x \geq -\dfrac{b}{2a}$.
24. $f(x) = \dfrac{ax+b}{cx+d}$, (See Exercise 33 below.)
25. In Example 29, the price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.
    (a) Find $p^{-1}(x)$ and state its domain.
    (b) Find and interpret $p^{-1}(105)$.
    (c) In Example 29, we determined that the profit (in dollars) made from producing and selling x dOpis per week is $P(x) = -15x^2 + 350x - 2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?
26. Show that the Fahrenheit to Celsius conversion function found in Exercise 35 in Section 3.1 is invertible and that its inverse is the Celsius to Fahrenheit conversion function.
27. Analytically show that the function $f(x) = x^3 + 3x + 1$ is one-to-one. Since finding a formula for its inverse is beyond the scope of this textbook, use Theorem 37 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$.
28. Let $f(x) = \dfrac{2x}{x^2-1}$. Using the techniques in Section 5.2, graph $y = f(x)$. Verify that f is one-to-one on the interval $(-1, 1)$. Use the procedure outlined on Page 241 and your graphing calculator to find the formula for $f^{-1}(x)$. Note that since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.
29. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.
30. If f is odd and invertible, prove that f^{-1} is also odd.
31. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.
32. What graphical feature must a function f possess for it to be its own inverse?
33. What conditions must you place on the values of a, b, c and d in Exercise 24 in order to guarantee that the function is invertible?

6.3 Algebraic Functions

This section serves as a watershed for functions which are combinations of polynomial, and more generally, rational functions, with the operations of radicals. It is business of Calculus to discuss these functions in all the detail they demand so our aim in this section is to help shore up the requisite skills needed so that the reader can answer Calculus's call when the time comes. We briefly recall the definition and some of the basic properties of radicals. Although we discussed imaginary numbers in Section 4.4, we restrict our attention to real numbers in this section. See the epilogue on page 189 for more details. Recall that we defined the principal n^{th} root in Definition 15. We repeat the definition here for convenience.

When n is even, it is necessary to specify that the principal n^{th} root is non-negative for it to be uniquely defined. For example, both $x = -2$ and $x = 2$ satisfy $x^4 = 16$, but $\sqrt[4]{16} = 2$, not -2.

Definition 48 Principal n^{th} root

Let x be a real number and n a natural number. If n is odd, the **principal n^{th} root** of x, denoted $\sqrt[n]{x}$ is the unique real number satisfying $\left(\sqrt[n]{x}\right)^n = x$. If n is even, $\sqrt[n]{x}$ is defined similarly provided $x \geq 0$ and $\sqrt[n]{x} \geq 0$. The **index** is the number n and the **radicand** is the number x. For $n = 2$, we write $\sqrt{x}$ instead of $\sqrt[2]{x}$.

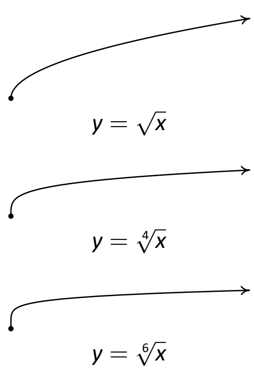


Figure 6.20: Graphs of the first three even root functions

It is worth remarking that, in light of Section 6.2, we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from Section 4.1 along with Theorem 38, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Figure 6.20 shows the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.

The odd-indexed radical functions also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$, as seen in Figure 6.21. In the exercises, you'll have a chance to graph some basic radical functions using the techniques presented in Section 2.6.

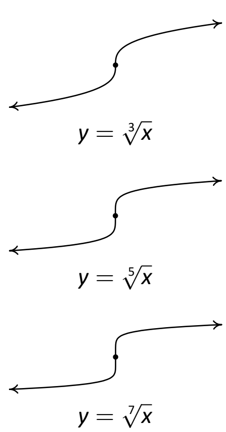


Figure 6.21: Graphs of the first three odd root functions

Next, we recall the properties of radicals given in Definition 3. We have used all of these properties at some point in the textbook for the case $n = 2$ (the square root), but we repeat them here in generality for completeness.

Theorem 41 Properties of Radicals

Let x and y be real numbers and m and n be natural numbers. If $\sqrt[n]{x}$, $\sqrt[n]{y}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
- **Powers of Radicals:** $\sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$
- **Quotient Rule:** $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$, provided $y \neq 0$.
- If n is odd, $\sqrt[n]{x^n} = x$; if n is even, $\sqrt[n]{x^n} = |x|$.

The proof of Theorem 41 is based on the definition of the principal roots and properties of exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{xy}$ is the unique real number such that $(\sqrt[n]{xy})^n = xy$. Given that $\left(\sqrt[n]{x}\,\sqrt[n]{y}\right)^n = \left(\sqrt[n]{x}\right)^n \left(\sqrt[n]{y}\right)^n = xy$, it must be the case that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. If n is even, then $\sqrt[n]{xy}$ is the unique non-negative real number such that $(\sqrt[n]{xy})^n = xy$. Also note that since n is even, $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are also non-negative and hence so is $\sqrt[n]{x}\sqrt[n]{y}$. Proceeding as above, we find that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{x}$ is a real number to start with.(Otherwise we'd run into the same paradox we did in Section 4.4.) The last property is an application of the power rule when n is odd, and the occurrence of the absolute value when n is even is due to the requirement that $\sqrt[n]{x} \geq 0$ in Definition 48. For instance, $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 = |-2|$, not -2. It's this last property which makes compositions of roots and powers delicate. This is especially true when we use exponential notation for radicals. Recall the following definition, first given in Definition 16.

Definition 49 Rational power function

Let x be a real number, m an integer and n a natural number.

- $x^{\frac{1}{n}} = \sqrt[n]{x}$ and is defined whenever $\sqrt[n]{x}$ is defined.
- $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$, whenever $\left(\sqrt[n]{x}\right)^m$ is defined.

The rational exponents defined in Definition 49 behave very similarly to the usual integer exponents from Elementary Algebra with one critical exception. Consider the expression $\left(x^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(x^{2/3}\right)^{3/2} = x^{\frac{2}{3}\cdot\frac{3}{2}} = x^1 = x$. However, if we substitute $x = -1$ and apply Definition 49, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. We see in this case that $\left(x^{2/3}\right)^{3/2} \neq x$. If we take the time to rewrite $\left(x^{2/3}\right)^{3/2}$ with radicals, we see

$$\left(x^{2/3}\right)^{3/2} = \left(\left(\sqrt[3]{x}\right)^2\right)^{3/2} = \left(\sqrt{\left(\sqrt[3]{x}\right)^2}\right)^3 = \left(\left|\sqrt[3]{x}\right|\right)^3 = \left|\left(\sqrt[3]{x}\right)^3\right| = |x|$$

In the play-by-play analysis, we see that when we cancelled the 2's in multiplying $\frac{2}{3} \cdot \frac{3}{2}$, we were, in fact, attempting to cancel a square with a square root. The fact that $\sqrt{x^2} = |x|$ and not simply x is the root of the trouble. (Pun intended.) It may amuse the reader to know that $\left(x^{3/2}\right)^{2/3} = x$, and this verification is left as an exercise. The moral of the story is that when simplifying fractional exponents, it's usually best to rewrite them as radicals. (In most other cases, though, rational exponents are preferred.) The last major property we will state, and leave to Calculus to prove, is that radical functions are continuous on their domains, so the Intermediate Value Theorem, Theorem 19, applies. This means that if we take combinations of radical functions with polynomial and rational functions to form what the authors consider the **algebraic functions**, we can make sign diagrams using the procedure set forth in Section 5.2.

As mentioned in Section 3.2, $f(x) = \sqrt{x^2} = |x|$ so that absolute value is also considered an algebraic function.

Key Idea 26 Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line with an '?' above them.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.

Our next example reviews quite a bit of Intermediate Algebra and demonstrates some of the new features of these graphs.

Example 109 Analyzing algebraic functions

For the following functions, state their domains and create sign diagrams. Check your answer graphically using your computer or calculator.

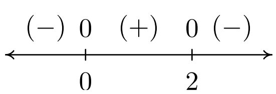
1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$
3. $h(x) = \sqrt[3]{\dfrac{8x}{x+1}}$
4. $k(x) = \dfrac{2x}{\sqrt{x^2-1}}$

Solution

1. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means its domain is $(-\infty, \infty)$. To create the sign diagram, we find the zeros of f.

$$\begin{aligned} f(x) &= 0 \\ 3x\sqrt[3]{2-x} &= 0 \\ 3x = 0 \quad &\text{or} \quad \sqrt[3]{2-x} = 0 \\ x = 0 \quad &\text{or} \quad \left(\sqrt[3]{2-x}\right)^3 = 0^3 \\ x = 0 \quad &\text{or} \quad 2 - x = 0 \\ x = 0 \quad &\text{or} \quad x = 2 \end{aligned}$$

The zeros 0 and 2 divide the real number line into three test intervals. The sign diagram and accompanying graph are below. Note that the intervals on which f is $(+)$ correspond to where the graph of f is above the x-axis, and where the graph of f is below the x-axis we have that f is $(-)$. Plotting the function in GeoGebra, we notice that the graph becomes nearly vertical near $x = 2$. You'll have to wait until Calculus to fully understand this phenomenon.



Sign diagram for $f(x)$

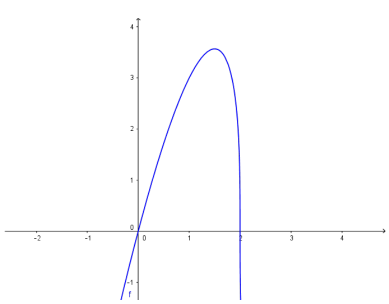

The graph $y = f(x)$

Figure 6.22: $f(x) = 3x\sqrt[3]{2-x}$

2. In $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$, we have two radicals both of which are even indexed. To satisfy $\sqrt[4]{x+3}$, we require $x + 3 \geq 0$ or $x \geq -3$. To satisfy $\sqrt{2 - \sqrt[4]{x+3}}$, we need $2 - \sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results. (For instance, $-2 \geq \sqrt[4]{x+3}$, which has no solution or $-2 \leq \sqrt[4]{x+3}$ whose solution is $[-3, \infty)$.) Instead, we solve $2 - \sqrt[4]{x+3} \geq 0$ using a sign diagram. If we let $r(x) = 2 - \sqrt[4]{x+3}$, we know $x \geq -3$, so we concern ourselves with only this portion of the number line. To find the zeros of r we set $r(x) = 0$ and solve $2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$ so that $\left(\sqrt[4]{x+3}\right)^4 = 2^4$ from which we obtain $x + 3 = 16$ or $x = 13$. Since we raised both sides of an equation to an even power, we need to check to see if $x = 13$ is an extraneous solution. (Recall that this means we have produced a candidate which doesn't satisfy the original equation. Do you remember how raising both sides of an equation to an even power could cause this?) We find $x = 13$ does check since $2 - \sqrt[4]{x+3} = 2 - \sqrt[4]{13+3} = 2 - \sqrt[4]{16} = 2 - 2 = 0$. Our sign diagram for r is given in Figure 6.23.

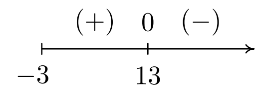
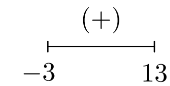
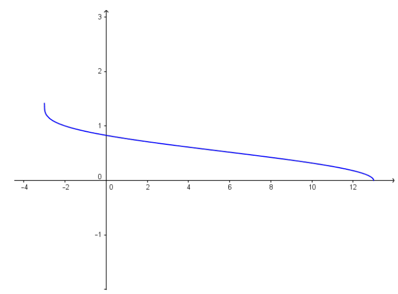


Figure 6.23: Sign diagram for $r(x) = 2 - \sqrt[4]{x+3}$

We find $2 - \sqrt[4]{x+3} \geq 0$ on $[-3, 13]$ so this is the domain of g. To find a sign diagram for g, we look for the zeros of g. Setting $g(x) = 0$ is equivalent to $\sqrt{2 - \sqrt[4]{x+3}} = 0$. After squaring both sides, we get $2 - \sqrt[4]{x+3} = 0$, whose solution we have found to be $x = 13$. Since we squared both sides, we double check and find $g(13)$ is, in fact, 0. Our sign diagram and graph of g are below. Since the domain of g is $[-3, 13]$, what we have below is not just a *portion* of the graph of g, but the *complete* graph. It is always above or on the x-axis, which verifies our sign diagram: see Figure 6.24.



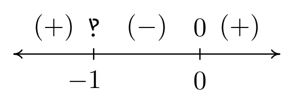
Sign diagram for $g(x)$



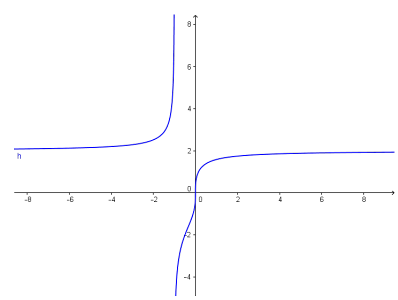
Complete graph of $y = g(x)$

Figure 6.24: $g(x) = \sqrt{2 - \sqrt[4]{x+3}}$

3. The radical in $h(x)$ is odd, so our only concern is the denominator. Setting $x + 1 = 0$ gives $x = -1$, so our domain is $(-\infty, -1) \cup (-1, \infty)$. To find the zeros of h, we set $h(x) = 0$. To solve $\sqrt[3]{\frac{8x}{x+1}} = 0$, we cube both sides to get $\frac{8x}{x+1} = 0$. We get $8x = 0$, or $x = 0$. Below is the resulting sign diagram and corresponding graph. From the graph, it appears as though $x = -1$ is a vertical asymptote. Carrying out an analysis as $x \to -1$ as in Section 5.2 confirms this. (We leave the details to the reader.) Near $x = 0$, we have a situation similar to $x = 2$ in the graph of f in number 1 above. Finally, it appears as if the graph of h has a horizontal asymptote $y = 2$. Using techniques from Section 5.2, we find as $x \to \pm\infty$, $\frac{8x}{x+1} \to 8$. From this, it is hardly surprising that as $x \to \pm\infty$, $h(x) = \sqrt[3]{\frac{8x}{x+1}} \approx \sqrt[3]{8} = 2$. The sign diagram and graph for h are given in Figure 6.25.



Sign diagram for $h(x)$



Graph of $y = h(x)$

Figure 6.25: $h(x) = \sqrt[3]{\dfrac{8x}{x+1}}$

4. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2 - 1 \geq 0$. Setting $r(x) = x^2 - 1$, we find the zeros of r to be $x = \pm 1$, and we find the sign diagram of r shown in Figure 6.26.

We find $x^2 - 1 \geq 0$ for $(-\infty, -1] \cup [1, \infty)$. To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2 - 1} = 0$. We leave it to the reader to verify the solutions are $x = \pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty, -1) \cup (1, \infty)$. To build the sign diagram for k, we need the zeros of k. Setting $k(x) = 0$ results in $\frac{2x}{\sqrt{x^2-1}} = 0$. We get $2x = 0$ or $x = 0$. However, $x = 0$ isn't in the domain of k, which means k has no zeros. We construct our sign diagram on the domain of k in Figure 6.27 along with the graph of k. It appears that the

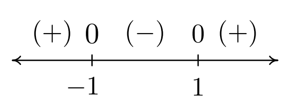


Figure 6.26: The sign diagram of $r(x) = x^2 - 1$

Sign diagram for $k(x)$

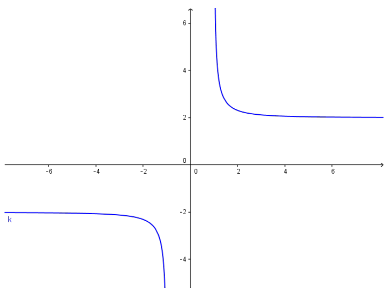

Graph of $y = k(x)$

Figure 6.27: $k(x) = \dfrac{2x}{\sqrt{x^2-1}}$

graph of k has two vertical asymptotes, one at $x = -1$ and one at $x = 1$. The gap in the graph between the asymptotes is because of the gap in the domain of k. Concerning end behaviour, there appear to be two horizontal asymptotes, $y = 2$ and $y = -2$. To see why this is the case, we think of $x \to \pm\infty$. The radicand of the denominator $x^2 - 1 \approx x^2$, and as such, $k(x) = \frac{2x}{\sqrt{x^2-1}} \approx \frac{2x}{\sqrt{x^2}} = \frac{2x}{|x|}$. As $x \to \infty$, we have $|x| = x$ so $k(x) \approx \frac{2x}{x} = 2$. On the other hand, as $x \to -\infty$, $|x| = -x$, and as such $k(x) \approx \frac{2x}{-x} = -2$. Finally, it appears as though the graph of k passes the Horizontal Line Test which means k is one to one and k^{-1} exists. Computing k^{-1} is left as an exercise.

As the previous example illustrates, the graphs of general algebraic functions can have features we've seen before, like vertical and horizontal asymptotes, but they can occur in new and exciting ways. For example, $k(x) = \frac{2x}{\sqrt{x^2-1}}$ had two distinct horizontal asymptotes. You'll recall that rational functions could have at most one horizontal asymptote. Also some new characteristics like 'unusual steepness' (the proper Calculus term for this is 'vertical tangent', but for now we'll be okay calling it 'unusual steepness') and cusps (see page 158 for the first reference to this feature) can appear in the graphs of arbitrary algebraic functions. Our next example first demonstrates how we can use sign diagrams to solve nonlinear inequalities. (Don't panic. The technique is very similar to the ones used in Chapters 3, 4 and 5.) We then check our answers graphically with a calculator and see some of the new graphical features of the functions in this extended family.

Example 110 Inequalities with algebraic functions

Solve the following inequalities. Check your answers graphically with a computer or calculator.

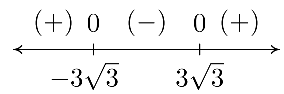
1. $x^{2/3} < x^{4/3} - 6$
2. $3(2-x)^{1/3} \le x(2-x)^{-2/3}$

Solution

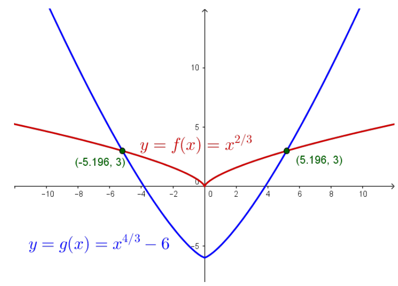
1. To solve $x^{2/3} < x^{4/3} - 6$, we get 0 on one side and attempt to solve $x^{4/3} - x^{2/3} - 6 > 0$. We set $r(x) = x^{4/3} - x^{2/3} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(x) = 0$ and attempt to solve $x^{4/3} - x^{2/3} - 6 = 0$. At this point, it may be unclear how to proceed. We could always try as a last resort converting back to radical notation, but in this case we can take a cue from Example 75. Since there are three terms, and the exponent on one of the variable terms, $x^{4/3}$, is exactly twice that of the other, $x^{2/3}$, we have ourselves a 'quadratic in disguise' and we can rewrite $x^{4/3} - x^{2/3} - 6 = 0$ as $\left(x^{2/3}\right)^2 - x^{2/3} - 6 = 0$. If we let $u = x^{2/3}$, then in terms of u, we get $u^2 - u - 6 = 0$. Solving for u, we obtain $u = -2$ or $u = 3$. Replacing $x^{2/3}$ back in for u, we get $x^{2/3} = -2$ or $x^{2/3} = 3$. To avoid the trouble we encountered in the discussion following Definition 16, we now convert back to radical notation. By interpreting $x^{2/3}$ as $\sqrt[3]{x^2}$ we have $\sqrt[3]{x^2} = -2$ or $\sqrt[3]{x^2} = 3$. Cubing both sides of these equations results in $x^2 = -8$, which admits no real solution, or $x^2 = 27$, which gives $x = \pm 3\sqrt{3}$. We construct a sign diagram and find $x^{4/3} - x^{2/3} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(x) = x^{2/3}$ and $g(x) = x^{4/3} - 6$.

The solution to $x^{2/3} < x^{4/3} - 6$ corresponds to the inequality $f(x) < g(x)$, which means we are looking for the x values for which the graph of f is below the graph of g. Using the 'Intersect' tool we confirm (or at least, confirm to a few decimal places) that the graphs cross at $x = \pm 3\sqrt{3}$. We see in Figure 6.28 that the graph of f (in red) is below the graph of g (in blue) on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$.

As a point of interest, if we take a closer look at the graphs of f and g near $x = 0$ with the axes off, we see in Figure 6.29 that despite the fact they both involve cube roots, they exhibit different behaviour near $x = 0$. The graph of f has a sharp turn, or cusp, while g does not. (Recall that we introduced this feature on page 158 as a feature which makes the graph of a function 'not smooth'.)



Sign diagram for $x^{4/3} - x^{2/3} - 6$



Graphs $y = f(x)$ and $y = g(x)$

Figure 6.28: Sign diagram and graph for Example 110.1

2. To solve $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$, we gather all the nonzero terms on one side and obtain $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$. We set $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$. As in number 1, the denominators of the rational exponents are odd, which means there are no domain concerns there. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}}$$

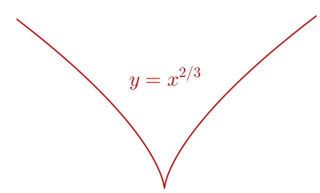
Setting the denominator equal to zero we get $(2-x)^{2/3} = 0$, or $\sqrt[3]{(2-x)^2} = 0$. After cubing both sides, and subsequently taking square roots, we get $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$. To find the zeros of r, we set $r(x) = 0$. There are two school of thought on how to proceed and we demonstrate both.

- *Factoring Approach.* From $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\frac{2}{3} < \frac{1}{3}$, we factor $(2-x)^{-2/3}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-2/3}$ *from* $(2-x)^{1/3}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents. (And we exercise special care when reducing the $\frac{3}{3}$ power to 1.)

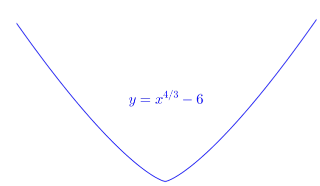
$$\begin{aligned} r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\ &= (2-x)^{-2/3}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{3/3} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{1} - x\right] && \text{since } \sqrt[3]{u^3} = \left(\sqrt[3]{u}\right)^3 = u \\ &= (2-x)^{-2/3}(6-4x) \\ &= (2-x)^{-2/3}(6-4x) \end{aligned}$$

To solve $r(x) = 0$, we set $(2-x)^{-2/3}(6-4x) = 0$, or $\frac{6-4x}{(2-x)^{2/3}} = 0$. We have $6 - 4x = 0$ or $x = \frac{3}{2}$.

- *Common Denominator Approach.* We rewrite

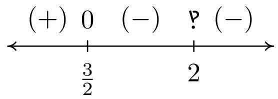


$y = f(x)$

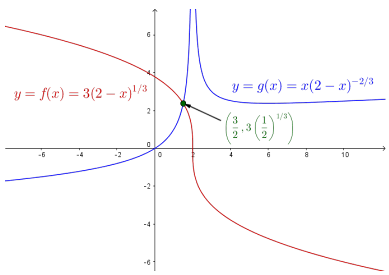


$y = g(x)$

Figure 6.29: The graphs of f and g near $x = 0$



Sign diagram for $3(2-x)^{1/3} - x(2-x)^{-2/3}$



Graphs $y = f(x)$ and $y = g(x)$

Figure 6.30: Sign diagram and graph for Example 110.2

$$\begin{aligned}
r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\
&= 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{1/3}(2-x)^{2/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{common denominator} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{3/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{since } \sqrt[3]{u^3} = \left(\sqrt[3]{u}\right)^3 = u \\
&= \frac{3(2-x) - x}{(2-x)^{2/3}} \\
&= \frac{6-4x}{(2-x)^{2/3}}
\end{aligned}$$

As before, when we set $r(x) = 0$ we obtain $x = \frac{3}{2}$.

We now create our sign diagram and find $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{1/3}$ (the red curve) and $g(x) = x(2-x)^{-2/3}$ (the blue curve) in Figure 6.30. We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x \geq \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.

One application of algebraic functions was given in Example 35 in Section 1.3. Our last example is a more sophisticated application of distance.

Example 111 Pricing cable installation

Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles downroad from the post, as indicated in Figure 6.31. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.

1. Express the total cost C of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

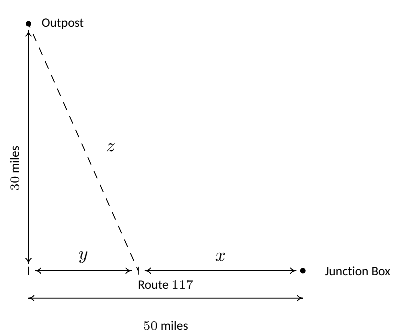


Figure 6.31: Diagram for Example 111

SOLUTION

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C =$

$15x + 20z$. Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Hence, $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$ so that our cost as a function of x only is given by

$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$$

From the context of the problem, we have $0 \leq x \leq 50$.

2. Graphing $y = C(x)$ on a calculator and using the 'Minimum' feature, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is $\$1146.86$. The ability to stream live SasquatchCasts? Priceless.

Exercises 6.3

Problems

For each function in Exercises 1 – 10below,

- **Find its domain.**
- **Create a sign diagram.**
- **Use your computer or calculator to help you sketch its graph and identify any vertical or horizontal asymptotes, 'unusual steepness' or cusps.**

1. $f(x) = \sqrt{1 - x^2}$

2. $f(x) = \sqrt{x^2 - 1}$

3. $f(x) = x\sqrt{1 - x^2}$

4. $f(x) = x\sqrt{x^2 - 1}$

5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$

6. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

7. $f(x) = x^{\frac{2}{3}}(x - 7)^{\frac{1}{3}}$

8. $f(x) = x^{\frac{3}{2}}(x - 7)^{\frac{1}{3}}$

9. $f(x) = \sqrt{x(x + 5)(x - 4)}$

10. $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$

In Exercises 11 – 16, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using the transformations presented in Section 2.6.

11. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x - 1} - 2$

12. $f(x) = \sqrt[3]{x}$, $g(x) = -2\sqrt[3]{x + 1} + 4$

13. $f(x) = \sqrt[4]{x}$, $g(x) = \sqrt[4]{x - 1} - 2$

14. $f(x) = \sqrt[4]{x}$, $g(x) = 3\sqrt[4]{x - 7} - 1$

15. $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{x + 2} + 3$

16. $f(x) = \sqrt[8]{x}$, $g(x) = \sqrt[8]{-x} - 2$

In Exercises 17 – 35, solve the equation or inequality.

17. $x + 1 = \sqrt{3x + 7}$

18. $2x + 1 = \sqrt{3 - 3x}$

19. $x + \sqrt{3x + 10} = -2$

20. $3x + \sqrt{6 - 9x} = 2$

21. $2x - 1 = \sqrt{x + 3}$

22. $x^{\frac{3}{2}} = 8$

23. $x^{\frac{2}{3}} = 4$

24. $\sqrt{x - 2} + \sqrt{x - 5} = 3$

25. $\sqrt{2x + 1} = 3 + \sqrt{4 - x}$

26. $5 - (4 - 2x)^{\frac{2}{3}} = 1$

27. $10 - \sqrt{x - 2} \leq 11$

28. $\sqrt[3]{x} \leq x$

29. $2(x - 2)^{-\frac{1}{3}} - \frac{2}{3}x(x - 2)^{-\frac{4}{3}} \leq 0$

30. $-\frac{4}{3}(x - 2)^{-\frac{4}{3}} + \frac{8}{9}x(x - 2)^{-\frac{7}{3}} \geq 0$

31. $2x^{-\frac{1}{3}}(x - 3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x - 3)^{-\frac{2}{3}} \geq 0$

32. $\sqrt[3]{x^3 + 3x^2 - 6x - 8} > x + 1$

33. $\frac{1}{3}x^{\frac{3}{4}}(x - 3)^{-\frac{2}{3}} + \frac{3}{4}x^{-\frac{1}{4}}(x - 3)^{\frac{1}{3}} < 0$

34. $x^{-\frac{1}{3}}(x - 3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x - 3)^{-\frac{5}{3}}(x^2 - 3x + 2) \geq 0$

35. $\frac{2}{3}(x + 4)^{\frac{3}{5}}(x - 2)^{-\frac{1}{3}} + \frac{3}{5}(x + 4)^{-\frac{2}{5}}(x - 2)^{\frac{2}{3}} \geq 0$

36. Rework Example 111 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.

37. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2 + h^2}$. Suppose a cone is to have a volume of 100 cubic centimetres.

    (a) Use the formula for volume to find the height h as a function of r.

    (b) Use the formula for surface area and your answer to 37a to find the surface area S as a function of r.

    (c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

38. The National Weather Service uses the following formula to calculate the wind chill:

    $$W = 35.74 + 0.6215\,T_a - 35.75\,V^{0.16} + 0.4275\,T_a\,V^{0.16}$$

    where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

(a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

(b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

39. As a follow-up to Exercise 38, suppose the air temperature is 28°F.

(a) Use the formula from Exercise 38 to find an expression for the wind chill temperature as a function of the wind speed, $W(V)$.

(b) Solve $W(V) = 0$, round your answer to two decimal places, and interpret.

(c) Graph the function W using your calculator and check your answer to part 39b.

40. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in metres and $g = 9.8$ metres per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

41. The Cobb-Douglas production model states that the yearly total dollar value of the production output P in an economy is a function of labour x (the total number of hours worked in a year) and capital y (the total dollar value of all of the stuff purchased in order to make things). Specifically, $P = ax^by^{1-b}$. By fixing P, we create what's known as an 'isoquant' and we can then solve for y as a function of x. Let's assume that the Cobb-Douglas production model for the country of Sasquatchia is $P = 1.23x^{0.4}y^{0.6}$.

(a) Let $P = 300$ and solve for y in terms of x. If $x = 100$, what is y?

(b) Graph the isoquant $300 = 1.23x^{0.4}y^{0.6}$. What information does an ordered pair (x, y) which makes $P = 300$ give you? With the help of your classmates, find several different combinations of labour and capital all of which yield $P = 300$. Discuss any patterns you may see.

42. According to Einstein's Theory of Special Relativity, the observed mass m of an object is a function of how fast the object is travelling. Specifically,

$$m(x) = \frac{m_r}{\sqrt{1 - \dfrac{x^2}{c^2}}}$$

where $m(0) = m_r$ is the mass of the object at rest, x is the speed of the object and c is the speed of light.

(a) Find the applied domain of the function.

(b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

(c) As $x \to c^-$, what happens to $m(x)$?

(d) How slowly must the object be travelling so that the observed mass is no greater than 100 times its mass at rest?

43. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

44. Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modelled by

$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1 + v_1/v_2} - \frac{x^{1-v_1/v_2}}{1 - v_1/v_2}\right) + \frac{v_1v_2}{v_2^2 - v_1^2}$$

(a) Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?

(b) Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use your calculator to graph this path for $x > 0$. Describe the behaviour of y as $x \to 0^+$ and interpret this physically.

(c) With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise 32 in Section 7.5.

45. Verify the Quotient Rule for Radicals in Theorem 3.

46. Show that $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = x$ for all $x \geq 0$.

47. Show that $\sqrt[3]{2}$ is an irrational number by first showing that it is a zero of $p(x) = x^3 - 2$ and then showing p has no rational zeros. (You'll need the Rational Zeros Theorem, Theorem 27, in order to show this last part.)

48. With the help of your classmates, generalize Exercise 47 to show that $\sqrt[n]{c}$ is an irrational number for any natural numbers $c \geq 2$ and $n \geq 2$ provided that $c \neq p^n$ for some natural number p.

7: Exponential and Logarithmic Functions

7.1 Introduction to Exponential and Logarithmic Functions

Exponential and logarithmic functions frequently occur in solutions to differential equations, which are used to produce mathematical models of phenomena throughout the physical, life, and social sciences. You'll see some examples if you continue on to Calculus I and II, and even more if you take Math 3600, our first course in differential equations.

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like x^2 or $x^{2/3}$, in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every textbook we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect the dots in a pleasing fashion: see Figure 7.1

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$

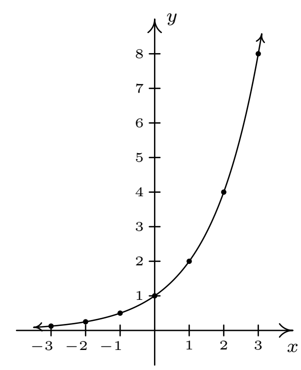


Figure 7.1: Plotting $f(x) = 2^x$

A few remarks about the graph of $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$ and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the x-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of f is $(0, \infty)$. The graph of f passes the Horizontal Line Test which means f is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous (recall that this means there are no holes or other kinds of breaks in the graph) and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation. The number $\sqrt{3} = 1.73205\ldots$ is an irrational number and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason (this is where Calculus and continuity come into play) that we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.

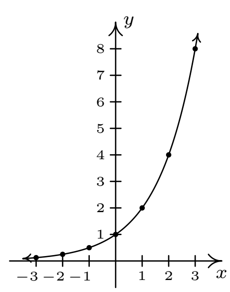
You (yes, you) can actually prove that $\sqrt{3}$ is irrational by considering the polynomial $p(x) = x^2 - 3$ and showing it has no rational zeros by applying Theorem 27.

To fully understand the argument we used to define 2^x when x is irrational, you'll have to proceed far enough through the Calculus sequence (Calculus III should do it) to encounter the topic of convergence of infinite sequences.

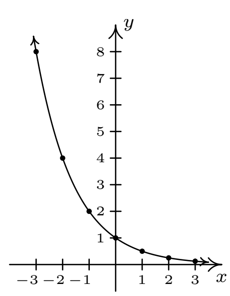
Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 50 Exponential function

A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called a **base *b* exponential function**.



(a) $y = f(x) = 2^x$



(b) $y = g(x) = f(-x) = 2^{-x}$

Figure 7.2: Reflecting $y = 2^x$ across the y-axis to obtain the graph $y = 2^{-x}$

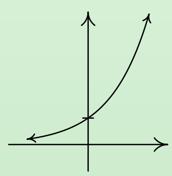
We leave it to the reader to verify (by graphing some more examples on your own) that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Thinking back to Section 2.6, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis. We get the graph in Figure 7.2 (b).

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty$, $g(x) \to \infty$, and on the flip side, as $x \to \infty$, $g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g. We summarize the basic properties of exponential functions in the following theorem. (The proof of which, like many things discussed in the text, requires Calculus.)

Theorem 42 Properties of Exponential Functions

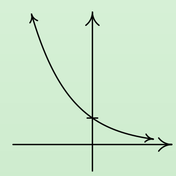
Suppose $f(x) = b^x$.

- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth (the graph of f has no sharp turns or corners).
- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x$, $0 < b < 1$

Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. You may encounter a more formal discussion of the number e in later Calculus courses. For now, it is enough to know that since $e > 1$, $f(x) = e^x$ is an increasing exponential function. The following examples give us an idea how these functions are used in the wild.

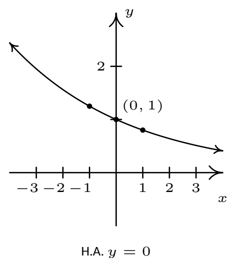
Example 112 Modelling vehicle depreciation

The value of a car can be modelled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

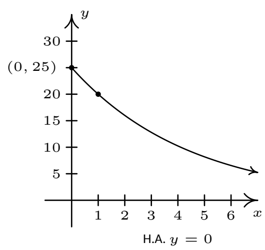
SOLUTION

1. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since x represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of \$25,000. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was \$25,000.



$y = f(x) = \left(\frac{4}{5}\right)^x$

↓



$y = V(x) = 25f(x), x \geq 0$

Figure 7.3: The graph $y = V(x)$ in Example 112

2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the y-intercept $(0, 1)$ and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from f by 25, in other words, $V(x) = 25f(x)$. In accordance with Theorem 10, this results in a vertical stretch by a factor of 25. We multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) and obtain the points $\left(-1, \frac{125}{4}\right)$, $(0, 25)$ and $(1, 20)$. The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us. We have the result in Figure 7.3.

3. We see from the graph of V that its horizontal asymptote is $y = 0$. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0.

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' many examples of which are encountered in applications of exponential functions. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations from Section 2.6.

Example 113 Newton's Law of Cooling

According to Newton's Law of Cooling the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modelled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph.

Solution

1. To find $T(0)$, we replace every occurrence of the independent variable t with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.

2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of f is an increasing exponential with y-intercept $(0, 1)$ and horizontal asymptote $y = 0$. The points $\left(-1, e^{-1}\right) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. Since the formula $T(t)$ looks rather complicated, we rewrite $T(t)$ in the form presented in Theorem 12 and use that result to track the changes to our three points and the horizontal asymptote. We have

$$T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$$

Multiplication of the input to f, t, by -0.1 results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis. We divide each of the x values of our points by -0.1 (which amounts to multiplying them by -10) to obtain $\left(10, e^{-1}\right)$, $(0, 1)$, and $(-10, e)$. Since none of these changes affected the y values, the horizontal asymptote remains $y = 0$. Next, we see that the output from f is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the y-coordinates by 90 to obtain $\left(10, 90e^{-1}\right)$, $(0, 90)$, and $(-10, 90e)$. We also multiply the y value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the y-coordinates, which shifts the graph upwards to obtain $\left(10, 90e^{-1} + 70\right) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of f and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is given in Figure 7.4.

3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically and to review some of the 'number sense' developed in Chapter 5. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big } (-)}$. Since $e > 1$,

$$e^{\text{very big } (-)} = \frac{1}{e^{\text{very big } (+)}} \approx \frac{1}{\text{very big } (+)} \approx \text{very small } (+)$$

The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70+$very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature.

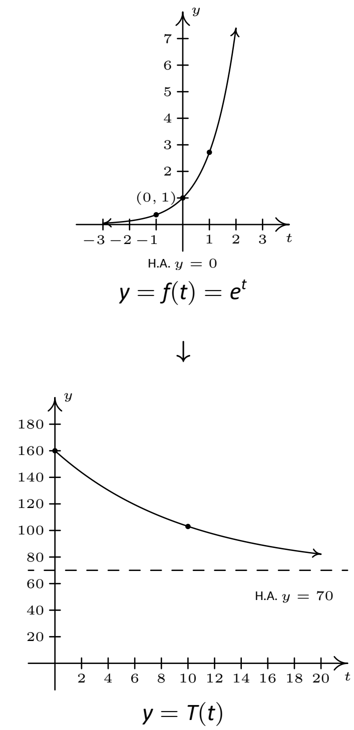


Figure 7.4: Graphing $T(t)$ in Example 113

As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 51 Logarithm function

The inverse of the exponential function $f(x) = b^x$ is called the **base** b **logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ We read '$\log_b(x)$' as 'log base b of x.'

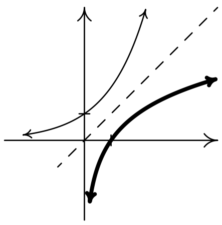
We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 52 Common and Natural Logarithms

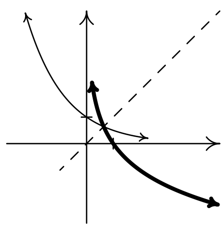
The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

The reader is cautioned that in more advanced mathematics textbooks, the notation $\log(x)$ is often used to denote the natural logarithm (or its generalization to the complex numbers). In mathematics, the natural logarithm is preferred since it is better behaved with respect to the operations of Calculus. The base 10 logarithm tends to appear in other science fields.

Since logs are defined as the inverses of exponential functions, we can use Theorems 37 and 38 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function,



$y = b^x, b > 1$
$\boldsymbol{y = \log_b(x)}, b > 1$
$y = b^x, b > 1$
$y = \log_b(x), b > 1$



$y = b^x, 0 < b < 1$
$\boldsymbol{y = \log_b(x)}, 0 < b < 1$
$y = b^x, 0 < b < 1$
$y = \log_b(x), 0 < b < 1$

Figure 7.5: The logarithm is the inverse of the exponential function

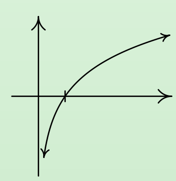
namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$ as shown below. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs: see Figure 7.5.

On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 43 Properties of Logarithmic Functions

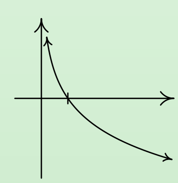
Suppose $f(x) = \log_b(x)$.

- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$
- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x), b > 1$
$y = \log_b(x)$, $b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x), 0 < b < 1$
$y = \log_b(x)$, $0 < b < 1$

As we have mentioned, Theorem 43 is a consequence of Theorems 37 and 38. However, it is worth the reader's time to understand Theorem 43 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$

is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 43 at this level instead of just merely memorizing the facts.

Example 114 Using properties of logarithms

Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\frac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

Solution

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.
2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.
3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.
4. First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.
5. Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.
6. We can use Theorem 43 directly to simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.
7. From Theorem 43, we know $117^{\log_{117}(6)} = 6$, but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$. (Can you see why?) At this point, we use a property of exponents followed by Theorem 43 to get

$$117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$$

It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$.

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals.

With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument of the log must be strictly positive.

Example 115 Domain for logarithmic functions

Find the domain of the following functions. Check your answers graphically using the computer or calculator.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Solution

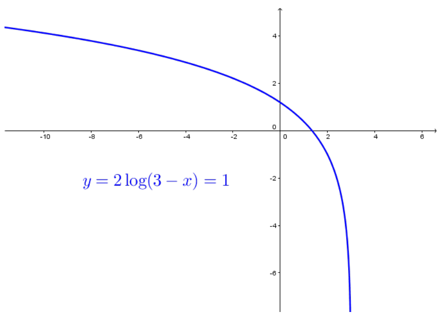


Figure 7.6: $y = f(x) = 2\log(3 - x) - 1$

1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph in Figure 7.6 verifies this. Note that we could have graphed f using transformations. Taking a cue from Theorem 12, we rewrite $f(x) = 2\log_{10}(-x + 3) - 1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x + 3) - 1$, Theorem 12 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multiplying by -1 gives us the vertical asymptote $x = 3$. Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.

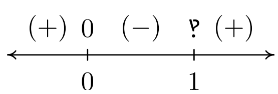


Figure 7.7: Sign diagram for $r(x) = \frac{x}{x-1}$

2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$. As usual, we proceed using a sign diagram. If we define $r(x) = \dfrac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram in Figure 7.7.

   We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of g. The graph of $y = g(x)$ in Figure 7.8 confirms this. We can tell from the graph of g that it is not the result of Section 2.6 transformations being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behaviour of g. The graph suggests that as $x \to \pm\infty$, $g(x) \to 0$. We can verify this analytically. Using results from Chapter 5 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.

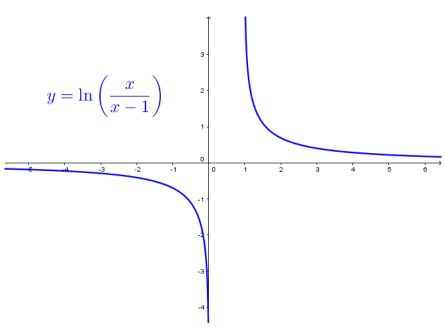


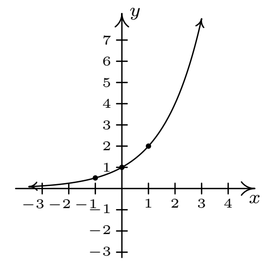
Figure 7.8: $y = g(x) = \ln\left(\frac{x}{x-1}\right)$

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

Example 116 Inverting an exponential function

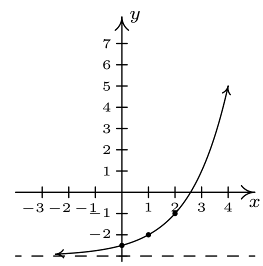
Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check the symmetry about the line $y = x$.



$y = h(x) = 2^x$

↓



$y = f(x) = 2^{x-1} - 3$

Figure 7.9: Graphing $f(x) = 2^{x-1} - 3$ in Example 116

Solution

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x-1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ on the graph of g along with the horizontal asymptote $y = 0$ to track through the transformations. By Theorem 12 we first add 1 to the x-coordinates of the points on the graph of g (shifting g to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1, 1)$ and $(2, 2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the y-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1, -2)$ and $(2, -1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of g, we get the bottom graph in Figure 7.9. We see that the domain of f is the same as g, namely $(-\infty, \infty)$, but that the range of f is $(-3, \infty)$.
2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find f takes an input x and applies the steps

   (a) subtract 1

   (b) put as an exponent on 2

   (c) subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should

   (a) add 3

   (b) take the logarithm base 2

   (c) add 1

   In symbols, $f^{-1}(x) = \log_2(x+3) + 1$.
3. To graph $f^{-1}(x) = \log_2(x+3) + 1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations using Theorem 12. Since $f^{-1}(x) = j(x+3) + 1$, we first subtract 3 from each of the x values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, $(-2, 0)$ and $(-1, 1)$ with a vertical asymptote $x = -3$. Next, we add 1 to

the y values on the graph and get $\left(-\frac{5}{2}, 0\right)$, $(-2, 1)$ and $(-1, 2)$. If you are experiencing *déjà vu*, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f.

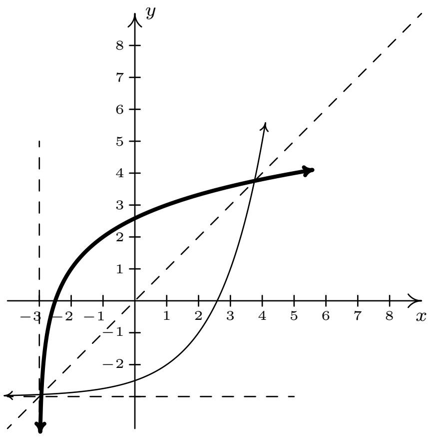
4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3) + 1$ satisfy the composition requirement for inverses. For all real numbers x,

$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(2^{x-1} - 3\right) \\ &= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\ &= \log_2\left(2^{x-1}\right) + 1 \\ &= (x-1) + 1 \quad \text{Since } \log_2\left(2^u\right) = u \text{ for all real numbers } u \\ &= x \checkmark\end{aligned}$$

For all real numbers $x > -3$, we have (pay attention - can you spot in which step below we need $x > -3$?)

$$\begin{aligned}\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\ &= f\left(\log_2(x+3) + 1\right) \\ &= 2^{(\log_2(x+3)+1)-1} - 3 \\ &= 2^{\log_2(x+3)} - 3 \\ &= (x+3) - 3 \quad \text{Since } 2^{\log_2(u)} = u \text{ for all real numbers } u > 0 \\ &= x \checkmark\end{aligned}$$

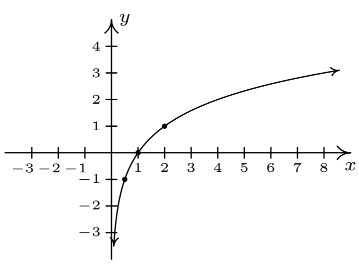
5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$ in Figure 7.11



$y = f(x) = 2^{x-1} - 3$

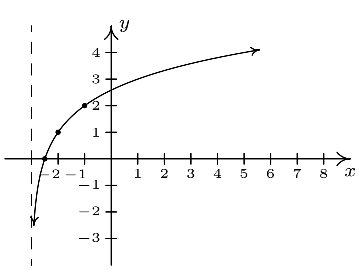
$\boldsymbol{y = f^{-1}(x) = \log_2(x+3) + 1}$

Figure 7.11: The graphs of f and f^{-1} in Example 116



$y = j(x) = \log_2(x)$

↓



$y = f^{-1}(x) = \log_2(x+3) + 1$

Figure 7.10: Graphing $f^{-1}(x) = \log_2(x+3) + 1$ in Example 116

Exercises 7.1

Problems

In Exercises 1 – 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 43 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$
12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 – 42, evaluate the expression.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}\left(\sqrt{13}\right)$
31. $\log_{36}\left(\sqrt[4]{36}\right)$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 – 57, find the domain of the function.

43. $f(x) = \ln(x^2 + 1)$
44. $f(x) = \log_7(4x + 8)$
45. $f(x) = \ln(4x - 20)$
46. $f(x) = \log\left(x^2 + 9x + 18\right)$
47. $f(x) = \log\left(\dfrac{x+2}{x^2-1}\right)$
48. $f(x) = \log\left(\dfrac{x^2+9x+18}{4x-20}\right)$

49. $f(x) = \ln(7 - x) + \ln(x - 4)$

50. $f(x) = \ln(4x - 20) + \ln\left(x^2 + 9x + 18\right)$

51. $f(x) = \log\left(x^2 + x + 1\right)$

52. $f(x) = \sqrt[4]{\log_4(x)}$

53. $f(x) = \log_9(|x + 3| - 4)$

54. $f(x) = \ln(\sqrt{x - 4} - 3)$

55. $f(x) = \dfrac{1}{3 - \log_5(x)}$

56. $f(x) = \dfrac{\sqrt{-1 - x}}{\log_{\frac{1}{2}}(x)}$

57. $f(x) = \ln(-2x^3 - x^2 + 13x - 6)$

In Exercises 58 – 63, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

58. $f(x) = 2^x$, $g(x) = 2^x - 1$

59. $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$

60. $f(x) = 3^x$, $g(x) = 3^{-x} + 2$

61. $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$

62. $f(x) = e^x$, $g(x) = 8 - e^{-x}$

63. $f(x) = e^x$, $g(x) = 10e^{-0.1x}$

In Exercises 64 – 69, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

64. $f(x) = \log_2(x)$, $g(x) = \log_2(x + 1)$

65. $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{3}}(x) + 1$

66. $f(x) = \log_3(x)$, $g(x) = -\log_3(x - 2)$

67. $f(x) = \log(x)$, $g(x) = 2\log(x + 20) - 1$

68. $f(x) = \ln(x)$, $g(x) = -\ln(8 - x)$

69. $f(x) = \ln(x)$, $g(x) = -10\ln\left(\frac{x}{10}\right)$

70. Verify that each function in Exercises 64 - 69 is the inverse of the corresponding function in Exercises 58 - 63. (Match up #58 and #64, and so on.)

In Exercises 71 – 74, find the inverse of the function from the 'procedural perspective' discussed in Example 116 and graph the function and its inverse on the same set of axes.

71. $f(x) = 3^{x+2} - 4$

72. $f(x) = \log_4(x - 1)$

73. $f(x) = -2^{-x} + 1$

74. $f(x) = 5\log(x) - 2$

(Logarithmic Scales) In Exercises 75 – 77, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

75. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[1] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimetres recorded on a seismometer 100 kilometres from the earthquake's epicentre. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimetres of the earthquake recorded 100 kilometres from the epicentre.

(a) Show that $M(0.001) = 0$.

(b) Compute $M(80,000)$.

(c) Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.

(d) Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

76. While the decibel scale can be used in many disciplines,[2] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound.[3] The

[1] Rock-solid, perhaps?

[2] See this webpage for more information.

[3] As of the writing of this exercise, the Wikipedia page given here states that it may not meet the "general notability guideline" nor does it cite any references or sources. I find this odd because it is this very usage of the decibel scale which shows up in every College Algebra book I have read. Perhaps those other books have been wrong all along and we're just blindly following tradition.

Sound Intensity Level L (measured in decibels) of a sound intensity I (measured in watts per square meter) is given by

$$L(I) = 10\log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\,\frac{W}{m^2}$ is the threshold of human hearing.

(a) Compute $L(10^{-6})$.

(b) Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?

(c) Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

77. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log[H^+]$ where $[H^+]$ is the hydrogen ion concentration in moles per litre. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.

    (a) The hydrogen ion concentration of pure water is $[H^+] = 10^{-7}$. Find its pH.

    (b) Find the pH of a solution with $[H^+] = 6.3 \times 10^{-13}$.

    (c) The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?

78. Show that $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0,\ b \neq 1$.

79. (Crazy bonus question) Without using your calculator, determine which is larger: e^π or π^e.

7.2 Properties of Logarithms

In Section 7.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 43 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 44 Inverse Properties of Exponential and Logarithmic Functions

Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b\left(b^x\right) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 45 One-to-one Properties of Exponential and Logarithmic Functions

Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0$, $w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base b is restricted to $b > 0$, $b \neq 1$.

Theorem 46 Algebraic Properties of Exponential Functions

Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 46 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 47 Algebraic Properties of Logarithmic Functions

Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0$, $b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g\left(u^w\right) = wg(u)$. In other words, $\log_b\left(u^w\right) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 47 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and $b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of b^x, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly. From a purely functional approach, we can see the properties in Theorem 47 as an example of how inverse functions interchange the roles of inputs in outputs. For instance, the Product Rule for exponential functions given in Theorem 46, $f(u + w) = f(u)f(w)$, says that adding inputs results in multiplying outputs. Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely what the Product Rule for Logarithmic functions states in Theorem 47: $g(uw) = g(u) + g(w)$. The reader is encouraged to view the remaining properties listed in Theorem 47 similarly. The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 47 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.

Example 117 Expanding logarithmic expressions

Expand the following using the properties of logarithms and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\log_2\left(\frac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\frac{3}{ex}\right)^2$
4. $\log\sqrt[3]{\frac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2-4\right)$

Interestingly enough, expanding logarithms is the exact *opposite* process (which we will practice later) that is most useful in Algebra. The utility of expanding logarithms becomes apparent in Calculus.

Solution

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

$$\begin{aligned} \log_2\left(\frac{8}{x}\right) &= \log_2(8) - \log_2(x) && \text{Quotient Rule} \\ &= 3 - \log_2(x) && \text{Since } 2^3 = 8 \\ &= -\log_2(x) + 3 \end{aligned}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product. In order to use the Product Rule, the *entire* quantity inside the logarithm must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u = 10$ and $w = x^2$. Once we get the x^2 by itself inside the log, we may apply the Power Rule with $u = x$ and $w = 2$ and simplify.

$$\begin{aligned} \log_{0.1}\left(10x^2\right) &= \log_{0.1}(10) + \log_{0.1}\left(x^2\right) && \text{Product Rule} \\ &= \log_{0.1}(10) + 2\log_{0.1}(x) && \text{Power Rule} \\ &= -1 + 2\log_{0.1}(x) && \text{Since } (0.1)^{-1} = 10 \\ &= 2\log_{0.1}(x) - 1 \end{aligned}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{ex}$ and $w = 2$. Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3) - \ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3)-\ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base e.

$$
\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule}\\
&= 2\left[\ln(3)-\ln(ex)\right] && \text{Quotient Rule}\\
&= 2\ln(3)-2\ln(ex)\\
&= 2\ln(3)-2\left[\ln(e)+\ln(x)\right] && \text{Product Rule}\\
&= 2\ln(3)-2\ln(e)-2\ln(x)\\
&= 2\ln(3)-2-2\ln(x) && \text{Since } e^1=e\\
&= -2\ln(x)+2\ln(3)-2
\end{aligned}
$$

4. In Theorem 47, there is no mention of how to deal with radicals. However, thinking back to Definition 16, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule, and we keep in mind that the common log is log base 10.

$$
\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3}\\
&= \frac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule}\\
&= \frac{1}{3}\left[\log\left(100x^2\right)-\log\left(yz^5\right)\right] && \text{Quotient Rule}\\
&= \frac{1}{3}\log\left(100x^2\right)-\frac{1}{3}\log\left(yz^5\right)\\
&= \frac{1}{3}\left[\log(100)+\log\left(x^2\right)\right]-\frac{1}{3}\left[\log(y)+\log\left(z^5\right)\right]\\
& && \text{Product Rule}\\
&= \frac{1}{3}\log(100)+\frac{1}{3}\log\left(x^2\right)-\frac{1}{3}\log(y)-\frac{1}{3}\log\left(z^5\right)\\
&= \frac{1}{3}\log(100)+\frac{2}{3}\log(x)-\frac{1}{3}\log(y)-\frac{5}{3}\log(z)\\
& && \text{Power Rule}\\
&= \frac{2}{3}+\frac{2}{3}\log(x)-\frac{1}{3}\log(y)-\frac{5}{3}\log(z) && \text{Since } 10^2=100\\
&= \frac{2}{3}\log(x)-\frac{1}{3}\log(y)-\frac{5}{3}\log(z)+\frac{2}{3}
\end{aligned}
$$

At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2-4=(x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule.

$$
\begin{aligned}
\log_{117}\left(x^2-4\right) &= \log_{117}\left[(x+2)(x-2)\right] && \text{Factor}\\
&= \log_{117}(x+2)+\log_{117}(x-2) && \text{Product Rule}
\end{aligned}
$$

A couple of remarks about Example 117 are in order. First, while not explicitly stated in the above example, a general rule of thumb to determine which log property to apply first to a complicated problem is 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. In a multi-step problem, this rule can give the required guidance on which log property to apply at each step. The reader is encouraged to look through the solutions to Example 117 to see this rule in action. Second, while we were instructed to assume when necessary that all quantities represented positive real numbers, the authors would be committing a sin of omission if we failed to point out that, for instance, the functions $f(x) = \log_{117}\left(x^2-4\right)$ and $g(x) = \log_{117}(x+2) + \log_{117}(x-2)$ have different domains, and, hence, are different functions. We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$. In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so. One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right) = \log_{117}\left(x^2\right) - \log_{117}(4)$, which simply isn't true, in general. The unwritten (the authors relish the irony involved in writing what follows) property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 118 Combining logarithmic expressions

Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1) - \log_3(x+1)$
2. $\log(x) + 2\log(y) - \log(z)$
3. $4\log_2(x) + 3$
4. $-\ln(x) - \frac{1}{2}$

SOLUTION Whereas in Example 117 we read the properties in Theorem 47 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1) - \log_3(x+1) = \log_3\left(\frac{x-1}{x+1}\right)$.

2. In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x) + 2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it all together, we have

$$\begin{aligned}\log(x)+2\log(y)-\log(z) &= \log(x)+\log\left(y^2\right)-\log(z) && \text{Power Rule}\\ &= \log\left(xy^2\right)-\log(z) && \text{Product Rule}\\ &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule}\end{aligned}$$

3. We can certainly get started rewriting $4\log_2(x)+3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule

to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 44, we know $3 = \log_2\left(2^3\right)$, so we get

$$\begin{aligned}
4\log_2(x) + 3 &= \log_2\left(x^4\right) + 3 && \text{Power Rule} \\
&= \log_2\left(x^4\right) + \log_2\left(2^3\right) && \text{Since } 3 = \log_2\left(2^3\right) \\
&= \log_2\left(x^4\right) + \log_2(8) \\
&= \log_2\left(8x^4\right) && \text{Product Rule}
\end{aligned}$$

4. To get started with $-\ln(x) - \frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x) = \ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 44 gives us $\frac{1}{2} = \ln\left(e^{1/2}\right) = \ln\left(\sqrt{e}\right)$. We have

$$\begin{aligned}
-\ln(x) - \frac{1}{2} &= (-1)\ln(x) - \frac{1}{2} \\
&= \ln\left(x^{-1}\right) - \frac{1}{2} && \text{Power Rule} \\
&= \ln\left(x^{-1}\right) - \ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2} = \ln\left(e^{1/2}\right) \\
&= \ln\left(x^{-1}\right) - \ln\left(\sqrt{e}\right) \\
&= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\
&= \ln\left(\frac{1}{x\sqrt{e}}\right)
\end{aligned}$$

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x) = \log_3(x-1) - \log_3(x+1)$ is $(1, \infty)$ but the domain of $g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty, -1) \cup (1, \infty)$.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 48 Change of Base Formulas

Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x\log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

The proofs of the Change of Base formulas are a result of the other properties studied in this section. If we start with $b^{x\log_b(a)}$ and use the Power Rule in the exponent to rewrite $x\log_b(a)$ as $\log_b(a^x)$ and then apply one of the Inverse Properties in Theorem 44, we get

$$b^{x\log_b(a)} = b^{\log_b(a^x)} = a^x,$$

as required. To verify the logarithmic form of the property, we also use the Power Rule and an Inverse Property. We note that

$$\log_a(x)\cdot\log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x),$$

and we get the result by dividing through by $\log_b(a)$. Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$. What Theorem 48 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base - be it 10, e, 42, or 117.

While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus. The authors feel so strongly about showing students that every property of logarithms comes from and corresponds to a property of exponents that we have broken tradition with the vast majority of other authors in this field. This isn't the first time this happened, and it certainly won't be the last.

Example 119 Using change of base formulas

Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a computer or calculator, as appropriate.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2\log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.
2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x\ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ (in red) and $g(x) = e^{x\ln(2)}$ (in blue). Their graphs are indistinguishable which provides evidence that they are the same function: see Figure 7.12.
3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The plot from GeoGebra in Figure 7.13 confirms this. (Which means if it is lying to us about the first answer it gave us, at least it is being consistent.)
4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.

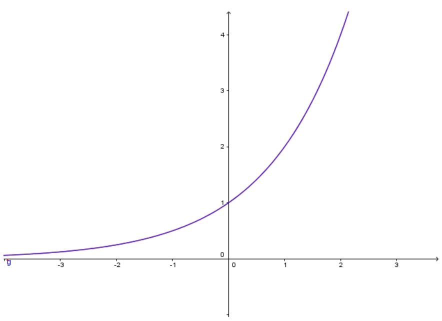


Figure 7.12: $y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

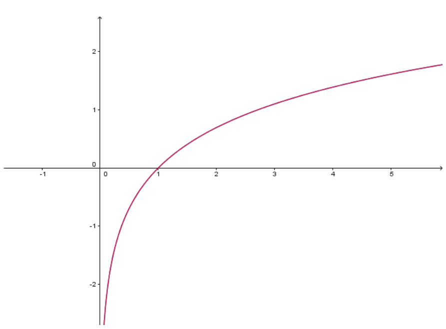


Figure 7.13: $y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

Exercises 7.2

Problems

In Exercises 1 – 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\dfrac{128}{x^2+4}\right)$
3. $\log_5\left(\dfrac{z}{25}\right)^3$
4. $\log(1.23\times 10^{37})$
5. $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2-25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}(9x(y^3-8))$
9. $\log\left(1000x^3y^5\right)$
10. $\log_3\left(\dfrac{x^2}{81y^4}\right)$
11. $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$
12. $\log_6\left(\dfrac{216}{x^3y}\right)^4$
13. $\log\left(\dfrac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$
14. $\log_{\frac{1}{2}}\left(\dfrac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$
15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 – 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$
17. $\log_2(x)+\log_2(y)-\log_2(z)$
18. $\log_3(x)-2\log_3(y)$
19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
20. $2\ln(x)-3\ln(y)-4\ln(z)$
21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
23. $\log_5(x)-3$
24. $3-\log(x)$
25. $\log_7(x)+\log_7(x-3)-2$
26. $\ln(x)+\frac{1}{2}$
27. $\log_2(x)+\log_4(x)$
28. $\log_2(x)+\log_4(x-1)$
29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 – 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e
31. $\log_3(x+2)$ to base 10
32. $\left(\dfrac{2}{3}\right)^x$ to base e
33. $\log(x^2+1)$ to base e

In Exercises 34 – 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$
35. $\log_5(80)$
36. $\log_6(72)$
37. $\log_4\left(\dfrac{1}{10}\right)$
38. $\log_{\frac{3}{5}}(1000)$
39. $\log_{\frac{2}{3}}(50)$
40. Compare and contrast the graphs of $y=\ln(x^2)$ and $y=2\ln(x)$.
41. Prove the Quotient Rule and Power Rule for Logarithms.
42. Give numerical examples to show that, in general,
    (a) $\log_b(x+y)\neq\log_b(x)+\log_b(y)$
    (b) $\log_b(x-y)\neq\log_b(x)-\log_b(y)$

(c) $\log_b\left(\frac{x}{y}\right) \neq \frac{\log_b(x)}{\log_b(y)}$

43. The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction

$$HA \rightleftharpoons H^+ + A^-.$$

The acid disassociation constant, K_a, is given by

$$K_\alpha = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

where the square brackets denote the concentrations just as they did in Exercise 77 in Section 7.1. The symbol pK_a is defined similarly to pH in that p$K_a = -\log(K_a)$. Using the definition of pH from Exercise 77 and the properties of logarithms, derive the Henderson-Hasselbalch Equation which states

$$\text{pH} = \text{p}K_a + \log\frac{[A^-]}{[HA]}$$

44. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?

45. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

7.3 Exponential Equations and Inequalities

In this section we will develop techniques for solving equations involving exponential functions. Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 45, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution. Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 44 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$. This makes sense because, after all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 48, to give us something more calculator friendly, say $\log_2(129) = \frac{\ln(129)}{\ln(2)}$. (You can use natural logs or common logs. We choose natural logs. When we reach Calculus we'll see that natural logs are the easiest to work with.) Another way to arrive at this answer is as follows

$$\begin{aligned} 2^x &= 129 \\ \ln\left(2^x\right) &= \ln(129) && \text{Take the natural log of both sides.} \\ x\ln(2) &= \ln(129) && \text{Power Rule} \\ x &= \frac{\ln(129)}{\ln(2)} \end{aligned}$$

Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

'Taking the natural log' of both sides is akin to squaring both sides: since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal. (This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 45.) Also note that we treat $\ln(2)$ as any other nonzero real number and divide it through to isolate the variable x. We summarize below the two common ways to solve exponential equations, motivated by our examples.

Key Idea 27 Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (a) If convenient, express both sides with a common base and equate the exponents.
   (b) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

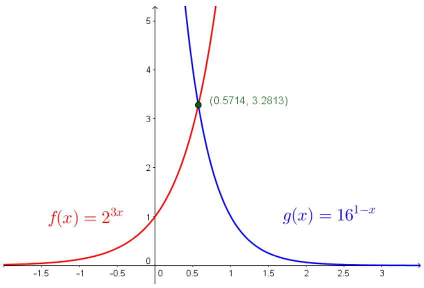


Figure 7.14: $y = f(x) = 2^{3x}$ and $y = g(x) = 16^{1-x}$

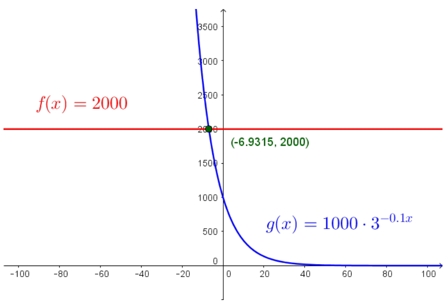


Figure 7.15: $y = f(x) = 2000$ and $y = g(x) = 1000 \cdot e^{-0.1x}$

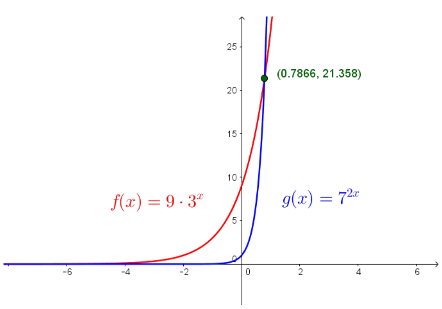


Figure 7.16: $y = f(x) = 9 \cdot 3^x$ and $y = g(x) = 7^{2x}$

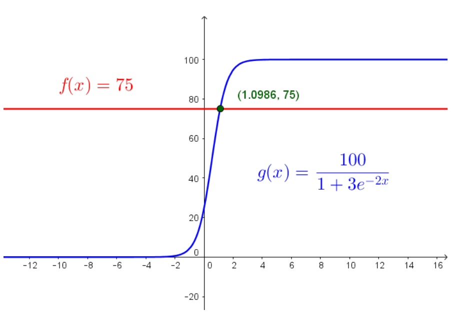


Figure 7.17: $y = f(x) = 75$ and $y = g(x) = \frac{100}{1+3e^{-2x}}$

Example 120 Solving exponential equations

Solve the following equations. Check your answer graphically using a computer or calculator.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

Solution

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$. Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$. To check graphically, we set $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.5714$: see Figure 7.14.

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$. Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$. Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ to get $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$. Using GeoGebra, we graph $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ and find that they intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.3093$: see Figure 7.15.

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$. Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$. The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants. The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{aligned} (x+2)\ln(3) &= 2x\ln(7) \\ x\ln(3) + 2\ln(3) &= 2x\ln(7) \\ 2\ln(3) &= 2x\ln(7) - x\ln(3) \\ 2\ln(3) &= x(2\ln(7) - \ln(3)) \quad \text{Factor.} \\ x &= \frac{2\ln(3)}{2\ln(7) - \ln(3)} \end{aligned}$$

Graphing $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ in GeoGebra, we see that these two graphs intersect at $x = \frac{2\ln(3)}{2\ln(7)-\ln(3)} \approx 0.7866$: see Figure 7.16.

4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential. To that end, we clear denominators and get $75\left(1 + 3e^{-2t}\right) = 100$. From this we get $75 + 225e^{-2t} = 100$, which leads to $225e^{-2t} = 25$, and finally, $e^{-2t} = \frac{1}{9}$. Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. We can also use the Power Rule to write $\ln\left(\frac{1}{9}\right) = -\ln(9)$. Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?)

GeoGebra confirms the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$ intersect at $x = \ln(3) \approx 1.099$: see Figure 7.17.

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$. Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs. If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'. Letting $u = 5^x$, we have $u^2 = (5^x)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$. Since $5^x = -2$ has no real solution, (Why not?) we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln\left(5^x\right) = \ln(3)$ so that $x \ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$. Using GeoGebra, we see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.6826$: see Figure 7.18.

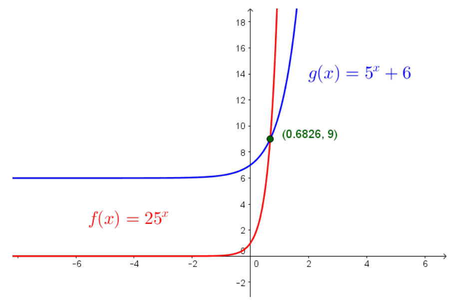


Figure 7.18: $y = f(x) = 25^x$ and $y = g(x) = 5^x + 6$

6. At first, it's unclear how to proceed with $\frac{e^x - e^{-x}}{2} = 5$, besides clearing the denominator to obtain $e^x - e^{-x} = 10$. Of course, if we rewrite $e^{-x} = \frac{1}{e^x}$, we see we have another denominator lurking in the problem: $e^x - \frac{1}{e^x} = 10$. Clearing this denominator gives us $e^{2x} - 1 = 10e^x$, and once again, we have an equation with three terms where the exponent on one term is exactly twice that of another - a 'quadratic in disguise.' If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$, we obtain by the quadratic formula $u = 5 \pm \sqrt{26}$. From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$, but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$. If we graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$, we see in Figure 7.19 that the graphs intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$.

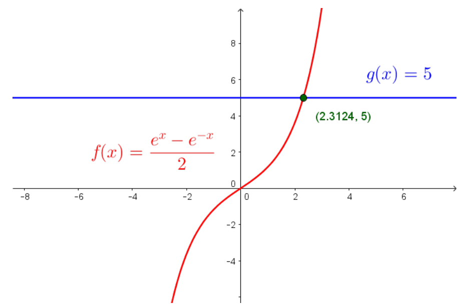


Figure 7.19: $y = f(x) = \dfrac{e^x - e^{-x}}{2}$ and $y = g(x) = 5$

The authors would be remiss not to mention that Example 120 still holds great educational value. Much can be learned about logarithms and exponentials by verifying the solutions obtained in Example 120 analytically. For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and obtain

$$\begin{array}{rcll} 2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\ 2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\ 2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\ 2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\ 2000 & \stackrel{\checkmark}{=} & 2000 & \end{array}$$

The other solutions can be verified by using a combination of log and inverse properties. Some fall out quite quickly, while others are more involved. We leave them to the reader.

Since exponential functions are continuous on their domains, the Intermediate Value Theorem 19 applies. As with the algebraic functions in Section 6.3, this allows us to solve inequalities using sign diagrams as demonstrated below.

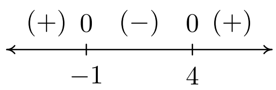
Example 121 Exponential inequalities

Solve the following inequalities. Check your answer graphically using a computer or calculator.

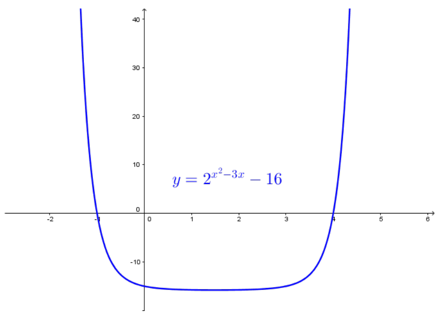
1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $xe^{2x} < 4x$

Solution

1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$. The domain of r is all real numbers, so in order to construct our sign diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by the one-to-one property of exponential functions, $x^2 - 3x = 4$. Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$, is on or above the x-axis: see Figure 7.20.



Sign diagram for $r(x) = 2^{x^2-3x} - 16$



$y = r(x) = 2^{x^2-3x} - 16$

Figure 7.20: Solving $2^{x^2-3x} - 16 \geq 0$

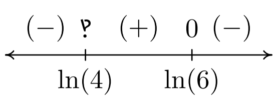
2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned} \frac{e^x}{e^x - 4} &\leq 3 \\ \frac{e^x}{e^x - 4} - 3 &\leq 0 \\ \frac{e^x}{e^x - 4} - \frac{3\left(e^x - 4\right)}{e^x - 4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12 - 2e^x}{e^x - 4} &\leq 0 \end{aligned}$$

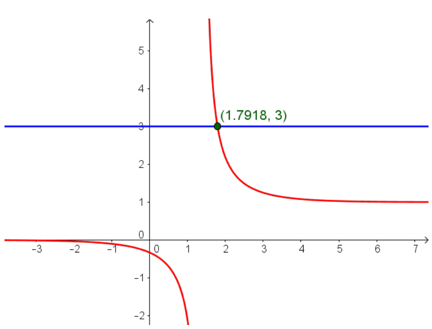
We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. (This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.) While the prospect of determining the sign of $r\left(\ln(3)\right)$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r\left(\ln(3)\right) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly. (We could, of course, use the calculator, but what fun would that be?) From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$. Using GeoGebra, we see the graph of $f(x) = \frac{e^x}{e^x-4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x = \ln(6) \approx 1.792$.



Sign diagram for $r(x) = \dfrac{12 - 2e^x}{e^x - 4}$



$y = f(x) = \dfrac{e^x}{e^x - 4}$

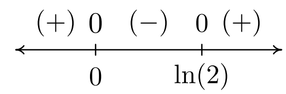
$y = g(x) = 3$

Figure 7.21: Solving $\dfrac{e^x}{e^x - 4} \leq 3$

3. As before, we start solving $xe^{2x} < 4x$ by getting 0 on one side of the inequality, $xe^{2x} - 4x < 0$. We set $r(x) = xe^{2x} - 4x$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers. Setting $r(x) = 0$ produces $xe^{2x} - 4x = 0$. We factor to get $x\left(e^{2x} - 4\right) = 0$ which gives $x = 0$ or $e^{2x} - 4 = 0$. To solve the latter, we isolate the exponential and take logs to get $2x = \ln(4)$, or $x = \frac{\ln(4)}{2} = \ln(2)$. (Can you explain the last equality using properties of logs?) As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating, we get

$$\begin{aligned} r\left(\ln\left(\tfrac{1}{2}\right)\right) &= \ln\left(\tfrac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\tfrac{1}{2}\right) \\ &= \ln\left(\tfrac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\tfrac{1}{2}\right) && \text{Power Rule} \\ &= \ln\left(\tfrac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{4}\ln\left(\tfrac{1}{2}\right) - 4\ln\left(\tfrac{1}{2}\right) = -\tfrac{15}{4}\ln\left(\tfrac{1}{2}\right) \end{aligned}$$

Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$, so $r(x) < 0$ on $(0, \ln(2))$. Plotting in GeoGebra confirms that the graph of $f(x) = xe^{2x}$ is below the graph of $g(x) = 4x$ on these intervals: see Figure 7.22. (Note: $\ln(2) \approx 0.693$.)



Sign diagram for $r(x) = xe^{2x} - 4x$

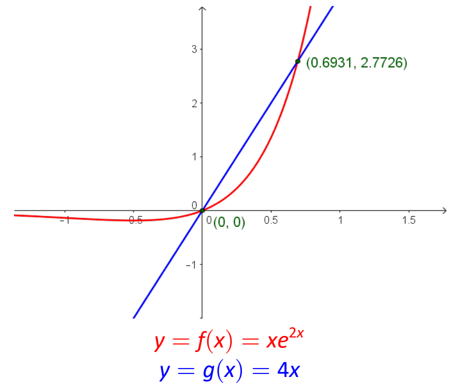


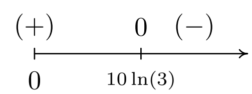
Figure 7.22: Solving $xe^{2x} < 4x$

Example 122 **Newton's Law of Cooling**

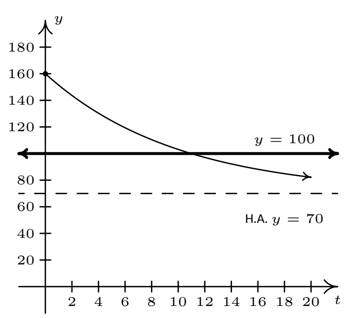
Recall from Example 113 that the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modelled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Solution We need to find when $T(t) > 100$, or in other words, we need to solve the inequality $70 + 90e^{-0.1t} > 100$. Getting 0 on one side of the inequality, we have $90e^{-0.1t} - 30 > 0$, and we set $r(t) = 90e^{-0.1t} - 30$. The domain of r is artificially restricted due to the context of the problem to $[0, \infty)$, so we proceed to find the zeros of r. Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which, after a quick application of the Power Rule leaves us with $t = 10\ln(3)$. If we wish to avoid using the calculator to choose test values, we note that since $1 < 3$, $0 = \ln(1) < \ln(3)$ so that $10\ln(3) > 0$. So we choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $10\ln(3) < 10\ln(4)$, so the latter is our choice of a test value for the interval $(10\ln(3), \infty)$. Our sign diagram is given in Figure 7.23, along with our graph of $y = T(t)$ from Example 113 with the horizontal line $y = 100$ shown.

In order to interpret what this means in the context of the real world, we need a reasonable approximation of the number $10\ln(3) \approx 10.986$. This means it takes approximately 11 minutes for the coffee to cool to 100°F. Until then, the coffee is warmer than that.



Sign diagram for $r(t) = 90e^{-0.1t} - 30$



$y = T(t)$ and $y = 100$

Figure 7.23: Solving $T(t) = 100$ in Example 122

We close this section by finding the inverse of a function which is a composition of a rational function with an exponential function.

Example 123 **Inverting a fractional exponential function**

The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$
\begin{aligned}
y &= \frac{5e^x}{e^x+1} \\
x &= \frac{5e^y}{e^y+1} && \text{Switch } x \text{ and } y \\
x\left(e^y+1\right) &= 5e^y \\
xe^y + x &= 5e^y \\
x &= 5e^y - xe^y \\
x &= e^y(5-x) \\
e^y &= \frac{x}{5-x} \\
\ln\left(e^y\right) &= \ln\left(\frac{x}{5-x}\right) \\
y &= \ln\left(\frac{x}{5-x}\right)
\end{aligned}
$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this to the reader. To verify our solution graphically, we graph $y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$ on the same set of axes and observe the symmetry about the line $y = x$ in Figure 7.24. Note the domain of f is the range of g and vice-versa.

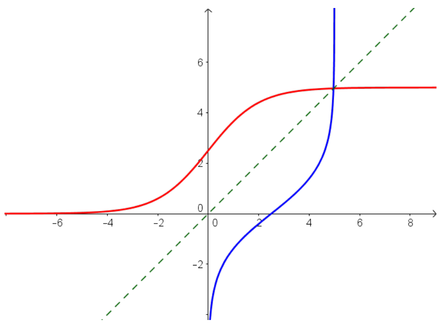


Figure 7.24: $y = f(x) = \frac{5e^x}{e^x+1}$
$y = g(x) = \ln\left(\frac{x}{5-x}\right)$

Exercises 7.3

Problems

In Exercises 1 – 33, solve the equation analytically.

1. $2^{4x} = 8$
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2x} = \frac{1}{2}$
5. $8^{x} = \frac{1}{128}$
6. $2^{(x^3-x)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-x} = 2$
11. $5^{x} = -2$
12. $3^{(x-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1 - e^{2x}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1x} = 20$
19. $\dfrac{100e^{x}}{e^{x}+2} = 50$
20. $\dfrac{5000}{1+2e^{-3t}} = 2500$
21. $\dfrac{150}{1+29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^{x} = 10$
23. $e^{2x} = 2e^{x}$
24. $7e^{2x} = 28e^{-6x}$
25. $3^{(x-1)} = 2^{x}$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2x} - 3e^{x} - 10 = 0$
29. $e^{2x} = e^{x} + 6$
30. $4^{x} + 2^{x} = 12$
31. $e^{x} - 3e^{-x} = 2$
32. $e^{x} + 15e^{-x} = 8$
33. $3^{x} + 25 \cdot 3^{-x} = 10$

In Exercises 34 – 39, solve the inequality analytically.

34. $e^{x} > 53$
35. $1000\,(1.005)^{12t} \geq 3000$
36. $2^{(x^3-x)} < 1$
37. $25\left(\frac{4}{5}\right)^{x} \geq 10$
38. $\dfrac{150}{1+29e^{-0.8t}} \leq 130$
39. $70 + 90e^{-0.1t} \leq 75$

In Exercises 40 – 45, use your computer or calculator to help you solve the equation or inequality.

40. $2^{x} = x^{2}$
41. $e^{x} = \ln(x) + 5$
42. $e^{\sqrt{x}} = x + 1$
43. $e^{-x} - xe^{-x} \geq 0$
44. $3^{(x-1)} < 2^{x}$
45. $e^{x} < x^{3} - x$
46. Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram.
47. Use the technique in Exercise 46 to solve the inequalities in Exercises 34 - 39. (NOTE: Isolate the exponential function first!)
48. Compute the inverse of $f(x) = \dfrac{e^{x} - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.

49. In Example 123, we found that the inverse of $f(x) = \frac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.

   (a) Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.

   (b) Find the range of f by finding the domain of f^{-1}.

   (c) Let $g(x) = \frac{5x}{x+1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise 31 in Section 6.2, but it's nice to see a specific example of the property.)

50. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

7.4 Logarithmic Equations and Inequalities

In Section 7.3 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from. For example, suppose we wish to solve $\log_2(x) = \log_2(5)$. Theorem 45 tells us that the *only* solution to this equation is $x = 5$. Now suppose we wish to solve $\log_2(x) = 3$. If we want to use Theorem 45, we need to rewrite 3 as a logarithm base 2. We can use Theorem 44 to do just that: $3 = \log_2\left(2^3\right) = \log_2(8)$. Our equation then becomes $\log_2(x) = \log_2(8)$ so that $x = 8$. However, we could have arrived at the same answer, in fewer steps, by using Theorem 44 to rewrite the equation $\log_2(x) = 3$ as $2^3 = x$, or $x = 8$. We summarize the two common ways to solve log equations below.

Key Idea 28 Steps for Solving an Equation involving Logarithmic Functions

1. Isolate the logarithmic function.
2. (a) If convenient, express both sides as logs with the same base and equate the arguments of the log functions.
   (b) Otherwise, rewrite the log equation as an exponential equation.

Example 124 Logarithmic equations

Solve the following equations. Check your solutions graphically using a computer or calculator.

1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(x-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2x) = 1 - \log_7(3-x)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(x+1) = 2\log_2(x)$

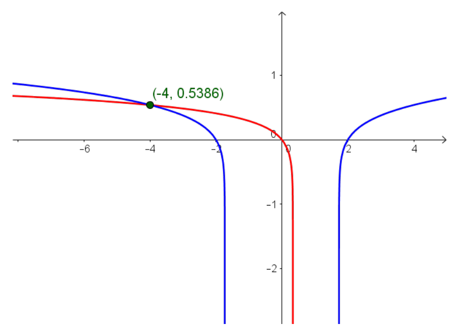


Figure 7.25: $y = f(x) = \log_{117}(1-3x)$ and $y = g(x) = \log_{117}(x^2-3)$

Solution

1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate what's inside the logs to get $1-3x = x^2-3$. Solving $x^2+3x-4 = 0$ gives $x = -4$ and $x = 1$. To check these answers using the calculator, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln\left(x^2-3\right)}{\ln(117)}$ and we see they intersect only at $x = -4$. To see what happened to the solution $x = 1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number, which means $x = 1$ is not in the domain of the original equation, and is not a solution. Using GeoGebra to solve the equation graphically gives us Figure 7.25.

2. Our first objective in solving $2 - \ln(x-3) = 1$ is to isolate the logarithm. We get $\ln(x-3) = 1$, which, as an exponential equation, is $e^1 = x-3$. We get our solution $x = e+3$. In Figure 7.26, we see the graph of $f(x) = 2-\ln(x-3)$ intersects the graph of $g(x) = 1$ at $x = e+3 \approx 5.718$.

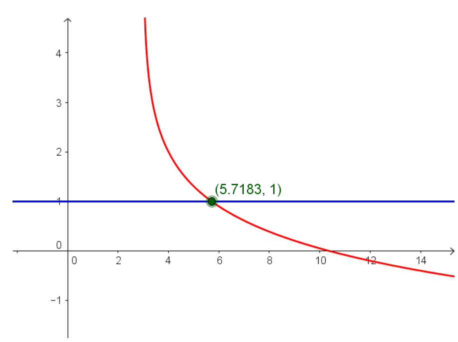


Figure 7.26: $y = f(x) = 2 - \ln(x-3)$ and $y = g(x) = 1$

3. We can start solving $\log_6(x+4)+\log_6(3-x)=1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6\left[(x+4)(3-x)\right]=1$. Rewriting this as an exponential equation, we get $6^1=(x+4)(3-x)$. This reduces to $x^2+x-6=0$, which gives $x=-3$ and $x=2$. Graphing $y=f(x)=\frac{\ln(x+4)}{\ln(6)}+\frac{\ln(3-x)}{\ln(6)}$ and $y=g(x)=1$, we see they intersect twice, at $x=-3$ and $x=2$ (Figure 7.27).

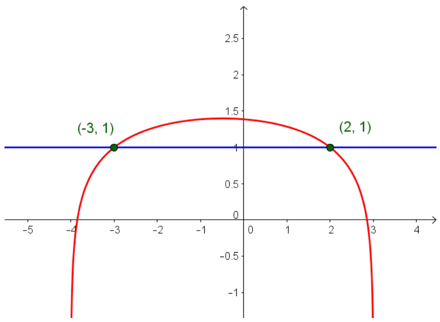


Figure 7.27: $y=f(x)=\log_6(x+4)+\log_6(3-x)$ and $y=g(x)=1$

4. Taking a cue from the previous problem, we begin solving $\log_7(1-2x)=1-\log_7(3-x)$ by first collecting the logarithms on the same side, $\log_7(1-2x)+\log_7(3-x)=1$, and then using the Product Rule to get $\log_7[(1-2x)(3-x)]=1$. Rewriting this as an exponential equation gives $7^1=(1-2x)(3-x)$ which gives the quadratic equation $2x^2-7x-4=0$. Solving, we find $x=-\frac{1}{2}$ and $x=4$. Graphing, we find $y=f(x)=\frac{\ln(1-2x)}{\ln(7)}$ and $y=g(x)=1-\frac{\ln(3-x)}{\ln(7)}$ intersect only at $x=-\frac{1}{2}$: see Figure 7.28. Checking $x=4$ in the original equation produces $\log_7(-7)=1-\log_7(-1)$, which is a clear domain violation.

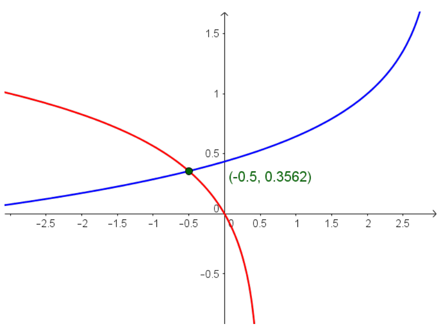


Figure 7.28: $y=f(x)=\log_7(1-2x)$ and $y=g(x)=1-\log_7(3-x)$

5. Starting with $\log_2(x+3)=\log_2(6-x)+3$, we gather the logarithms to one side and get $\log_2(x+3)-\log_2(6-x)=3$. We then use the Quotient Rule and convert to an exponential equation

$$\log_2\left(\frac{x+3}{6-x}\right)=3 \iff 2^3=\frac{x+3}{6-x}$$

This reduces to the linear equation $8(6-x)=x+3$, which gives us $x=5$. When we graph $f(x)=\frac{\ln(x+3)}{\ln(2)}$ and $g(x)=\frac{\ln(6-x)}{\ln(2)}+3$, we find they intersect at $x=5$: see Figure 7.29.

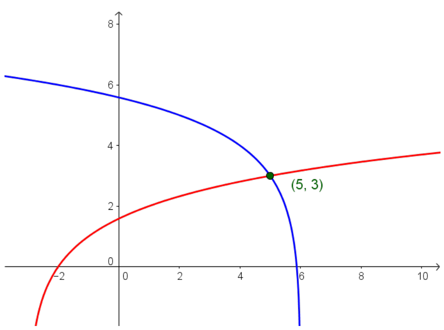


Figure 7.29: $y=f(x)=\log_2(x+3)$ and $y=g(x)=log_2(6-x)+3$

6. Starting with $1+2\log_4(x+1)=2\log_2(x)$, we gather the logs to one side to get the equation $1=2\log_2(x)-2\log_4(x+1)$. Before we can combine the logarithms, however, we need a common base. Since 4 is a power of 2, we use change of base to convert

$$\log_4(x+1)=\frac{\log_2(x+1)}{\log_2(4)}=\frac{1}{2}\log_2(x+1)$$

Hence, our original equation becomes

$$\begin{array}{rcll} 1 & = & 2\log_2(x)-2\left(\frac{1}{2}\log_2(x+1)\right) & \\ 1 & = & 2\log_2(x)-\log_2(x+1) & \\ 1 & = & \log_2\left(x^2\right)-\log_2(x+1) & \text{Power Rule} \\ 1 & = & \log_2\left(\dfrac{x^2}{x+1}\right) & \text{Quotient Rule} \end{array}$$

Rewriting this in exponential form, we get $\frac{x^2}{x+1}=2$ or $x^2-2x-2=0$. Using the quadratic formula, we get $x=1\pm\sqrt{3}$. Graphing $f(x)=1+\frac{2\ln(x+1)}{\ln(4)}$ and $g(x)=\frac{2\ln(x)}{\ln(2)}$, we see in Figure 7.30 that the graphs intersect only at $x=1+\sqrt{3}\approx 2.732$. The solution $x=1-\sqrt{3}<0$, which means if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined.

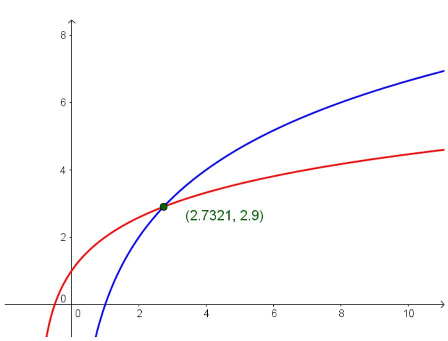


Figure 7.30: $y=f(x)=1+2\log_4(x+1)$ and $y=g(x)=2\log_2(x)$

If nothing else, Example 124 demonstrates the importance of checking for extraneous solutions when solving equations involving logarithms. (Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.) Even though we checked our answers graphically, extraneous solutions are easy to spot - any supposed solution which causes a negative number inside a logarithm needs to be discarded. As with the equations in Example 120, much can be learned from checking all of the answers in Example 124 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

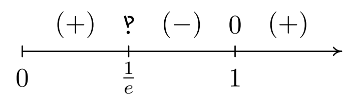
Example 125 Logarithmic inequalities

Solve the following inequalities. Check your answer graphically using a computer or calculator.

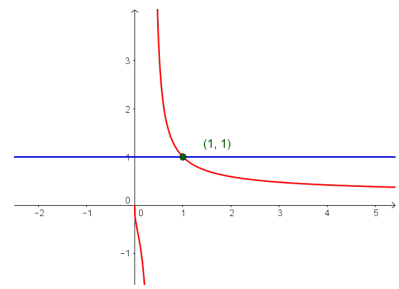
1. $\dfrac{1}{\ln(x)+1} \leq 1$
2. $(\log_2(x))^2 < 2\log_2(x)+3$
3. $x\log(x+1) \geq x$

Solution

1. We start solving $\dfrac{1}{\ln(x)+1} \leq 1$ by getting 0 on one side of the inequality: $\dfrac{1}{\ln(x)+1} - 1 \leq 0$. Getting a common denominator yields $\dfrac{1}{\ln(x)+1} - \dfrac{\ln(x)+1}{\ln(x)+1} \leq 0$ which reduces to $\dfrac{-\ln(x)}{\ln(x)+1} \leq 0$, or $\dfrac{\ln(x)}{\ln(x)+1} \geq 0$. We define $r(x) = \dfrac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x > 0$. In order to keep the denominator away from zero, we solve $\ln(x)+1 = 0$ so $\ln(x) = -1$, so $x = e^{-1} = \frac{1}{e}$. Hence, the domain of r is $\left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$. To find the zeros of r, we set $r(x) = \dfrac{\ln(x)}{\ln(x)+1} = 0$ so that $\ln(x) = 0$, and we find $x = e^0 = 1$. In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e \approx 2.718 > 1$, $0 < \frac{1}{e^2} < \frac{1}{e} < \frac{1}{\sqrt{e}} < 1 < e$. To determine the sign of $r\left(\frac{1}{e^2}\right)$, we use the fact that $\ln\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) = -2$, and find $r\left(\frac{1}{e^2}\right) = \frac{-2}{-2+1} = 2$, which is $(+)$. The rest of the test values are determined similarly. From our sign diagram, we find the solution to be $\left(0, \frac{1}{e}\right) \cup [1, \infty)$. Graphing $f(x) = \frac{1}{\ln(x)+1}$ and $g(x) = 1$, we see in Figure 7.31 the graph of f is below the graph of g on the solution intervals, and that the graphs intersect at $x = 1$.



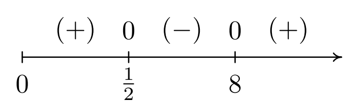
Sign diagram for $r(x) = \dfrac{\ln(x)}{\ln(x)+1}$



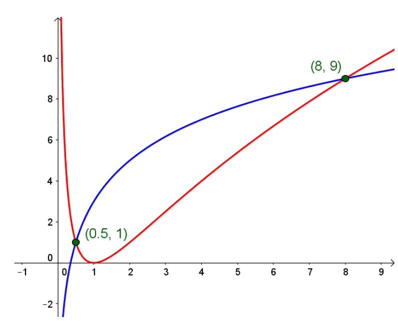
$y = f(x) = \dfrac{1}{\ln(x)+1}$ and $y = g(x) = 1$

Figure 7.31: Solving $\dfrac{1}{\ln(x)+1} \leq 1$

2. Moving all of the nonzero terms of $(\log_2(x))^2 < 2\log_2(x)+3$ to one side of the inequality, we have $(\log_2(x))^2 - 2\log_2(x) - 3 < 0$. Defining $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3 = 0$ which results in a 'quadratic in disguise.' We set $u = \log_2(x)$ so our equation becomes $u^2 - 2u - 3 = 0$ which gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, which gives us $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which yields $x = 2^3 = 8$. We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$, and from our sign diagram, we see $r(x) < 0$ on $\left(\frac{1}{2}, 8\right)$. Geometrically, we see the

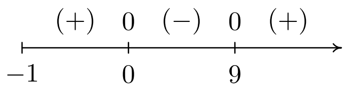


Sign diagram for $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$

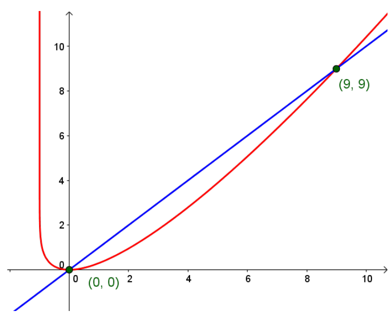


$y = f(x) = (\log_2(x))^2$ and $y = g(x) = 2\log_2(x)+3$

Figure 7.32: Solving $(\log_2(x))^2 < 2\log_2(x)+3$



Sign diagram for $r(x) = x\log(x+1) - x$



$y = f(x) = x\log(x+1)$
and $y = g(x) = x$

Figure 7.33: Solving $x\log(x+1) \geq x$

graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on the solution interval: see Figure 7.32.

3. We begin to solve $x\log(x+1) \geq x$ by subtracting x from both sides to get $x\log(x+1) - x \geq 0$. We define $r(x) = x\log(x+1) - x$ and due to the presence of the logarithm, we require $x + 1 > 0$, or $x > -1$. To find the zeros of r, we set $r(x) = x\log(x+1) - x = 0$. Factoring, we get $x\left(\log(x+1) - 1\right) = 0$, which gives $x = 0$ or $\log(x+1) - 1 = 0$. The latter gives $\log(x+1) = 1$, or $x + 1 = 10^1$, which admits $x = 9$. We select test values x so that $x + 1$ is a power of 10, and we obtain $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$. Our sign diagram gives the solution to be $(-1, 0] \cup [9, \infty)$. Figure 7.33 indicates the graph of $y = f(x) = x\log(x+1)$ is above $y = g(x) = x$ on the solution intervals, and the graphs intersect at $x = 0$ and $x = 9$.

Our next example revisits the concept of pH first seen in Exercise 77 in Section 7.1.

Example 126 Calculating pH range

In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration, and check your answer using a calculator.

Solution Recall from Exercise 77 in Section 7.1 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-7.8 \geq \log[\text{H}^+] \geq -8.5$. To solve this compound inequality we solve $-7.8 \geq \log[\text{H}^+]$ and $\log[\text{H}^+] \geq -8.5$ and take the intersection of the solution sets. (Refer to page 2 for a discussion of what this means.) The former inequality yields $0 < [\text{H}^+] \leq 10^{-7.8}$ and the latter yields $[\text{H}^+] \geq 10^{-8.5}$. Taking the intersection gives us our final answer $10^{-8.5} \leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) After carefully adjusting the viewing window on GeoGebra we see that the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.16 \times 10^{-9}, 1.58 \times 10^{-8}]$: see Figure 7.34.

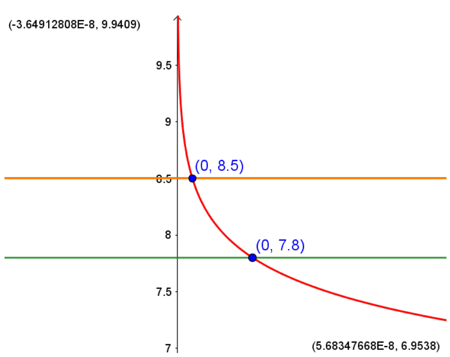


Figure 7.34: The graphs of $y = f(x) = -\log(x)$, $y = 7.8$ and $y = 8.5$

We close this section by finding an inverse of a one-to-one function which involves logarithms.

Example 127 Inverting a fractional logarithmic function

The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution We first write $y = f(x)$ then interchange the x and y and solve for y.

$$
\begin{aligned}
y &= f(x) \\
y &= \frac{\log(x)}{1-\log(x)} \\
x &= \frac{\log(y)}{1-\log(y)} && \text{Interchange } x \text{ and } y. \\
x\,(1-\log(y)) &= \log(y) \\
x - x\log(y) &= \log(y) \\
x &= x\log(y) + \log(y) \\
x &= (x+1)\log(y) \\
\frac{x}{x+1} &= \log(y) \\
y &= 10^{\frac{x}{x+1}} && \text{Rewrite as an exponential equation.}
\end{aligned}
$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} in GeoGebra gives us Figure 7.35.

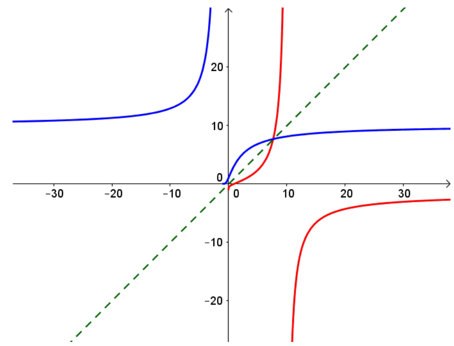


Figure 7.35: $y = f(x) = \dfrac{\log(x)}{1-\log(x)}$ and $y = g(x) = 10^{\frac{x}{x+1}}$

Exercises 7.4

Problems

In Exercises 1 – 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-x^2\right) = \ln(2-x)$
4. $\log_5\left(18-x^2\right) = \log_5(6-x)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(x^2-99\right) = 0$
8. $\log(x^2-3x) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6 - 3\log_5(2x) = 0$
14. $3\ln(x) - 2 = 1 - \ln(x)$
15. $\log_3(x-4) + \log_3(x+4) = 2$
16. $\log_5(2x+1) + \log_5(x+2) = 1$
17. $\log_{169}(3x+7) - \log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1) - \ln(x) = 3$
19. $2\log_7(x) = \log_7(2) + \log_7(x+12)$
20. $\log(x) - \log(2) = \log(x+8) - \log(x+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$
22. $\ln(\ln(x)) = 3$
23. $(\log(x))^2 = 2\log(x) + 15$
24. $\ln(x^2) = (\ln(x))^2$

In Exercises 25 – 30, solve the inequality analytically.

25. $\dfrac{1-\ln(x)}{x^2} < 0$
26. $x\ln(x) - x > 0$
27. $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$
28. $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$
29. $2.3 < -\log(x) < 5.4$
30. $\ln(x^2) \leq (\ln(x))^2$

In Exercises 31 – 34, use your calculator or computer to help you solve the equation or inequality.

31. $\ln(x) = e^{-x}$
32. $\ln(x) = \sqrt[4]{x}$
33. $\ln(x^2+1) \geq 5$
34. $\ln(-2x^3 - x^2 + 13x - 6) < 0$
35. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x+1) < 3$ without a sign diagram.
36. Use the technique from Exercise 35 to solve the inequalities in Exercises 27 - 29. (Compare this to Exercise 46 in Section 7.3.)
37. Solve $\ln(3-y) - \ln(y) = 2x + \ln(5)$ for y.
38. In Example 127 we found the inverse of $f(x) = \dfrac{\log(x)}{1-\log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.
    (a) Show that $\left(f^{-1}\circ f\right)(x) = x$ for all x in the domain of f and that $\left(f\circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
    (b) Find the range of f by finding the domain of f^{-1}.
    (c) Let $g(x) = \dfrac{x}{1-x}$ and $h(x) = \log(x)$. Show that $f = g\circ h$ and $(g\circ h)^{-1} = h^{-1}\circ g^{-1}$. (We know this is true in general by Exercise 31 in Section 6.2, but it's nice to see a specific example of the property.)
39. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.
40. Explain the equation in Exercise 10 and the inequality in Exercise 28 above in terms of the Richter scale for earthquake magnitude. (See Exercise 75 in Section 7.1.)

41. Explain the equation in Exercise 12 and the inequality in Exercise 27 above in terms of sound intensity level as measured in decibels. (See Exercise 76 in Section 7.1.)

42. Explain the equation in Exercise 11 and the inequality in Exercise 29 above in terms of the pH of a solution. (See Exercise 77 in Section 7.1.)

7.5 Applications of Exponential and Logarithmic Functions

As we mentioned in Section 7.1, exponential and logarithmic functions are used to model a wide variety of behaviours in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, the calculator is often used to express our answers as decimal approximations.

7.5.1 Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have $\$100$ to invest at your local bank and they are offering a whopping 5 $\%$ annual percentage interest rate. This means that after one year, the bank will pay *you* 5% of that $\$100$, or $\$100(0.05) = \$5$ in interest, so you now have $\$105$. (How generous of them!) This is in accordance with the formula for *simple interest* which you have undoubtedly run across at some point before.

Key Idea 29 Simple Interest

The amount of interest I accrued at an annual rate r on an investment (called the **principal**) P after t years is

$$I = Prt$$

The amount A in the account after t years is given by

$$A = P + I = P + Prt = P(1 + rt)$$

Suppose, however, that six months into the year, you hear of a better deal at a rival bank. (Some restrictions may apply.) Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial $\$100$ yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest. You take your $\$102.50$ off to the competitor and find out that those restrictions which *may* apply actually do apply to you, and you return to your bank which happily accepts your $\$102.50$ for the remaining six months of the year. To your surprise and delight, at the end of the year your statement reads $\$105.06$, not $\$105$ as you had expected. (Actually, the final balance should be $\$105.0625$.) Where did those extra six cents come from? For the first six months of the year, interest was earned on the original principal of $\$100$, but for the second six months, interest was earned on $\$102.50$, that is, you earned interest on your interest. This is the basic concept behind **compound interest**. In the previous discussion, we would say that the interest was compounded twice, or semiannually. (Using this convention, simple interest after one year is the same as compounding the interest only once.) If more money can be earned by earning interest on interest already earned, a natural question to ask is what happens if the interest is compounded more often, say 4 times a year, which is every three months, or 'quarterly.' In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1+rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the $\$101.25$ for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is $\$103.79$, and, at last, we obtain $\$105.08$. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound. In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{th}}$ of a year between compoundings. Let A_k denote the amount in the account after the k^{th} compounding. Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$. Since we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

Key Idea 30 Compounded Interest

If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount A in the account after t years is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 30 becomes $A(t) = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A(t) = 100(1.0125)^{4t}$. To check this new formula against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 128 Computing compound interest

Suppose $\$2000$ is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. (See Definition 32 in Section 3.1.)

Solution

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly*) into Equation 30 yields $A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$.
2. Since t represents the length of the investment in years, we substitute $t = 5$ into $A(t)$ to find $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$. After 5 years, we have approximately $\$2852.92$.

3. Our initial investment is $\$2000$, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$. Taking natural logs as in Section 7.3, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.

4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute $\frac{A(5)-A(4)}{5-4} \approx 195.63$. Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$. This means that the value of the investment is increasing at a rate of approximately $\$195.63$ per year between the end of the fourth and fifth years, while that rate jumps to $\$1648.21$ per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.

In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions. We'll have more to say about this once we reach Calculus.

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit. Consider an investment of $\$1$ invested at 100% interest for 1 year compounded n times a year. Equation 30 tells us that the amount of money in the account after 1 year is $A = \left(1+\frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing. We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest. With Calculus, we can show (or define, depending on your point of view) that as $n \to \infty$, $A = \left(1+\frac{1}{n}\right)^n \to e$, where e is the natural base first presented in Section 7.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 49 An interesting definition of e

If you invest $\$1$ at 100% interest compounded continuously, then you will have $\$e$ at the end of one year.

Using this definition of e and a little Calculus, we can take Equation 30 and produce a formula for continuously compounded interest.

Key Idea 31 Continuously Compounded Interest

If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount A in the account after t years is

$$A(t) = Pe^{rt}$$

If we take the scenario of Example 128 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about $\$24{,}035.28$, whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$ which is about $\$24{,}213.18$ - a difference of less than 1%.

Equations 30 and 31 both use exponential functions to describe the growth of an investment. Curiously enough, the same principles which govern compound interest are also used to model short term growth of populations. In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time. In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Its solution is stated below.

The average rate of change of a function over an interval was first introduced in Section 3.1. *Instantaneous* rates of change are the business of Calculus, as is mentioned on Page 117.

Key Idea 32 Uninhibited growth

If a population increases according to The Law of Uninhibited Growth, the number of organisms N at time t is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)$$

It is worth taking some time to compare Equations 31 and 32. In Equation 31, we use P to denote the initial investment; in Equation 32, we use N_0 to denote the initial population. In Equation 31, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 32 corresponds to a growth rate as well. While Equations 31 and 32 look entirely different, they both represent the same mathematical concept.

Example 129 Modelling cell growth

In order to perform atherosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, N, in thousands, after t days.

SOLUTION We begin with $N(t) = N_0 e^{kt}$. Since N is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$. In order to complete the formula, we need to determine the growth rate k. We know that after one week, the number of cells has grown to five million. Since t measures days and the units of N are in thousands, this translates mathematically to $N(7) = 5000$. We get the equation $12e^{7k} = 5000$ which gives $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$. Hence, $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$. Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells.

Whereas Equations 31 and 32 model the growth of quantities, we can use equations like them to describe the decline of quantities. One example we've seen already is Example 112 in Section 7.1. There, the value of a car declined from its purchase price of $\$25{,}000$ to nothing at all. Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays. This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 32 with the exception that the rate constant k is negative.

Key Idea 33 Radioactive Decay

The amount of a radioactive element A at time t is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } A(t) \text{ at time } t) = k\,A(t)$$

Example 130 Radioactive decay of iodine

Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 33, and that the half-life (the time it takes for half of the substance to decay) of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

Solution Since we start with 5 grams initially, Equation 33 gives $A(t) = 5e^{kt}$. Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Hence, $A(8) = 2.5$ which means $5e^{8k} = 2.5$. Solving, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily. Hence, $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$.

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 113 of Section 7.1. In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where t was measured in minutes. In this situation, we know the physical limit of the temperature of the coffee is room temperature, and the differential equation which gives rise to our formula for $T(t)$ takes this into account. Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature gap exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly. Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

Key Idea 34 Newton's Law of Cooling (Warming)

The temperature T of an object at time t is given by the formula

$$T(t) = T_a + (T_0 - T_a)\, e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature (that is, the temperature of the surroundings) and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } T(t) \text{ at time } t) = k\ (T(t) - T_a)$$

The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

If we re-examine the situation in Example 113 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 34, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given. The rate constant $k = 0.1$ indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings. Note in Equation 34 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 131 Newton's Law of warming

A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast T as a function of its time in the oven, t, in hours.

2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming tells us $T(t) = 350 + (40 - 350)e^{-kt}$, or $T(t) = 350 - 310e^{-kt}$. To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature

function is

$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$

2. To determine when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. It takes roughly 3 hours and 15 minutes to cook the roast completely.

If we had taken the time to graph $y = T(t)$ in Example 131, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion by applying a bit of 'number sense'. As $t \to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350-$very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F (at which point it would be more toast than roast). The function T is sometimes called a **limited** growth model, since the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate. The **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Key Idea 35 Logistic Growth

If a population behaves according to the assumptions of logistic growth, the number of organisms N at time t is given by the equation

$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$

where $N(0) = N_0$ is the initial population, L is the limiting population, (that is, as $t \to \infty$, $N(t) \to L$) C is a measure of how much room there is to grow given by

$$C = \frac{L}{N_0} - 1.$$

and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)\,(L - N(t))$$

The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumours.

Example 132 Modelling spread of rumours

The number of people N, in hundreds, at a local community college who have heard the rumour 'Carl is afraid of Virginia Woolf' can be modelled using the logistic equation

$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

where $t \geq 0$ is the number of days after April 1, 2009.

1. Find and interpret $N(0)$.
2. Find and interpret the end behaviour of $N(t)$.
3. How long until 4200 people have heard the rumour?
4. Check your answers to 2 and 3 using your computer or calculator.

Solution

1. We find $N(0) = \frac{84}{1+2799e^0} = \frac{84}{2800} = \frac{3}{100}$. Since $N(t)$ measures the number of people who have heard the rumour in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2009, we may conclude that on that day, 3 people have heard the rumour.(Or, more likely, three people started the rumour. I'd wager Jeff, Jamie, and Jason started it. So much for telling your best friends something in confidence!)

2. We could simply note that $N(t)$ is written in the form of Equation 35, and identify $L = 84$. However, to see why the answer is 84, we proceed analytically. Since the domain of N is restricted to $t \geq 0$, the only end behaviour of significance is $t \to \infty$. As we've seen before, (see, for example, Example 113) as $t \to \infty$, we have $1997e^{-t} \to 0^+$ and so $N(t) \approx \frac{84}{1+\text{very small }(+)} \approx 84$. Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumour approaches 8400.

3. To find how long it takes until 4200 people have heard the rumour, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$. It takes around 8 days until 4200 people have heard the rumour.

4. We graph $y = N(x)$ using the calculator and see in Figure 7.36 that the line $y = 84$ is the horizontal asymptote of the graph, confirming our answer to part 2, and the graph intersects the line $y = 42$ at $x = \ln(2799) \approx 7.937$ in Figure 7.37, which confirms our answer to part 3.

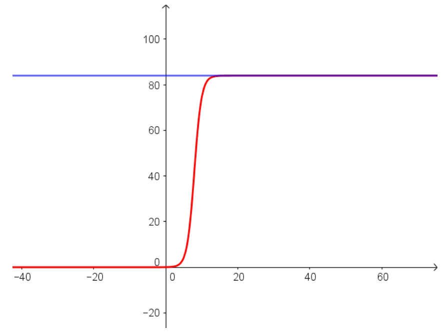


Figure 7.36: $y = \dfrac{84}{1+2799e^{-x}}$ and $y = 84$

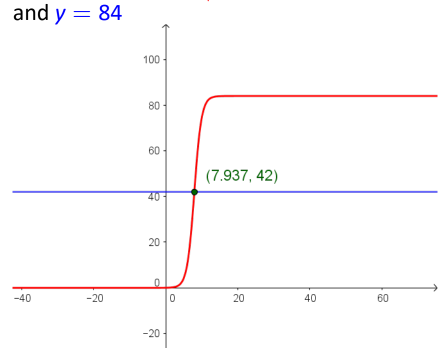


Figure 7.37: $y = \dfrac{84}{1+2799e^{-x}}$ and $y = 42$

If we take the time to analyze the graph of $y = N(x)$ above, we can see graphically how logistic growth combines features of uninhibited and limited growth. The curve seems to rise steeply, then at some point, begins to level off. The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. At this point, even though the function is still increasing, the rate at which it does so begins to decline. It turns out the point of diminishing returns always occurs at half the limiting population. (In our case, when $y = 42$.) While these concepts are more precisely quantified using Calculus, Figures 7.38 and 7.39 give two views of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth; the latter like limited growth.

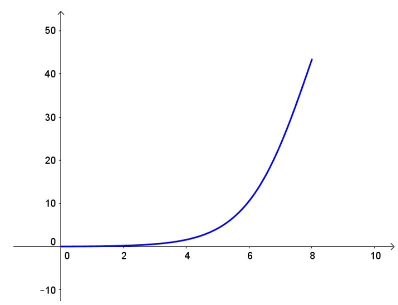


Figure 7.38: $y = \dfrac{84}{1+2799e^{-x}}$ for $0 \leq x \leq 8$

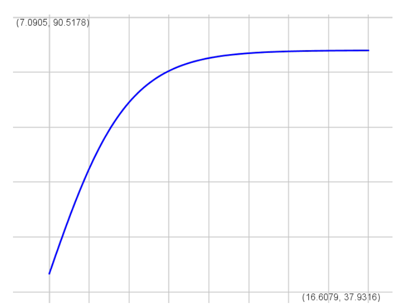


Figure 7.39: $y = \dfrac{84}{1+2799e^{-x}}$ for $8 \leq x \leq 16$

7.5.2 Applications of Logarithms

Just as many physical phenomena can be modelled by exponential functions, the same is true of logarithmic functions. In Exercises 75, 76 and 77 of Section 7.1, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 133 Password strengh

The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive (that is, upper and lower case letters are treated as different characters) password is comprised of letters and numbers only, find the associated information entropy.
2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.
2. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from. (Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increased.)

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 77 in Section 7.1.

Example 134 Buffer solutions

Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr.

The equation for blood pH in Example 134 is derived from the Henderson-Hasselbalch Equation. See Exercise 43 in Section 7.2. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

Exercises 7.5

Problems

For each of the scenarios given in Exercises 1 – 6,

- **Find the amount A in the account as a function of the term of the investment t in years.**
- **Determine how much is in the account after 5 years, 10 years, 30 years and 35 years. Round your answers to the nearest cent.**
- **Determine how long will it take for the initial investment to double. Round your answer to the nearest year.**
- **Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.**

1. $\$500$ is invested in an account which offers 0.75%, compounded monthly.

2. $\$500$ is invested in an account which offers 0.75%, compounded continuously.

3. $\$1000$ is invested in an account which offers 1.25%, compounded monthly.

4. $\$1000$ is invested in an account which offers 1.25%, compounded continuously.

5. $\$5000$ is invested in an account which offers 2.125%, compounded monthly.

6. $\$5000$ is invested in an account which offers 2.125%, compounded continuously.

7. Look back at your answers to Exercises 1 - 6. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.

8.

9. How much money needs to be invested now to obtain $\$5000$ in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.

10. On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation 30 to answer the following.

    (a) If $P = 2000$ what is $A(8)$?

    (b) Solve the equation $A(t) = 4000$ for t.

    (c) What principal P should be invested so that the account balance is \$2000 is three years?

11. Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

    (a) If $P = 2000$ what is $A(8)$?

    (b) Solve the equation $A(t) = 4000$ for t.

    (c) What principal P should be invested so that the account balance is \$2000 in three years?

    (d) The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

12. A finance company offers a promotion on $\$5000$ loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?

13. Use Equation 30 to show that the time it takes for an investment to double in value does not depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested (pun intended!) in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

In Exercises 14 – 18, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- **Find the decay constant k. Round your answer to four decimal places.**
- **Find a function which gives the amount of isotope A which remains after time t. (Keep the units of A and t the same as the given data.)**
- **Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)**

14. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

15. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

16. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

17. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

18. Uranium 235, used for nuclear power, initial amount 1 kg grams, half-life 704 million years.

19. With the help of your classmates, show that the time it takes for 90$\%$ of each isotope listed in Exercises 14 - 18 to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k. Find a formula, in terms of k only, to determine how long it takes for 90$\%$ of a radioactive isotope to decay.

20. In Example 112 in Section 7.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use the properties of logs and/or exponents to rewrite the model in the form $V(t) = 25e^{kt}$.

21. The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modelled by:

$$G(t) = 9743.77e^{0.0514t}$$

    (a) Find and interpret $G(0)$.
    (b) According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was $\$14,369.1$ billion and the 2010 GDP was $\$14,657.8$ billion.)

22. The diameter D of a tumour, in millimetres, t days after it is detected is given by:

$$D(t) = 15e^{0.0277t}$$

    (a) What was the diameter of the tumour when it was originally detected?
    (b) How long until the diameter of the tumour doubles?

23. Under optimal conditions, the growth of a certain strain of *E. Coli* is modelled by the Law of Uninhibited Growth $N(t) = N_0e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

    (a) Find the growth constant k. Round your answer to four decimal places.
    (b) Find a function which gives the number of bacteria $N(t)$ after t minutes.
    (c) How long until there are 9000 bacteria? Round your answer to the nearest minute.

24. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimetre (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0e^{kt}$.

    (a) Find the growth constant k. Round your answer to four decimal places.
    (b) Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.
    (c) What is the doubling time for this strain of yeast?

25. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

26. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $N(t) = N_0e^{kt}$ were $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy?

27. The population of Sasquatch in Bigfoot county is modelled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

    where $P(t)$ is the population of Sasquatch t years after 2010.

    (a) Find and interpret $P(0)$.
    (b) Find the population of Sasquatch in Bigfoot county in 2013. Round your answer to the nearest Sasquatch.
    (c) When will the population of Sasquatch in Bigfoot county reach 60? Round your answer to the nearest year.
    (d) Find and interpret the end behaviour of the graph of $y = P(t)$. Check your answer using a graphing utility.

28. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

    (a) Use Equation 33 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.
    (b) What percentage of the original amount of Carbon-14 is left after 20,000 years?

(c) If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

(d) Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

29. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 33 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

30. Use Equation 33 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

31. A pork roast[4] was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F. Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 34),

(a) Express the temperature T (in °F) as a function of time t (in minutes).

(b) Find the time at which the roast would have dropped to 140°F had it not been carved and eaten.

32. In reference to Exercise 44 in Section 6.3, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

Use your calculator to graph this path for $x > 0$. Describe the behaviour of y as $x \to 0^+$ and interpret this physically.

33. The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine the value of i as $t \to \infty$. (This is called the **steady state** current.)

34. If the voltage in the circuit in Exercise 33 above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if} \quad 0 \leq t < 30 \\ \left(2 - 2e^{-300}\right) e^{-10t+300} & \text{if} \quad t \geq 30 \end{cases}$$

With the help of your calculator, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \leq t < 30$ and $t \geq 30$.

35. In Exercise 26 in Section 3.3, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a catenary and its basic shape is given by $y = \frac{1}{2}\left(e^x + e^{-x}\right)$. Use your calculator to graph this function. What are its domain and range? What is its end behaviour? Is it invertible? How do you think it is related to the function given in Exercise 48 in Section 7.3 and the one given in the answer to Exercise 39 in Section 7.4? When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2}\left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}}\right)$$

where x and y are measured in feet and $-315 \leq x \leq 315$. Find the highest point on the arch.

[4]This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

8: Foundations of Trigonometry

8.1 Angles and their Measure

This section begins our study of Trigonometry and to get started, we recall some basic definitions from Geometry. A **ray** is usually described as a 'half-line' and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other, as pictured in Figure 8.3. The point from which the ray originates is called the **initial point** of the ray.

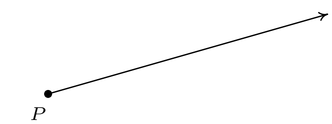
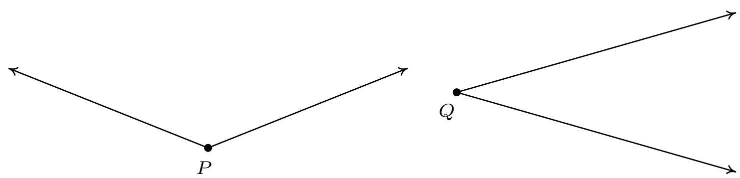


Figure 8.3: A ray with initial point *P*

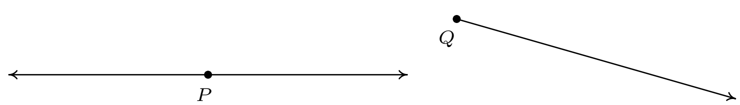
When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Two examples of what are commonly thought of as angles are given in Figure 8.1



An angle with vertex *P* An angle with vertex *Q*

Figure 8.1: Typical angles

However, the two figures in Figure 8.2 also depict angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**; in the second, the rays are identical so the 'angle' is indistinguishable from the ray itself.



A straight angle

Figure 8.2: Less typical angles

The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured in Figure 8.4.

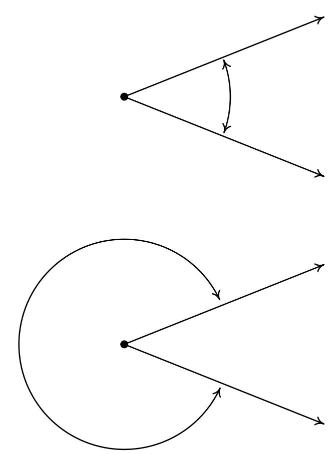


Figure 8.4: Two ways to measure an angle

Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram. (The phrase 'at least' will be justified in short order.) Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have the labels in Figure 8.5.

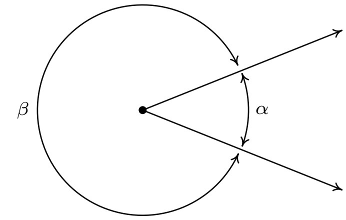


Figure 8.5: Labelling angles

One commonly used system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the familiar '$^\circ$' symbol. One complete revolution as shown below is 360°, and parts of a revolution are measured proportionately. Thus half of a revolution (a straight angle) measures $\frac{1}{2}\left(360^\circ\right) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4}\left(360^\circ\right) = 90^\circ$ and so on.

The choice of '360' is most often attributed to the Babylonians.

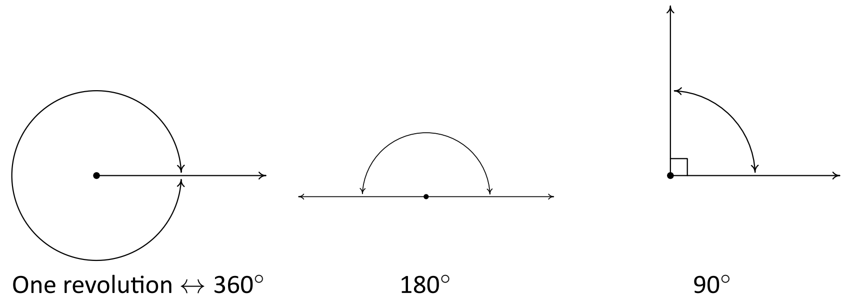


Figure 8.6: Defining degree measure

Note that in Figure 8.6 above, we have used the small square '□' to denote a right angle, as is commonplace in Geometry. Recall that if an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of entire revolution. For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}\left(360^\circ\right) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}\left(360^\circ\right) = 30^\circ$ and an angle which indicates no rotation at all is measured as 0°: see Figure 8.7.

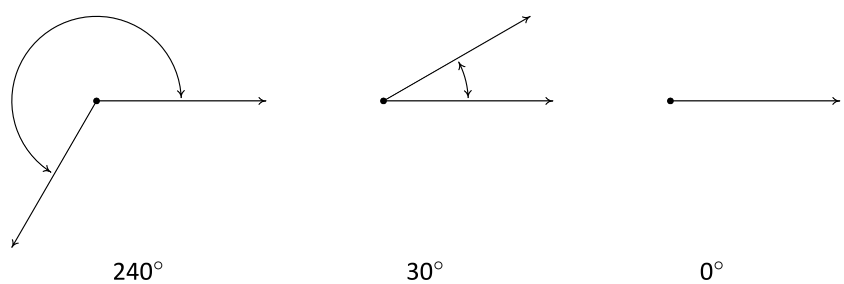


Figure 8.7: Measuring angles in degrees

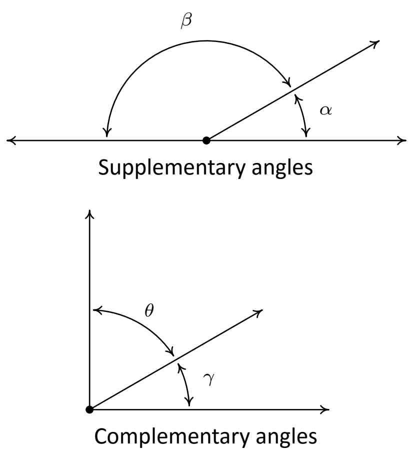


Figure 8.8: Supplementary and complementary angles

Two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In Figure 8.8, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.

In practice, the distinction between the angle itself and its measure is blurred so that the sentence 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$.'

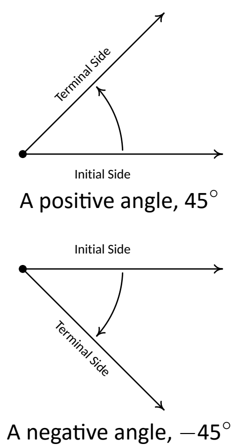


Figure 8.9: The sign of an angle

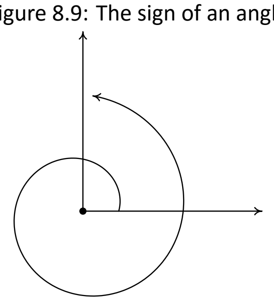


Figure 8.10: Angles can comprise more than one revolution

Up to this point, we have discussed only angles which measure between 0° and 360°, inclusive. Ultimately, we want to use the arsenal of Algebra which we have stockpiled in Chapters 2 through 5 to not only solve geometric problems involving angles, but also to extend their applicability to other real-world phenomena. A first step in this direction is to extend our notion of 'angle' from merely measuring an extent of rotation to quantities which can be associated with real numbers. To that end, we introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown in Figure 8.9. When the rotation is counterclockwise from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

At this point, we also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the 'first' 360°) then continue with an additional 90° counter-clockwise rotation, as seen in Figure 8.10.

To further connect angles with the Algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive *x*-axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side lies in Quadrant I is called a 'Quadrant I angle'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side. (Note that by being in standard position they automatically share the same initial side which is the positive *x*-axis.) In Figure 8.11, $\alpha = 120°$ and $\beta = -240°$ are two coterminal Quadrant II angles drawn in standard position. Note that $\alpha = \beta + 360°$, or equivalently, $\beta = \alpha - 360°$. We leave it as an exercise to the reader to verify that coterminal angles always differ by a multiple of 360°. (It is worth noting that all of the pathologies of Analytic Trigonometry result from this innocuous fact.) More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360° \cdot k$ where k is an integer.

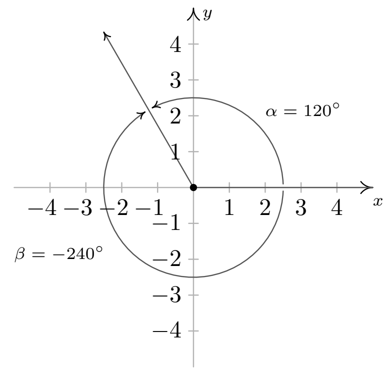


Figure 8.11: Two coterminal angles, $\alpha = 120°$ and $\beta = -240°$, in standard position.

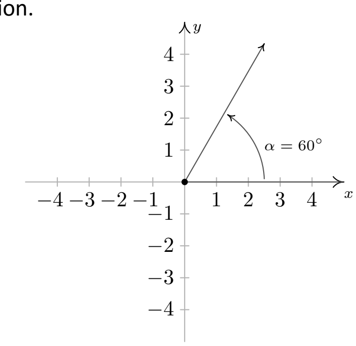


Figure 8.12: $\alpha = 60°$ in standard position

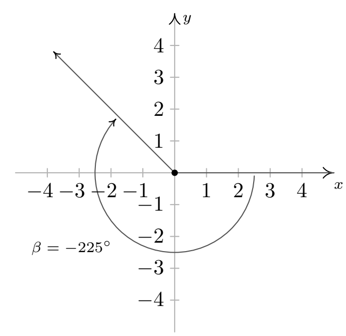


Figure 8.13: $\beta = -225°$ in standard position

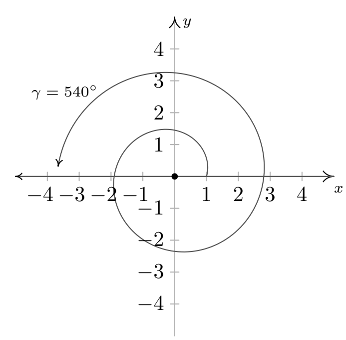


Figure 8.14: $\gamma = 540°$ in standard position

Example 135 **Plotting and classifying angles**

Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = 60°$
2. $\beta = -225°$
3. $\gamma = 540°$
4. $\phi = -750°$

SOLUTION

1. To graph $\alpha = 60°$, we draw an angle with its initial side on the positive *x*-axis and rotate counter-clockwise $\frac{60°}{360°} = \frac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360° \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60° + 360° = 420°$. Substituting $k = -1$ gives $\theta = 60° - 360° = -300°$. Finally, if we let $k = 2$, we get $\theta = 60° + 720° = 780°$: see Figure 8.12.

2. Since $\beta = -225°$ is negative, we start at the positive *x*-axis and rotate *clockwise* $\frac{225°}{360°} = \frac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225° + 360° \cdot k$ for integer values of k. We find 135°, −585° and 495° are all coterminal with −225°: see Figure 8.13.

3. Since $\gamma = 540°$ is positive, we rotate counter-clockwise from the positive *x*-axis. One full revolution accounts for 360°, with 180°, or $\frac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative *x*-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540° + 360° \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: 180°, −180° and 900°: see Figure 8.14.

4. The Greek letter ϕ is pronounced 'fee' or 'fie' and since ϕ is negative, we begin our rotation clockwise from the positive *x*-axis. Two full revolutions

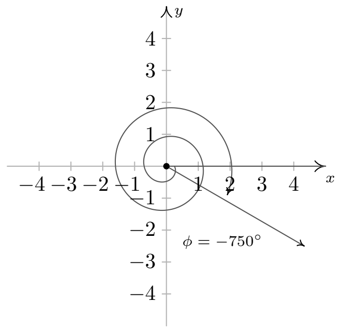


Figure 8.15: $\phi = -750^\circ$ in standard position

account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°: see Figure 8.15.

Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in Example 135 to see this. We are now just one step away from completely marrying angles with the real numbers and the rest of Algebra. To that end, we recall the following definition.

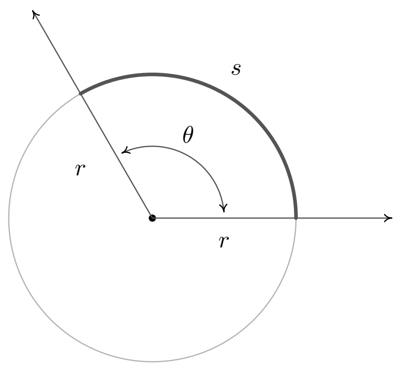


Figure 8.16: The radian measure of θ is $\dfrac{s}{r}$

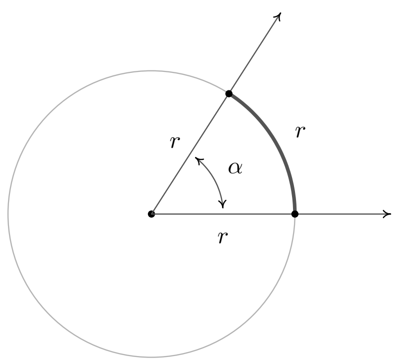
Definition 53 The number π

The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

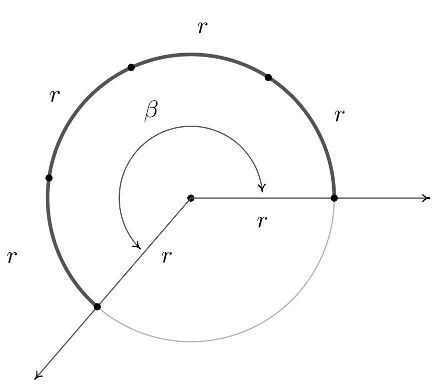
$$\pi = \frac{C}{d}$$

While Definition 53 is quite possibly the 'standard' definition of π, the authors would be remiss if we didn't mention that buried in this definition is actually a theorem. As the reader is probably aware, the number π is a mathematical constant - that is, it doesn't matter *which* circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle. While this is indeed true, it is far from obvious. (If you think it *is* obvious, try to come up with a rigorous proof of this fact!) Since the diameter of a circle is twice its radius, we can quickly rearrange the equation in Definition 53 to get a formula more useful for our purposes, namely: $2\pi = \dfrac{C}{r}$

This tells us that for any circle, the ratio of its circumference to its radius is also always constant; in this case the constant is 2π. Suppose now we take a **portion** of the circle, so instead of comparing the entire circumference C to the radius, we compare some arc measuring s units in length to the radius, as depicted in Figure 8.16. Let θ be the **central angle** subtended by this arc, that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc. Using proportionality arguments, it stands to reason that the ratio $\dfrac{s}{r}$ should also be a constant among all circles, and it is this ratio which defines the **radian measure** of an angle.



α has radian measure 1



β has radian measure 4

Figure 8.17: An angle of k radians subtends an arc of length $k \cdot r$

To get a better feel for radian measure, we note that an angle with radian measure 1 means the corresponding arc length s equals the radius of the circle r, hence $s = r$. When the radian measure is 2, we have $s = 2r$; when the radian measure is 3, $s = 3r$, and so forth. Thus the radian measure of an angle θ tells us how many 'radius lengths' we need to sweep out along the circle to subtend the angle θ: see Figure 8.17.

Since one revolution sweeps out the entire circumference $2\pi r$, one revolution has radian measure $\dfrac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$, a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered 'pure' numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word 'radians' to denote these dimensionless units as needed. For instance, we say one revolution measures '2π radians,' half of a revolution measures 'π radians,' and so forth.

As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write '$\theta = \frac{\pi}{2}$', we mean θ is an angle which measures $\frac{\pi}{2}$ radians. (The authors are well aware that we are now identifying radians with real numbers. We will justify this shortly.) We extend radian measure to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation. Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

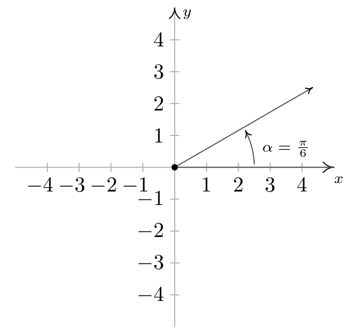


Figure 8.18: $\alpha = \frac{\pi}{6}$ in standard position.

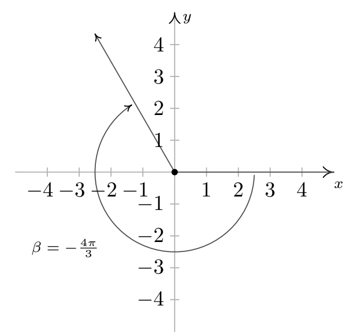


Figure 8.19: $\beta = -\frac{4\pi}{3}$ in standard position.

Example 136 Plotting and classifying angles

Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \dfrac{\pi}{6}$
2. $\beta = -\dfrac{4\pi}{3}$
3. $\gamma = \dfrac{9\pi}{4}$
4. $\phi = -\dfrac{5\pi}{2}$

Solution

1. The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle. Coterminal angles θ are of the form $\theta = \alpha + 2\pi \cdot k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$, thus when $k = 1$, we get $\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Substituting $k = -1$ gives $\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$ and when we let $k = 2$, we get $\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$: see Figure 8.18.

2. Since $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle. To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles: see Figure 8.19.

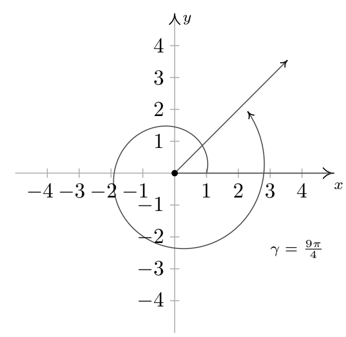


Figure 8.20: $\gamma = \frac{9\pi}{4}$ in standard position.

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a Quadrant I angle. All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where k is an integer. Working through the arithmetic, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$: see Figure 8.20.

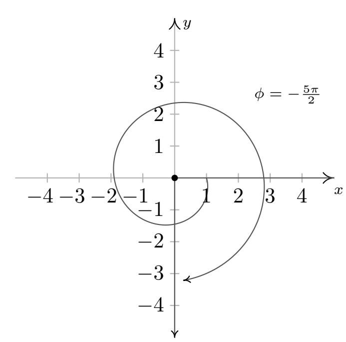


Figure 8.21: $\phi = -\frac{5\pi}{2}$ in standard position.

4. To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of ϕ lies on the negative y-axis, ϕ is a quadrantal angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers k and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$: see Figure 8.21.

It is worth mentioning that we could have plotted the angles in Example 136 by first converting them to degree measure and following the procedure set forth in Example 135. While converting back and forth from degrees and radians is certainly a good skill to have, it is best that you learn to 'think in radians' as well as you can 'think in degrees'. The authors would, however, be derelict in our duties if we ignored the basic conversion between these systems altogether. Since one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems. For example, to convert 60° to radians we find $60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3}$ radians, or simply $\frac{\pi}{3}$. To convert from radian measure back to degrees, we multiply by the ratio $\frac{180^\circ}{\pi \text{ radian}}$. For example, $-\frac{5\pi}{6}$ radians is equal to $\left(-\frac{5\pi}{6} \text{ radians}\right)\left(\frac{180^\circ}{\pi \text{ radians}}\right) = -150^\circ$. Of particular interest is the fact that an angle which measures 1 in radian measure is equal to $\frac{180^\circ}{\pi} \approx 57.2958^\circ$.

We summarize these conversions below.

Key Idea 36 Degree - Radian Conversion

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$

In light of Example 136 and Equation 36, the reader may well wonder what the allure of radian measure is. The numbers involved are, admittedly, much more complicated than degree measure. The answer lies in how easily angles in radian measure can be identified with real numbers. Consider the Unit Circle, $x^2 + y^2 = 1$, as drawn below, the angle θ in standard position and the corresponding arc measuring s units in length. By definition, and the fact that the Unit Circle has radius 1, the radian measure of θ is $\dfrac{s}{r} = \dfrac{s}{1} = s$ so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$. In order to identify real numbers with oriented angles, we make good use of this fact by essentially 'wrapping' the real number line around the Unit Circle and associating to each real number t an *oriented* arc on the Unit Circle with initial point $(1, 0)$. This identification between angles and real numbers will also be essential once we begin our study of trigonometric functions in Calculus.

Viewing the vertical line $x = 1$ as another real number line demarcated like the y-axis, given a real number $t > 0$, we 'wrap' the (vertical) interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of t units and therefore the corresponding angle has radian measure equal to t. If $t < 0$, we wrap the interval $[t, 0]$ *clockwise* around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has radian measure equal to t. If $t = 0$, we are at the point $(1, 0)$ on the x-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number t with the corresponding angle with radian measure t.

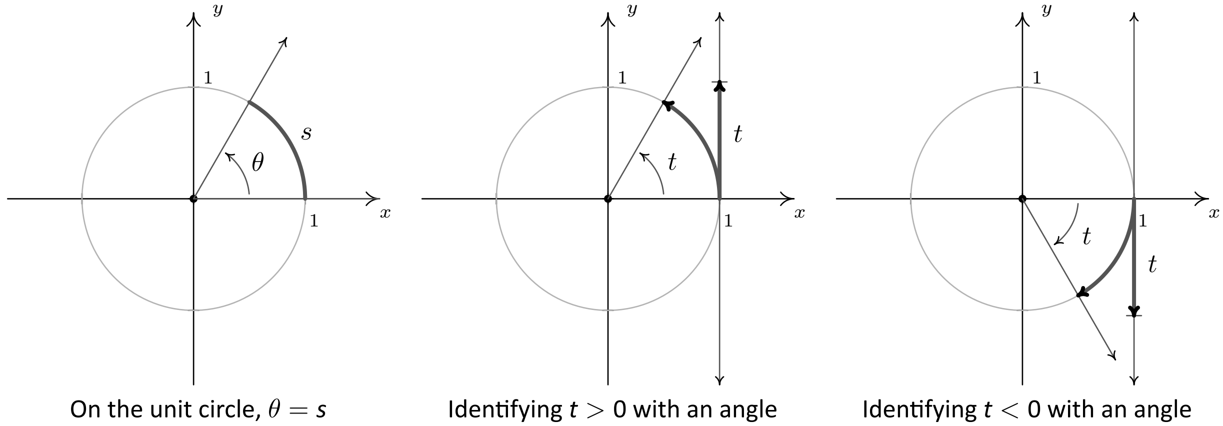


Figure 8.22: Identifying real numnbers with angles

Example 137 Angles corresponding to real numbers

Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t = \dfrac{3\pi}{4}$
2. $t = -2\pi$
3. $t = -2$
4. $t = 117$

SOLUTION

1. The arc associated with $t = \frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1, 0)$ proceeds counter-clockwise up to midway through Quadrant II: see Figure 8.20.

2. Since one revolution is 2π radians, and $t = -2\pi$ is negative, we graph the arc which begins at $(1, 0)$ and proceeds *clockwise* for one full revolution: see Figure 8.20.

3. Like $t = -2\pi$, $t = -2$ is negative, so we begin our arc at $(1, 0)$ and proceed clockwise around the unit circle. Since $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find that rotating 2 radians clockwise from the point $(1, 0)$ lands us in Quadrant III. To more accurately place the endpoint, we successively halve the angle measure until we find $\frac{5\pi}{8} \approx 1.96$ which tells us our arc extends just a bit beyond the quarter mark into Quadrant III: see Figure 8.20.

4. Since 117 is positive, the arc corresponding to $t = 117$ begins at $(1, 0)$ and proceeds counter-clockwise. As 117 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{117}{2\pi}$ as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of $\frac{5}{8}$ of a revolution to spare. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III: see Figure 8.20.

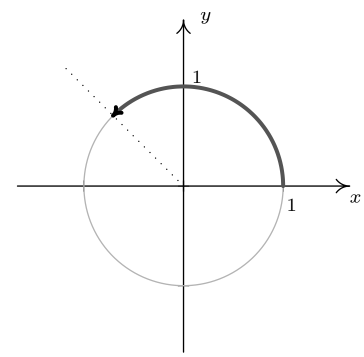


Figure 8.23: $t = \frac{3\pi}{4}$

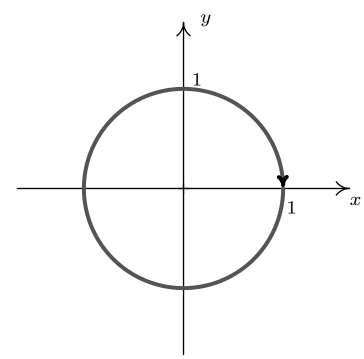


Figure 8.24: $t = -2\pi$

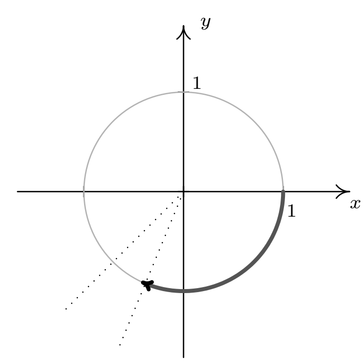


Figure 8.25: $t = -2$

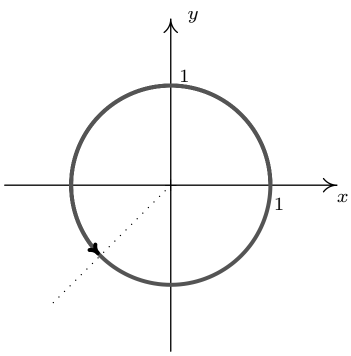


Figure 8.26: $t = 117$

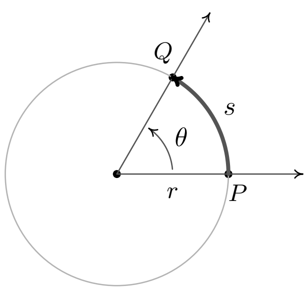


Figure 8.27: Circular motion

8.1.1 Applications of Radian Measure: Circular Motion

Now that we have paired angles with real numbers via radian measure, a whole world of applications awaits us. Our first excursion into this realm comes by way of circular motion. Suppose an object is moving as pictured in Figure 8.27 along a circular path of radius r from the point P to the point Q in an amount of time t.

Here s represents a *displacement* so that $s > 0$ means the object is travelling in a counter-clockwise direction and $s < 0$ indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation.

Borrowing terminology from Physics, if we imagine the circular motion of our object taking place over a duration of time t, we can define the quantity $\frac{\theta}{t}$, called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$. If the circular motion is *uniform*, meaning that the rate at which the angle θ changes with time is constant, then the average angular velocity $\overline{\omega}$ is the same as the *instantaneous* angular velocity ω. (If the rate is not constant, we can't define ω without calculus.)

If the path of the object were 'uncurled' from a circle to form a line segment, then we could discuss the average *linear* velocity of the object, given by $\overline{v} = \frac{s}{t}$. Note that since $s = r\theta$, we obtain

$$\overline{v} = \frac{s}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\overline{\omega}.$$

One note of caution is needed here: the true motion of our object is, of course, **not** linear – it's circular. Lest we draw the ire of any students with high school Physics under their belts, we should point out that motion in the plane is best described as a *vector* quantity (we will *not* be discussing vectors in this text), and the relationship $\overline{v} = r\overline{\omega}$ describes not the velocity of the object, but its *speed*.

Example 138 Finding speed of rotation

Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object travelling on a circle which completes one revolution in (approximately) 24 hours. The path traced out by the point during this 24 hour period is the Latitude of that point. Lakeland Community College is at 41.628° north latitude, and it can be shown that the radius of the earth at this Latitude is approximately 2960 miles. (We will discuss how we arrived at this approximation in Example 144.) Find the linear speed, in miles per hour, of Lakeland Community College as the world turns.

SOLUTION To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The earth makes one revolution in 24 hours, and one revolution is 2π radians, so $\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}}$, where, once again, we are using the fact that radians are real numbers and are dimensionless. (For simplicity's sake, we are also assuming that we are viewing the rotation of the earth as counter-clockwise so $\omega > 0$.) Hence, the linear velocity is

$$v = 2960 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \approx 775 \, \frac{\text{miles}}{\text{hour}}$$

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example 138 is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of Example 138, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the 'magnification factor' which relates ω and v. While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: if an object is moving on a circular path of radius r with a fixed angular velocity (frequency) ω, what is the position of the object at time t? The answer to this question is the very heart of Trigonometry and is answered in the next section.

Exercises 8.1

Problems

In Exercises 1 – 20, graph the oriented angle in standard position. Classify each angle according to where its terminal side lies and then give two coterminal angles, one of which is positive and the other negative.

1. 330°
2. -135°
3. 120°
4. 405°
5. -270°
6. $\frac{5\pi}{6}$
7. $-\frac{11\pi}{3}$
8. $\frac{5\pi}{4}$
9. $\frac{3\pi}{4}$
10. $-\frac{\pi}{3}$
11. $\frac{7\pi}{2}$
12. $\frac{\pi}{4}$
13. $-\frac{\pi}{2}$
14. $\frac{7\pi}{6}$
15. $-\frac{5\pi}{3}$
16. 3π
17. -2π
18. $-\frac{\pi}{4}$
19. $\frac{15\pi}{4}$
20. $-\frac{13\pi}{6}$

In Exercises 21 – 28, convert the angle from degree measure into radian measure, giving the exact value in terms of π.

21. 0°
22. 240°
23. 135°
24. -270°
25. -315°
26. 150°
27. 45°
28. -225°

In Exercises 29 – 36, convert the angle from radian measure into degree measure.

29. π
30. $-\frac{2\pi}{3}$
31. $\frac{7\pi}{6}$
32. $\frac{11\pi}{6}$
33. $\frac{\pi}{3}$
34. $\frac{5\pi}{3}$
35. $-\frac{\pi}{6}$
36. $\frac{\pi}{2}$

In Exercises 37 – 41, sketch the oriented arc on the Unit Circle which corresponds to the given real number.

37. $t = \frac{5\pi}{6}$
38. $t = -\pi$
39. $t = 6$
40. $t = -2$
41. $t = 12$
42. A yo-yo which is 2.25 inches in diameter spins at a rate of 4500 revolutions per minute. How fast is the edge of the yo-yo spinning in miles per hour? Round your answer to two decimal places.

43. How many revolutions per minute would the yo-yo in exercise 42 have to complete if the edge of the yo-yo is to be spinning at a rate of 42 miles per hour? Round your answer to two decimal places.

44. In the yo-yo trick 'Around the World,' the performer throws the yo-yo so it sweeps out a vertical circle whose radius is the yo-yo string. If the yo-yo string is 28 inches long and the yo-yo takes 3 seconds to complete one revolution of the circle, compute the speed of the yo-yo in miles per hour. Round your answer to two decimal places.

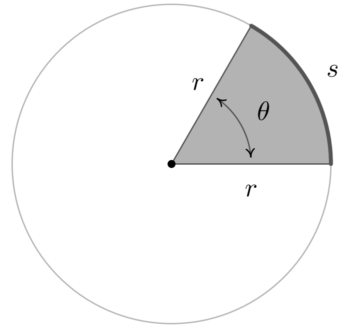
45. A computer hard drive contains a circular disk with diameter 2.5 inches and spins at a rate of 7200 RPM (revolutions per minute). Find the linear speed of a point on the edge of the disk in miles per hour.

46. A rock got stuck in the tread of my tire and when I was driving 70 miles per hour, the rock came loose and hit the inside of the wheel well of the car. How fast, in miles per hour, was the rock travelling when it came out of the tread? (The tire has a diameter of 23 inches.)

47. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet. It completes two revolutions in 2 minutes and 7 seconds.[1] Assuming the riders are at the edge of the circle, how fast are they traveling in miles per hour?

48. Consider the circle of radius r pictured below with central angle θ, measured in radians, and subtended arc of length s. Prove that the area of the shaded sector is $A = \frac{1}{2}r^2\theta$.

(Hint: Use the proportion $\frac{A}{\text{area of the circle}} = \frac{s}{\text{circumference of the circle}}$.)



In Exercises 49 – 54, use the result of Exercise 48 to compute the areas of the circular sectors with the given central angles and radii.

49. $\theta = \frac{\pi}{6},\ r = 12$

50. $\theta = \frac{5\pi}{4},\ r = 100$

51. $\theta = 330^\circ,\ r = 9.3$

52. $\theta = \pi,\ r = 1$

53. $\theta = 240^\circ,\ r = 5$

54. $\theta = 1^\circ,\ r = 117$

55. Imagine a rope tied around the Earth at the equator. Show that you need to add only 2π feet of length to the rope in order to lift it one foot above the ground around the entire equator. (You do NOT need to know the radius of the Earth to show this.)

56. With the help of your classmates, look for a proof that π is indeed a constant.

[1] Source: Cedar Point's webpage.

8.2 The Unit Circle: Sine and Cosine

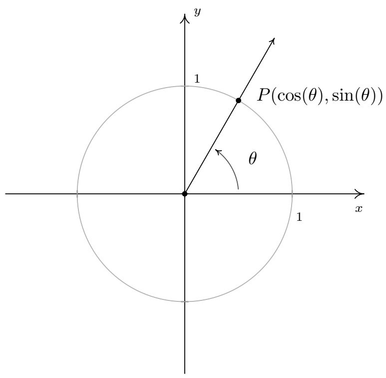


Figure 8.29: Defining $\cos(\theta)$ and $\sin(\theta)$

In Section 8.1.1, we introduced circular motion and derived a formula which describes the linear velocity of an object moving on a circular path at a constant angular velocity. One of the goals of this section is describe the *position* of such an object. To that end, consider an angle θ in standard position and let P denote the point where the terminal side of θ intersects the Unit Circle, as in Figure 8.29. By associating the point P with the angle θ, we are assigning a *position* on the Unit Circle to the angle θ. The x-coordinate of P is called the **cosine** of θ, written $\cos(\theta)$, while the y-coordinate of P is called the **sine** of θ, written $\sin(\theta)$. The reader is encouraged to verify that these rules used to match an angle with its cosine and sine do, in fact, satisfy the definition of a function. That is, for each angle θ, there is only one associated value of $\cos(\theta)$ and only one associated value of $\sin(\theta)$.

The etymology of the name 'sine' is quite colourful, and the interested reader is invited to research it; the 'co' in 'cosine' is explained in Section 8.4.

Example 139 Evaluating $\cos(\theta)$ and $\sin(\theta)$

Find the cosine and sine of the following angles.

1. $\theta = 270°$
2. $\theta = -\pi$
3. $\theta = \frac{\pi}{4}$
4. $\theta = \frac{\pi}{6}$
5. $\theta = \frac{\pi}{3}$

Solution

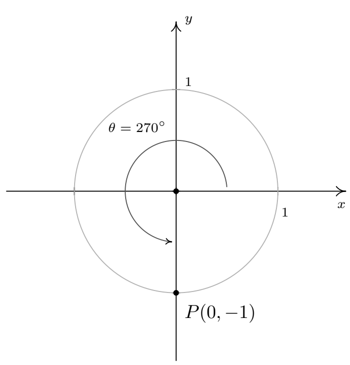


Figure 8.30: Finding $\cos(270°)$ and $\sin(270°)$

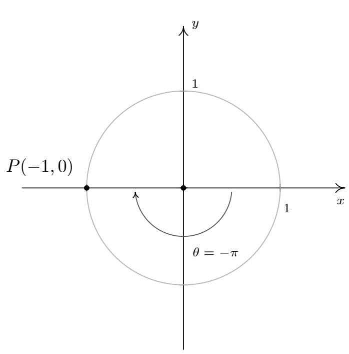
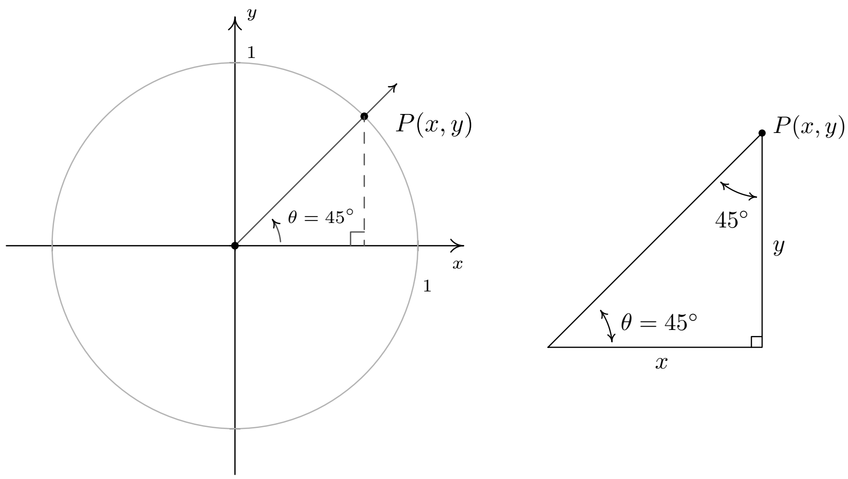


Figure 8.31: Finding $\cos(-\pi)$ and $\sin(-\pi)$

1. To find $\cos(270°)$ and $\sin(270°)$, we plot the angle $\theta = 270°$ in standard position in Figure 8.30 and find the point on the terminal side of θ which lies on the Unit Circle. Since $270°$ represents $\frac{3}{4}$ of a counter-clockwise revolution, the terminal side of θ lies along the negative y-axis. Hence, the point we seek is $(0, -1)$ so that $\cos(270°) = 0$ and $\sin(270°) = -1$.

2. The angle $\theta = -\pi$ represents one half of a clockwise revolution so its terminal side lies on the negative x-axis. The point on the Unit Circle that lies on the negative x-axis is $(-1, 0)$ which means $\cos(-\pi) = -1$ and $\sin(-\pi) = 0$.

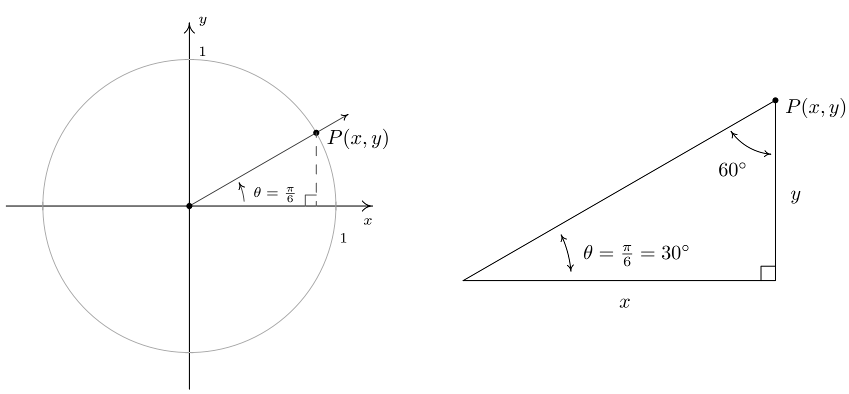
3. When we sketch $\theta = \frac{\pi}{4}$ in standard position, we see in Figure 8.28 that its terminal does not lie along any of the coordinate axes which makes our job of finding the cosine and sine values a bit more difficult. Let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. By definition, $x = \cos\left(\frac{\pi}{4}\right)$ and $y = \sin\left(\frac{\pi}{4}\right)$. If we drop a perpendicular line segment from P to the x-axis, we obtain a $45° - 45° - 90°$ right triangle whose legs have lengths x and y units. From Geometry, we get $y = x$. (Can you show this?) Since $P(x, y)$ lies on the Unit Circle, we have $x^2 + y^2 = 1$. Substituting $y = x$ into this equation yields $2x^2 = 1$, or $x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Since $P(x, y)$ lies in the first quadrant, $x > 0$, so $x = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and with $y = x$ we have $y = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.



$\theta = \frac{\pi}{4}$ in standard position $\qquad$ $45^\circ - 45^\circ - 90^\circ$ triangle

Figure 8.28: Finding $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$

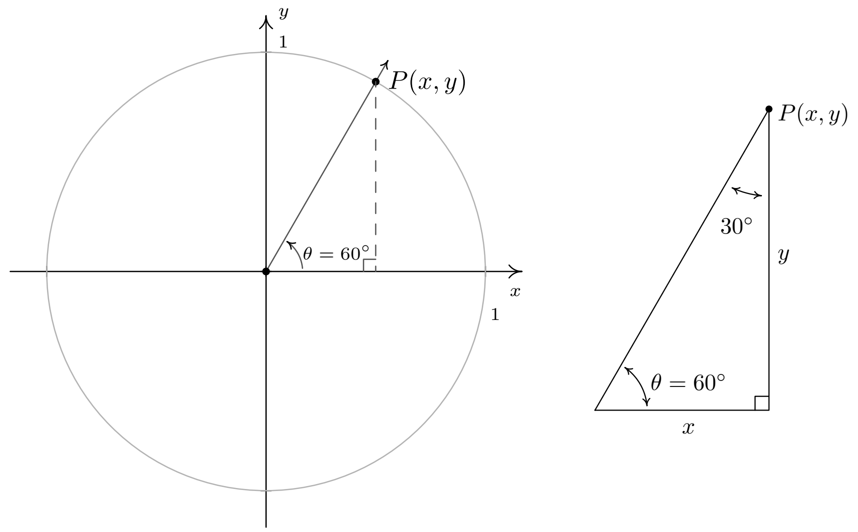
4. As before, the terminal side of $\theta = \frac{\pi}{6}$ does not lie on any of the coordinate axes, so we proceed using a triangle approach. Letting $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle, we drop a perpendicular line segment from P to the x-axis to form a $30^\circ - 60^\circ - 90^\circ$ right triangle: see Figure 8.32. After a bit of Geometry (again, can you show this?) we find $y = \frac{1}{2}$ so $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. Since $P(x, y)$ lies on the Unit Circle, we substitute $y = \frac{1}{2}$ into $x^2 + y^2 = 1$ to get $x^2 = \frac{3}{4}$, or $x = \pm\frac{\sqrt{3}}{2}$. Here, $x > 0$ so $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.



$\theta = \frac{\pi}{6}$ in standard position $\qquad$ $30^\circ - 60^\circ - 90^\circ$ triangle

Figure 8.32: Finding $\cos\left(\frac{\pi}{6}\right)$ and $\sin\left(\frac{\pi}{6}\right)$

5. Plotting $\theta = \frac{\pi}{3}$ in standard position, we find it is not a quadrantal angle and set about using a triangle approach. Once again, we get a $30^\circ - 60^\circ - 90^\circ$ right triangle and, after the usual computations, find $x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $y = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.



$\theta = \frac{\pi}{3}$ in standard position $\qquad$ $30^\circ - 60^\circ - 90^\circ$ triangle

Figure 8.33: Finding $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$

In Example 139, it was quite easy to find the cosine and sine of the quadrantal angles, but for non-quadrantal angles, the task was much more involved. In these latter cases, we made good use of the fact that the point $P(x, y) = (\cos(\theta), \sin(\theta))$ lies on the Unit Circle, $x^2 + y^2 = 1$. If we substitute $x = \cos(\theta)$ and $y = \sin(\theta)$ into $x^2 + y^2 = 1$, we get $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. An unfortunate convention, which the authors are compelled to perpetuate, is to write $(\cos(\theta))^2$ as $\cos^2(\theta)$ and $(\sin(\theta))^2$ as $\sin^2(\theta)$. (This is unfortunate from a 'function notation' perspective, as you will see once you encounter the inverse trigonometric functions.) Rewriting the identity using this convention results in the following theorem, which is without a doubt one of the most important results in Trigonometry.

Theorem 50 The Pythagorean Identity

For any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$.

The moniker 'Pythagorean' brings to mind the Pythagorean Theorem, from which both the Distance Formula and the equation for a circle are ultimately derived. The word 'Identity' reminds us that, regardless of the angle θ, the equation in Theorem 50 is always true. If one of $\cos(\theta)$ or $\sin(\theta)$ is known, Theorem 50 can be used to determine the other, up to a $(\pm)$ sign. If, in addition, we know where the terminal side of θ lies when in standard position, then we can remove the ambiguity of the $(\pm)$ and completely determine the missing value as the next example illustrates.

Example 140 Using the Pythagorean Identity

Using the given information about θ, find the indicated value.

1. If θ is a Quadrant II angle with $\sin(\theta) = \frac{3}{5}$, find $\cos(\theta)$.
2. If $\pi < \theta < \frac{3\pi}{2}$ with $\cos(\theta) = -\frac{\sqrt{5}}{5}$, find $\sin(\theta)$.
3. If $\sin(\theta) = 1$, find $\cos(\theta)$.

Solution

1. When we substitute $\sin(\theta) = \frac{3}{5}$ into The Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\cos^2(\theta) + \frac{9}{25} = 1$. Solving, we find $\cos(\theta) = \pm\frac{4}{5}$. Since θ is a Quadrant II angle, its terminal side, when plotted in standard position, lies in Quadrant II. Since the *x*-coordinates are negative in Quadrant II, $\cos(\theta)$ is too. Hence, $\cos(\theta) = -\frac{4}{5}$.

2. Substituting $\cos(\theta) = -\frac{\sqrt{5}}{5}$ into $\cos^2(\theta) + \sin^2(\theta) = 1$ gives $\sin(\theta) = \pm\frac{2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$. Since we are given that $\pi < \theta < \frac{3\pi}{2}$, we know θ is a Quadrant III angle. Hence both its sine and cosine are negative and we conclude $\sin(\theta) = -\frac{2\sqrt{5}}{5}$.

3. When we substitute $\sin(\theta) = 1$ into $\cos^2(\theta) + \sin^2(\theta) = 1$, we find $\cos(\theta) = 0$.

Another tool which helps immensely in determining cosines and sines of angles is the symmetry inherent in the Unit Circle. Suppose, for instance, we wish to know the cosine and sine of $\theta = \frac{5\pi}{6}$. We plot θ in standard position below and, as usual, let $P(x, y)$ denote the point on the terminal side of θ which lies on the Unit Circle. Note that the terminal side of θ lies $\frac{\pi}{6}$ radians short of one half revolution. In Example 139, we determined that $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. This means that the point on the terminal side of the angle $\frac{\pi}{6}$, when plotted in standard position, is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. From Figure 8.34, it is clear that the point $P(x, y)$ we seek can be obtained by reflecting that point about the *y*-axis. Hence, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$.

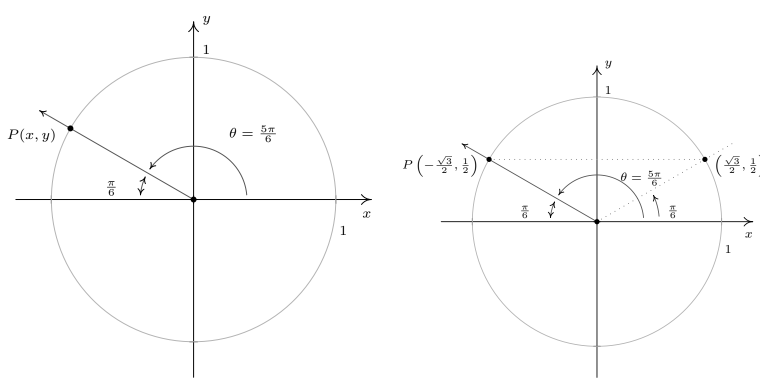


Figure 8.34: Refelcting $P(x, y)$ across the *y*-axis to obtain a Quadrant I angle

In the above scenario, the angle $\frac{\pi}{6}$ is called the **reference angle** for the angle $\frac{5\pi}{6}$. In general, for a non-quadrantal angle θ, the reference angle for θ (usually denoted α) is the *acute* angle made between the terminal side of θ and the *x*-axis. If θ is a Quadrant I or IV angle, α is the angle between the terminal side of θ and the *positive* *x*-axis; if θ is a Quadrant II or III angle, α is the angle between the terminal side of θ and the *negative* *x*-axis. If we let *P* denote the point $(\cos(\theta), \sin(\theta))$, then *P* lies on the Unit Circle. Since the Unit Circle possesses symmetry with respect to the *x*-axis, *y*-axis and origin, regardless of where the terminal side of θ lies, there is a point *Q* symmetric with *P* which determines θ's reference angle, α as seen below.

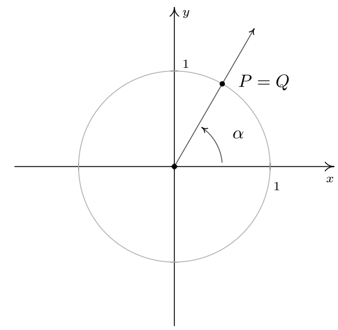


Figure 8.35: Reference angle α for a Quadrant I angle

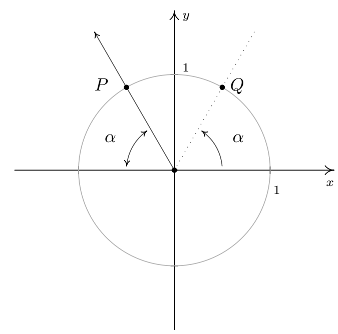


Figure 8.36: Reference angle α for a Quadrant II angle

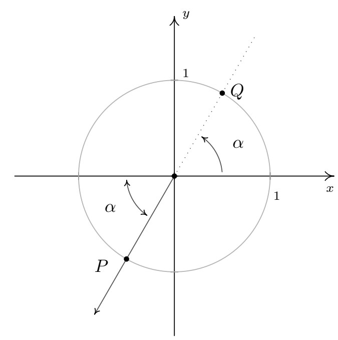


Figure 8.37: Reference angle α for a Quadrant III angle

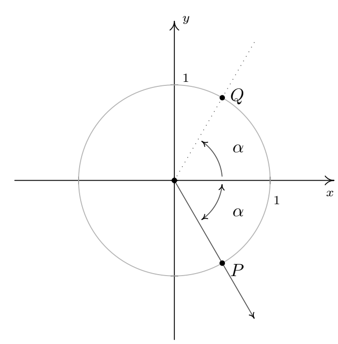


Figure 8.38: Reference angle α for a Quadrant IV angle

We have just outlined the proof of the following theorem.

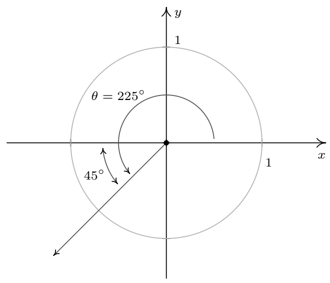


Figure 8.39: Finding $\cos\left(\frac{5\pi}{4}\right)$ and $\sin\left(\frac{5\pi}{4}\right)$

Theorem 51 Reference Angle Theorem

Suppose α is the reference angle for θ. Then $\cos(\theta) = \pm\cos(\alpha)$ and $\sin(\theta) = \pm\sin(\alpha)$, where the choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

In light of Theorem 51, it pays to know the cosine and sine values for certain common angles. In the table below, we summarize the values which we consider essential and must be memorized.

Cosine and Sine Values of Common Angles

θ(degrees)	θ(radians)	$\cos(\theta)$	$\sin(\theta)$
0°	0	1	0
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
90°	$\frac{\pi}{2}$	0	1

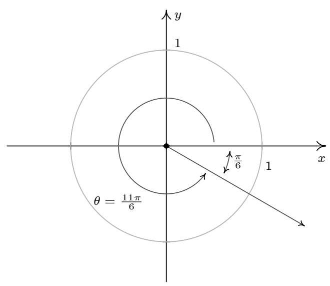


Figure 8.40: Finding $\cos\left(\frac{11\pi}{6}\right)$ and $\sin\left(\frac{11\pi}{6}\right)$

Example 141 Using reference angles

Find the cosine and sine of the following angles.

1. $\theta = \frac{5\pi}{4}$
2. $\theta = \frac{11\pi}{6}$
3. $\theta = -\frac{5\pi}{4}$
4. $\theta = \frac{7\pi}{3}$

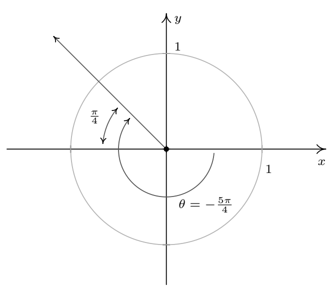


Figure 8.41: Finding $\cos\left(-\frac{5\pi}{4}\right)$ and $\sin\left(-\frac{5\pi}{4}\right)$

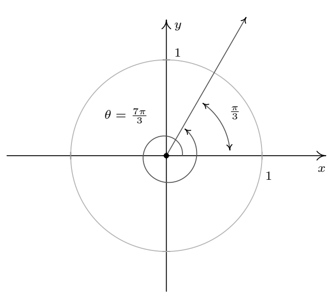


Figure 8.42: Finding $\cos\left(\frac{7\pi}{3}\right)$ and $\sin\left(\frac{7\pi}{3}\right)$

Solution

1. We begin by plotting $\theta = \frac{5\pi}{4}$ in standard position and find its terminal side overshoots the negative *x*-axis to land in Quadrant III. Hence, we obtain θ's reference angle α by subtracting: $\alpha = \theta - \pi = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$. Since θ is a Quadrant III angle, both $\cos(\theta) < 0$ and $\sin(\theta) < 0$. The Reference Angle Theorem yields: $\cos\left(\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.
2. The terminal side of $\theta = \frac{11\pi}{6}$, when plotted in standard position, lies in Quadrant IV, just shy of the positive *x*-axis. To find θ's reference angle α, we subtract: $\alpha = 2\pi - \theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$ and $\sin(\theta) < 0$, so the Reference Angle Theorem gives: $\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(\frac{11\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$.
3. To plot $\theta = -\frac{5\pi}{4}$, we rotate *clockwise* an angle of $\frac{5\pi}{4}$ from the positive *x*-axis. The terminal side of θ, therefore, lies in Quadrant II making an angle of $\alpha = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$ radians with respect to the negative *x*-axis. Since θ is a Quadrant II angle, the Reference Angle Theorem gives: $\cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.
4. Since the angle $\theta = \frac{7\pi}{3}$ measures more than $2\pi = \frac{6\pi}{3}$, we find the terminal side of θ by rotating one full revolution followed by an additional $\alpha = \frac{7\pi}{3} - 2\pi = \frac{\pi}{3}$ radians. Since θ and α are coterminal, $\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{7\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

The reader may have noticed that when expressed in radian measure, the reference angle for a non-quadrantal angle is easy to spot. Reduced fraction multiples of π with a denominator of 6 have $\frac{\pi}{6}$ as a reference angle, those with a denominator of 4 have $\frac{\pi}{4}$ as their reference angle, and those with a denominator of 3 have $\frac{\pi}{3}$ as their reference angle. (For once, we have something convenient about using radian measure in contrast to the abstract theoretical nonsense about using them as a 'natural' way to match oriented angles with real numbers!) The Reference Angle Theorem in conjunction with the table of cosine and sine values on Page 324 can be used to generate the following figure, which the authors feel should be committed to memory. (At the very least, one should memorize the first quadrant and learn to make use of Theorem 51.)

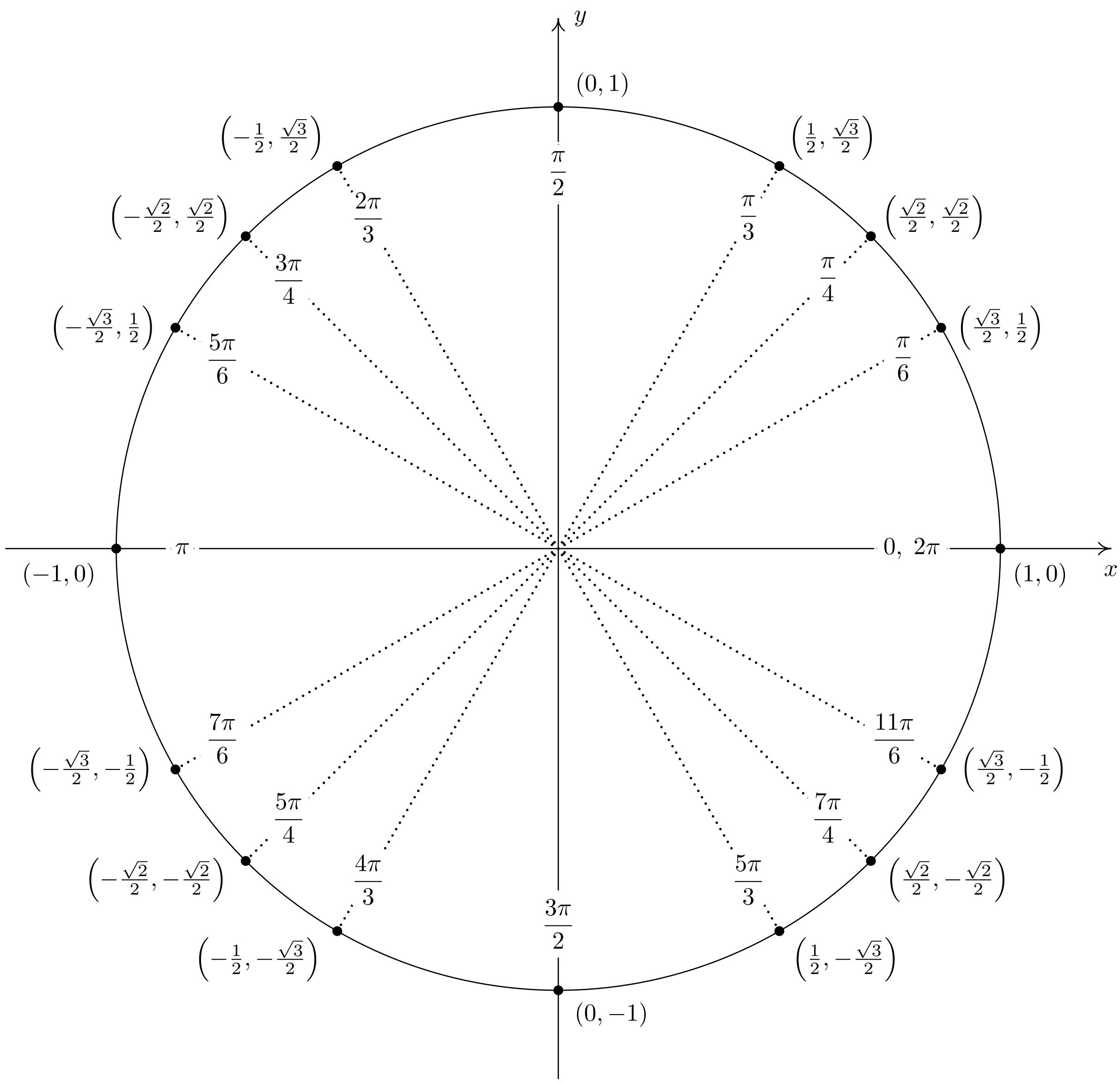


Figure 8.43: Important Points on the Unit Circle

The next example summarizes all of the important ideas discussed thus far in the section.

Example 142 Using reference angles

Suppose α is an acute angle with $\cos(\alpha) = \frac{5}{13}$.

1. Find $\sin(\alpha)$ and use this to plot α in standard position.

2. Find the sine and cosine of the following angles:

   (a) $\theta = \pi + \alpha$
   (b) $\theta = 2\pi - \alpha$
   (c) $\theta = 3\pi - \alpha$
   (d) $\theta = \frac{\pi}{2} + \alpha$

SOLUTION

1. Proceeding as in Example 140, we substitute $\cos(\alpha) = \frac{5}{13}$ into $\cos^2(\alpha) + \sin^2(\alpha) = 1$ and find $\sin(\alpha) = \pm\frac{12}{13}$. Since α is an acute (and therefore Quadrant I) angle, $\sin(\alpha)$ is positive. Hence, $\sin(\alpha) = \frac{12}{13}$. To plot α in standard position, we begin our rotation on the positive *x*-axis to the ray which contains the point $(\cos(\alpha), \sin(\alpha)) = \left(\frac{5}{13}, \frac{12}{13}\right)$: see Figure 8.46.

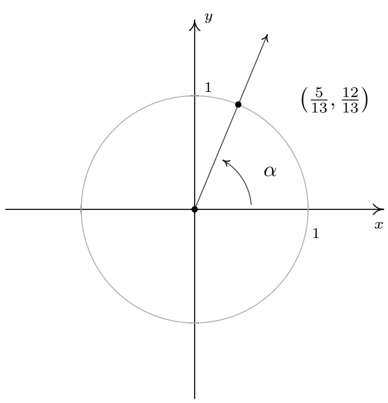


Figure 8.46: Sketching α

2. (a) To find the cosine and sine of $\theta = \pi + \alpha$, we first plot θ in standard position. We can imagine the sum of the angles $\pi + \alpha$ as a sequence of two rotations: a rotation of π radians followed by a rotation of α radians. (Since $\pi + \alpha = \alpha + \pi$, θ may be plotted by reversing the order of rotations given here. You should do this.) We see that α is the reference angle for θ, so by The Reference Angle Theorem, $\cos(\theta) = \pm\cos(\alpha) = \pm\frac{5}{13}$ and $\sin(\theta) = \pm\sin(\alpha) = \pm\frac{12}{13}$. Since the terminal side of θ falls in Quadrant III, both $\cos(\theta)$ and $\sin(\theta)$ are negative, hence, $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.

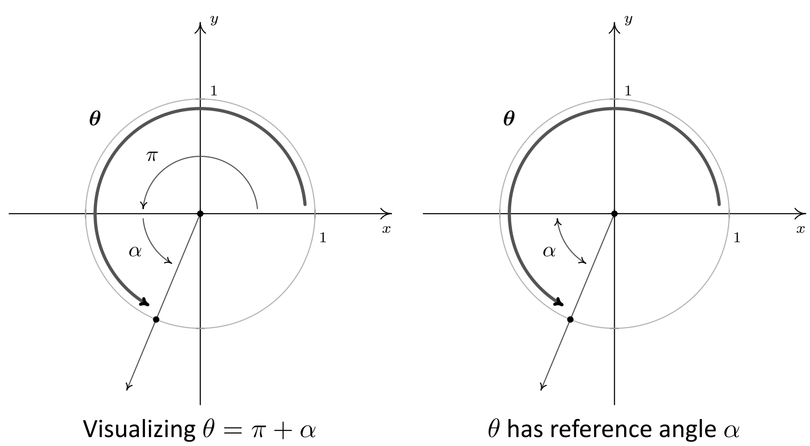
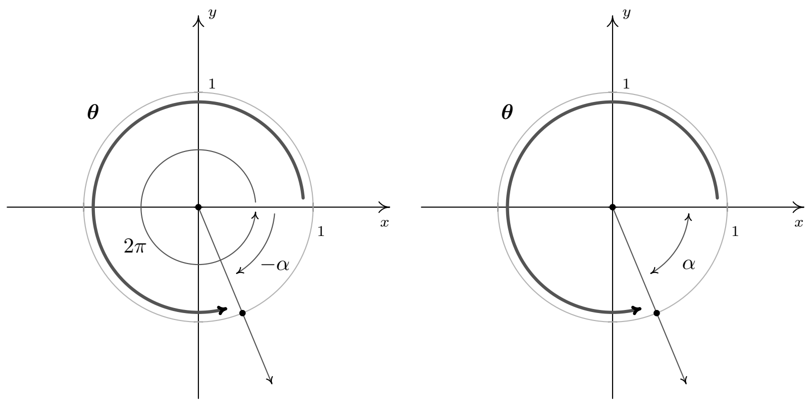


Figure 8.44: Finding $\cos(\theta)$ and $\sin(\theta)$ in Example 142.2(a)

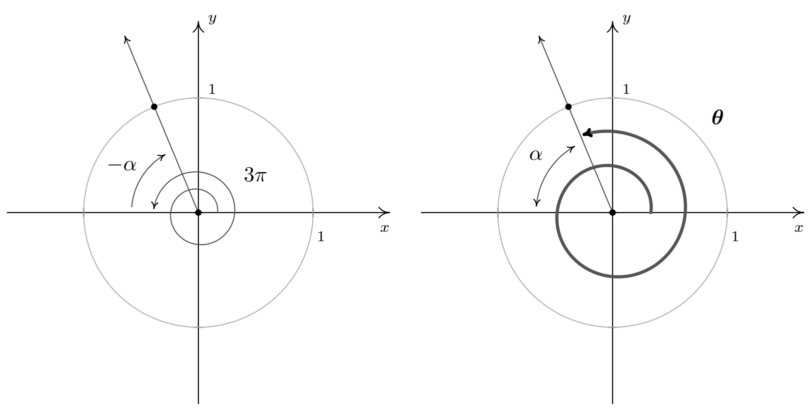
   (b) Rewriting $\theta = 2\pi - \alpha$ as $\theta = 2\pi + (-\alpha)$, we can plot θ by visualizing one complete revolution counter-clockwise followed by a *clockwise* revolution, or 'backing up,' of α radians. We see that α is θ's reference angle, and since θ is a Quadrant IV angle, the Reference Angle Theorem gives: $\cos(\theta) = \frac{5}{13}$ and $\sin(\theta) = -\frac{12}{13}$.



Visualizing $\theta = 2\pi - \alpha$ — θ has reference angle α

Figure 8.45: Finding $\cos(\theta)$ and $\sin(\theta)$ in Example 142.2(b)

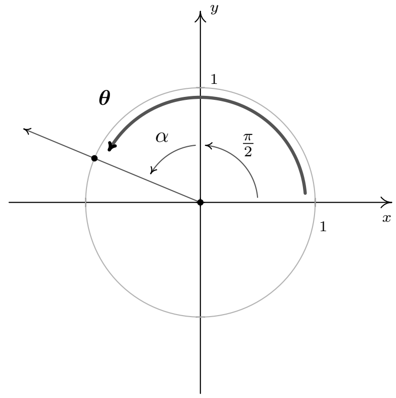
(c) Taking a cue from the previous problem, we rewrite $\theta = 3\pi - \alpha$ as $\theta = 3\pi + (-\alpha)$. The angle 3π represents one and a half revolutions counter-clockwise, so that when we 'back up' α radians, we end up in Quadrant II. Using the Reference Angle Theorem, we get $\cos(\theta) = -\frac{5}{13}$ and $\sin(\theta) = \frac{12}{13}$.



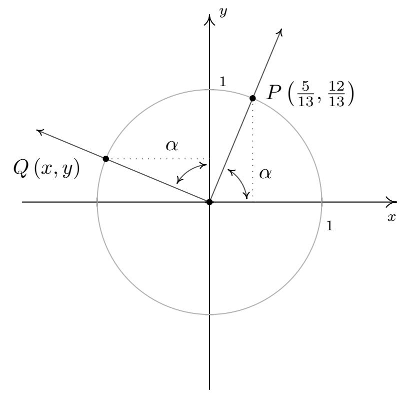
Visualizing $\theta = 3\pi - \alpha$ — θ has reference angle α

Figure 8.47: Finding $\cos(\theta)$ and $\sin(\theta)$ in Example 142.2(c)

(d) To plot $\theta = \frac{\pi}{2} + \alpha$, we first rotate $\frac{\pi}{2}$ radians and follow up with α radians. The reference angle here is *not* α, so The Reference Angle Theorem is not immediately applicable. (It's important that you see why this is the case. Take a moment to think about this before reading on.) Let $Q(x, y)$ be the point on the terminal side of θ which lies on the Unit Circle so that $x = \cos(\theta)$ and $y = \sin(\theta)$. Once we graph α in standard position, we use the fact that equal angles subtend equal chords to show that the dotted lines in the figure below are equal. Hence, $x = \cos(\theta) = -\frac{12}{13}$. Similarly, we find $y = \sin(\theta) = \frac{5}{13}$.



Visualizing $\theta = \frac{\pi}{2} + \alpha$



Using symmetry to determine $Q(x, y)$

Figure 8.48: Finding $\cos(\theta)$ and $\sin(\theta)$ in Example 142.2(a)

Our next example asks us to solve some very basic trigonometric equations.

Example 143 Solving basic trigonometric equations

Find all of the angles which satisfy the given equation.

1. $\cos(\theta) = \dfrac{1}{2}$
2. $\sin(\theta) = -\dfrac{1}{2}$
3. $\cos(\theta) = 0$.

Solution Since there is no context in the problem to indicate whether to use degrees or radians, we will default to using radian measure in our answers to each of these problems. This choice will be justified later in the text when we study what is known as Analytic Trigonometry. In those sections to come, radian measure will be the *only* appropriate angle measure so it is worth the time to become "fluent in radians" now.

1. If $\cos(\theta) = \frac{1}{2}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $x = \frac{1}{2}$. This means θ is a Quadrant I or IV angle with reference angle $\frac{\pi}{3}$.

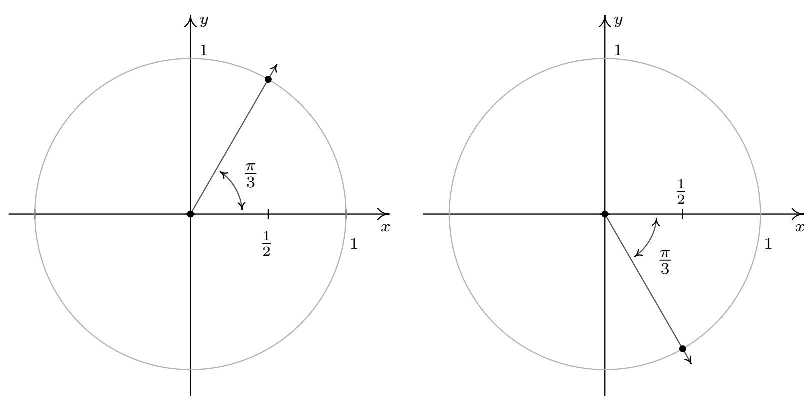


Figure 8.49: Angles with $\cos(\theta) = \frac{1}{2}$

One solution in Quadrant I is $\theta = \frac{\pi}{3}$, and since all other Quadrant I solutions must be coterminal with $\frac{\pi}{3}$, we find $\theta = \frac{\pi}{3} + 2\pi k$ for integers k. Proceeding similarly for the Quadrant IV case, we find the solution to $\cos(\theta) = \frac{1}{2}$ here is $\frac{5\pi}{3}$, so our answer in this Quadrant is $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

Recall from Section 8.1 that two angles in radian measure are coterminal if and only if they differ by an integer multiple of 2π. Hence to describe all angles coterminal with a given angle, we add $2\pi k$ for integers $k = 0, \pm 1, \pm 2, \ldots$.

2. If $\sin(\theta) = -\frac{1}{2}$, then when θ is plotted in standard position, its terminal side intersects the Unit Circle at $y = -\frac{1}{2}$. From this, we determine θ is a Quadrant III or Quadrant IV angle with reference angle $\frac{\pi}{6}$.

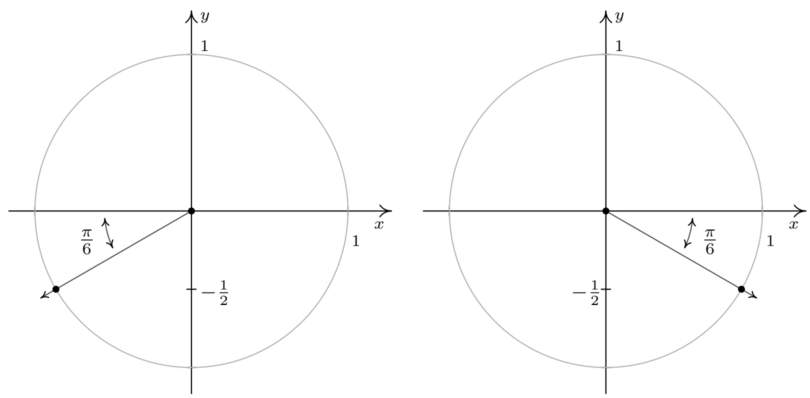


Figure 8.50: Angles with $\sin(\theta) = -\frac{1}{2}$

In Quadrant III, one solution is $\frac{7\pi}{6}$, so we capture all Quadrant III solutions by adding integer multiples of 2π: $\theta = \frac{7\pi}{6} + 2\pi k$. In Quadrant IV, one solution is $\frac{11\pi}{6}$ so all the solutions here are of the form $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k.

3. The angles with $\cos(\theta) = 0$ are quadrantal angles whose terminal sides, when plotted in standard position, lie along the y-axis.

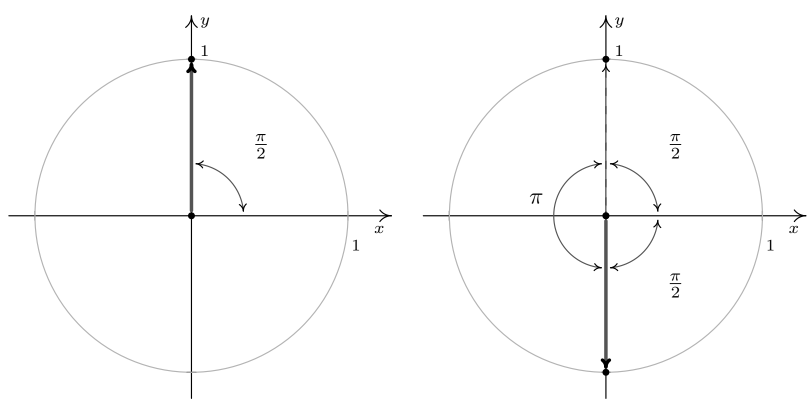


Figure 8.51: Angles with $\cos(\theta) = 0$

While, technically speaking, $\frac{\pi}{2}$ isn't a reference angle we can nonetheless use it to find our answers. If we follow the procedure set forth in the previous examples, we find $\theta = \frac{\pi}{2} + 2\pi k$ and $\theta = \frac{3\pi}{2} + 2\pi k$ for integers, k. While this solution is correct, it can be shortened to $\theta = \frac{\pi}{2} + \pi k$ for integers k. (Can you see why this works from the diagram?)

One of the key items to take from Example 143 is that, in general, solutions to trigonometric equations consist of infinitely many answers. The reader is encouraged write out as many of these answers as necessary to get a feel for them. This is especially important when checking answers to the exercises. For example, another Quadrant IV solution to $\sin(\theta) = -\frac{1}{2}$ is $\theta = -\frac{\pi}{6}$. Hence, the family of Quadrant IV answers to number 2 above could just have easily been written $\theta = -\frac{\pi}{6} + 2\pi k$ for integers k. While on the surface, this family may look different than the stated solution of $\theta = \frac{11\pi}{6} + 2\pi k$ for integers k, we leave it to the reader to show they represent the same list of angles.

8.2.1 Beyond the Unit Circle

We began the section with a quest to describe the position of a particle experiencing circular motion. In defining the cosine and sine functions, we assigned to each angle a position on the *Unit* Circle. In this subsection, we broaden our scope to include circles of radius r centered at the origin. Consider for the moment the *acute* angle θ drawn below in standard position. Let $Q(x, y)$ be the point on the terminal side of θ which lies on the circle $x^2 + y^2 = r^2$, and let $P(x', y')$ be the point on the terminal side of θ which lies on the Unit Circle. Now consider dropping perpendiculars from P and Q to create two right triangles, $\triangle OPA$ and $\triangle OQB$. These triangles are similar, (do you remember why?) thus it follows that $\frac{x}{x'} = \frac{r}{1} = r$, so $x = rx'$ and, similarly, we find $y = ry'$. Since, by definition, $x' = \cos(\theta)$ and $y' = \sin(\theta)$, we get the coordinates of Q to be $x = r\cos(\theta)$ and $y = r\sin(\theta)$. By reflecting these points through the x-axis, y-axis and origin, we obtain the result for all non-quadrantal angles θ, and we leave it to the reader to verify these formulas hold for the quadrantal angles.

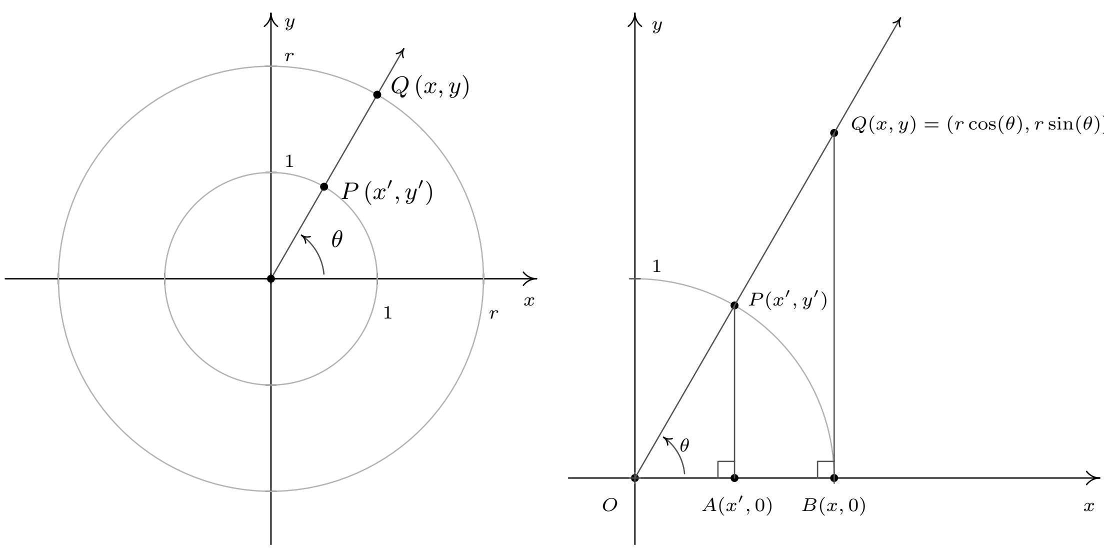


Figure 8.52: Determining coordinates of $Q(x, y)$ in terms of $\cos(\theta)$ and $\sin(\theta)$

Not only can we describe the coordinates of Q in terms of $\cos(\theta)$ and $\sin(\theta)$ but since the radius of the circle is $r = \sqrt{x^2 + y^2}$, we can also express $\cos(\theta)$ and $\sin(\theta)$ in terms of the coordinates of Q. These results are summarized in the following theorem.

Theorem 52 Generalized sine and cosine

If $Q(x,y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2+y^2=r^2$ then $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Moreover,

$$\cos(\theta)=\frac{x}{r}=\frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(\theta)=\frac{y}{r}=\frac{y}{\sqrt{x^2+y^2}}$$

Note that in the case of the Unit Circle we have $r=\sqrt{x^2+y^2}=1$, so Theorem 52 reduces to our definitions of $\cos(\theta)$ and $\sin(\theta)$.

Example 144 Finding $cos(\theta)$ and $\sin(\theta)$ beyond the unit circle

1. Suppose that the terminal side of an angle θ, when plotted in standard position, contains the point $Q(4,-2)$. Find $\sin(\theta)$ and $\cos(\theta)$.

2. In Example 138 in Section 8.1, we approximated the radius of the earth at 41.628° north latitude to be 2960 miles. Justify this approximation if the radius of the Earth at the Equator is approximately 3960 miles.

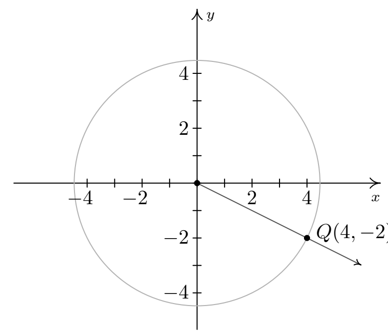


Figure 8.53: The terminal side of θ contains $Q(4,-2)$

SOLUTION

1. Using Theorem 52 with $x=4$ and $y=-2$, we find $r=\sqrt{(4)^2+(-2)^2}=\sqrt{20}=2\sqrt{5}$ so that $\cos(\theta)=\frac{x}{r}=\frac{4}{2\sqrt{5}}=\frac{2\sqrt{5}}{5}$ and $\sin(\theta)=\frac{y}{r}=\frac{-2}{2\sqrt{5}}=-\frac{\sqrt{5}}{5}$: see Figure 8.53.

2. Assuming the Earth is a sphere, a cross-section through the poles produces a circle of radius 3960 miles. Viewing the Equator as the x-axis, the value we seek is the x-coordinate of the point $Q(x,y)$ indicated in Figure 8.54

   Using Theorem 52, we get $x=3960\cos(41.628^\circ)$. Using a calculator in 'degree' mode, we find $3960\cos(41.628^\circ)\approx 2960$. Hence, the radius of the Earth at North Latitude 41.628° is approximately 2960 miles.

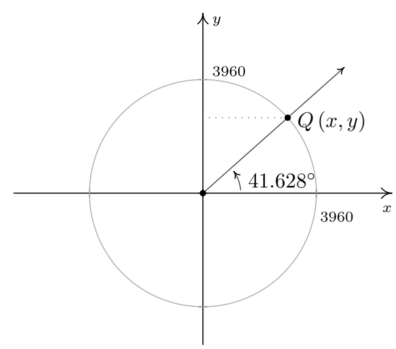


Figure 8.54: A point on the Earth at 41.628°N

Theorem 52 gives us what we need to describe the position of an object traveling in a circular path of radius r with constant angular velocity ω. Suppose that at time t, the object has swept out an angle measuring θ radians. If we assume that the object is at the point $(r,0)$ when $t=0$, the angle θ is in standard position. By definition, $\omega=\frac{\theta}{t}$ which we rewrite as $\theta=\omega t$. According to Theorem 52, the location of the object $Q(x,y)$ on the circle is found using the equations $x=r\cos(\theta)=r\cos(\omega t)$ and $y=r\sin(\theta)=r\sin(\omega t)$. Hence, at time t, the object is at the point $(r\cos(\omega t), r\sin(\omega t))$. We have just argued the following.

Theorem 53 Equations for circular motion

Suppose an object is travelling in a circular path of radius r centred at the origin with constant angular velocity ω. If $t = 0$ corresponds to the point $(r, 0)$, then the x and y coordinates of the object are functions of t and are given by $x = r\cos(\omega t)$ and $y = r\sin(\omega t)$. Here, $\omega > 0$ indicates a counter-clockwise direction and $\omega < 0$ indicates a clockwise direction.

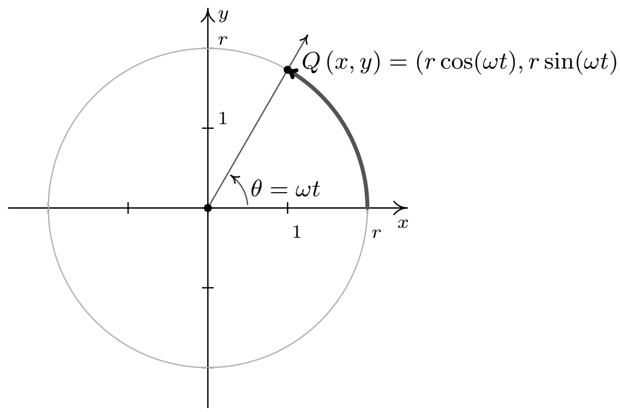


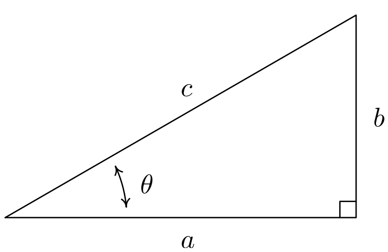
Figure 8.55: Equations for circular motion

Example 145 Motion on the Earth's surface

Suppose we are in the situation of Example 138. Find the equations of motion of Lakeland Community College as the earth rotates.

Solution From Example 138, we take $r = 2960$ miles and and $\omega = \frac{\pi}{12 \text{ hours}}$. Hence, the equations of motion are $x = r\cos(\omega t) = 2960\cos\left(\frac{\pi}{12}t\right)$ and $y = r\sin(\omega t) = 2960\sin\left(\frac{\pi}{12}t\right)$, where x and y are measured in miles and t is measured in hours.

In addition to circular motion, Theorem 52 is also the key to developing what is usually called 'right triangle' trigonometry. (You were probably exposed to this in High School.) As we shall see in the sections to come, many applications in trigonometry involve finding the measures of the angles in, and lengths of the sides of, right triangles. Indeed, we made good use of some properties of right triangles to find the exact values of the cosine and sine of many of the angles in Example 139, so the following development shouldn't be that much of a surprise. Consider the generic right triangle below with corresponding acute angle θ. The side with length a is called the side of the triangle *adjacent* to θ; the side with length b is called the side of the triangle *opposite* θ; and the remaining side of length c (the side opposite the right angle) is called the hypotenuse. We now imagine drawing this triangle in Quadrant I so that the angle θ is in standard position with the adjacent side to θ lying along the positive x-axis.



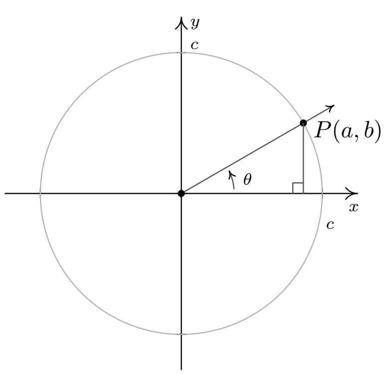


Figure 8.56: Relating triangular and circular trigonometry

According to the Pythagorean Theorem, $a^2 + b^2 = c^2$, so that the point $P(a, b)$ lies on a circle of radius c. Theorem 52 tells us that $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, so we have determined the cosine and sine of θ in terms of the lengths of the sides of the right triangle. Thus we have the following theorem.

Theorem 54 Sine and cosine for right triangles

Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a, the length of the side opposite θ is b, and the length of the hypotenuse is c, then $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$.

Example 146 Using triangular trigonometry

Find the measure of the missing angle and the lengths of the missing sides of the triangle in Figure 8.57.

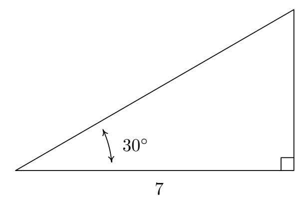


Figure 8.57: The triangle for Example 146

Solution The first and easiest task is to find the measure of the missing angle. Since the sum of angles of a triangle is 180°, we know that the missing angle has measure $180^\circ - 30^\circ - 90^\circ = 60^\circ$. We now proceed to find the lengths of the remaining two sides of the triangle. Let c denote the length of the hypotenuse of the triangle. By Theorem 54, we have $\cos(30^\circ) = \frac{7}{c}$, or $c = \frac{7}{\cos(30^\circ)}$. Since $\cos(30^\circ) = \frac{\sqrt{3}}{2}$, we have, after the usual fraction gymnastics, $c = \frac{14\sqrt{3}}{3}$. At this point, we have two ways to proceed to find the length of the side opposite the 30° angle, which we'll denote b. We know the length of the adjacent side is 7 and the length of the hypotenuse is $\frac{14\sqrt{3}}{3}$, so we could use the Pythagorean Theorem to find the missing side and solve $(7)^2 + b^2 = \left(\frac{14\sqrt{3}}{3}\right)^2$ for b. Alternatively, we could use Theorem 54, namely that $\sin(30^\circ) = \frac{b}{c}$. Choosing the latter, we find $b = c\sin(30^\circ) = \frac{14\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{7\sqrt{3}}{3}$. The triangle with all of its data is recorded in Figure 8.58

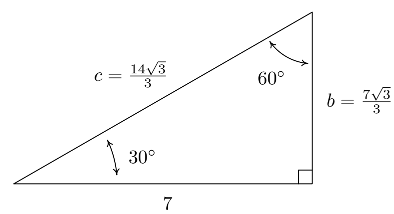
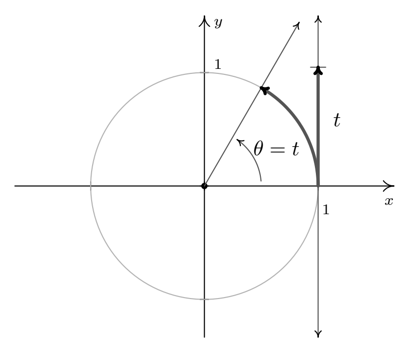


Figure 8.58: The completed triangle for Example 146

We close this section by noting that we can easily extend the functions cosine and sine to real numbers by identifying a real number t with the angle $\theta = t$ radians. Using this identification, we define $\cos(t) = \cos(\theta)$ and $\sin(t) = \sin(\theta)$. In practice this means expressions like $\cos(\pi)$ and $\sin(2)$ can be found by regarding the inputs as angles in radian measure or real numbers; the choice is the reader's. If we trace the identification of real numbers t with angles θ in radian measure to its roots on page 315, we can spell out this correspondence more precisely. For each real number t, we associate an oriented arc t units in length with initial point $(1, 0)$ and endpoint $P(\cos(t), \sin(t))$.

In the same way we studied polynomial, rational, exponential, and logarithmic functions, we will study the trigonometric functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$. The first order of business is to find the domains and ranges of these functions. Whether we think of identifying the real number t with the angle $\theta = t$ radians, or think of wrapping an oriented arc around the Unit Circle to find coordinates on the Unit Circle, it should be clear that both the cosine and sine functions are defined for all real numbers t. In other words, the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is $(-\infty, \infty)$. Since $\cos(t)$ and $\sin(t)$ represent x- and y-coordinates, respectively, of points on the Unit Circle, they both take on all of the values between -1 an 1, inclusive. In other words, the range of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is the interval $[-1, 1]$. To summarize:



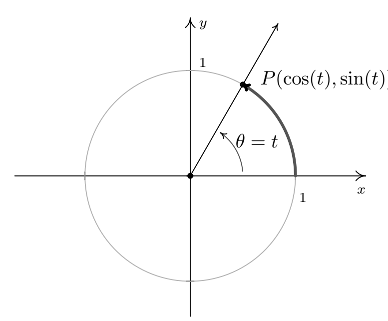


Figure 8.59: Defining $\cos(t)$ and $\sin(t)$ as functions of a real variable

Theorem 55 Domain and Range of the Cosine and Sine Functions

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

Suppose, as in the Exercises, we are asked to solve an equation such as $\sin(t) = -\frac{1}{2}$. As we have already mentioned, the distinction between t as a real number and as an angle $\theta = t$ radians is often blurred. Indeed, we solve $\sin(t) = -\frac{1}{2}$ in the exact same manner as we did in Example 143 number 2. Our solution is only cosmetically different in that the variable used is t rather than θ: $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for integers, k. We will study the cosine and sine functions in greater detail in Section 8.5. Until then, keep in mind that any properties of cosine and sine developed in the following sections which regard them as functions of *angles* in *radian* measure apply equally well if the inputs are regarded as *real numbers*.

Exercises 8.2

Problems

In Exercises 1 – 20, find the exact value of the cosine and sine of the given angle.

1. $\theta = 0$
2. $\theta = \frac{\pi}{4}$
3. $\theta = \frac{\pi}{3}$
4. $\theta = \frac{\pi}{2}$
5. $\theta = \frac{2\pi}{3}$
6. $\theta = \frac{3\pi}{4}$
7. $\theta = \pi$
8. $\theta = \frac{7\pi}{6}$
9. $\theta = \frac{5\pi}{4}$
10. $\theta = \frac{4\pi}{3}$
11. $\theta = \frac{3\pi}{2}$
12. $\theta = \frac{5\pi}{3}$
13. $\theta = \frac{7\pi}{4}$
14. $\theta = \frac{23\pi}{6}$
15. $\theta = -\frac{13\pi}{2}$
16. $\theta = -\frac{43\pi}{6}$
17. $\theta = -\frac{3\pi}{4}$
18. $\theta = -\frac{\pi}{6}$
19. $\theta = \frac{10\pi}{3}$
20. $\theta = 117\pi$

In Exercises 21 – 30, use the results developed throughout the section to find the requested value.

21. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?
22. If $\cos(\theta) = \frac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?
23. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?
24. If $\cos(\theta) = -\frac{2}{11}$ with θ in Quadrant III, what is $\sin(\theta)$?
25. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, what is $\cos(\theta)$?
26. If $\cos(\theta) = \frac{28}{53}$ with θ in Quadrant IV, what is $\sin(\theta)$?
27. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, what is $\cos(\theta)$?
28. If $\cos(\theta) = \frac{\sqrt{10}}{10}$ and $2\pi < \theta < \frac{5\pi}{2}$, what is $\sin(\theta)$?
29. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, what is $\cos(\theta)$?
30. If $\cos(\theta) = -0.98$ and $\frac{\pi}{2} < \theta < \pi$, what is $\sin(\theta)$?

In Exercises 31 – 39, find all of the angles which satisfy the given equation.

31. $\sin(\theta) = \frac{1}{2}$
32. $\cos(\theta) = -\frac{\sqrt{3}}{2}$
33. $\sin(\theta) = 0$
34. $\cos(\theta) = \frac{\sqrt{2}}{2}$
35. $\sin(\theta) = \frac{\sqrt{3}}{2}$
36. $\cos(\theta) = -1$
37. $\sin(\theta) = -1$
38. $\cos(\theta) = \frac{\sqrt{3}}{2}$
39. $\cos(\theta) = -1.001$

In Exercises 40 – 48, solve the equation for t. (See the comments following Theorem 55.)

40. $\cos(t) = 0$

41. $\sin(t) = -\frac{\sqrt{2}}{2}$

42. $\cos(t) = 3$

43. $\sin(t) = -\frac{1}{2}$

44. $\cos(t) = \frac{1}{2}$

45. $\sin(t) = -2$

46. $\cos(t) = 1$

47. $\sin(t) = 1$

48. $\cos(t) = -\frac{\sqrt{2}}{2}$

In Exercises 49 – 54, use your calculator to approximate the given value to three decimal places. Make sure your calculator is in the proper angle measurement mode!

49. $\sin(78.95^\circ)$

50. $\cos(-2.01)$

51. $\sin(392.994)$

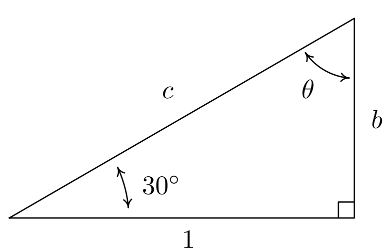
52. $\cos(207^\circ)$

53. $\sin(\pi^\circ)$

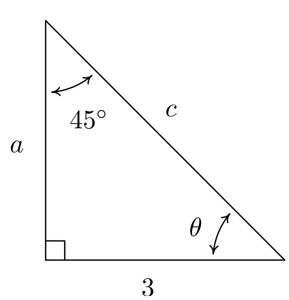
54. $\cos(e)$

In Exercises 55 – 58, find the measurement of the missing angle and the lengths of the missing sides. (See Example 146)

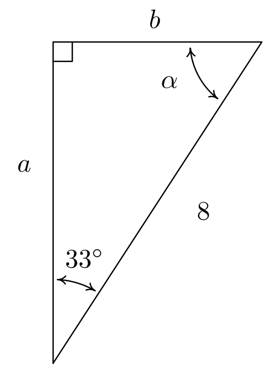
55. Find θ, b, and c.



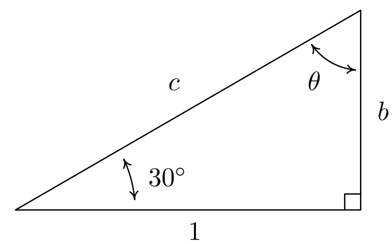
56. Find θ, a, and c.



57. Find α, a, and b.



58. Find β, a, and c.



In Exercises 59 – 64, assume that θ is an acute angle in a right triangle and use Theorem 54 to find the requested side.

59. If $\theta = 12^\circ$ and the side adjacent to θ has length 4, how long is the hypotenuse?

60. If $\theta = 78.123^\circ$ and the hypotenuse has length 5280, how long is the side adjacent to θ?

61. If $\theta = 59^\circ$ and the side opposite θ has length 117.42, how long is the hypotenuse?

62. If $\theta = 5^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?

63. If $\theta = 5^\circ$ and the hypotenuse has length 10, how long is the side adjacent to θ?

64. If $\theta = 37.5^\circ$ and the side opposite θ has length 306, how long is the side adjacent to θ?

In Exercises 65 – 68, let θ be the angle in standard position whose terminal side contains the given point then compute $\cos(\theta)$ **and** $\sin(\theta)$**.**

65. $P(-7, 24)$

66. $Q(3, 4)$

67. $R(5, -9)$

68. $T(-2, -11)$

In Exercises 69 – 72, find the equations of motion for the given scenario. Assume that the center of the motion is the origin, the motion is counter-clockwise and that $t = 0$ corresponds to a position along the positive x-axis. (See Equation 53 and Example 138.)

69. A point on the edge of the spinning yo-yo in Exercise 42 from Section 8.1.

    Recall: The diameter of the yo-yo is 2.25 inches and it spins at 4500 revolutions per minute.

70. The yo-yo in exercise 44 from Section 8.1.

    Recall: The radius of the circle is 28 inches and it completes one revolution in 3 seconds.

71. A point on the edge of the hard drive in Exercise 45 from Section 8.1.

    Recall: The diameter of the hard disk is 2.5 inches and it spins at 7200 revolutions per minute.

72. A passenger on the Big Wheel in Exercise 47 from Section 8.1.

    Recall: The diameter is 128 feet and completes 2 revolutions in 2 minutes, 7 seconds.

73. A passenger on the Big Wheel in Exercise 47 from Section 8.1.

    Recall: The diameter is 128 feet and completes 2 revolutions in 2 minutes, 7 seconds.

74. Let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sin(\alpha) = \cos(\beta)$ and $\sin(\beta) = \cos(\alpha)$. The fact that co-functions of complementary angles are equal in this case is not an accident and a more general result will be given in Section 8.4.

75. In the scenario of Theorem 53, we assumed that at $t = 0$, the object was at the point $(r, 0)$. If this is not the case, we can adjust the equations of motion by introducing a 'time delay.' If $t_0 > 0$ is the first time the object passes through the point $(r, 0)$, show, with the help of your classmates, the equations of motion are $x = r\cos(\omega(t - t_0))$ and $y = r\sin(\omega(t - t_0))$.

8.3 The Six Circular Functions and Fundamental Identities

In section 8.2, we defined $\cos(\theta)$ and $\sin(\theta)$ for angles θ using the coordinate values of points on the Unit Circle. As such, these functions earn the moniker **circular functions**. It turns out that cosine and sine are just two of the six commonly used circular functions which we define below.

In Theorem 54 we also showed cosine and sine to be functions of an angle residing in a right triangle so we could just as easily call them *trigonometric* functions. In later sections, you will find that we do indeed use the phrase 'trigonometric function' interchangeably with the term 'circular function'.

Definition 54 The Circular Functions

Suppose θ is an angle plotted in standard position and $P(x, y)$ is the point on the terminal side of θ which lies on the Unit Circle.

- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided $y \neq 0$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **cotangent** of θ, denoted $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

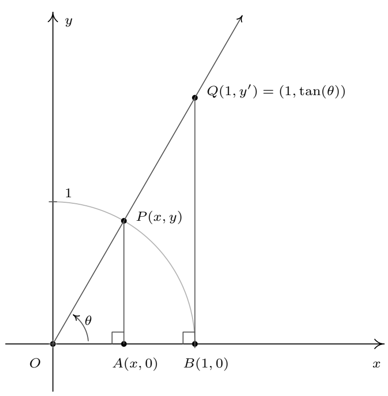


Figure 8.60: Explaining the tangent and secant functions

While we left the history of the name 'sine' as an interesting research project in Section 8.2, the names 'tangent' and 'secant' can be explained using the diagram below. Consider the acute angle θ below in standard position. Let $P(x, y)$ denote, as usual, the point on the terminal side of θ which lies on the Unit Circle and let $Q(1, y')$ denote the point on the terminal side of θ which lies on the vertical line $x = 1$, as in Figure 8.60.

The word 'tangent' comes from the Latin meaning 'to touch,' and for this reason, the line $x = 1$ is called a *tangent* line to the Unit Circle since it intersects, or 'touches', the circle at only one point, namely $(1, 0)$. Dropping perpendiculars from P and Q creates a pair of similar triangles $\triangle OPA$ and $\triangle OQB$. Thus $\frac{y'}{y} = \frac{1}{x}$ which gives $y' = \frac{y}{x} = \tan(\theta)$, where this last equality comes from applying Definition 54. We have just shown that for acute angles θ, $\tan(\theta)$ is the y-coordinate of the point on the terminal side of θ which lies on the line $x = 1$ which is *tangent* to the Unit Circle. Now the word 'secant' means 'to cut', so a secant line is any line that 'cuts through' a circle at two points. (Compare this with the definition given in Section 3.1.) The line containing the terminal side of θ is a secant line since it intersects the Unit Circle in Quadrants I and III. With the point P lying on the Unit Circle, the length of the hypotenuse of $\triangle OPA$ is 1. If we let h denote the length of the hypotenuse of $\triangle OQB$, we have from similar triangles that $\frac{h}{1} = \frac{1}{x}$, or $h = \frac{1}{x} = \sec(\theta)$. Hence for an acute angle θ, $\sec(\theta)$ is the length of the line segment which lies on the secant line determined by the terminal side of θ and 'cuts off' the tangent line $x = 1$. Not only do these observations help explain the names of these functions, they serve as the basis for a fundamental inequality needed for Calculus which we'll explore in the Exercises.

Of the six circular functions, only cosine and sine are defined for all angles. Since $\cos(\theta) = x$ and $\sin(\theta) = y$ in Definition 54, it is customary to rephrase the remaining four circular functions in terms of cosine and sine. The following theorem is a result of simply replacing x with $\cos(\theta)$ and y with $\sin(\theta)$ in Definition 54.

Theorem 56 Reciprocal and Quotient Identities

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

Example 147 Evaluating circular functions

Find the indicated value, if it exists.

1. $\sec\left(60^\circ\right)$
2. $\csc\left(\frac{7\pi}{4}\right)$
3. $\cot(3)$
4. $\tan\left(\theta\right)$, where θ is any angle coterminal with $\frac{3\pi}{2}$.
5. $\cos\left(\theta\right)$, where $\csc(\theta) = -\sqrt{5}$ and θ is a Quadrant IV angle.
6. $\sin\left(\theta\right)$, where $\tan(\theta) = 3$ and $\pi < \theta < \frac{3\pi}{2}$.

Solution

1. According to Theorem 56, $\sec\left(60^\circ\right) = \frac{1}{\cos(60^\circ)}$. Hence, $\sec\left(60^\circ\right) = \frac{1}{(1/2)} = 2$.
2. Since $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\csc\left(\frac{7\pi}{4}\right) = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.
3. Since $\theta = 3$ radians is not one of the 'common angles' from Section 8.2, we resort to the calculator for a decimal approximation. Ensuring that the calculator is in radian mode, we find $\cot(3) = \frac{\cos(3)}{\sin(3)} \approx -7.015$.
4. If θ is coterminal with $\frac{3\pi}{2}$, then $\cos(\theta) = \cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin(\theta) = \sin\left(\frac{3\pi}{2}\right) = -1$. Attempting to compute $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ results in $\frac{-1}{0}$, so $\tan(\theta)$ is undefined.

5. We are given that $\csc(\theta) = \frac{1}{\sin(\theta)} = -\sqrt{5}$ so $\sin(\theta) = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$. As we saw in Section 8.2, we can use the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$, to find $\cos(\theta)$ by knowing $\sin(\theta)$. Substituting, we get $\cos^2(\theta) + \left(-\frac{\sqrt{5}}{5}\right)^2 = 1$, which gives $\cos^2(\theta) = \frac{4}{5}$, or $\cos(\theta) = \pm\frac{2\sqrt{5}}{5}$. Since θ is a Quadrant IV angle, $\cos(\theta) > 0$, so $\cos(\theta) = \frac{2\sqrt{5}}{5}$.

6. If $\tan(\theta) = 3$, then $\frac{\sin(\theta)}{\cos(\theta)} = 3$. Be careful - this does **NOT** mean we can take $\sin(\theta) = 3$ and $\cos(\theta) = 1$. Instead, from $\frac{\sin(\theta)}{\cos(\theta)} = 3$ we get: $\sin(\theta) = 3\cos(\theta)$. To relate $\cos(\theta)$ and $\sin(\theta)$, we once again employ the Pythagorean Identity, $\cos^2(\theta)+\sin^2(\theta) = 1$. Solving $\sin(\theta) = 3\cos(\theta)$ for $\cos(\theta)$, we find $\cos(\theta) = \frac{1}{3}\sin(\theta)$. Substituting this into the Pythagorean Identity, we find $\sin^2(\theta) + \left(\frac{1}{3}\sin(\theta)\right)^2 = 1$. Solving, we get $\sin^2(\theta) = \frac{9}{10}$ so $\sin(\theta) = \pm\frac{3\sqrt{10}}{10}$. Since $\pi < \theta < \frac{3\pi}{2}$, θ is a Quadrant III angle. This means $\sin(\theta) < 0$, so our final answer is $\sin(\theta) = -\frac{3\sqrt{10}}{10}$.

While the Reciprocal and Quotient Identities presented in Theorem 56 allow us to always reduce problems involving secant, cosecant, tangent and cotangent to problems involving cosine and sine, it is not always convenient to do so. It is worth taking the time to memorize the tangent and cotangent values of the common angles summarized below.

As we shall see shortly, when solving equations involving secant and cosecant, we usually convert back to cosines and sines. However, when solving for tangent or cotangent, we usually stick with what we're dealt.

Tangent and Cotangent Values of Common Angles

θ(degrees)	θ(radians)	$\tan(\theta)$	$\cot(\theta)$
0°	0	0	undefined
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	1	1
60°	$\frac{\pi}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	undefined	0

Coupling Theorem 56 with the Reference Angle Theorem, Theorem 51, we get the following.

Theorem 57 Generalized Reference Angle Theorem

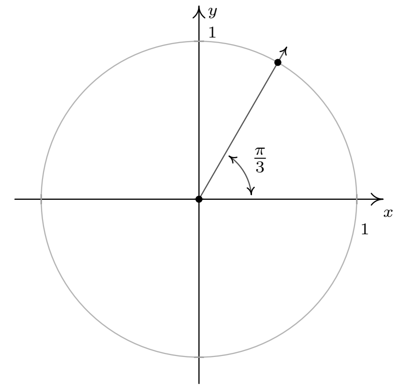
The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle. More specifically, if α is the reference angle for θ, then: $\cos(\theta) = \pm\cos(\alpha)$, $\sin(\theta) = \pm\sin(\alpha)$, $\sec(\theta) = \pm\sec(\alpha)$, $\csc(\theta) = \pm\csc(\alpha)$, $\tan(\theta) = \pm\tan(\alpha)$ and $\cot(\theta) = \pm\cot(\alpha)$. The choice of the $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

We put Theorem 57 to good use in the following example.

Example 148 Solving basic trigonometric equations

Find all angles which satisfy the given equation.

1. $\sec(\theta) = 2$
2. $\tan(\theta) = \sqrt{3}$
3. $\cot(\theta) = -1$.



SOLUTION

1. To solve $\sec(\theta) = 2$, we convert to cosines and get $\frac{1}{\cos(\theta)} = 2$ or $\cos(\theta) = \frac{1}{2}$. This is the exact same equation we solved in Example 143, number 1, so we know the answer is: $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$ for integers k.

2. From the table of common values, we see $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$. According to Theorem 57, we know the solutions to $\tan(\theta) = \sqrt{3}$ must, therefore, have a reference angle of $\frac{\pi}{3}$. Our next task is to determine in which quadrants the solutions to this equation lie. Since tangent is defined as the ratio $\frac{y}{x}$ of points (x, y) on the Unit Circle with $x \neq 0$, tangent is positive when x and y have the same sign (i.e., when they are both positive or both negative.) This happens in Quadrants I and III. In Quadrant I, we get the solutions: $\theta = \frac{\pi}{3} + 2\pi k$ for integers k, and for Quadrant III, we get $\theta = \frac{4\pi}{3} + 2\pi k$ for integers k. While these descriptions of the solutions are correct, they can be combined into one list as $\theta = \frac{\pi}{3} + \pi k$ for integers k. The latter form of the solution is best understood looking at the geometry of the situation in Figure 8.61. (See Example 143 number 3 in Section 8.2 for another example of this kind of simplification of the solution.)

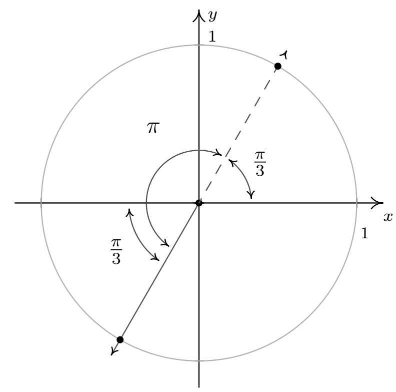
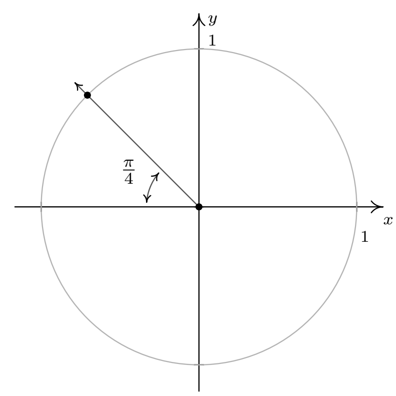


Figure 8.61: Solving $\tan(\theta) = \sqrt{3}$

3. From the table of common values, we see that $\frac{\pi}{4}$ has a cotangent of 1, which means the solutions to $\cot(\theta) = -1$ have a reference angle of $\frac{\pi}{4}$. To find the quadrants in which our solutions lie, we note that $\cot(\theta) = \frac{x}{y}$ for a point (x, y) on the Unit Circle where $y \neq 0$. If $\cot(\theta)$ is negative, then x and y must have different signs (i.e., one positive and one negative.) Hence, our solutions lie in Quadrants II and IV: see Figure 8.62. Our Quadrant II solution is $\theta = \frac{3\pi}{4} + 2\pi k$, and for Quadrant IV, we get $\theta = \frac{7\pi}{4} + 2\pi k$ for integers k. Can these lists be combined? Indeed they can - one such way to capture all the solutions is: $\theta = \frac{3\pi}{4} + \pi k$ for integers k.



We have already seen the importance of identities in trigonometry. Our next task is to use use the Reciprocal and Quotient Identities found in Theorem 56 coupled with the Pythagorean Identity found in Theorem 50 to derive new Pythagorean-like identities for the remaining four circular functions. Assuming $\cos(\theta) \neq 0$, we may start with $\cos^2(\theta) + \sin^2(\theta) = 1$ and divide both sides by $\cos^2(\theta)$ to obtain $1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$. Using properties of exponents along with the Reciprocal and Quotient Identities, this reduces to $1 + \tan^2(\theta) = \sec^2(\theta)$. If $\sin(\theta) \neq 0$, we can divide both sides of the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$, apply Theorem 56 once again, and obtain $\cot^2(\theta) + 1 = \csc^2(\theta)$. These three Pythagorean Identities are worth memorizing and they, along with some of their other common forms, are summarized in the following theorem.

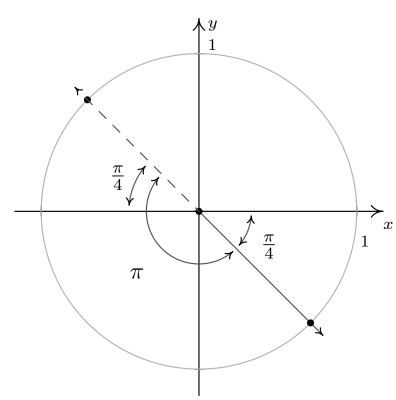


Figure 8.62: Solving $\cot(\theta) = -1$

Theorem 58 The Pythagorean Identities

1. $\cos^2(\theta) + \sin^2(\theta) = 1.$

   Common Alternate Forms:

   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$

2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.

   Common Alternate Forms:

   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$

3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.

   Common Alternate Forms:

   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

Trigonometric identities play an important role in not just Trigonometry, but in Calculus as well. We'll use them in this book to find the values of the circular functions of an angle and solve equations and inequalities. In Calculus, they are needed to simplify otherwise complicated expressions. In the next example, we make good use of the Theorems 56 and 58.

Example 149 Verifying trigonometric identities

Verify the following identities. Assume that all quantities are defined.

1. $\dfrac{1}{\csc(\theta)} = \sin(\theta)$
2. $\tan(\theta) = \sin(\theta)\sec(\theta)$
3. $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = 1$
4. $\dfrac{\sec(\theta)}{1-\tan(\theta)} = \dfrac{1}{\cos(\theta)-\sin(\theta)}$
5. $6\sec(\theta)\tan(\theta) = \dfrac{3}{1-\sin(\theta)} - \dfrac{3}{1+\sin(\theta)}$
6. $\dfrac{\sin(\theta)}{1-\cos(\theta)} = \dfrac{1+\cos(\theta)}{\sin(\theta)}$

Solution In verifying identities, we typically start with the more complicated side of the equation and use known identities to *transform* it into the other side of the equation.

1. To verify $\frac{1}{\csc(\theta)} = \sin(\theta)$, we start with the left side. Using $\csc(\theta) = \frac{1}{\sin(\theta)}$, we get:

$$\frac{1}{\csc(\theta)} = \frac{1}{\frac{1}{\sin(\theta)}} = \sin(\theta),$$

   which is what we were trying to prove.

2. Starting with the right hand side of $\tan(\theta) = \sin(\theta)\sec(\theta)$, we use $\sec(\theta) = \frac{1}{\cos(\theta)}$ and find:

$$\sin(\theta)\sec(\theta) = \sin(\theta)\frac{1}{\cos(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta),$$

where the last equality is courtesy of Theorem 56.

3. Expanding the left hand side of the equation gives: $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta)) = \sec^2(\theta)-\tan^2(\theta)$. According to Theorem 58, $\sec^2(\theta)-\tan^2(\theta) = 1$. Putting it all together,

$$(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1.$$

4. While both sides of our last identity contain fractions, the left side affords us more opportunities to use our identities. (Or, to put to another way, earn more partial credit if this were an exam question!) Substituting $\sec(\theta) = \frac{1}{\cos(\theta)}$ and $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, we get:

$$\begin{aligned} \frac{\sec(\theta)}{1-\tan(\theta)} &= \frac{\frac{1}{\cos(\theta)}}{1-\frac{\sin(\theta)}{\cos(\theta)}} = \frac{\frac{1}{\cos(\theta)}}{1-\frac{\sin(\theta)}{\cos(\theta)}} \cdot \frac{\cos(\theta)}{\cos(\theta)} \\ &= \frac{\left(\frac{1}{\cos(\theta)}\right)(\cos(\theta))}{\left(1-\frac{\sin(\theta)}{\cos(\theta)}\right)(\cos(\theta))} = \frac{1}{(1)(\cos(\theta)) - \left(\frac{\sin(\theta)}{\cos(\theta)}\right)(\cos(\theta))} \\ &= \frac{1}{\cos(\theta)-\sin(\theta)}, \end{aligned}$$

which is exactly what we had set out to show.

5. The right hand side of the equation seems to hold more promise. We get common denominators and add:

$$\begin{aligned} \frac{3}{1-\sin(\theta)} - \frac{3}{1+\sin(\theta)} &= \frac{3(1+\sin(\theta))}{(1-\sin(\theta))(1+\sin(\theta))} - \frac{3(1-\sin(\theta))}{(1+\sin(\theta))(1-\sin(\theta))} \\ &= \frac{3+3\sin(\theta)}{1-\sin^2(\theta)} - \frac{3-3\sin(\theta)}{1-\sin^2(\theta)} \\ &= \frac{(3+3\sin(\theta))-(3-3\sin(\theta))}{1-\sin^2(\theta)} \\ &= \frac{6\sin(\theta)}{1-\sin^2(\theta)} \end{aligned}$$

At this point, it is worth pausing to remind ourselves of our goal. We wish to transform this expression into $6\sec(\theta)\tan(\theta)$. Using a reciprocal and quotient identity, we find $6\sec(\theta)\tan(\theta) = 6\left(\frac{1}{\cos(\theta)}\right)\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$. In other words, we need to get cosines in our denominator. Theorem 58 tells us $1-\sin^2(\theta) = \cos^2(\theta)$ so we get:

$$\begin{aligned}\frac{3}{1-\sin(\theta)}-\frac{3}{1+\sin(\theta)} &= \frac{6\sin(\theta)}{1-\sin^2(\theta)}=\frac{6\sin(\theta)}{\cos^2(\theta)}\\ &= 6\left(\frac{1}{\cos(\theta)}\right)\left(\frac{\sin(\theta)}{\cos(\theta)}\right)=6\sec(\theta)\tan(\theta)\end{aligned}$$

6. It is debatable which side of the identity is more complicated. One thing which stands out is that the denominator on the left hand side is $1-\cos(\theta)$, while the numerator of the right hand side is $1+\cos(\theta)$. This suggests the strategy of starting with the left hand side and multiplying the numerator and denominator by the quantity $1+\cos(\theta)$:

$$\begin{aligned}\frac{\sin(\theta)}{1-\cos(\theta)} &= \frac{\sin(\theta)}{(1-\cos(\theta))}\cdot\frac{(1+\cos(\theta))}{(1+\cos(\theta))}=\frac{\sin(\theta)(1+\cos(\theta))}{(1-\cos(\theta))(1+\cos(\theta))}\\ &= \frac{\sin(\theta)(1+\cos(\theta))}{1-\cos^2(\theta)}=\frac{\sin(\theta)(1+\cos(\theta))}{\sin^2(\theta)}\\ &= \frac{\cancel{\sin(\theta)}(1+\cos(\theta))}{\cancel{\sin(\theta)}\sin(\theta)}=\frac{1+\cos(\theta)}{\sin(\theta)}\end{aligned}$$

In Example 149.6 above, we see that multiplying $1-\cos(\theta)$ by $1+\cos(\theta)$ produces a difference of squares that can be simplified to one term using Theorem 58. This is exactly the same kind of phenomenon that occurs when we multiply expressions such as $1-\sqrt{2}$ by $1+\sqrt{2}$ or $3-4i$ by $3+4i$. (Can you recall instances from earlier chapters where we did such things?) For this reason, the quantities $(1-\cos(\theta))$ and $(1+\cos(\theta))$ are called 'Pythagorean Conjugates.' Below is a list of other common Pythagorean Conjugates.

Key Idea 37 Pythagorean Conjugates

- $1-\cos(\theta)$ and $1+\cos(\theta)$: $(1-\cos(\theta))(1+\cos(\theta))=1-\cos^2(\theta)=\sin^2(\theta)$
- $1-\sin(\theta)$ and $1+\sin(\theta)$: $(1-\sin(\theta))(1+\sin(\theta))=1-\sin^2(\theta)=\cos^2(\theta)$
- $\sec(\theta)-1$ and $\sec(\theta)+1$: $(\sec(\theta)-1)(\sec(\theta)+1)=\sec^2(\theta)-1=\tan^2(\theta)$
- $\sec(\theta)-\tan(\theta)$ and $\sec(\theta)+\tan(\theta)$: $(\sec(\theta)-\tan(\theta))(\sec(\theta)+\tan(\theta))=\sec^2(\theta)-\tan^2(\theta)=1$
- $\csc(\theta)-1$ and $\csc(\theta)+1$: $(\csc(\theta)-1)(\csc(\theta)+1)=\csc^2(\theta)-1=\cot^2(\theta)$
- $\csc(\theta)-\cot(\theta)$ and $\csc(\theta)+\cot(\theta)$: $(\csc(\theta)-\cot(\theta))(\csc(\theta)+\cot(\theta))=\csc^2(\theta)-\cot^2(\theta)=1$

Verifying trigonometric identities requires a healthy mix of tenacity and inspiration. You will need to spend many hours struggling with them just to become proficient in the basics. Like many things in life, there is no short-cut here – there is no complete algorithm for verifying identities. Nevertheless, a summary of some strategies which may be helpful (depending on the situation) is provided below and ample practice is provided for you in the Exercises.

Key Idea 38 Strategies for Verifying Identities

- Try working on the more complicated side of the identity.
- Use the Reciprocal and Quotient Identities in Theorem 56 to write functions on one side of the identity in terms of the functions on the other side of the identity. Simplify the resulting complex fractions.
- Add rational expressions with unlike denominators by obtaining common denominators.
- Use the Pythagorean Identities in Theorem 58 to 'exchange' sines and cosines, secants and tangents, cosecants and cotangents, and simplify sums or differences of squares to one term.
- Multiply numerator **and** denominator by Pythagorean Conjugates in order to take advantage of the Pythagorean Identities in Theorem 58.
- If you find yourself stuck working with one side of the identity, try starting with the other side of the identity and see if you can find a way to bridge the two parts of your work.

8.3.1 Beyond the Unit Circle

In Section 8.2, we generalized the cosine and sine functions from coordinates on the Unit Circle to coordinates on circles of radius r. Using Theorem 52 in conjunction with Theorem 58, we generalize the remaining circular functions in kind.

Theorem 59 Generalized circular functios

Suppose $Q(x, y)$ is the point on the terminal side of an angle θ (plotted in standard position) which lies on the circle of radius r, $x^2 + y^2 = r^2$. Then:

- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2+y^2}}{x}$, provided $x \neq 0$.
- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2+y^2}}{y}$, provided $y \neq 0$.
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

We may choose *any* values x and y so long as $x > 0$, $y < 0$ and $\frac{x}{y} = -4$. For example, we could choose $x = 8$ and $y = -2$. The fact that all such points lie on the terminal side of θ is a consequence of the fact that the terminal side of θ is the portion of the line with slope $-\frac{1}{4}$ which extends from the origin into Quadrant IV.

Example 150 Evaluating circular functions

1. Suppose the terminal side of θ, when plotted in standard position, contains the point $Q(3, -4)$. Find the values of the six circular functions of θ.

2. Suppose θ is a Quadrant IV angle with $\cot(\theta) = -4$. Find the values of the five remaining circular functions of θ.

Solution

1. Since $x = 3$ and $y = -4$, $r = \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2} = \sqrt{25} = 5$. Theorem 59 tells us $\cos(\theta) = \frac{3}{5}$, $\sin(\theta) = -\frac{4}{5}$, $\sec(\theta) = \frac{5}{3}$, $\csc(\theta) = -\frac{5}{4}$, $\tan(\theta) = -\frac{4}{3}$ and $\cot(\theta) = -\frac{3}{4}$.

2. In order to use Theorem 59, we need to find a point $Q(x, y)$ which lies on the terminal side of θ, when θ is plotted in standard position. We have that $\cot(\theta) = -4 = \frac{x}{y}$, and since θ is a Quadrant IV angle, we also know $x > 0$ and $y < 0$. Viewing $-4 = \frac{4}{-1}$, we may choose $x = 4$ and $y = -1$ so that $r = \sqrt{x^2+y^2} = \sqrt{(4)^2+(-1)^2} = \sqrt{17}$. Applying Theorem 59 once more, we find $\cos(\theta) = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17}$, $\sin(\theta) = -\frac{1}{\sqrt{17}} = -\frac{\sqrt{17}}{17}$, $\sec(\theta) = \frac{\sqrt{17}}{4}$, $\csc(\theta) = -\sqrt{17}$ and $\tan(\theta) = -\frac{1}{4}$.

We may also specialize Theorem 59 to the case of acute angles θ which reside in a right triangle, as visualized in Figure 8.63.

> **Theorem 60 Circular functions defined by a right-angled triangle**
>
> Suppose θ is an acute angle residing in a right triangle. If the length of the side adjacent to θ is a, the length of the side opposite θ is b, and the length of the hypotenuse is c, then
>
> $$\tan(\theta) = \frac{b}{a} \qquad \sec(\theta) = \frac{c}{a} \qquad \csc(\theta) = \frac{c}{b} \qquad \cot(\theta) = \frac{a}{b}$$

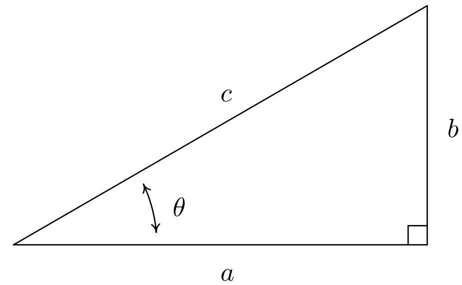


Figure 8.63: A right-angled triangle

The following example uses Theorem 60 as well as the concept of an 'angle of inclination.' The angle of inclination (or angle of elevation) of an object refers to the angle whose initial side is some kind of base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. This is represented schematically in Figure 8.64.

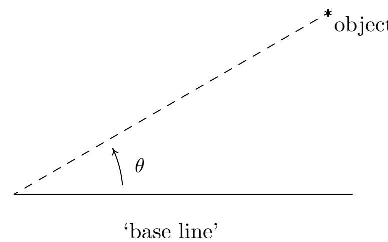


Figure 8.64: The angle of inclination from the base line to the object is θ

Example 151 Using angle of inclination

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland's Armington Clocktower is 60°. Find the height of the Clocktower to the nearest foot.

2. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

Solution

1. We can represent the problem situation using a right triangle as shown in Figure 8.65. If we let h denote the height of the tower, then Theorem 60 gives $\tan(60^\circ) = \frac{h}{30}$. From this we get $h = 30\tan(60^\circ) = 30\sqrt{3} \approx 51.96$. Hence, the Clocktower is approximately 52 feet tall.

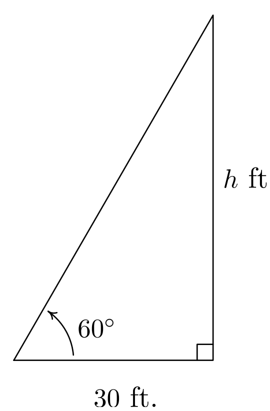


Figure 8.65: Finding the height of the Clocktower

2. Sketching the problem situation in Figure 8.66, we find ourselves with two unknowns: the height h of the tree and the distance x from the base of the tree to the first observation point.

   Using Theorem 60, we get a pair of equations: $\tan(45^\circ) = \frac{h}{x}$ and $\tan(30^\circ) = \frac{h}{x+200}$. Since $\tan(45^\circ) = 1$, the first equation gives $\frac{h}{x} = 1$, or $x = h$. Substituting this into the second equation gives $\frac{h}{h+200} = \tan(30^\circ) = \frac{\sqrt{3}}{3}$. Clearing fractions, we get $3h = (h+200)\sqrt{3}$. The result is a linear equation for h, so we proceed to expand the right hand side and gather all the terms involving h to one side.

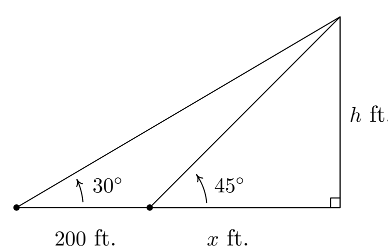
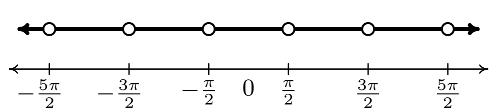


Figure 8.66: Finding the height of a California Redwood

$$\begin{aligned}
3h &= (h+200)\sqrt{3} \\
3h &= h\sqrt{3} + 200\sqrt{3} \\
3h - h\sqrt{3} &= 200\sqrt{3} \\
(3-\sqrt{3})h &= 200\sqrt{3} \\
h &= \frac{200\sqrt{3}}{3-\sqrt{3}} \approx 273.20
\end{aligned}$$

Hence, the tree is approximately 273 feet tall.

As we did in Section 8.2.1, we may consider all six circular functions as functions of real numbers. At this stage, there are three equivalent ways to define the functions $\sec(t)$, $\csc(t)$, $\tan(t)$ and $\cot(t)$ for real numbers t. First, we could go through the formality of the wrapping function on page 315 and define these functions as the appropriate ratios of x and y coordinates of points on the Unit Circle; second, we could define them by associating the real number t with the angle $\theta = t$ radians so that the value of the trigonometric function of t coincides with that of θ; lastly, we could simply define them using the Reciprocal and Quotient Identities as combinations of the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$. Presently, we adopt the last approach. We now set about determining the domains and ranges of the remaining four circular functions. Consider the function $F(t) = \sec(t)$ defined as $F(t) = \sec(t) = \frac{1}{\cos(t)}$. We know F is undefined whenever $\cos(t) = 0$. From Example 143 number 3, we know $\cos(t) = 0$ whenever $t = \frac{\pi}{2} + \pi k$ for integers k. Hence, our domain for $F(t) = \sec(t)$, in set builder notation is $\{t : t \neq \frac{\pi}{2} + \pi k, \text{for integers } k\}$. To get a better understanding what set of real numbers we're dealing with, it pays to write out and graph this set. Running through a few values of k, we find the domain to be $\{t : t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\}$. Graphing this set on the number line we get



Using interval notation to describe this set, we get

$$\ldots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$$

This is cumbersome, to say the least! In order to write this in a more compact way, we note that from the set-builder description of the domain, the kth point excluded from the domain, which we'll call x_k, can be found by the formula $x_k = \frac{\pi}{2} + \pi k$. Getting a common denominator and factoring out the π in the numerator, we get $x_k = \frac{(2k+1)\pi}{2}$. The domain consists of the intervals determined by successive points x_k: $(x_k, x_{k+1}) = \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2}\right)$. In order to capture all of the intervals in the domain, k must run through all of the integers, that is, $k = 0, \pm 1, \pm 2, \ldots$.

The way we denote taking the union of infinitely many intervals like this is to use what we call in this text **extended interval notation**. The domain of $F(t) = \sec(t)$ can now be written as

$$\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$$

The reader who has previously encountered **summation notation** should find it useful to compare the it with our extended interval notation. In the same way the index *k* in the geometric series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

can never equal the upper limit ∞, but rather, ranges through all of the natural numbers, the index *k* in the union

$$\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$$

can never actually be ∞ or $-\infty$, but rather, this conveys the idea that *k* ranges through all of the integers. If you have never seen summation notation before, don't worry. You won't need to work with it (or, for that matter the extended interval notation) in this class.

Now that we have painstakingly determined the domain of $F(t) = \sec(t)$, it is time to discuss the range. Once again, we appeal to the definition $F(t) = \sec(t) = \frac{1}{\cos(t)}$. The range of $f(t) = \cos(t)$ is $[-1, 1]$, and since $F(t) = \sec(t)$ is undefined when $\cos(t) = 0$, we split our discussion into two cases: when $0 < \cos(t) \leq 1$ and when $-1 \leq \cos(t) < 0$. If $0 < \cos(t) \leq 1$, then we can divide the inequality $\cos(t) \leq 1$ by $\cos(t)$ to obtain $\sec(t) = \frac{1}{\cos(t)} \geq 1$. Moreover, using the notation introduced in Section 5.2, we have that as $\cos(t) \to 0^+$, $\sec(t) = \frac{1}{\cos(t)} \approx \frac{1}{\text{very small } (+)} \approx \text{very big } (+)$. In other words, as $\cos(t) \to 0^+, \sec(t) \to \infty$. If, on the other hand, if $-1 \leq \cos(t) < 0$, then dividing by $\cos(t)$ causes a reversal of the inequality so that $\sec(t) = \frac{1}{\sec(t)} \leq -1$. In this case, as $\cos(t) \to 0^-$, $\sec(t) = \frac{1}{\cos(t)} \approx \frac{1}{\text{very small } (-)} \approx \text{very big } (-)$, so that as $\cos(t) \to 0^-$, we get $\sec(t) \to -\infty$. Since $f(t) = \cos(t)$ admits all of the values in $[-1, 1]$, the function $F(t) = \sec(t)$ admits all of the values in $(-\infty, -1] \cup [1, \infty)$. Using set-builder notation, the range of $F(t) = \sec(t)$ can be written as $\{u : u \leq -1 \text{ or } u \geq 1\}$, or, more succinctly, (using Theorem 18 from Section 3.4) as $\{u : |u| \geq 1\}$. Similar arguments can be used to determine the domains and ranges of the remaining three circular functions: $\csc(t)$, $\tan(t)$ and $\cot(t)$. The reader is encouraged to do so. (See the Exercises.) For now, we gather these facts into the theorem below.

Notice we have used the variable '*u*' as the 'dummy variable' to describe the range elements. While there is no mathematical reason to do this (we are describing a set of real numbers, and, as such, could use *t* again) we choose *u* to help solidify the idea that these real numbers are the outputs from the inputs, which we have been calling *t*.

Theorem 61 Domains and Ranges of the Circular Functions

- The function $f(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $F(t) = \sec(t) = \dfrac{1}{\cos(t)}$
  - has domain $\{t : t \neq \frac{\pi}{2} + \pi k, \text{ for integers } k\} = \displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $\{u : |u| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- The function $G(t) = \csc(t) = \dfrac{1}{\sin(t)}$
  - has domain $\{t : t \neq \pi k, \text{ for integers } k\} = \displaystyle\bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $\{u : |u| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
- The function $J(t) = \tan(t) = \dfrac{\sin(t)}{\cos(t)}$
  - has domain $\{t : t \neq \frac{\pi}{2} + \pi k, \text{ for integers } k\} = \displaystyle\bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $(-\infty, \infty)$
- The function $K(t) = \cot(t) = \dfrac{\cos(t)}{\sin(t)}$
  - has domain $\{t : t \neq \pi k, \text{ for integers } k\} = \displaystyle\bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, \infty)$

We close this section with a few notes about solving equations which involve the circular functions. First, the discussion on page 334 in Section 8.2.1 concerning solving equations applies to all six circular functions, not just $f(t) = \cos(t)$ and $g(t) = \sin(t)$. In particular, to solve the equation $\cot(t) = -1$ for real numbers t, we can use the same thought process we used in Example 148, number 3 to solve $\cot(\theta) = -1$ for angles θ in radian measure – we just need to remember to write our answers using the variable t as opposed to θ. Next, it is critical that you know the domains and ranges of the six circular functions so that you know which equations have no solutions. For example, $\sec(t) = \frac{1}{2}$ has no solution because $\frac{1}{2}$ is not in the range of secant. Finally, you will need to review the notions of reference angles and coterminal angles so that you can see why $\csc(t) = -42$ has an infinite set of solutions in Quadrant III and another infinite set of solutions in Quadrant IV.

Exercises 8.3

Problems

In Exercises 1 – 20, find the exact value of the cosine and sine of the given angle.

1. $\theta = 0$
2. $\theta = \frac{\pi}{4}$
3. $\theta = \frac{\pi}{3}$
4. $\theta = \frac{\pi}{2}$
5. $\theta = \frac{2\pi}{3}$
6. $\theta = \frac{3\pi}{4}$
7. $\theta = \pi$
8. $\theta = \frac{7\pi}{6}$
9. $\theta = \frac{5\pi}{4}$
10. $\theta = \frac{4\pi}{3}$
11. $\theta = \frac{3\pi}{2}$
12. $\theta = \frac{5\pi}{3}$
13. $\theta = \frac{7\pi}{4}$
14. $\theta = \frac{23\pi}{6}$
15. $\theta = -\frac{13\pi}{2}$
16. $\theta = -\frac{43\pi}{6}$
17. $\theta = -\frac{3\pi}{4}$
18. $\theta = -\frac{\pi}{6}$
19. $\theta = \frac{10\pi}{3}$
20. $\theta = 117\pi$

In Exercises 21 – 34, use the given the information to find the exact values of the remaining circular functions of θ.

21. $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II
22. $\tan(\theta) = \frac{12}{5}$ with θ in Quadrant III
23. $\csc(\theta) = \frac{25}{24}$ with θ in Quadrant I
24. $\sec(\theta) = 7$ with θ in Quadrant IV
25. $\csc(\theta) = -\frac{10\sqrt{91}}{91}$ with θ in Quadrant III
26. $\cot(\theta) = -23$ with θ in Quadrant II
27. $\tan(\theta) = -2$ with θ in Quadrant IV.
28. $\sec(\theta) = -4$ with θ in Quadrant II.
29. $\cot(\theta) = \sqrt{5}$ with θ in Quadrant III.
30. $\cos(\theta) = \frac{1}{3}$ with θ in Quadrant I.
31. $\cot(\theta) = 2$ with $0 < \theta < \frac{\pi}{2}$.
32. $\csc(\theta) = 5$ with $\frac{\pi}{2} < \theta < \pi$.
33. $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \frac{3\pi}{2}$.
34. $\sec(\theta) = 2\sqrt{5}$ with $\frac{3\pi}{2} < \theta < 2\pi$.

In Exercises 35 – 42, use your calculator to approximate the given value to three decimal places. Make sure your calculator is in the proper angle measurement mode!

35. $\csc(78.95^\circ)$
36. $\tan(-2.01)$
37. $\cot(392.994)$
38. $\sec(207^\circ)$
39. $\csc(5.902)$
40. $\tan(39.672^\circ)$
41. $\cot(3^\circ)$
42. $\sec(0.45)$

In Exercises 43 – 57, find all of the angles which satisfy the equation.

43. $\tan(\theta) = \sqrt{3}$

44. $\sec(\theta) = 2$

45. $\csc(\theta) = -1$

46. $\cot(\theta) = \dfrac{\sqrt{3}}{3}$

47. $\tan(\theta) = 0$

48. $\sec(\theta) = 1$

49. $\csc(\theta) = 2$

50. $\cot(\theta) = 0$

51. $\tan(\theta) = -1$

52. $\sec(\theta) = 0$

53. $\csc(\theta) = -\dfrac{1}{2}$

54. $\sec(\theta) = -1$

55. $\tan(\theta) = -\sqrt{3}$

56. $\csc(\theta) = -2$

57. $\cot(\theta) = -1$

In Exercises 58 – 65, solve the equation for t. Give exact values.

58. $\cot(t) = 1$

59. $\tan(t) = \dfrac{\sqrt{3}}{3}$

60. $\sec(t) = -\dfrac{2\sqrt{3}}{3}$

61. $\csc(t) = 0$

62. $\cot(t) = -\sqrt{3}$

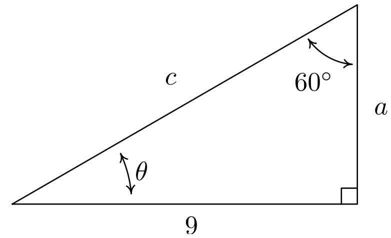
63. $\tan(t) = -\dfrac{\sqrt{3}}{3}$

64. $\sec(t) = \dfrac{2\sqrt{3}}{3}$

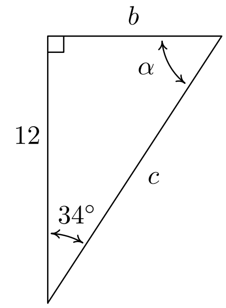
65. $\csc(t) = \dfrac{2\sqrt{3}}{3}$

In Exercises 66 – 69, use Theorem 60 to find the requested quantities.

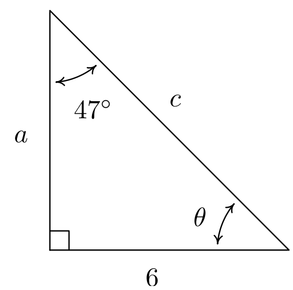
66. Find θ, a, and c.



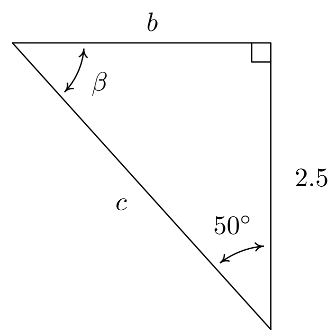
67. Find α, b, and c.



68. Find θ, a, and c.



69. Find β, b, and c.



In Exercises 70 – 75, use Theorem 60 to answer the question. Assume that θ is an angle in a right triangle.use Theorem 60 to find the requested quantities.

70. If $\theta = 30^\circ$ and the side opposite θ has length 4, how long is the side adjacent to θ?

71. If $\theta = 15^\circ$ and the hypotenuse has length 10, how long is the side opposite θ?

72. If $\theta = 87^\circ$ and the side adjacent to θ has length 2, how long is the side opposite θ?

73. If $\theta = 38.2^\circ$ and the side opposite θ has length 14, how long is the hypotenuse?

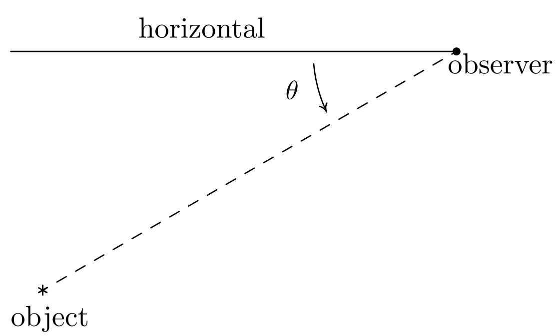
74. If $\theta = 2.05^\circ$ and the hypotenuse has length 3.98, how long is the side adjacent to θ?

75. If $\theta = 42^\circ$ and the side adjacent to θ has length 31, how long is the side opposite θ?

76. A tree standing vertically on level ground casts a 120 foot long shadow. The angle of elevation from the end of the shadow to the top of the tree is 21.4°. Find the height of the tree to the nearest foot. With the help of your classmates, research the term *umbra versa* and see what it has to do with the shadow in this problem.

77. The broadcast tower for radio station WSAZ (Home of "Algebra in the Morning with Carl and Jeff") has two enormous flashing red lights on it: one at the very top and one a few feet below the top. From a point 5000 feet away from the base of the tower on level ground the angle of elevation to the top light is 7.970° and to the second light is 7.125°. Find the distance between the lights to the nearest foot.

78. On page 347 we defined the angle of inclination (also known as the angle of elevation) and in this exercise we introduce a related angle - the angle of depression (also known as the angle of declination). The angle of depression of an object refers to the angle whose initial side is a horizontal line above the object and whose terminal side is the line-of-sight to the object below the horizontal. This is represented schematically below.



The angle of depression from the horizontal to the object is θ

(a) Show that if the horizontal is above and parallel to level ground then the angle of depression (from observer to object) and the angle of inclination (from object to observer) will be congruent because they are alternate interior angles.

(b) From a firetower 200 feet above level ground in the Sasquatch National Forest, a ranger spots a fire off in the distance. The angle of depression to the fire is 2.5°. How far away from the base of the tower is the fire?

(c) The ranger in part 78b sees a Sasquatch running directly from the fire towards the firetower. The ranger takes two sightings. At the first sighting, the angle of depression from the tower to the Sasquatch is 6°. The second sighting, taken just 10 seconds later, gives the the angle of depression as 6.5°. How far did the Saquatch travel in those 10 seconds? Round your answer to the nearest foot. How fast is it running in miles per hour? Round your answer to the nearest mile per hour. If the Sasquatch keeps up this pace, how long will it take for the Sasquatch to reach the firetower from his location at the second sighting? Round your answer to the nearest minute.

79. When I stand 30 feet away from a tree at home, the angle of elevation to the top of the tree is 50° and the angle of depression to the base of the tree is 10°. What is the height of the tree? Round your answer to the nearest foot.

80. From the observation deck of the lighthouse at Sasquatch Point 50 feet above the surface of Lake Ippizuti, a lifeguard spots a boat out on the lake sailing directly toward the lighthouse. The first sighting had an angle of depression of 8.2° and the second sighting had an angle of depression of 25.9°. How far had the boat travelled between the sightings?

81. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it makes a 43° angle with the ground. How tall is the tower? How far away from the base of the tower does the wire hit the ground?

In Exercises 82 – 128, verify the identity. Assume that all quantities are defined.

82. $\cos(\theta)\sec(\theta) = 1$

83. $\tan(\theta)\cos(\theta) = \sin(\theta)$

84. $\sin(\theta)\csc(\theta) = 1$

85. $\tan(\theta)\cot(\theta) = 1$

86. $\csc(\theta)\cos(\theta) = \cot(\theta)$

87. $\dfrac{\sin(\theta)}{\cos^2(\theta)} = \sec(\theta)\tan(\theta)$

88. $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

89. $\dfrac{1+\sin(\theta)}{\cos(\theta)} = \sec(\theta)+\tan(\theta)$

90. $\dfrac{1-\cos(\theta)}{\sin(\theta)} = \csc(\theta)-\cot(\theta)$

91. $\dfrac{\cos(\theta)}{1-\sin^2(\theta)} = \sec(\theta)$

92. $\dfrac{\sin(\theta)}{1-\cos^2(\theta)} = \csc(\theta)$

93. $\dfrac{\sec(\theta)}{1+\tan^2(\theta)} = \cos(\theta)$

94. $\dfrac{\csc(\theta)}{1+\cot^2(\theta)} = \sin(\theta)$

95. $\dfrac{\tan(\theta)}{\sec^2(\theta)-1} = \cot(\theta)$

96. $\dfrac{\cot(\theta)}{\csc^2(\theta)-1} = \tan(\theta)$

97. $4\cos^2(\theta) + 4\sin^2(\theta) = 4$

98. $9 - \cos^2(\theta) - \sin^2(\theta) = 8$

99. $\tan^3(\theta) = \tan(\theta)\sec^2(\theta) - \tan(\theta)$

100. $\sin^5(\theta) = \left(1-\cos^2(\theta)\right)^2 \sin(\theta)$

101. $\sec^{10}(\theta) = \left(1+\tan^2(\theta)\right)^4 \sec^2(\theta)$

102. $\cos^2(\theta)\tan^3(\theta) = \tan(\theta) - \sin(\theta)\cos(\theta)$

103. $\sec^4(\theta) - \sec^2(\theta) = \tan^2(\theta) + \tan^4(\theta)$

104. $\dfrac{\cos(\theta)+1}{\cos(\theta)-1} = \dfrac{1+\sec(\theta)}{1-\sec(\theta)}$

105. $\dfrac{\sin(\theta)+1}{\sin(\theta)-1} = \dfrac{1+\csc(\theta)}{1-\csc(\theta)}$

106. $\dfrac{1-\cot(\theta)}{1+\cot(\theta)} = \dfrac{\tan(\theta)-1}{\tan(\theta)+1}$

107. $\dfrac{1-\tan(\theta)}{1+\tan(\theta)} = \dfrac{\cos(\theta)-\sin(\theta)}{\cos(\theta)+\sin(\theta)}$

108. $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

109. $\csc(\theta) - \sin(\theta) = \cot(\theta)\cos(\theta)$

110. $\cos(\theta) - \sec(\theta) = -\tan(\theta)\sin(\theta)$

111. $\cos(\theta)(\tan(\theta) + \cot(\theta)) = \csc(\theta)$

112. $\sin(\theta)(\tan(\theta) + \cot(\theta)) = \sec(\theta)$

113. $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

114. $\dfrac{1}{\sec(\theta)+1} + \dfrac{1}{\sec(\theta)-1} = 2\csc(\theta)\cot(\theta)$

115. $\dfrac{1}{\csc(\theta)+1} + \dfrac{1}{\csc(\theta)-1} = 2\sec(\theta)\tan(\theta)$

116. $\dfrac{1}{\csc(\theta)-\cot(\theta)} - \dfrac{1}{\csc(\theta)+\cot(\theta)} = 2\cot(\theta)$

117. $\dfrac{\cos(\theta)}{1-\tan(\theta)} + \dfrac{\sin(\theta)}{1-\cot(\theta)} = \sin(\theta) + \cos(\theta)$

118. $\dfrac{1}{\sec(\theta)+\tan(\theta)} = \sec(\theta) - \tan(\theta)$

119. $\dfrac{1}{\sec(\theta)-\tan(\theta)} = \sec(\theta) + \tan(\theta)$

120. $\dfrac{1}{\csc(\theta)-\cot(\theta)} = \csc(\theta) + \cot(\theta)$

121. $\dfrac{1}{\csc(\theta)+\cot(\theta)} = \csc(\theta) - \cot(\theta)$

122. $\dfrac{1}{1-\sin(\theta)} = \sec^2(\theta) + \sec(\theta)\tan(\theta)$

123. $\dfrac{1}{1+\sin(\theta)} = \sec^2(\theta) - \sec(\theta)\tan(\theta)$

124. $\dfrac{1}{1-\cos(\theta)} = \csc^2(\theta) + \csc(\theta)\cot(\theta)$

125. $\dfrac{1}{1+\cos(\theta)} = \csc^2(\theta) - \csc(\theta)\cot(\theta)$

126. $\dfrac{\cos(\theta)}{1+\sin(\theta)} = \dfrac{1-\sin(\theta)}{\cos(\theta)}$

127. $\csc(\theta) - \cot(\theta) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$

128. $\dfrac{1-\sin(\theta)}{1+\sin(\theta)} = (\sec(\theta) - \tan(\theta))^2$

In Exercises 129 – 132, verify the identity. You may need to consult Sections 3.2 and 7.2 for a review of the properties of absolute value and logarithms before proceeding.

129. $\ln|\sec(\theta)| = -\ln|\cos(\theta)|$

130. $-\ln|\csc(\theta)| = \ln|\sin(\theta)|$

131. $-\ln|\sec(\theta) - \tan(\theta)| = \ln|\sec(\theta) + \tan(\theta)|$

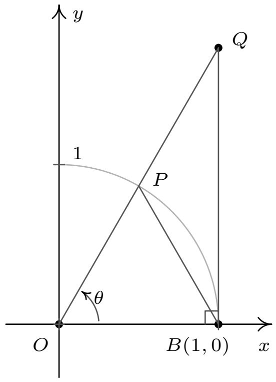
132. $-\ln|\csc(\theta) + \cot(\theta)| = \ln|\csc(\theta) - \cot(\theta)|$

133. Verify the domains and ranges of the tangent, cosecant and cotangent functions as presented in Theorem 61.

134. As we did in Exercise 74 in Section 8.2, let α and β be the two acute angles of a right triangle. (Thus α and β are complementary angles.) Show that $\sec(\alpha) = \csc(\beta)$ and $\tan(\alpha) = \cot(\beta)$. The fact that co-functions of complementary angles are equal in this case is not an accident and a more general result will be given in Section 8.4.

135. We wish to establish the inequality $\cos(\theta) < \dfrac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \dfrac{\pi}{2}$. Use the diagram from the beginning of the

section, partially reproduced below, to answer the following.



(a) Show that triangle *OPB* has area $\dfrac{1}{2}\sin(\theta)$.

(b) Show that the circular sector *OPB* with central angle θ has area $\dfrac{1}{2}\theta$.

(c) Show that triangle *OQB* has area $\dfrac{1}{2}\tan(\theta)$.

(d) Comparing areas, show that $\sin(\theta) < \theta < \tan(\theta)$ for $0 < \theta < \dfrac{\pi}{2}$.

(e) Use the inequality $\sin(\theta) < \theta$ to show that $\dfrac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \dfrac{\pi}{2}$.

(f) Use the inequality $\theta < \tan(\theta)$ to show that $\cos(\theta) < \dfrac{\sin(\theta)}{\theta}$ for $0 < \theta < \dfrac{\pi}{2}$. Combine this with the previous part to complete the proof.

136. Show that $\cos(\theta) < \dfrac{\sin(\theta)}{\theta} < 1$ also holds for $-\dfrac{\pi}{2} < \theta < 0$.

137. Explain why the fact that $\tan(\theta) = 3 = \frac{3}{1}$ does not mean $\sin(\theta) = 3$ and $\cos(\theta) = 1$? (See the solution to number 6 in Example 147.)

8.4 Trigonometric Identities

As mentioned at the end of Section 8.2, properties of the circular functions when thought of as functions of angles in radian measure hold equally well if we view these functions as functions of real numbers. Not surprisingly, the Even / Odd properties of the circular functions are so named because they identify cosine and secant as even functions, while the remaining four circular functions are odd. (See Section 2.5.)

In Section 8.3, we saw the utility of the Pythagorean Identities in Theorem 58 along with the Quotient and Reciprocal Identities in Theorem 56. Not only did these identities help us compute the values of the circular functions for angles, they were also useful in simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond. Our first set of identities is the 'Even / Odd' identities.

Theorem 62 Even / Odd Identities

For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

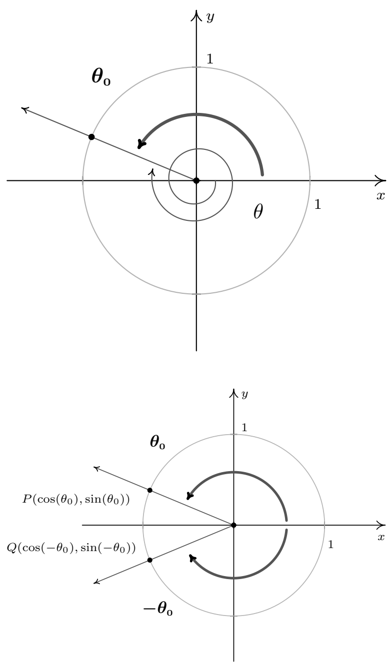


Figure 8.67: Establishing Theorem 62

In light of the Quotient and Reciprocal Identities, Theorem 56, it suffices to show $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. The remaining four circular functions can be expressed in terms of $\cos(\theta)$ and $\sin(\theta)$ so the proofs of their Even / Odd Identities are left as exercises. Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \leq \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive *x*-axis to the terminal side of θ as pictured in Figure 8.67.) Since θ and θ_0 are coterminal, $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.

We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k so that $\theta = \theta_0 + 2\pi \cdot k$. Therefore, $-\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k)$. Since k is an integer, so is $(-k)$, which means $-\theta$ is coterminal with $-\theta_0$. Hence, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$. Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$. Since θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the *x*-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$. Since the cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required. The Even / Odd Identities are readily demonstrated using any of the 'common angles' noted in Section 8.2. Their true utility, however, lies not in computation, but in simplifying expressions involving the circular functions. In fact, our next batch of identities makes heavy use of the Even / Odd Identities.

Theorem 63 Sum and Difference Identities for Cosine

For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd

Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $\alpha - \beta$ is coterminal with $\alpha_0 - \beta_0$. Consider the case in Figure 8.68 where $\alpha_0 \geq \beta_0$.

Since the angles *POQ* and *AOB* are congruent, the distance between *P* and *Q* is equal to the distance between *A* and *B*. The distance formula, Equation 6, yields

$$\begin{aligned}&\sqrt{(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2} \\ &\qquad = \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2}\end{aligned}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}(\cos(\alpha_0) - \cos(\beta_0))^2 &+ (\sin(\alpha_0) - \sin(\beta_0))^2 \\ &= \cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) \\ &\quad + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0) \\ &= \cos^2(\alpha_0) + \sin^2(\alpha_0) + \cos^2(\beta_0) + \sin^2(\beta_0) \\ &\quad - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

From the Pythagorean Identities, $\cos^2(\alpha_0) + \sin^2(\alpha_0) = 1$ and $\cos^2(\beta_0) + \sin^2(\beta_0) = 1$, so

$$\begin{aligned}(\cos(\alpha_0) - \cos(\beta_0))^2 &+ (\sin(\alpha_0) - \sin(\beta_0))^2 \\ &= 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0 - \beta_0) - 1)^2 &+ (\sin(\alpha_0 - \beta_0) - 0)^2 \\ &= \cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 + \sin^2(\alpha_0 - \beta_0) \\ &= 1 + \cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) = 1$, so that

$$(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 \quad = \quad 2 - 2\cos(\alpha_0 - \beta_0)$$

Putting it all together, we get $2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = 2 - 2\cos(\alpha_0 - \beta_0)$, which simplifies to: $\cos(\alpha_0 - \beta_0) = \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0)$. Since α and α_0, β and β_0 and $\alpha - \beta$ and $\alpha_0 - \beta_0$ are all coterminal pairs of angles, we have $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. For the case where $\alpha_0 \leq \beta_0$, we can apply the above argument to the angle $\beta_0 - \alpha_0$ to obtain the identity $\cos(\beta_0 - \alpha_0) = \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0)$. Applying the Even Identity of cosine, we get $\cos(\beta_0 - \alpha_0) = \cos(-(\alpha_0 - \beta_0)) = \cos(\alpha_0 - \beta_0)$, and we get the identity in this case, too.

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\begin{aligned}\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\end{aligned}$$

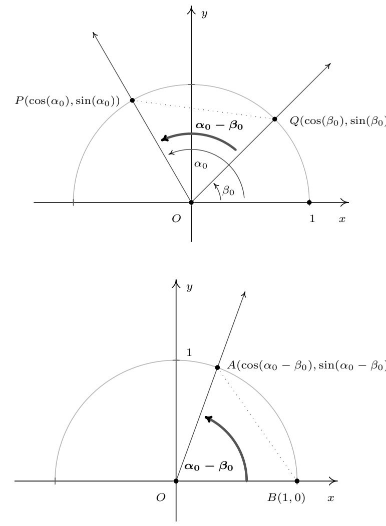


Figure 8.68: Establishing Theorem 63

In Figure 8.68, the triangles *POQ* and *AOB* are congruent, which is even better. However, $\alpha_0 - \beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

We put these newfound identities to good use in the following example.

Example 152 Using Theorem 63

1. Find the exact value of $\cos(15^\circ)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.

Solution

1. In order to use Theorem 63 to find $\cos(15^\circ)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ = 45^\circ - 30^\circ$.

$$\begin{aligned} \cos(15^\circ) &= \cos(45^\circ - 30^\circ) \\ &= \cos(45^\circ)\cos(30^\circ) + \sin(45^\circ)\sin(30^\circ) \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4} \end{aligned}$$

2. In a straightforward application of Theorem 63, we find

$$\begin{aligned} \cos\left(\frac{\pi}{2} - \theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta) + \sin\left(\frac{\pi}{2}\right)\sin(\theta) \\ &= (0)(\cos(\theta)) + (1)(\sin(\theta)) \\ &= \sin(\theta) \end{aligned}$$

The identity verified in Example 152, namely, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$, is the first of the celebrated 'cofunction' identities. These identities were first hinted at in Exercise 74 in Section 8.2. From $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$, we get:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - \left[\frac{\pi}{2} - \theta\right]\right) = \cos(\theta),$$

which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 64 Cofunction Identities

For all applicable angles θ,

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

With the Cofunction Identities in place, we are now in the position to derive the sum and difference formulas for sine. To derive the sum formula for sine, we convert to cosines using a cofunction identity, then expand using the difference formula for cosine

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right) \\
&= \cos\left(\left[\frac{\pi}{2}-\alpha\right]-\beta\right) \\
&= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta) \\
&= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)
\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 65 Sum and Difference Identities for Sine

For all angles α and β,

- $\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$

Example 153 Using Theorem 65

1. Find the exact value of $\sin\left(\frac{19\pi}{12}\right)$
2. If α is a Quadrant II angle with $\sin(\alpha)=\frac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta)=2$, find $\sin(\alpha-\beta)$.
3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution

1. As in Example 152, we need to write the angle $\frac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination is $\frac{19\pi}{12}=\frac{4\pi}{3}+\frac{\pi}{4}$. Applying Theorem 65, we get

$$\begin{aligned}
\sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right) \\
&= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\
&= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{-\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

2. In order to find $\sin(\alpha-\beta)$ using Theorem 65, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$. To find $\cos(\alpha)$, we use the Pythagorean Identity

$\cos^2(\alpha)+\sin^2(\alpha)=1$. Since $\sin(\alpha)=\frac{5}{13}$, we have $\cos^2(\alpha)+\left(\frac{5}{13}\right)^2=1$, or $\cos(\alpha)=\pm\frac{12}{13}$. Since α is a Quadrant II angle, $\cos(\alpha)=-\frac{12}{13}$. We now set about finding $\cos(\beta)$ and $\sin(\beta)$. We have several ways to proceed, but the Pythagorean Identity $1+\tan^2(\beta)=\sec^2(\beta)$ is a quick way to get $\sec(\beta)$, and hence, $\cos(\beta)$. With $\tan(\beta)=2$, we get $1+2^2=\sec^2(\beta)$ so that $\sec(\beta)=\pm\sqrt{5}$. Since β is a Quadrant III angle, we choose $\sec(\beta)=-\sqrt{5}$ so $\cos(\beta)=\frac{1}{\sec(\beta)}=\frac{1}{-\sqrt{5}}=-\frac{\sqrt{5}}{5}$. We now need to determine $\sin(\beta)$. We could use The Pythagorean Identity $\cos^2(\beta)+\sin^2(\beta)=1$, but we opt instead to use a quotient identity. From $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, we have $\sin(\beta)=\tan(\beta)\cos(\beta)$ so we get $\sin(\beta)=(2)\left(-\frac{\sqrt{5}}{5}\right)=-\frac{2\sqrt{5}}{5}$. We now have all the pieces needed to find $\sin(\alpha-\beta)$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ &= \left(\frac{5}{13}\right)\left(-\frac{\sqrt{5}}{5}\right)-\left(-\frac{12}{13}\right)\left(-\frac{2\sqrt{5}}{5}\right)\\ &= -\frac{29\sqrt{5}}{65}\end{aligned}$$

3. We can start expanding $\tan(\alpha+\beta)$ using a quotient identity and our sum formulas

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\end{aligned}$$

Since $\tan(\alpha)=\frac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta)=\frac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\cdot\frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}+\frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\frac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}}+\frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\frac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}-\frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}\\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}$$

Naturally, this formula is limited to those cases where all of the tangents are defined.

The formula developed in Exercise 153 for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$, and the reader is encouraged to fill in the details. Below we summarize all of the sum and difference formulas for cosine, sine and tangent.

Theorem 66　　Sum and Difference Identities

For all applicable angles α and β,

- $\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta)$
- $\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$
- $\tan(\alpha\pm\beta)=\dfrac{\tan(\alpha)\pm\tan(\beta)}{1\mp\tan(\alpha)\tan(\beta)}$

In the statement of Theorem 66, we have combined the cases for the sum '$+$' and difference '$-$' of angles into one formula. The convention here is that if you want the formula for the sum '$+$' of two angles, you use the top sign in the formula; for the difference, '$-$', use the bottom sign. For example,

$$\tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

If we specialize the sum formulas in Theorem 66 to the case when $\alpha=\beta$, we obtain the following 'Double Angle' Identities.

Theorem 67　　Double Angle Identities

For all applicable angles θ,

- $\cos(2\theta)=\begin{cases}\cos^2(\theta)-\sin^2(\theta)\\ 2\cos^2(\theta)-1\\ 1-2\sin^2(\theta)\end{cases}$
- $\sin(2\theta)=2\sin(\theta)\cos(\theta)$
- $\tan(2\theta)=\dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity $\cos^2(\theta)+\sin^2(\theta)=1$ and we leave the details to the reader. It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

Example 154 Using Theorem 67

1. Suppose $P(-3,4)$ lies on the terminal side of θ when θ is plotted in standard position. Find $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.

2. If $\sin(\theta)=x$ for $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

3. Verify the identity: $\sin(2\theta)=\dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

SOLUTION

1. Using Theorem 52 from Section 8.2 with $x=-3$ and $y=4$, we find $r=\sqrt{x^2+y^2}=5$. Hence, $\cos(\theta)=-\frac{3}{5}$ and $\sin(\theta)=\frac{4}{5}$. Applying Theorem 67, we get $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=\left(-\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2=-\frac{7}{25}$, and $\sin(2\theta)=2\sin(\theta)\cos(\theta)=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)=-\frac{24}{25}$. Since both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III.

2. If your first reaction to '$\sin(\theta)=x$' is 'No it's not, $\cos(\theta)=x$!' then you have indeed learned something, and we take comfort in that. However, context is everything. Here, 'x' is just a variable - it does not necessarily represent the x-coordinate of the point on The Unit Circle which lies on the terminal side of θ, assuming θ is drawn in standard position. Here, x represents the quantity $\sin(\theta)$, and what we wish to know is how to express $\sin(2\theta)$ in terms of x. Since $\sin(2\theta)=2\sin(\theta)\cos(\theta)$, we need to write $\cos(\theta)$ in terms of x to finish the problem. We substitute $x=\sin(\theta)$ into the Pythagorean Identity, $\cos^2(\theta)+\sin^2(\theta)=1$, to get $\cos^2(\theta)+x^2=1$, or $\cos(\theta)=\pm\sqrt{1-x^2}$. Since $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$, $\cos(\theta)\geq 0$, and thus $\cos(\theta)=\sqrt{1-x^2}$. Our final answer is $\sin(2\theta)=2\sin(\theta)\cos(\theta)=2x\sqrt{1-x^2}$.

3. We start with the right hand side of the identity and note that $1+\tan^2(\theta)=\sec^2(\theta)$. From this point, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$:

$$\begin{aligned}\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)}=\frac{2\left(\dfrac{\sin(\theta)}{\cos(\theta)}\right)}{\dfrac{1}{\cos^2(\theta)}}=2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta)\\ &= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta)=2\sin(\theta)\cos(\theta)=\sin(2\theta)\end{aligned}$$

4. In Theorem 67, the formula $\cos(2\theta)=2\cos^2(\theta)-1$ expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$. Using the sum formula for cosine, we begin with

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta+\theta)\\ &= \cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta)\end{aligned}$$

Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. We substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\
&= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - \left(2\sin(\theta)\cos(\theta)\right)\sin(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)
\end{aligned}$$

Finally, we exchange $\sin^2(\theta)$ for $1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}
\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta) \\
&= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta) \\
&= 4\cos^3(\theta) - 3\cos(\theta)
\end{aligned}$$

and we are done.

In the last problem in Example 154, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine. Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 68 Power Reduction Formulas

For all angles θ,

- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$

Example 155 Using Theorem 68

Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution We begin with a straightforward application of Theorem 68

$$\begin{aligned}
\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1 - \cos(2\theta)}{2}\right)\left(\frac{1 + \cos(2\theta)}{2}\right) \\
&= \frac{1}{4}\left(1 - \cos^2(2\theta)\right) \\
&= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta)
\end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta) \\ &= \frac{1}{4} - \frac{1}{4}\left(\frac{1+\cos(2(2\theta))}{2}\right) \\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta) \\ &= \frac{1}{8} - \frac{1}{8}\cos(4\theta)\end{aligned}$$

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to $\cos^2\left(\frac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1+\cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent. We summarize these formulas below.

Theorem 69 Half Angle Formulas

For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 156 Using Theorem 69

1. Use a half angle formula to find the exact value of $\cos(15^\circ)$.
2. Suppose $-\pi \leq \theta \leq 0$ with $\cos(\theta) = -\frac{3}{5}$. Find $\sin\left(\frac{\theta}{2}\right)$.
3. Use the identity given in number 3 of Example 154 to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution

1. To use the half angle formula, we note that $15^\circ = \frac{30^\circ}{2}$ and since 15° is a Quadrant I angle, its cosine is positive. Thus we have

$$\begin{aligned} \cos\left(15^\circ\right) &= +\sqrt{\frac{1+\cos\left(30^\circ\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2} \end{aligned}$$

Back in Example 152, we found $\cos\left(15^\circ\right)$ by using the difference formula for cosine. In that case, we determined $\cos\left(15^\circ\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$. The reader is encouraged to prove that these two expressions are equal.

2. If $-\pi \leq \theta \leq 0$, then $-\frac{\pi}{2} \leq \frac{\theta}{2} \leq 0$, which means $\sin\left(\frac{\theta}{2}\right) < 0$. Theorem 69 gives

$$\begin{aligned} \sin\left(\frac{\theta}{2}\right) &= -\sqrt{\frac{1-\cos\left(\theta\right)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5} \end{aligned}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number 3 of Example 154 and manipulate it into the identity we are asked to prove. The identity we are asked to start with is $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\frac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\frac{\theta}{2}$

$$\begin{aligned} \sin\left(2\left(\tfrac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \end{aligned}$$

We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. To get cosines involved, recall that $1+\tan^2\left(\frac{\theta}{2}\right) = \sec^2\left(\frac{\theta}{2}\right)$. We continue to manipulate our given identity by converting secants to cosines and using a power reduction formula

$$\begin{aligned} \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right) \\ \sin(\theta) &= 2\tan\left(\tfrac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\ \sin(\theta) &= \tan\left(\tfrac{\theta}{2}\right)\left(1+\cos(\theta)\right) \\ \tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)} \end{aligned}$$

Our next batch of identities, the Product to Sum Formulas, are easily verified by expanding each of the right hand sides in accordance with Theorem 66 and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 70 Product to Sum Formulas

For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which come in handy when attempting to solve equations involving trigonometric functions. These are easily verified using the Product to Sum Formulas, and as such, their proofs are left as exercises.

The identities in Theorem 70 are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

Theorem 71 Sum to Product Formulas

For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$

Example 157 Using Theorems 70 and 71

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

Solution

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned}\cos(2\theta)\cos(6\theta) &= \tfrac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right]\\ &= \tfrac{1}{2}\cos(-4\theta)+\tfrac{1}{2}\cos(8\theta)\\ &= \tfrac{1}{2}\cos(4\theta)+\tfrac{1}{2}\cos(8\theta),\end{aligned}$$

where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned}\sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta),\end{aligned}$$

where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$.

This section and the one before it present a rather large volume of trigonometric identities, leading to a very common student question: "Do I have to memorize **all** of these?" The answer, of course, is no. The indispensable identities are the Pythagorean identities (Theorem 50), and the sum/difference identities (Theorems 63 and 65). They are the most common, and all other identities can be derived from them. That said, there are a number of topics in Calculus (trig integration comes to mind) where having other identities like the power reduction formulas in Theorem 68 at your fingertips will come in handy.

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers. In Exercises 38 - 43 in Section 8.5, we see how some of these identities manifest themselves geometrically as we study the graphs of the these functions. In the upcoming Exercises, however, you need to do all of your work analytically without graphs.

Exercises 8.4

Problems

In Exercises 1 – 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$

2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$

3. $\tan(-t^2 + 1) = -\tan(t^2 - 1)$

4. $\csc(-\theta - 5) = -\csc(\theta + 5)$

5. $\sec(-6t) = \sec(6t)$

6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 – 21, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

7. $\cos(75^\circ)$

8. $\sec(165^\circ)$

9. $\sin(105^\circ)$

10. $\csc(195^\circ)$

11. $\cot(255^\circ)$

12. $\tan(375^\circ)$

13. $\cos\left(\frac{13\pi}{12}\right)$

14. $\sin\left(\frac{11\pi}{12}\right)$

15. $\tan\left(\frac{13\pi}{12}\right)$

16. $\cos\left(\frac{7\pi}{12}\right)$

17. $\tan\left(\frac{17\pi}{12}\right)$

18. $\sin\left(\frac{\pi}{12}\right)$

19. $\cot\left(\frac{11\pi}{12}\right)$

20. $\csc\left(\frac{5\pi}{12}\right)$

21. $\sec\left(-\frac{\pi}{12}\right)$

22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, find
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$

24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, find
    (a) $\sin(\alpha + \beta)$
    (b) $\cos(\alpha - \beta)$
    (c) $\tan(\alpha - \beta)$

25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, find
    (a) $\csc(\alpha - \beta)$
    (b) $\sec(\alpha + \beta)$
    (c) $\cot(\alpha + \beta)$

In Exercises 26 – 38, verify the identity.

26. $\cos(\theta - \pi) = -\cos(\theta)$

27. $\sin(\pi - \theta) = \sin(\theta)$

28. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

29. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

30. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

31. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

32. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

33. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

34. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

35. $\dfrac{\tan(\alpha+\beta)}{\tan(\alpha-\beta)} = \dfrac{\sin(\alpha)\cos(\alpha)+\sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha)-\sin(\beta)\cos(\beta)}$

36. $\dfrac{\sin(t+h)-\sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h)-1}{h}\right)$

37. $\dfrac{\cos(t+h)-\cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h)-1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

38. $\dfrac{\tan(t+h)-\tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1-\tan(t)\tan(h)}\right)$

In Exercises 39 – 48, use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.

39. $\cos(75^\circ)$ (compare with Exercise 7)

40. $\sin(105^\circ)$ (compare with Exercise 9)

41. $\cos(67.5^\circ)$

42. $\sin(157.5^\circ)$

43. $\tan(112.5^\circ)$

44. $\cos\left(\dfrac{7\pi}{12}\right)$ (compare with Exercise 16)

45. $\sin\left(\dfrac{\pi}{12}\right)$ (compare with Exercise 18)

46. $\cos\left(\dfrac{\pi}{8}\right)$

47. $\sin\left(\dfrac{5\pi}{8}\right)$

48. $\tan\left(\dfrac{7\pi}{8}\right)$

In Exercises 49 – 58, use the given information about θ to find the exact values of

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\dfrac{\theta}{2}\right)$
- $\cos\left(\dfrac{\theta}{2}\right)$
- $\tan\left(\dfrac{\theta}{2}\right)$

49. $\sin(\theta) = -\dfrac{7}{25}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

50. $\cos(\theta) = \dfrac{28}{53}$ where $0 < \theta < \dfrac{\pi}{2}$

51. $\tan(\theta) = \dfrac{12}{5}$ where $\pi < \theta < \dfrac{3\pi}{2}$

52. $\csc(\theta) = 4$ where $\dfrac{\pi}{2} < \theta < \pi$

53. $\cos(\theta) = \dfrac{3}{5}$ where $0 < \theta < \dfrac{\pi}{2}$

54. $\sin(\theta) = -\dfrac{4}{5}$ where $\pi < \theta < \dfrac{3\pi}{2}$

55. $\cos(\theta) = \dfrac{12}{13}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

56. $\sin(\theta) = \dfrac{5}{13}$ where $\dfrac{\pi}{2} < \theta < \pi$

57. $\sec(\theta) = \sqrt{5}$ where $\dfrac{3\pi}{2} < \theta < 2\pi$

58. $\tan(\theta) = -2$ where $\dfrac{\pi}{2} < \theta < \pi$

In Exercises 59 – 73, verify the identity. Assume all quantities are defined.

59. $(\cos(\theta)+\sin(\theta))^2 = 1+\sin(2\theta)$

60. $(\cos(\theta)-\sin(\theta))^2 = 1-\sin(2\theta)$

61. $\tan(2\theta) = \dfrac{1}{1-\tan(\theta)} - \dfrac{1}{1+\tan(\theta)}$

62. $\csc(2\theta) = \dfrac{\cot(\theta)+\tan(\theta)}{2}$

63. $8\sin^4(\theta) = \cos(4\theta) - 4\cos(2\theta) + 3$

64. $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

65. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

66. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

67. $32\sin^2(\theta)\cos^4(\theta) = 2+\cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)$

68. $32\sin^4(\theta)\cos^2(\theta) = 2-\cos(2\theta) - 2\cos(4\theta) + \cos(6\theta)$

69. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

70. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 69.)

71. $\sec(2\theta) = \dfrac{\cos(\theta)}{\cos(\theta)+\sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta)-\sin(\theta)}$

72. $\dfrac{1}{\cos(\theta)-\sin(\theta)} + \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

73. $\dfrac{1}{\cos(\theta)-\sin(\theta)} - \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

In Exercises 74 – 79, write the given product as a sum. You may need to use an Even/Odd Identity.

74. $\cos(3\theta)\cos(5\theta)$

75. $\sin(2\theta)\sin(7\theta)$

76. $\sin(9\theta)\cos(\theta)$

77. $\cos(2\theta)\cos(6\theta)$

78. $\sin(3\theta)\sin(2\theta)$

79. $\cos(\theta)\sin(3\theta)$

In Exercises 80 – 85, write the given sum as a product. You may need to use an Even/Odd or Cofunction Identity.

80. $\cos(3\theta) + \cos(5\theta)$

81. $\sin(2\theta) - \sin(7\theta)$

82. $\cos(5\theta) - \cos(6\theta)$

83. $\sin(9\theta) - \sin(-\theta)$

84. $\sin(\theta) + \cos(\theta)$

85. $\cos(\theta) - \sin(\theta)$

86. Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas
    (a) $\cos(\theta) = \sqrt{1 - x^2}$
    (b) $\sin(2\theta) = 2x\sqrt{1 - x^2}$
    (c) $\cos(2\theta) = 1 - 2x^2$

87. Discuss with your classmates how each of the formulas, if any, in Exercise 86 change if we change assume θ is a Quadrant II, III, or IV angle.

88. Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas
    (a) $\cos(\theta) = \dfrac{1}{\sqrt{x^2 + 1}}$
    (b) $\sin(\theta) = \dfrac{x}{\sqrt{x^2 + 1}}$
    (c) $\sin(2\theta) = \dfrac{2x}{x^2 + 1}$
    (d) $\cos(2\theta) = \dfrac{1 - x^2}{x^2 + 1}$

89. Discuss with your classmates how each of the formulas, if any, in Exercise 88 change if we change assume θ is a Quadrant II, III, or IV angle.

90. If $\sin(\theta) = \dfrac{x}{2}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\cos(2\theta)$ in terms of x.

91. If $\tan(\theta) = \dfrac{x}{7}$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.

92. If $\sec(\theta) = \dfrac{x}{4}$ for $0 < \theta < \dfrac{\pi}{2}$, find an expression for $\ln|\sec(\theta) + \tan(\theta)|$ in terms of x.

93. Show that $\cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$ for all θ.

94. Let θ be a Quadrant III angle with $\cos(\theta) = -\dfrac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\dfrac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \dfrac{7\pi}{2}$ and then assuming $\pi < \theta < \dfrac{3\pi}{2}$ and computing $\sin\left(\dfrac{\theta}{2}\right)$ in both cases.

95. Without using your calculator, show that $\dfrac{\sqrt{2 + \sqrt{3}}}{2} = \dfrac{\sqrt{6} + \sqrt{2}}{4}$

96. In part 4 of Example 154, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise 69, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?

97. In Exercise 65, we has you verify an identity which expresses $\sin(3\theta)$ as a polynomial in terms of $\sin(\theta)$. Can you do the same for $\sin(5\theta)$? What about for $\sin(4\theta)$? If not, what goes wrong?

98. Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.

99. Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.

100. Verify the Difference Identities for sine and tangent.

101. Verify the Product to Sum Identities.

102. Verify the Sum to Product Identities.

8.5 Graphs of the Trigonometric Functions

In this section, we return to our discussion of the circular (trigonometric) functions as functions of real numbers and pick up where we left off in Sections 8.2.1 and 8.3.1. As usual, we begin our study with the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$.

8.5.1 Graphs of the Cosine and Sine Functions

From Theorem 55 in Section 8.2.1, we know that the domain of $f(t) = \cos(t)$ and of $g(t) = \sin(t)$ is all real numbers, $(-\infty, \infty)$, and the range of both functions is $[-1, 1]$. The Even / Odd Identities in Theorem 62 tell us $\cos(-t) = \cos(t)$ for all real numbers t and $\sin(-t) = -\sin(t)$ for all real numbers t. This means $f(t) = \cos(t)$ is an even function, while $g(t) = \sin(t)$ is an odd function. (See section 2.5 for a review of these concepts.) Another important property of these functions is that for coterminal angles α and β, $\cos(\alpha) = \cos(\beta)$ and $\sin(\alpha) = \sin(\beta)$. Said differently, $\cos(t + 2\pi k) = \cos(t)$ and $\sin(t + 2\pi k) = \sin(t)$ for all real numbers t and any integer k. This last property is given a special name.

Definition 55 Periodic Function

A function f is said to be **periodic** if there is a real number c so that $f(t + c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t + p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

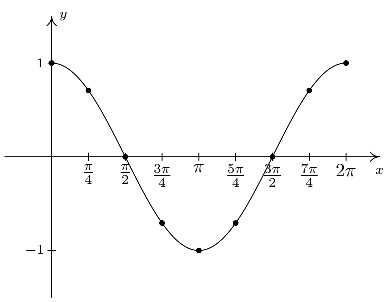
We have already seen a family of periodic functions in Section 3.1: the constant functions. However, despite being periodic, a constant function has no period. (We'll leave that odd gem as an exercise for you.) Returning to the circular functions, we see that by Definition 55, $f(t) = \cos(t)$ is periodic, since $\cos(t + 2\pi k) = \cos(t)$ for any integer k. To determine the period of f, we need to find the smallest real number p so that $f(t + p) = f(t)$ for all real numbers t or, said differently, the smallest positive real number p such that $\cos(t + p) = \cos(t)$ for all real numbers t. We know that $\cos(t + 2\pi) = \cos(t)$ for all real numbers t but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\cos(t + p) = \cos(t)$ for all real numbers t. Then, in particular, $\cos(0 + p) = \cos(0)$ so that $\cos(p) = 1$. From this we know p is a multiple of 2π and, since the smallest positive multiple of 2π is 2π itself, we have the result. Similarly, we can show $g(t) = \sin(t)$ is also periodic with 2π as its period. (Alternatively, we can use the Cofunction Identities in Theorem 64 to show that $g(t) = \sin(t)$ is periodic with period 2π since $g(t) = \sin(t) = \cos\left(\frac{\pi}{2} - t\right) = f\left(\frac{\pi}{2} - t\right)$.) Having period 2π essentially means that we can completely understand everything about the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$ by studying one interval of length 2π, say $[0, 2\pi]$.

Technically, we should study the interval $[0, 2\pi)$, since whatever happens at $t = 2\pi$ is the same as what happens at $t = 0$. As we will see shortly, $t = 2\pi$ gives us an extra 'check' when we go to graph these functions. In some texts, the interval of choice is $[-\pi, \pi)$.

One last property of the functions $f(t) = \cos(t)$ and $g(t) = \sin(t)$ is worth pointing out: both of these functions are continuous and smooth. Recall from Section 4.1 that geometrically this means the graphs of the cosine and sine functions have no jumps, gaps, holes in the graph, asymptotes, corners or cusps. As we shall see, the graphs of both $f(t) = \cos(t)$ and $g(t) = \sin(t)$ meander nicely and don't cause any trouble. We summarize these facts in the following theorem.

x	$\cos(x)$	$(x, \cos(x))$
0	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi, 1)$

Values of $f(x) = \cos(x)$ on $[0, 2\pi]$

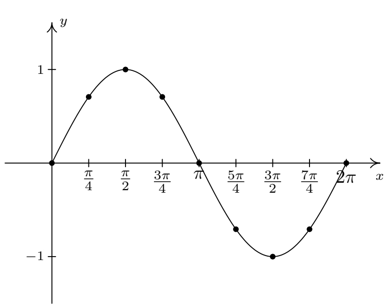


The ‘fundamental cycle’ of $y = \cos(x)$.

Figure 8.71: Graphing $y = \cos(x)$

x	$\sin(x)$	$(x, \sin(x))$
0	0	$(0, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$

Values of $f(x) = \sin(x)$ on $[0, 2\pi]$



The ‘fundamental cycle’ of $y = \sin(x)$

Figure 8.72: Graphing $y = \sin(x)$

Theorem 72 Properties of the Cosine and Sine Functions

- The function $f(x) = \cos(x)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π
- The function $f(x) = \sin(x)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π

In this section, we follow the convention established in Section 2.5 and use x as the independent variable and y as the dependent variable. This allows us to turn our attention to graphing the cosine and sine functions in the Cartesian Plane. (**Caution**: the use of x and y in this context is not to be confused with the x- and y-coordinates of points on the Unit Circle which define cosine and sine. Using the term ‘trigonometric function’ as opposed to ‘circular function’ can help with that, but one could then ask, “Hey, where’s the triangle?”) To graph $y = \cos(x)$, we make a table as we did in Section 2.5 using some of the ‘common values’ of x in the interval $[0, 2\pi]$. This generates a *portion* of the cosine graph, which we call the ‘**fundamental cycle**’ of $y = \cos(x)$.

A few things about the graph above are worth mentioning. First, this graph represents only part of the graph of $y = \cos(x)$. To get the entire graph, we imagine ‘copying and pasting’ this graph end to end infinitely in both directions (left and right) on the x-axis. Secondly, the vertical scale here has been greatly exaggerated for clarity and aesthetics. Below is an accurate-to-scale graph of $y = \cos(x)$ showing several cycles with the ‘fundamental cycle’ plotted thicker than the others. The graph of $y = \cos(x)$ is usually described as ‘wavelike’ – indeed, many of the applications involving the cosine and sine functions feature modelling wavelike phenomena.

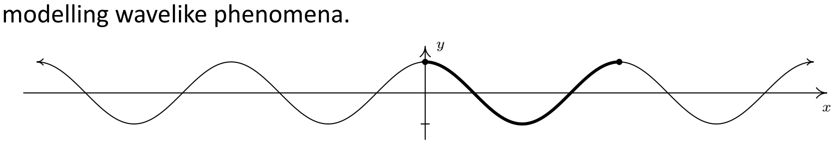


Figure 8.69: An accurately scaled graph of $y = \cos(x)$.

We can plot the fundamental cycle of the graph of $y = \sin(x)$ similarly, with similar results.

As with the graph of $y = \cos(x)$, we provide an accurately scaled graph of $y = \sin(x)$ below with the fundamental cycle highlighted.

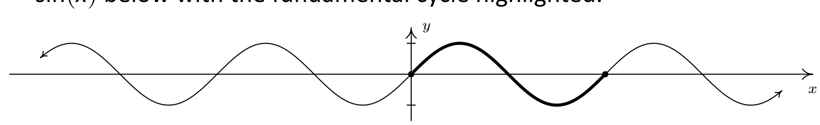


Figure 8.70: An accurately scaled graph of $y = \sin(x)$.

It is no accident that the graphs of $y = \cos(x)$ and $y = \sin(x)$ are so similar. Using a cofunction identity along with the even property of cosine, we have

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(-\left(x - \frac{\pi}{2}\right)\right) = \cos\left(x - \frac{\pi}{2}\right)$$

Recalling Section 2.6, we see from this formula that the graph of $y = \sin(x)$ is the result of shifting the graph of $y = \cos(x)$ to the right $\frac{\pi}{2}$ units. A visual inspection confirms this.

Now that we know the basic shapes of the graphs of $y = \cos(x)$ and $y = \sin(x)$, we can use Theorem 12 in Section 2.6 to graph more complicated curves. To do so, we need to keep track of the movement of some key points on the original graphs. We choose to track the values $x = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π. These 'quarter marks' correspond to quadrantal angles, and as such, mark the location of the zeros and the local extrema of these functions over exactly one period. Before we begin our next example, we need to review the concept of the 'argument' of a function as first introduced in Section 2.3. For the function $f(x) = 1 - 5\cos(2x - \pi)$, the argument of f is x. We shall have occasion, however, to refer to the argument of the *cosine*, which in this case is $2x - \pi$. Loosely stated, the argument of a trigonometric function is the expression 'inside' the function.

Example 158 **Plotting cosine and sine functions**

Graph one cycle of the following functions. State the period of each.

1. $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$
2. $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$

Solution

1. We set the argument of the cosine, $\frac{\pi x - \pi}{2}$, equal to each of the values: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π and solve for x. We summarize the results in Figure 8.74.

a	$\frac{\pi x - \pi}{2} = a$	x
0	$\frac{\pi x - \pi}{2} = 0$	1
$\frac{\pi}{2}$	$\frac{\pi x - \pi}{2} = \frac{\pi}{2}$	2
π	$\frac{\pi x - \pi}{2} = \pi$	3
$\frac{3\pi}{2}$	$\frac{\pi x - \pi}{2} = \frac{3\pi}{2}$	4
2π	$\frac{\pi x - \pi}{2} = 2\pi$	5

Figure 8.74: Reference points for $f(x)$ in Example 158

Next, we substitute each of these x values into $f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1$ to determine the corresponding y-values and connect the dots in a pleasing wavelike fashion.

x	$f(x)$	$(x, f(x))$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	1	$(4, 1)$
5	4	$(5, 4)$

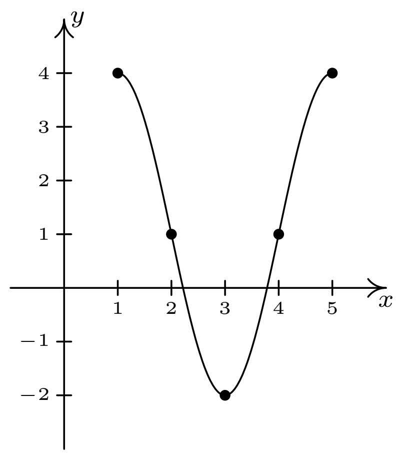


Figure 8.73: Plotting one cycle of $y = f(x)$ in Example 158

One cycle is graphed on $[1, 5]$ so the period is the length of that interval which is 4.

2. Proceeding as above, we set the argument of the sine, $\pi - 2x$, equal to each of our quarter marks and solve for x in Figure 8.75.

a	$\pi - 2x = a$	x
0	$\pi - 2x = 0$	$\frac{\pi}{2}$
$\frac{\pi}{2}$	$\pi - 2x = \frac{\pi}{2}$	$\frac{\pi}{4}$
π	$\pi - 2x = \pi$	0
$\frac{3\pi}{2}$	$\pi - 2x = \frac{3\pi}{2}$	$-\frac{\pi}{4}$
2π	$\pi - 2x = 2\pi$	$-\frac{\pi}{2}$

Figure 8.75: Reference points for $g(x)$ in Example 158

We now find the corresponding y-values on the graph by substituting each of these x-values into $g(x) = \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2}$. Once again, we connect the dots in a wavelike fashion.

x	$g(x)$	$(x, g(x))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{\pi}{4}$	2	$\left(\frac{\pi}{4}, 2\right)$
0	$\frac{3}{2}$	$\left(0, \frac{3}{2}\right)$
$-\frac{\pi}{4}$	1	$\left(-\frac{\pi}{4}, 1\right)$
$-\frac{\pi}{2}$	$\frac{3}{2}$	$\left(-\frac{\pi}{2}, \frac{3}{2}\right)$

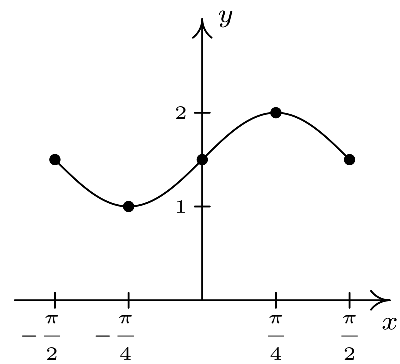


Figure 8.76: Plotting one cycle of $y = g(x)$ in Example 158

One cycle was graphed on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so the period is $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

The functions in Example 158 are examples of **sinusoids**. Roughly speaking, a sinusoid is the result of taking the basic graph of $f(x) = \cos(x)$ or $g(x) = \sin(x)$ and performing any of the transformations mentioned in Section 2.6. Sinusoids can be characterized by four properties: period, amplitude, phase shift and vertical shift. We have already discussed period, that is, how long it takes for the sinusoid to complete one cycle. The standard period of both $f(x) = \cos(x)$ and $g(x) = \sin(x)$ is 2π, but horizontal scalings will change the period of the resulting sinusoid. The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this: see Figure 8.77.

The **phase shift** of the sinusoid is the horizontal shift experienced by the fundamental cycle. We have seen that a phase (horizontal) shift of $\frac{\pi}{2}$ to the right takes $f(x) = \cos(x)$ to $g(x) = \sin(x)$ since $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$. As the reader can verify, a phase shift of $\frac{\pi}{2}$ to the left takes $g(x) = \sin(x)$ to $f(x) = \cos(x)$. The vertical shift of a sinusoid is exactly the same as the vertical shifts in Section 2.6. In most contexts, the vertical shift of a sinusoid is assumed to be 0, but we state the more general case below. The following theorem, which is reminiscent of Theorem 12 in Section 2.6, shows how to find these four fundamental quantities from the formula of the given sinusoid.

We have already seen how the Even/Odd and Cofunction Identities can be used to rewrite $g(x) = \sin(x)$ as a transformed version of $f(x) = \cos(x)$, so of course, the reverse is true: $f(x) = \cos(x)$ can be written as a transformed version of $g(x) = \sin(x)$. The authors have seen some instances where sinusoids are always converted to cosine functions while in other disciplines, the sinusoids are always written in terms of sine functions.

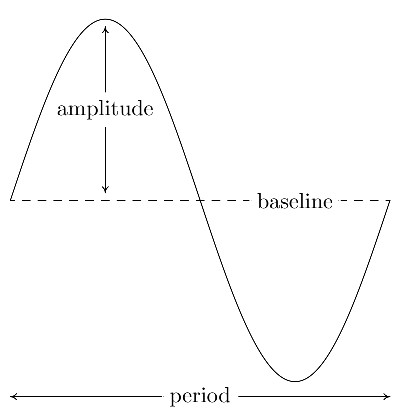


Figure 8.77: Properties of sinusoids

Theorem 73 Standard form of sinusoids

For $\omega > 0$, the functions

$$C(x) = A\cos(\omega x + \phi) + B \quad \text{and} \quad S(x) = A\sin(\omega x + \phi) + B$$

- have period $\dfrac{2\pi}{\omega}$
- have phase shift $-\dfrac{\phi}{\omega}$
- have amplitude $|A|$
- have vertical shift B

We note that in some scientific and engineering circles, the quantity ϕ mentioned in Theorem 73 is called the **phase** of the sinusoid. Since our interest in this book is primarily with *graphing* sinusoids, we focus our attention on the horizontal shift $-\frac{\phi}{\omega}$ induced by ϕ.

The proof of Theorem 73 is a direct application of Theorem 12 in Section 2.6 and is left to the reader. The parameter ω, which is stipulated to be positive, is called the (**angular**) **frequency** of the sinusoid and is the number of cycles the sinusoid completes over a 2π interval. We can always ensure $\omega > 0$ using the Even/Odd Identities. (Try using the formulas in Theorem 73 applied to $C(x) = \cos(-x + \pi)$ to see why we need $\omega > 0$.) We now test out Theorem 73 using the functions f and g featured in Example 158. First, we write $f(x)$ in the form prescribed in Theorem 73,

$$f(x) = 3\cos\left(\frac{\pi x - \pi}{2}\right) + 1 = 3\cos\left(\frac{\pi}{2}x + \left(-\frac{\pi}{2}\right)\right) + 1,$$

so that $A = 3$, $\omega = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$ and $B = 1$. According to Theorem 73, the period of f is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$, the amplitude is $|A| = |3| = 3$, the phase shift is $-\frac{\phi}{\omega} = -\frac{-\pi/2}{\pi/2} = 1$ (indicating a shift to the *right* 1 unit) and the vertical shift is $B = 1$ (indicating a shift *up* 1 unit.) All of these match with our graph of $y = f(x)$. Moreover, if we start with the basic shape of the cosine graph, shift it 1 unit to the right, 1 unit up, stretch the amplitude to 3 and shrink the period to 4, we will have reconstructed one period of the graph of $y = f(x)$. In other words, instead of tracking the five 'quarter marks' through the transformations to plot $y = f(x)$, we can use five other pieces of information: the phase shift, vertical shift, amplitude, period and basic shape of the cosine curve. Turning our attention now to the function g in Example 158, we first need to use the odd property of the sine function to write it in the form required by Theorem 73.

$$\begin{aligned} g(x) &= \frac{1}{2}\sin(\pi - 2x) + \frac{3}{2} = \frac{1}{2}\sin(-(2x - \pi)) + \frac{3}{2} \\ &= -\frac{1}{2}\sin(2x - \pi) + \frac{3}{2} = -\frac{1}{2}\sin(2x + (-\pi)) + \frac{3}{2} \end{aligned}$$

We find $A = -\frac{1}{2}$, $\omega = 2$, $\phi = -\pi$ and $B = \frac{3}{2}$. The period is then $\frac{2\pi}{2} = \pi$, the amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$, the phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$ (indicating a shift *right* $\frac{\pi}{2}$ units) and the vertical shift is *up* $\frac{3}{2}$. Note that, in this case, all of the data match our graph of $y = g(x)$ with the exception of the phase shift. Instead of the graph *starting* at $x = \frac{\pi}{2}$, it ends there. Remember, however, that the graph presented in Example 158 is only one portion of the graph of $y = g(x)$. Indeed, another complete cycle begins at $x = \frac{\pi}{2}$, and this is the cycle Theorem 73 is detecting. The reason for the discrepancy is that, in order to apply Theorem 73, we had to rewrite the formula for $g(x)$ using the odd property of the sine function. Note that whether we graph $y = g(x)$ using the 'quarter marks' approach or using the Theorem 73, we get one complete cycle of the graph, which means we have completely determined the sinusoid.

Example 159 Fitting a sinusoid to given data

Figure 8.78 shows the graph of one complete cycle of a sinusoid $y = f(x)$.

1. Find a cosine function whose graph matches the graph of $y = f(x)$.

2. Find a sine function whose graph matches the graph of $y = f(x)$.

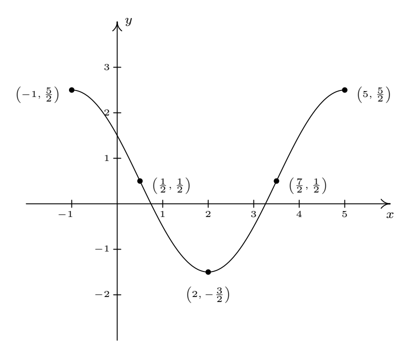


Figure 8.78: One cycle of $y = f(x)$ in Example 159

Solution

1. We fit the data to a function of the form $C(x) = A\cos(\omega x + \phi) + B$. Since one cycle is graphed over the interval $[-1, 5]$, its period is $5 - (-1) = 6$. According to Theorem 73, $6 = \frac{2\pi}{\omega}$, so that $\omega = \frac{\pi}{3}$. Next, we see that the phase shift is -1, so we have $-\frac{\phi}{\omega} = -1$, or $\phi = \omega = \frac{\pi}{3}$. To find the amplitude, note that the range of the sinusoid is $\left[-\frac{3}{2}, \frac{5}{2}\right]$. As a result, the amplitude $A = \frac{1}{2}\left[\frac{5}{2} - \left(-\frac{3}{2}\right)\right] = \frac{1}{2}(4) = 2$. Finally, to determine the vertical shift, we average the endpoints of the range to find $B = \frac{1}{2}\left[\frac{5}{2} + \left(-\frac{3}{2}\right)\right] = \frac{1}{2}(1) = \frac{1}{2}$. Our final answer is $C(x) = 2\cos\left(\frac{\pi}{3}x + \frac{\pi}{3}\right) + \frac{1}{2}$.
2. Most of the work to fit the data to a function of the form $S(x) = A\sin(\omega x + \phi) + B$ is done. The period, amplitude and vertical shift are the same as before with $\omega = \frac{\pi}{3}$, $A = 2$ and $B = \frac{1}{2}$. The trickier part is finding the phase shift. To that end, we imagine extending the graph of the given sinusoid as in Figure 8.79 below so that we can identify a cycle beginning at $\left(\frac{7}{2}, \frac{1}{2}\right)$. Taking the phase shift to be $\frac{7}{2}$, we get $-\frac{\phi}{\omega} = \frac{7}{2}$, or $\phi = -\frac{7}{2}\omega = -\frac{7}{2}\left(\frac{\pi}{3}\right) = -\frac{7\pi}{6}$. Hence, our answer is $S(x) = 2\sin\left(\frac{\pi}{3}x - \frac{7\pi}{6}\right) + \frac{1}{2}$.

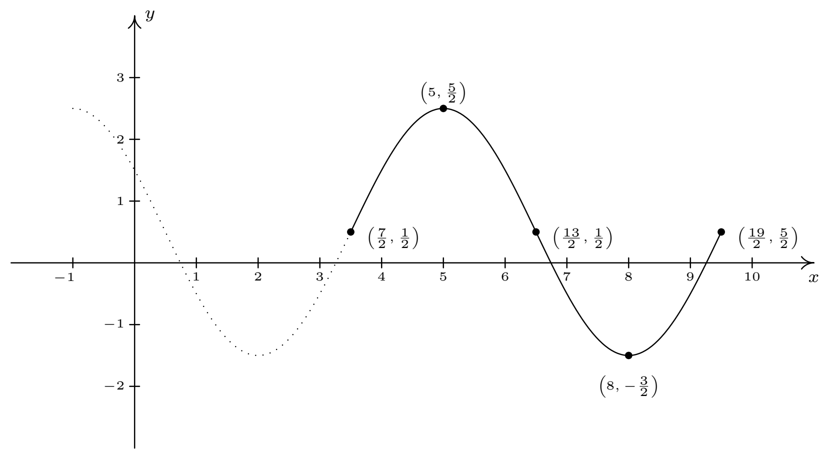


Figure 8.79: Extending the graph of $y = f(x)$

Note that each of the answers given in Example 159 is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at $\left(\frac{1}{2}, \frac{1}{2}\right)$ taking $A = -2$. In this case, the phase shift is $\frac{1}{2}$ so $\phi = -\frac{\pi}{6}$ for an answer of $S(x) = -2\sin\left(\frac{\pi}{3}x - \frac{\pi}{6}\right) + \frac{1}{2}$. Alternatively, we could have extended the graph of $y = f(x)$ to the left and considered a sine function starting at $\left(-\frac{5}{2}, \frac{1}{2}\right)$, and so on. Each of these formulas determine the same sinusoid curve and their formulas are all equivalent using identities. Speaking of identities, if we use the sum identity for cosine, we can expand the formula to yield

$$C(x) = A\cos(\omega x + \phi) + B = A\cos(\omega x)\cos(\phi) - A\sin(\omega x)\sin(\phi) + B.$$

Similarly, using the sum identity for sine, we get

$$S(x) = A\sin(\omega x + \phi) + B = A\sin(\omega x)\cos(\phi) + A\cos(\omega x)\sin(\phi) + B.$$

Making these observations allows us to recognize (and graph) functions as sinusoids which, at first glance, don't appear to fit the forms of either $C(x)$ or $S(x)$.

Example 160 Converting a sinusoid to standard form

Consider the function $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$. Find a formula for $f(x)$:

1. in the form $C(x) = A\cos(\omega x + \phi) + B$ for $\omega > 0$
2. in the form $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$

Solution

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. Equating $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $C(x) = A\cos(\omega x + \phi) + B$, we get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\cos(\omega x)\cos(\phi) - A\sin(\omega x)\sin(\phi) + B$$

It should be clear that we can take $\omega = 2$ and $B = 0$ to get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\cos(2x)\cos(\phi) - A\sin(2x)\sin(\phi)$$

To determine A and ϕ, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation. On the left hand side, the coefficient of $\cos(2x)$ is 1, while on the right hand side, it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have that $A\cos(\phi) = 1$. Similarly, we find by equating the coefficients of $\sin(2x)$ that $A\sin(\phi) = \sqrt{3}$. What we have here is a system of nonlinear equations! We can temporarily eliminate the dependence on ϕ by using the Pythagorean Identity. We know $\cos^2(\phi) + \sin^2(\phi) = 1$, so multiplying this by A^2 gives $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$. Since $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, we get $A^2 = 1^2 + (\sqrt{3})^2 = 4$ or $A = \pm 2$. Choosing $A = 2$, we have $2\cos(\phi) = 1$ and $2\sin(\phi) = \sqrt{3}$ or, after some rearrangement, $\cos(\phi) = \frac{1}{2}$ and $\sin(\phi) = \frac{\sqrt{3}}{2}$. One such angle ϕ which satisfies this criteria is $\phi = \frac{\pi}{3}$. Hence, one way to write $f(x)$ as a sinusoid is $f(x) = 2\cos\left(2x + \frac{\pi}{3}\right)$. We can easily check our answer using the sum formula for cosine

$$\begin{aligned} f(x) &= 2\cos\left(2x + \tfrac{\pi}{3}\right) \\ &= 2\left[\cos(2x)\cos\left(\tfrac{\pi}{3}\right) - \sin(2x)\sin\left(\tfrac{\pi}{3}\right)\right] \\ &= 2\left[\cos(2x)\left(\tfrac{1}{2}\right) - \sin(2x)\left(\tfrac{\sqrt{3}}{2}\right)\right] \\ &= \cos(2x) - \sqrt{3}\sin(2x) \end{aligned}$$

2. Proceeding as before, we equate $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ with the expanded form of $S(x) = A\sin(\omega x + \phi) + B$ to get

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(\omega x)\cos(\phi) + A\cos(\omega x)\sin(\phi) + B$$

Once again, we may take $\omega = 2$ and $B = 0$ so that

$$\cos(2x) - \sqrt{3}\sin(2x) = A\sin(2x)\cos(\phi) + A\cos(2x)\sin(\phi)$$

We equate (be careful here!) the coefficients of $\cos(2x)$ on either side and get $A\sin(\phi) = 1$ and $A\cos(\phi) = -\sqrt{3}$. Using $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$. This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, which means $\cos(\phi) =$

$-\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$. Hence, we have $f(x) = 2\sin\left(2x + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$\begin{aligned} f(x) &= 2\sin\left(2x + \tfrac{5\pi}{6}\right) \\ &= 2\left[\sin(2x)\cos\left(\tfrac{5\pi}{6}\right) + \cos(2x)\sin\left(\tfrac{5\pi}{6}\right)\right] \\ &= 2\left[\sin(2x)\left(-\tfrac{\sqrt{3}}{2}\right) + \cos(2x)\left(\tfrac{1}{2}\right)\right] \\ &= \cos(2x) - \sqrt{3}\sin(2x) \end{aligned}$$

In Section 8.3.1, we argued the range of $F(x) = \sec(x)$ is $(-\infty, -1] \cup [1, \infty)$. We can now see this graphically.

It is important to note that in order for the technique presented in Example 160 to fit a function into one of the forms in Theorem 73, the arguments of the cosine and sine function much match. That is, while $f(x) = \cos(2x) - \sqrt{3}\sin(2x)$ is a sinusoid, $g(x) = \cos(2x) - \sqrt{3}\sin(3x)$ is not.(This graph does, however, exhibit sinusoid-like characteristics! Check it out!) It is also worth mentioning that, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example 160, our final answers would have *looked* different. The reader is encouraged to rework Example 160 using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent. The general equations to fit a function of the form $f(x) = a\cos(\omega x) + b\sin(\omega x) + B$ into one of the forms in Theorem 73 are explored in Exercise 35.

8.5.2 Graphs of the Secant and Cosecant Functions

Note: provided that $\sec(\alpha)$ and $\sec(\beta)$ are defined, $\sec(\alpha) = \sec(\beta)$ if and only if $\cos(\alpha) = \cos(\beta)$. Hence, $\sec(x)$ inherits its period from $\cos(x)$.

We now turn our attention to graphing $y = \sec(x)$. Since $\sec(x) = \frac{1}{\cos(x)}$, we can use our table of values for the graph of $y = \cos(x)$ and take reciprocals. We know from Section 8.3.1 that the domain of $F(x) = \sec(x)$ excludes all odd multiples of $\frac{\pi}{2}$, and sure enough, we run into trouble at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ since $\cos(x) = 0$ at these values. Using the notation introduced in Section 5.2, we have that as $x \to \frac{\pi}{2}^-$, $\cos(x) \to 0^+$, so $\sec(x) \to \infty$. (See Section 8.3.1 for a more detailed analysis.) Similarly, we find that as $x \to \frac{\pi}{2}^+$, $\sec(x) \to -\infty$; as $x \to \frac{3\pi}{2}^-$, $\sec(x) \to -\infty$; and as $x \to \frac{3\pi}{2}^+$, $\sec(x) \to \infty$. This means we have a pair of vertical asymptotes to the graph of $y = \sec(x)$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Since $\cos(x)$ is periodic with period 2π, it follows that $\sec(x)$ is also. Below we graph a fundamental cycle of $y = \sec(x)$ along with a more complete graph obtained by the usual 'copying and pasting.'

x	$\cos(x)$	$\sec(x)$	$(x, \sec(x))$
0	1	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	0	undefined	
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{3\pi}{4}, -\sqrt{2}\right)$
π	-1	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	0	undefined	
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{7\pi}{4}, \sqrt{2}\right)$
2π	1	1	$(2\pi, 1)$

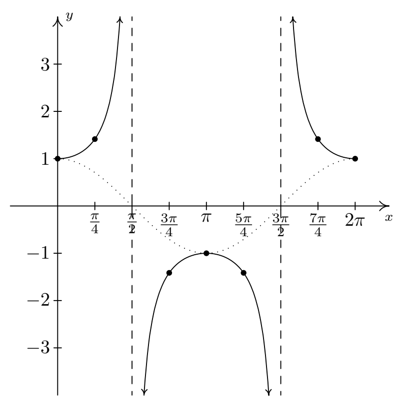


Figure 8.80: The 'fundamental cycle' of $y = \sec(x)$.

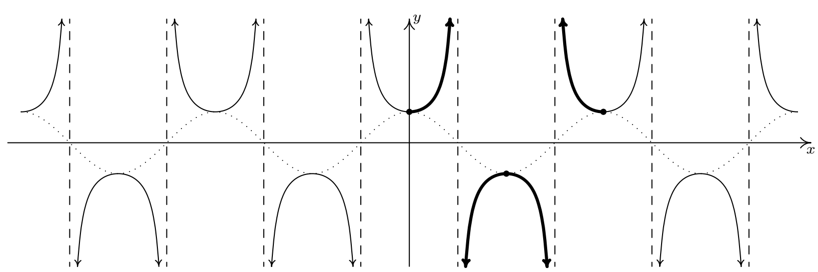


Figure 8.81: The graph of $y = \sec x$

As one would expect, to graph $y = \csc(x)$ we begin with $y = \sin(x)$ and take reciprocals of the corresponding y-values. Here, we encounter issues at $x = 0$, $x = \pi$ and $x = 2\pi$. Proceeding with the usual analysis, we graph the fundamental cycle of $y = \csc(x)$ below along with the dotted graph of $y = \sin(x)$ for reference. Since $y = \sin(x)$ and $y = \cos(x)$ are merely phase shifts of each other, so too are $y = \csc(x)$ and $y = \sec(x)$.

x	$\sin(x)$	$\csc(x)$	$(x, \csc(x))$
0	0	undefined	
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{\pi}{4}, \sqrt{2}\right)$
$\frac{\pi}{2}$	1	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$	$\left(\frac{3\pi}{4}, \sqrt{2}\right)$
π	0	undefined	
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{5\pi}{4}, -\sqrt{2}\right)$
$\frac{3\pi}{2}$	-1	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$	$\left(\frac{7\pi}{4}, -\sqrt{2}\right)$
2π	0	undefined	

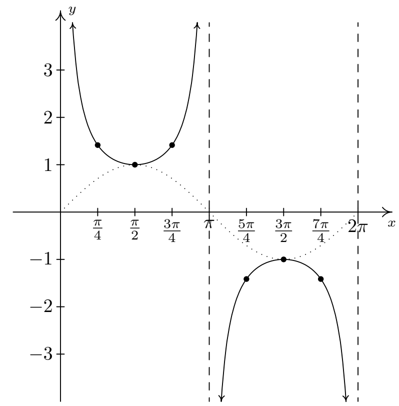


Figure 8.82: The 'fundamental cycle' of $y = \csc(x)$.

Just like the rational functions in Chapter 5 are continuous and smooth on their domains because polynomials are continuous and smooth everywhere, the secant and cosecant functions are continuous and smooth on their domains since the cosine and sine functions are continuous and smooth everywhere.

Once again, our domain and range work in Section 8.3.1 is verified geometrically in the graph of $y = G(x) = \csc(x)$.

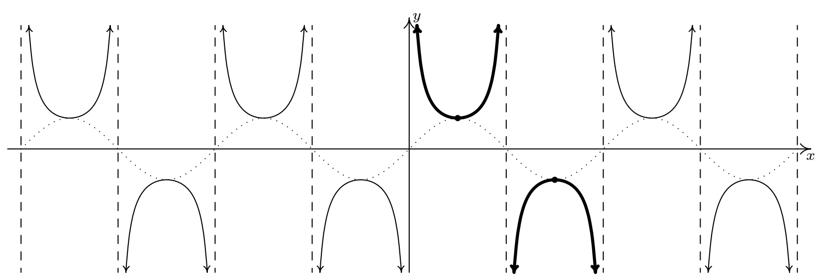


Figure 8.83: The graph of $y = \csc x$

Note that, on the intervals between the vertical asymptotes, both $F(x) = \sec(x)$ and $G(x) = \csc(x)$ are continuous and smooth. In other words, they are continuous and smooth *on their domains*. The following theorem summarizes the properties of the secant and cosecant functions. Note that all of these properties are direct results of them being reciprocals of the cosine and sine functions, respectively.

Theorem 74 Properties of the Secant and Cosecant Functions

- The function $F(x) = \sec(x)$
  - has domain $\{x : x \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $\{y : |y| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is even
  - has period 2π
- The function $G(x) = \csc(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $\{y : |y| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period 2π

In the next example, we discuss graphing more general secant and cosecant curves.

Example 161 Graphing secant and cosecant curves
Graph one cycle of the following functions. State the period of each.

1. $f(x) = 1 - 2\sec(2x)$
2. $g(x) = \dfrac{\csc(\pi - \pi x) - 5}{3}$

Solution

1. To graph $y = 1 - 2\sec(2x)$, we follow the same procedure as in Example 158. First, we set the argument of secant, $2x$, equal to the 'quarter marks' 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π and solve for x in Figure 8.85.

   Next, we substitute these x values into $f(x)$. If $f(x)$ exists, we have a point on the graph; otherwise, we have found a vertical asymptote. In addition to these points and asymptotes, we have graphed the associated cosine curve – in this case $y = 1 - 2\cos(2x)$ – dotted in the picture below. Since one cycle is graphed over the interval $[0, \pi]$, the period is $\pi - 0 = \pi$.

a	$2x = a$	x
0	$2x = 0$	0
$\frac{\pi}{2}$	$2x = \frac{\pi}{2}$	$\frac{\pi}{4}$
π	$2x = \pi$	$\frac{\pi}{2}$
$\frac{3\pi}{2}$	$2x = \frac{3\pi}{2}$	$\frac{3\pi}{4}$
2π	$2x = 2\pi$	π

Figure 8.85: Reference points for $f(x)$ in Example 161

x	$f(x)$	$(x, f(x))$
0	-1	$(0, -1)$
$\frac{\pi}{4}$	undefined	
$\frac{\pi}{2}$	3	$\left(\frac{\pi}{2}, 3\right)$
$\frac{3\pi}{4}$	undefined	
π	-1	$(\pi, -1)$

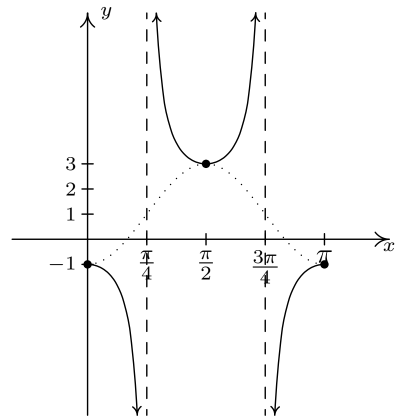


Figure 8.84: Plotting one cycle of $y = f(x)$ in Example 161

2. Proceeding as before, we set the argument of cosecant in $g(x) = \frac{\csc(\pi - \pi x) - 5}{3}$ equal to the quarter marks and solve for x in Figure 8.87.

   Substituting these x-values into $g(x)$, we generate the graph below and find the period to be $1 - (-1) = 2$. The associated sine curve, $y = \frac{\sin(\pi - \pi x) - 5}{3}$, is dotted in as a reference.

a	$\pi - \pi x = a$	x
0	$\pi - \pi x = 0$	1
$\frac{\pi}{2}$	$\pi - \pi x = \frac{\pi}{2}$	$\frac{1}{2}$
π	$\pi - \pi x = \pi$	0
$\frac{3\pi}{2}$	$\pi - \pi x = \frac{3\pi}{2}$	$-\frac{1}{2}$
2π	$\pi - \pi x = 2\pi$	-1

Figure 8.87: Reference points for $g(x)$ in Example 161

x	$g(x)$	$(x, g(x))$
1	undefined	
$\frac{1}{2}$	$-\frac{4}{3}$	$\left(\frac{1}{2}, -\frac{4}{3}\right)$
0	undefined	
$-\frac{1}{2}$	-2	$\left(-\frac{1}{2}, -2\right)$
-1	undefined	

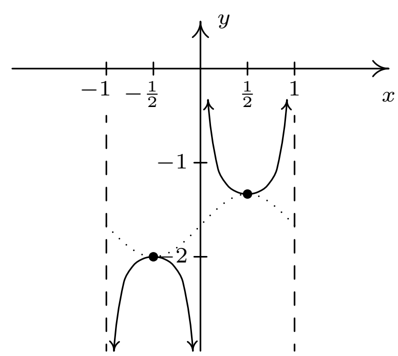


Figure 8.86: Plotting one cycle of $y = g(x)$ in Example 161

Before moving on, we note that it is possible to speak of the period, phase shift and vertical shift of secant and cosecant graphs and use even/odd identities to put them in a form similar to the sinusoid forms mentioned in Theorem 73. Since these quantities match those of the corresponding cosine and sine curves, we do not spell this out explicitly. Finally, since the ranges of secant and cosecant are unbounded, there is no amplitude associated with these curves.

8.5.3 Graphs of the Tangent and Cotangent Functions

Finally, we turn our attention to the graphs of the tangent and cotangent functions. When constructing a table of values for the tangent function, we see that $J(x) = \tan(x)$ is undefined at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, in accordance with our findings in Section 8.3.1. As $x \to \frac{\pi}{2}^-$, $\sin(x) \to 1^-$ and $\cos(x) \to 0^+$, so that $\tan(x) = \frac{\sin(x)}{\cos(x)} \to \infty$ producing a vertical asymptote at $x = \frac{\pi}{2}$. Using a similar analysis, we get that as $x \to \frac{\pi}{2}^+$, $\tan(x) \to -\infty$; as $x \to \frac{3\pi}{2}^-$, $\tan(x) \to \infty$; and as $x \to \frac{3\pi}{2}^+$, $\tan(x) \to -\infty$. Plotting this information and performing the usual 'copy and paste' produces Figures 8.88 and 8.89 below.

x	$\tan(x)$	$(x, \tan(x))$
0	0	$(0,0)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4},1\right)$
$\frac{\pi}{2}$	undefined	
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4},-1\right)$
π	0	$(\pi,0)$
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4},1\right)$
$\frac{3\pi}{2}$	undefined	
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4},-1\right)$
2π	0	$(2\pi,0)$

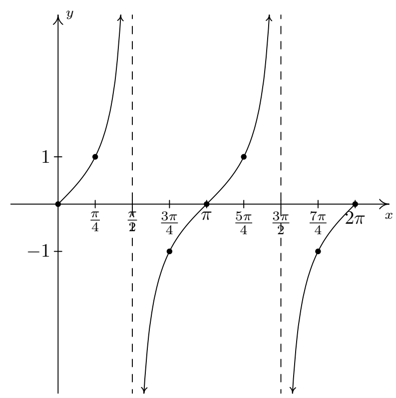


Figure 8.88: The graph of $y = \tan(x)$ over $[0, 2\pi]$

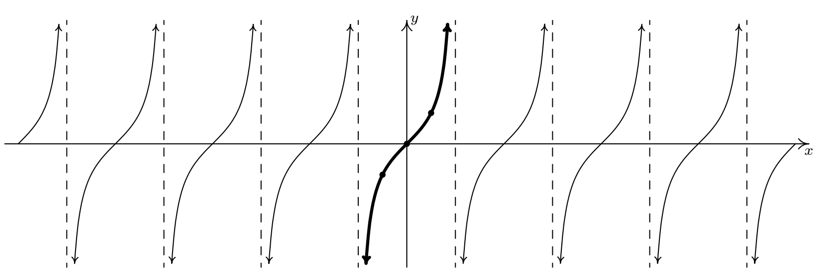


Figure 8.89: The graph of $y = \tan(x)$

From the graph, it appears as if the tangent function is periodic with period π. To prove that this is the case, we appeal to the sum formula for tangents. We have:

$$\tan(x+\pi) = \frac{\tan(x)+\tan(\pi)}{1-\tan(x)\tan(\pi)} = \frac{\tan(x)+0}{1-(\tan(x))(0)} = \tan(x),$$

which tells us the period of $\tan(x)$ is at most π. To show that it is exactly π, suppose p is a positive real number so that $\tan(x+p) = \tan(x)$ for all real numbers x. For $x = 0$, we have $\tan(p) = \tan(0+p) = \tan(0) = 0$, which means p is a multiple of π. The smallest positive multiple of π is π itself, so we have established the result. We take as our fundamental cycle for $y = \tan(x)$ the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and use as our 'quarter marks' $x = -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}$ and $\frac{\pi}{2}$. From the graph, we see confirmation of our domain and range work in Section 8.3.1.

It should be no surprise that $K(x) = \cot(x)$ behaves similarly to $J(x) = \tan(x)$. Plotting $\cot(x)$ over the interval $[0, 2\pi]$ results in the graph in Figure 8.90 below.

x	$\cot(x)$	$(x, \cot(x))$
0	undefined	
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
π	undefined	
$\frac{5\pi}{4}$	1	$\left(\frac{5\pi}{4}, 1\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	-1	$\left(\frac{7\pi}{4}, -1\right)$
2π	undefined	

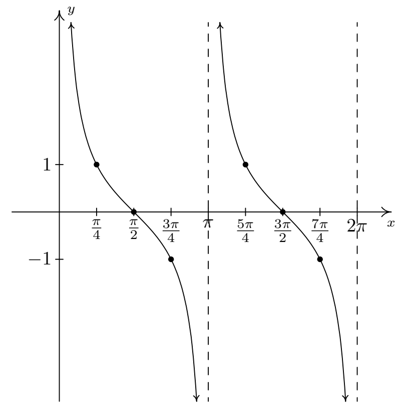


Figure 8.90: The graph of $y = \cot(x)$ over $[0, 2\pi]$

From these data, it clearly appears as if the period of $\cot(x)$ is π, and we leave it to the reader to prove this. (Certainly, mimicking the proof that the period of $\tan(x)$ is an option; for another approach, consider transforming $\tan(x)$ to $\cot(x)$ using identities.) We take as one fundamental cycle the interval $(0, \pi)$ with quarter marks: $x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$ and π. A more complete graph of $y = \cot(x)$ is below, along with the fundamental cycle highlighted as usual. Once again, we see the domain and range of $K(x) = \cot(x)$ as read from the graph matches with what we found analytically in Section 8.3.1.

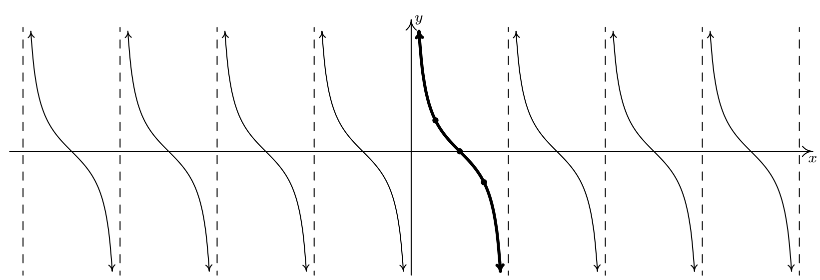


Figure 8.91: The graph of $y = \cot(x)$

The properties of the tangent and cotangent functions are summarized below. As with Theorem 74, each of the results below can be traced back to properties of the cosine and sine functions and the definition of the tangent and cotangent functions as quotients thereof.

Theorem 75 Properties of the Tangent and Cotangent Functions

- The function $J(x) = \tan(x)$
  - has domain $\left\{x : x \neq \frac{\pi}{2} + \pi k,\ k \text{ is an integer}\right\} = \bigcup_{k=-\infty}^{\infty} \left(\frac{(2k+1)\pi}{2}, \frac{(2k+3)\pi}{2} \right)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π
- The function $K(x) = \cot(x)$
  - has domain $\{x : x \neq \pi k,\ k \text{ is an integer}\} = \bigcup_{k=-\infty}^{\infty} (k\pi, (k+1)\pi)$
  - has range $(-\infty, \infty)$
  - is continuous and smooth on its domain
  - is odd
  - has period π

Example 162 Plotting tangent and cotangent curves

Graph one cycle of the following functions. Find the period.

1. $f(x) = 1 - \tan\left(\frac{x}{2}\right)$.

2. $g(x) = 2\cot\left(\frac{\pi}{2}x + \pi\right) + 1$.

Solution

1. We proceed as we have in all of the previous graphing examples by setting the argument of tangent in $f(x) = 1 - \tan\left(\frac{x}{2}\right)$, namely $\frac{x}{2}$, equal to each of the 'quarter marks' $-\frac{\pi}{2}$, $-\frac{\pi}{4}$, 0, $\frac{\pi}{4}$ and $\frac{\pi}{2}$, and solving for x: see Figure 8.93.

   Substituting these x-values into $f(x)$, we find points on the graph and the vertical asymptotes.

a	$\frac{x}{2} = a$	x
$-\frac{\pi}{2}$	$\frac{x}{2} = -\frac{\pi}{2}$	$-\pi$
$-\frac{\pi}{4}$	$\frac{x}{2} = -\frac{\pi}{4}$	$-\frac{\pi}{2}$
0	$\frac{x}{2} = 0$	0
$\frac{\pi}{4}$	$\frac{x}{2} = \frac{\pi}{4}$	$\frac{\pi}{2}$
$\frac{\pi}{2}$	$\frac{x}{2} = \frac{\pi}{2}$	π

Figure 8.93: Reference points for $f(x)$ in Example 162

x	$f(x)$	$(x, f(x))$
$-\pi$	undefined	
$-\frac{\pi}{2}$	2	$\left(-\frac{\pi}{2}, 2\right)$
0	1	$(0, 1)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
π	undefined	

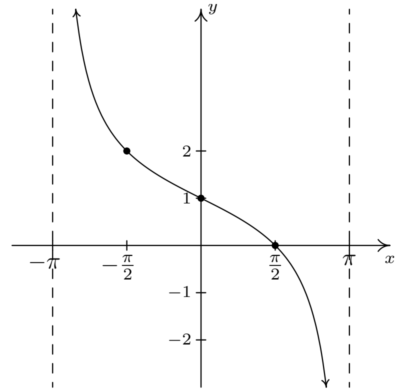


Figure 8.92: Plotting one cycle of $y = f(x)$ in Example 162

We see that the period is $\pi - (-\pi) = 2\pi$.

2. The 'quarter marks' for the fundamental cycle of the cotangent curve are 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$ and π. To graph $g(x) = 2\cot\left(\frac{\pi}{2}x + \pi\right) + 1$, we begin by setting $\frac{\pi}{2}x + \pi$ equal to each quarter mark and solving for x in Figure 8.95.

   We now use these x-values to generate our graph.

a	$\frac{\pi}{2}x + \pi = a$	x
0	$\frac{\pi}{2}x + \pi = 0$	-2
$\frac{\pi}{4}$	$\frac{\pi}{2}x + \pi = \frac{\pi}{4}$	$-\frac{3}{2}$
$\frac{\pi}{2}$	$\frac{\pi}{2}x + \pi = \frac{\pi}{2}$	-1
$\frac{3\pi}{4}$	$\frac{\pi}{2}x + \pi = \frac{3\pi}{4}$	$-\frac{1}{2}$
π	$\frac{\pi}{2}x + \pi = \pi$	0

Figure 8.95: Reference points for $g(x)$ in Example 162

x	$g(x)$	$(x, g(x))$
-2	undefined	
$-\frac{3}{2}$	3	$\left(-\frac{3}{2}, 3\right)$
-1	1	$(-1, 1)$
$-\frac{1}{2}$	-1	$\left(-\frac{1}{2}, -1\right)$
0	undefined	

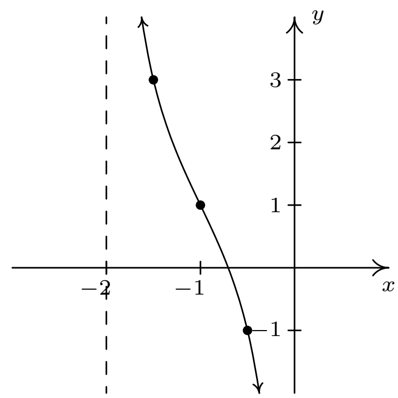


Figure 8.94: Plotting one cycle of $y = g(x)$ in Example 162

We find the period to be $0 - (-2) = 2$.

As with the secant and cosecant functions, it is possible to extend the notion of period, phase shift and vertical shift to the tangent and cotangent functions as we did for the cosine and sine functions in Theorem 73. Since the number of classical applications involving sinusoids far outnumber those involving tangent and cotangent functions, we omit this. The ambitious reader is invited to formulate such a theorem, however.

Exercises 8.5

Problems

In Exercises 1 – 12, graph one cycle of the given function. State the period, amplitude, phase shift and vertical shift of the function.

1. $y = 3\sin(x)$
2. $y = \sin(3x)$
3. $y = -2\cos(x)$
4. $y = \cos\left(x - \dfrac{\pi}{2}\right)$
5. $y = -\sin\left(x + \dfrac{\pi}{3}\right)$
6. $y = \sin(2x - \pi)$
7. $y = -\dfrac{1}{3}\cos\left(\dfrac{1}{2}x + \dfrac{\pi}{3}\right)$
8. $y = \cos(3x - 2\pi) + 4$
9. $y = \sin\left(-x - \dfrac{\pi}{4}\right) - 2$
10. $y = \dfrac{2}{3}\cos\left(\dfrac{\pi}{2} - 4x\right) + 1$
11. $y = -\dfrac{3}{2}\cos\left(2x + \dfrac{\pi}{3}\right) - \dfrac{1}{2}$
12. $y = 4\sin(-2\pi x + \pi)$

In Exercises 13 – 24, graph one cycle of the given function. State the period of the function.

13. $y = \tan\left(x - \dfrac{\pi}{3}\right)$
14. $y = 2\tan\left(\dfrac{1}{4}x\right) - 3$
15. $y = \dfrac{1}{3}\tan(-2x - \pi) + 1$
16. $y = \sec\left(x - \dfrac{\pi}{2}\right)$
17. $y = -\csc\left(x + \dfrac{\pi}{3}\right)$
18. $y = -\dfrac{1}{3}\sec\left(\dfrac{1}{2}x + \dfrac{\pi}{3}\right)$
19. $y = \csc(2x - \pi)$
20. $y = \sec(3x - 2\pi) + 4$
21. $y = \csc\left(-x - \dfrac{\pi}{4}\right) - 2$
22. $y = \cot\left(x + \dfrac{\pi}{6}\right)$
23. $y = -11\cot\left(\dfrac{1}{5}x\right)$
24. $y = \dfrac{1}{3}\cot\left(2x + \dfrac{3\pi}{2}\right) + 1$

In Exercises 25 – 34, use Example 160 as a guide to show that the function is a sinusoid by rewriting it in the forms $C(x) = A\cos(\omega x + \phi) + B$ and $S(x) = A\sin(\omega x + \phi) + B$ for $\omega > 0$ and $0 \le \phi < 2\pi$.

25. $f(x) = \sqrt{2}\sin(x) + \sqrt{2}\cos(x) + 1$
26. $f(x) = 3\sqrt{3}\sin(3x) - 3\cos(3x)$
27. $f(x) = -\sin(x) + \cos(x) - 2$
28. $f(x) = -\dfrac{1}{2}\sin(2x) - \dfrac{\sqrt{3}}{2}\cos(2x)$
29. $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x)$
30. $f(x) = \dfrac{3}{2}\cos(2x) - \dfrac{3\sqrt{3}}{2}\sin(2x) + 6$
31. $f(x) = -\dfrac{1}{2}\cos(5x) - \dfrac{\sqrt{3}}{2}\sin(5x)$
32. $f(x) = -6\sqrt{3}\cos(3x) - 6\sin(3x) - 3$
33. $f(x) = \dfrac{5\sqrt{2}}{2}\sin(x) - \dfrac{5\sqrt{2}}{2}\cos(x)$
34. $f(x) = 3\sin\left(\dfrac{x}{6}\right) - 3\sqrt{3}\cos\left(\dfrac{x}{6}\right)$
35. you should have noticed a relationship between the phases ϕ for the $S(x)$ and $C(x)$. Show that if $f(x) = A\sin(\omega x + \alpha) + B$, then $f(x) = A\cos(\omega x + \beta) + B$ where $\beta = \alpha - \dfrac{\pi}{2}$.
36. Let ϕ be an angle measured in radians and let $P(a, b)$ be a point on the terminal side of ϕ when it is drawn in standard position. Use Theorem 52 and the sum identity for sine in Theorem 65 to show that $f(x) = a\sin(\omega x) + b\cos(\omega x) + B$ (with $\omega > 0$) can be rewritten as $f(x) = \sqrt{a^2 + b^2}\sin(\omega x + \phi) + B$.
37. With the help of your classmates, express the domains of the functions in Examples 161 and 162 using extended interval notation.

In Exercises 38 – 43, verify the identity by graphing the right and left hand sides on a calculator.

38. $\sin^2(x) + \cos^2(x) = 1$

39. $\sec^2(x) - \tan^2(x) = 1$

40. $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$

41. $\tan(x + \pi) = \tan(x)$

42. $\sin(2x) = 2\sin(x)\cos(x)$

43. $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 + \cos(x)}$

In Exercises 44 – 50, graph the function with the help of your computer or calculator and discuss the given questions with your classmates.

44. $f(x) = \cos(3x) + \sin(x)$. Is this function periodic? If so, what is the period?

45. $f(x) = \frac{\sin(x)}{x}$. What appears to be the horizontal asymptote of the graph?

46. $f(x) = x\sin(x)$. Graph $y = \pm x$ on the same set of axes and describe the behaviour of f.

47. $f(x) = \sin\left(\frac{1}{x}\right)$. What's happening as $x \to 0$?

48. $f(x) = x - \tan(x)$. Graph $y = x$ on the same set of axes and describe the behaviour of f.

49. $f(x) = e^{-0.1x}(\cos(2x) + \sin(2x))$. Graph $y = \pm e^{-0.1x}$ on the same set of axes and describe the behaviour of f.

50. $f(x) = e^{-0.1x}(\cos(2x) + 2\sin(x))$. Graph $y = \pm e^{-0.1x}$ on the same set of axes and describe the behaviour of f.

51. Show that a constant function f is periodic by showing that $f(x + 117) = f(x)$ for all real numbers x. Then show that f has no period by showing that you cannot find a *smallest* number p such that $f(x + p) = f(x)$ for all real numbers x. Said another way, show that $f(x + p) = f(x)$ for all real numbers x for ALL values of $p > 0$, so no smallest value exists to satisfy the definition of 'period'.

9: Further Topics in Trigonometry

9.1 Inverse Trigonometric Functions

As the title indicates, in this section we concern ourselves with finding inverses of the (circular) trigonometric functions. Our immediate problem is that, owing to their periodic nature, none of the six circular functions is one-to-one. To remedy this, we restrict the domains of the circular functions in the same way we restricted the domain of the quadratic function in Example 107 in Section 6.2 to obtain a one-to-one function. We first consider $f(x) = \cos(x)$. Choosing the interval $[0, \pi]$ allows us to keep the range as $[-1, 1]$ as well as the properties of being smooth and continuous.

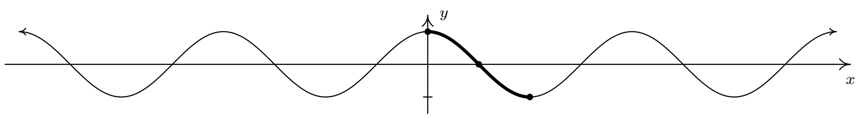


Figure 9.1: Restricting the domain of $f(x) = \cos(x)$ to $[0, \pi]$.

Recall from Section 6.2 that the inverse of a function f is typically denoted f^{-1}. For this reason, some textbooks use the notation $f^{-1}(x) = \cos^{-1}(x)$ for the inverse of $f(x) = \cos(x)$. The obvious pitfall here is our convention of writing $(\cos(x))^2$ as $\cos^2(x)$, $(\cos(x))^3$ as $\cos^3(x)$ and so on. It is far too easy to confuse $\cos^{-1}(x)$ with $\frac{1}{\cos(x)} = \sec(x)$ so we will not use this notation in our text. (But be aware that many books do! As always, be sure to check the context!) Instead, we use the notation $f^{-1}(x) = \arccos(x)$, read 'arc-cosine of x'. To understand the 'arc' in 'arccosine', recall that an inverse function, by definition, reverses the process of the original function. The function $f(t) = \cos(t)$ takes a real number input t, associates it with the angle $\theta = t$ radians, and returns the value $\cos(\theta)$. Digging deeper, we have that $\cos(\theta) = \cos(t)$ is the x-coordinate of the terminal point on the Unit Circle of an oriented arc of length $|t|$ whose initial point is $(1, 0)$. Hence, we may view the inputs to $f(t) = \cos(t)$ as oriented arcs and the outputs as x-coordinates on the Unit Circle. The function f^{-1}, then, would take x-coordinates on the Unit Circle and return oriented arcs, hence the 'arc' in arccosine. Figure 9.3 shows the graphs of $f(x) = \cos(x)$ and $f^{-1}(x) = \arccos(x)$, where we obtain the latter from the former by reflecting it across the line $y = x$, in accordance with Theorem 38.

We restrict $g(x) = \sin(x)$ in a similar manner, although the interval of choice is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

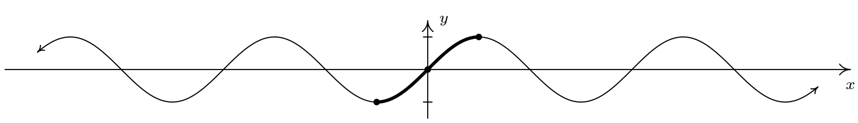


Figure 9.2: Restricting the domain of $f(x) = \sin(x)$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

It should be no surprise that we call $g^{-1}(x) = \arcsin(x)$, which is read 'arcsine of x'.

We list some important facts about the arccosine and arcsine functions in the following theorem.

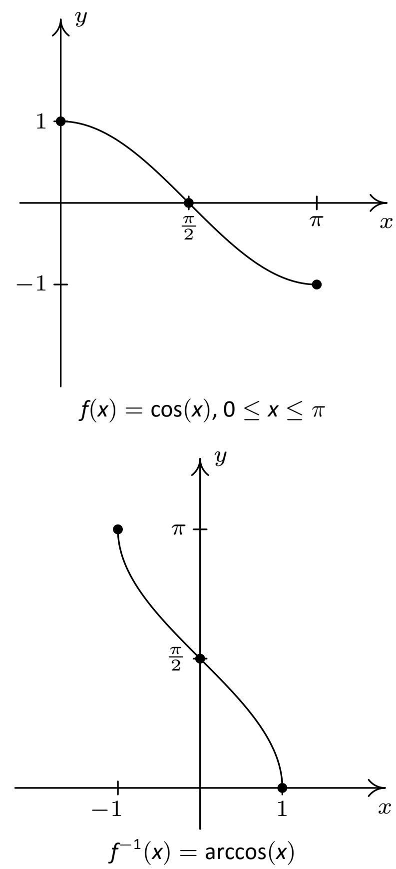
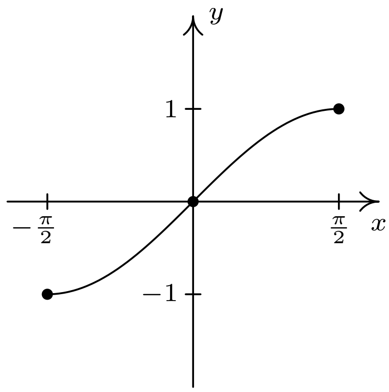
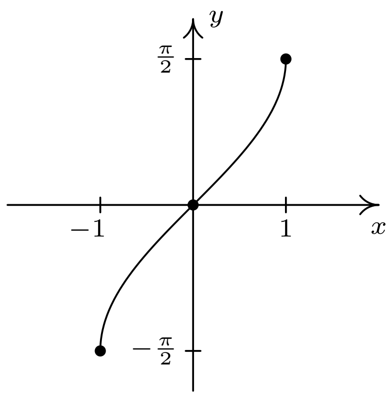


Figure 9.3: Reflecting $y = \cos(x)$ across $y = x$ yields $y = \arccos(x)$



$g(x) = \sin(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$g^{-1}(x) = \arcsin(x)$

Figure 9.4: Reflecting $y = \sin(x)$ across $y = x$ yields $y = \arcsin(x)$

Theorem 76 Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $0 \le t \le \pi$ and $\cos(t) = x$
  - $\cos(\arccos(x)) = x$ provided $-1 \le x \le 1$
  - $\arccos(\cos(x)) = x$ provided $0 \le x \le \pi$
- Properties of $G(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and $\sin(t) = x$
  - $\sin(\arcsin(x)) = x$ provided $-1 \le x \le 1$
  - $\arcsin(\sin(x)) = x$ provided $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
  - additionally, arcsine is odd

Everything in Theorem 76 is a direct consequence of the facts that $f(x) = \cos(x)$ for $0 \le x \le \pi$ and $F(x) = \arccos(x)$ are inverses of each other as are $g(x) = \sin(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $G(x) = \arcsin(x)$. It's about time for an example.

Example 163 Evaluating the arcsine and arccosine functions

1. Find the exact values of the following.

   (a) $\arccos\left(\frac{1}{2}\right)$

   (b) $\arcsin\left(\frac{\sqrt{2}}{2}\right)$

   (c) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

   (d) $\arcsin\left(-\frac{1}{2}\right)$

   (e) $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right)$

   (f) $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right)$

   (g) $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$

   (h) $\sin\left(\arccos\left(-\frac{3}{5}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan\left(\arccos(x)\right)$

   (b) $\cos\left(2\arcsin(x)\right)$

Solution

1. (a) To find $\arccos\left(\frac{1}{2}\right)$, we need to find the real number t (or, equivalently, an angle measuring t radians) which lies between 0 and π with $\cos(t) = \frac{1}{2}$. We know $t = \frac{\pi}{3}$ meets these criteria, so $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) The value of $\arcsin\left(\frac{\sqrt{2}}{2}\right)$ is a real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\sin(t) = \frac{\sqrt{2}}{2}$. The number we seek is $t = \frac{\pi}{4}$. Hence, $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

(c) The number $t = \arccos\left(-\frac{\sqrt{2}}{2}\right)$ lies in the interval $[0, \pi]$ with $\cos(t) = -\frac{\sqrt{2}}{2}$. Our answer is $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(d) To find $\arcsin\left(-\frac{1}{2}\right)$, we seek the number t in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = -\frac{1}{2}$. The answer is $t = -\frac{\pi}{6}$ so that $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

(e) Since $0 \leq \frac{\pi}{6} \leq \pi$, one option would be to simply invoke Theorem 76 to get $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$. However, in order to make sure we understand *why* this is the case, we choose to work the example through using the definition of arccosine. Working from the inside out, $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. Now, $\arccos\left(\frac{\sqrt{3}}{2}\right)$ is the real number t with $0 \leq t \leq \pi$ and $\cos(t) = \frac{\sqrt{3}}{2}$. We find $t = \frac{\pi}{6}$, so that $\arccos\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$.

(f) Since $\frac{11\pi}{6}$ does not fall between 0 and π, Theorem 76 does not apply. We are forced to work through from the inside out starting with $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \arccos\left(\frac{\sqrt{3}}{2}\right)$. From the previous problem, we know $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$. Hence, $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$.

(g) One way to simplify $\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$ is to use Theorem 76 directly. Since $-\frac{3}{5}$ is between -1 and 1, we have that $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = -\frac{3}{5}$ and we are done. However, as before, to really understand *why* this cancellation occurs, we let $t = \arccos\left(-\frac{3}{5}\right)$. Then, by definition, $\cos(t) = -\frac{3}{5}$. Hence, $\cos\left(\arccos\left(-\frac{3}{5}\right)\right) = \cos(t) = -\frac{3}{5}$, and we are finished in (nearly) the same amount of time.

(h) As in the previous example, we let $t = \arccos\left(-\frac{3}{5}\right)$ so that $\cos(t) = -\frac{3}{5}$ for some t where $0 \leq t \leq \pi$. Since $\cos(t) < 0$, we can narrow this down a bit and conclude that $\frac{\pi}{2} < t < \pi$, so that t corresponds to an angle in Quadrant II. In terms of t, then, we need to find $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \sin(t)$. Using the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, we get $\left(-\frac{3}{5}\right)^2 + \sin^2(t) = 1$ or $\sin(t) = \pm\frac{4}{5}$. Since t corresponds to a Quadrants II angle, we choose $\sin(t) = \frac{4}{5}$. Hence, $\sin\left(\arccos\left(-\frac{3}{5}\right)\right) = \frac{4}{5}$.

2. (a) We begin this problem in the same manner we began the previous two problems. To help us see the forest for the trees, we let $t = \arccos(x)$, so our goal is to find a way to express $\tan(\arccos(x)) = \tan(t)$ in terms of x. Since $t = \arccos(x)$, we know $\cos(t) = x$ where $0 \leq t \leq \pi$, but since we are after an expression for $\tan(t)$, we know we need to throw out $t = \frac{\pi}{2}$ from consideration. Hence, either $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$ so that, geometrically, t corresponds to an angle in Quadrant I or Quadrant II. One approach to finding $\tan(t)$ is to use the quotient identity $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Substituting $\cos(t) = x$ into the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$ gives $x^2 + \sin^2(t) = 1$, from which we get $\sin(t) = \pm\sqrt{1 - x^2}$. Since t corresponds to angles in Quadrants I and II, $\sin(t) \geq 0$, so we choose $\sin(t) = \sqrt{1 - x^2}$. Thus,

$$\tan(t) = \frac{\sin(t)}{\cos(t)} = \frac{\sqrt{1 - x^2}}{x}$$

To determine the values of x for which this equivalence is valid, we consider our substitution $t = \arccos(x)$. Since the domain of $\arccos(x)$

An alternative approach to finding $\tan(t)$ is to use the identity $1 + \tan^2(t) = \sec^2(t)$. Since $x = \cos(t)$, $\sec(t) = \frac{1}{\cos(t)} = \frac{1}{x}$. The reader is invited to work through this approach to see what, if any, difficulties arise.

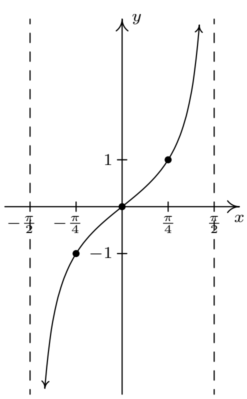
is $[-1,1]$, we know we must restrict $-1 \leq x \leq 1$. Additionally, since we had to discard $t = \frac{\pi}{2}$, we need to discard $x = \cos\left(\frac{\pi}{2}\right) = 0$. Hence, $\tan(\arccos(x)) = \frac{\sqrt{1-x^2}}{x}$ is valid for x in $[-1,0) \cup (0,1]$.

(b) We proceed as in the previous problem by writing $t = \arcsin(x)$ so that t lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin(t) = x$. We aim to express $\cos(2\arcsin(x)) = \cos(2t)$ in terms of x. Since $\cos(2t)$ is defined everywhere, we get no additional restrictions on t as we did in the previous problem. We have three choices for rewriting $\cos(2t)$: $\cos^2(t) - \sin^2(t)$, $2\cos^2(t) - 1$ and $1 - 2\sin^2(t)$. Since we know $x = \sin(t)$, it is easiest to use the last form:

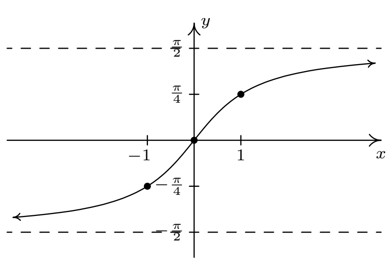
$$\cos(2\arcsin(x)) = \cos(2t) = 1 - 2\sin^2(t) = 1 - 2x^2$$

To find the restrictions on x, we once again appeal to our substitution $t = \arcsin(x)$. Since $\arcsin(x)$ is defined only for $-1 \leq x \leq 1$, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid only on $[-1,1]$.

A few remarks about Example 163 are in order. Most of the common errors encountered in dealing with the inverse circular functions come from the need to restrict the domains of the original functions so that they are one-to-one. One instance of this phenomenon is the fact that $\arccos\left(\cos\left(\frac{11\pi}{6}\right)\right) = \frac{\pi}{6}$ as opposed to $\frac{11\pi}{6}$. This is the exact same phenomenon discussed in Section 6.2 when we saw $\sqrt{(-2)^2} = 2$ as opposed to -2. Additionally, even though the expression we arrived at in part 2b above, namely $1 - 2x^2$, is defined for all real numbers, the equivalence $\cos(2\arcsin(x)) = 1 - 2x^2$ is valid for only $-1 \leq x \leq 1$. This is akin to the fact that while the expression x is defined for all real numbers, the equivalence $\left(\sqrt{x}\right)^2 = x$ is valid only for $x \geq 0$. For this reason, it pays to be careful when we determine the intervals where such equivalences are valid.



$f(x) = \tan(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$



$f^{-1}(x) = \arctan(x)$

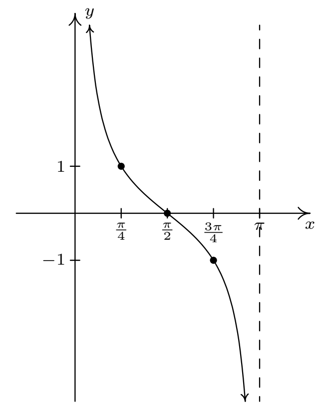
Figure 9.5: Reflecting $y = \tan(x)$ across $y = x$ yields $y = \arctan(x)$

The next pair of functions we wish to discuss are the inverses of tangent and cotangent, which are named arctangent and arccotangent, respectively. First, we restrict $f(x) = \tan(x)$ to its fundamental cycle on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to obtain $f^{-1}(x) = \arctan(x)$. Among other things, note that the *vertical* asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ of the graph of $f(x) = \tan(x)$ become the *horizontal* asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ of the graph of $f^{-1}(x) = \arctan(x)$: see Figure 9.5.

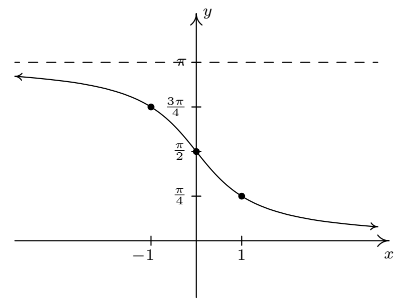
Next, we restrict $g(x) = \cot(x)$ to its fundamental cycle on $(0,\pi)$ to obtain $g^{-1}(x) = \operatorname{arccot}(x)$. Once again, the vertical asymptotes $x = 0$ and $x = \pi$ of the graph of $g(x) = \cot(x)$ become the horizontal asymptotes $y = 0$ and $y = \pi$ of the graph of $g^{-1}(x) = \operatorname{arccot}(x)$. We show these graphs in Figure 9.6; the basic properties of the arctangent and arccotangent functions are given in the following theorem.

Theorem 77 Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$
  - $\arctan(x) = \text{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(x)) = x$ provided $-\frac{\pi}{2} < x < \frac{\pi}{2}$
  - additionally, arctangent is odd
- Properties of $G(x) = \text{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty$, $\text{arccot}(x) \to \pi^{-}$; as $x \to \infty$, $\text{arccot}(x) \to 0^{+}$
  - $\text{arccot}(x) = t$ if and only if $0 < t < \pi$ and $\cot(t) = x$
  - $\text{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\text{arccot}(x)) = x$ for all real numbers x
  - $\text{arccot}(\cot(x)) = x$ provided $0 < x < \pi$



$g(x) = \cot(x)$, $0 < x < \pi$



$gf^{-1}(x) = \text{arccot}(x)$

Figure 9.6: Reflecting $y = \cot(x)$ across $y = x$ yields $y = \text{arccot}(x)$

Example 164 Evaluating the arctangent and arccotangent functions

1. Find the exact values of the following.

   (a) $\arctan(\sqrt{3})$
   (b) $\text{arccot}(-\sqrt{3})$
   (c) $\cot(\text{arccot}(-5))$
   (d) $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(2\arctan(x))$
   (b) $\cos(\text{arccot}(2x))$

Solution

1. (a) We know $\arctan(\sqrt{3})$ is the real number t between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with $\tan(t) = \sqrt{3}$. We find $t = \frac{\pi}{3}$, so $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

   (b) The real number $t = \text{arccot}(-\sqrt{3})$ lies in the interval $(0, \pi)$ with $\cot(t) = -\sqrt{3}$. We get $\text{arccot}(-\sqrt{3}) = \frac{5\pi}{6}$.

(c) We can apply Theorem 77 directly and obtain $\cot(\text{arccot}(-5)) = -5$. However, working it through provides us with yet another opportunity to understand why this is the case. Letting $t = \text{arccot}(-5)$, we have that t belongs to the interval $(0, \pi)$ and $\cot(t) = -5$. Hence, $\cot(\text{arccot}(-5)) = \cot(t) = -5$.

(d) We start simplifying $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$ by letting $t = \arctan\left(-\frac{3}{4}\right)$. Then $\tan(t) = -\frac{3}{4}$ for some $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know, in fact, $-\frac{\pi}{2} < t < 0$. One way to proceed is to use The Pythagorean Identity, $1 + \cot^2(t) = \csc^2(t)$, since this relates the reciprocals of $\tan(t)$ and $\sin(t)$ and is valid for all t under consideration. From $\tan(t) = -\frac{3}{4}$, we get $\cot(t) = -\frac{4}{3}$. Substituting, we get $1 + \left(-\frac{4}{3}\right)^2 = \csc^2(t)$ so that $\csc(t) = \pm\frac{5}{3}$. Since $-\frac{\pi}{2} < t < 0$, we choose $\csc(t) = -\frac{5}{3}$, so $\sin(t) = -\frac{3}{5}$. Hence, $\sin\left(\arctan\left(-\frac{3}{4}\right)\right) = -\frac{3}{5}$.

2. (a) If we let $t = \arctan(x)$, then $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\tan(t) = x$. We look for a way to express $\tan(2\arctan(x)) = \tan(2t)$ in terms of x. Before we get started using identities, we note that $\tan(2t)$ is undefined when $2t = \frac{\pi}{2} + \pi k$ for integers k. Dividing both sides of this equation by 2 tells us we need to exclude values of t where $t = \frac{\pi}{4} + \frac{\pi}{2}k$, where k is an integer. The only members of this family which lie in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are $t = \pm\frac{\pi}{4}$, which means the values of t under consideration are $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Returning to $\arctan(2t)$, we note the double angle identity $\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)}$, is valid for all the values of t under consideration, hence we get

$$\tan(2\arctan(x)) = \tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)} = \frac{2x}{1-x^2}$$

To find where this equivalence is valid we check back with our substitution $t = \arctan(x)$. Since the domain of $\arctan(x)$ is all real numbers, the only exclusions come from the values of t we discarded earlier, $t = \pm\frac{\pi}{4}$. Since $x = \tan(t)$, this means we exclude $x = \tan\left(\pm\frac{\pi}{4}\right) = \pm 1$. Hence, the equivalence $\tan(2\arctan(x)) = \frac{2x}{1-x^2}$ holds for all x in $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

It's always a good idea to make sure the identities used in these situations are valid for all values t under consideration. Check our work back in Example 163. Were the identities we used there valid for all t under consideration? A pedantic point, to be sure, but what else do you expect from this book?

(b) To get started, we let $t = \text{arccot}(2x)$ so that $\cot(t) = 2x$ where $0 < t < \pi$. In terms of t, $\cos(\text{arccot}(2x)) = \cos(t)$, and our goal is to express the latter in terms of x. Since $\cos(t)$ is always defined, there are no additional restrictions on t, so we can begin using identities to relate $\cot(t)$ to $\cos(t)$. The identity $\cot(t) = \frac{\cos(t)}{\sin(t)}$ is valid for t in $(0, \pi)$, so our strategy is to obtain $\sin(t)$ in terms of x, then write $\cos(t) = \cot(t)\sin(t)$. The identity $1 + \cot^2(t) = \csc^2(t)$ holds for all t in $(0, \pi)$ and relates $\cot(t)$ and $\csc(t) = \frac{1}{\sin(t)}$. Substituting $\cot(t) = 2x$, we get $1 + (2x)^2 = \csc^2(t)$, or $\csc(t) = \pm\sqrt{4x^2+1}$. Since t is between 0 and π, $\csc(t) > 0$, so $\csc(t) = \sqrt{4x^2+1}$ which gives $\sin(t) = \frac{1}{\sqrt{4x^2+1}}$. Hence,

$$\cos(\text{arccot}(2x)) = \cos(t) = \cot(t)\sin(t) = \frac{2x}{\sqrt{4x^2+1}}$$

Since $\text{arccot}(2x)$ is defined for all real numbers x and we encountered no additional restrictions on t, we have $\cos(\text{arccot}(2x)) = \frac{2x}{\sqrt{4x^2+1}}$ for all real numbers x.

The last two functions to invert are secant and cosecant. A portion of each of their graphs, which were first discussed in Subsection 8.5.2, are given in Figure 9.7 below with the fundamental cycles highlighted.

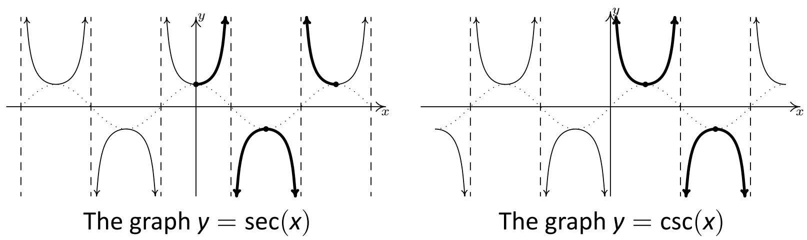


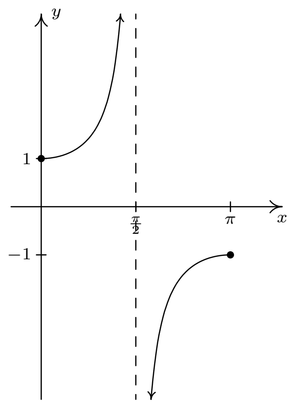
Figure 9.7: The fundamental cycles of $f(x) = \sec(x)$ and $g(x) = \csc(x)$

It is clear from the graph of secant that we cannot find one single continuous piece of its graph which covers its entire range of $(-\infty, -1] \cup [1, \infty)$ and restricts the domain of the function so that it is one-to-one. The same is true for cosecant. Thus in order to define the arcsecant and arccosecant functions, we must settle for a piecewise approach wherein we choose one piece to cover the top of the range, namely $[1, \infty)$, and another piece to cover the bottom, namely $(-\infty, -1]$. There are two generally accepted ways make these choices which restrict the domains of these functions so that they are one-to-one. One approach simplifies the Trigonometry associated with the inverse functions, but complicates the Calculus; the other makes the Calculus easier, but the Trigonometry less so. We present both points of view.

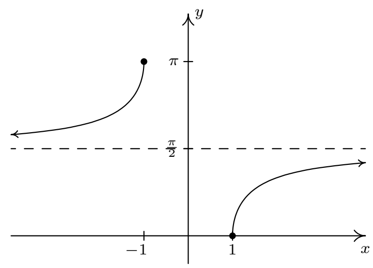
9.1.1 Inverses of Secant and Cosecant: Trigonometry Friendly Approach

In this subsection, we restrict the secant and cosecant functions to coincide with the restrictions on cosine and sine, respectively. For $f(x) = \sec(x)$, we restrict the domain to $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ (Figure 9.8) and we restrict $g(x) = \csc(x)$ to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ (Figure 9.9.

Note that for both arcsecant and arccosecant, the domain is $(-\infty, -1] \cup [1, \infty)$. Taking a page from Section 3.2, we can rewrite this as $\{x : |x| \geq 1\}$. This is often done in Calculus textbooks, so we include it here for completeness. Using these definitions, we get the following properties of the arcsecant and arccosecant functions.

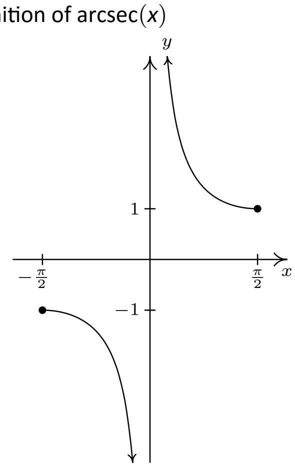


$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

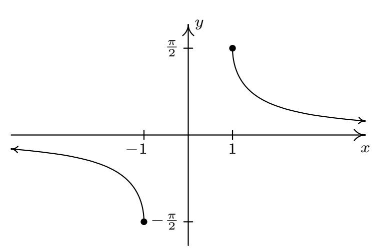


$f^{-1}(x) = \text{arcsec}(x)$

Figure 9.8: The "Trigonometry Friendly" definition of $\text{arcsec}(x)$



$g(x) = \csc(x)$ on $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



$g^{-1}(x) = \text{arccsc}(x)$

Figure 9.9: The "Trigonometry Friendly" definition of $\text{arccsc}(x)$

Theorem 78 Properties of the Arcsecant and Arccosecant Functions ("Trigonometry Friendly" version)

- Properties of $F(x) = \text{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
  - as $x \to -\infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^+$; as $x \to \infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^-$
  - $\text{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\frac{\pi}{2} < t \leq \pi$ and $\sec(t) = x$
  - $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\sec(\text{arcsec}(x)) = x$ provided $|x| \geq 1$
  - $\text{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x \leq \pi$
- Properties of $G(x) = \text{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
  - as $x \to -\infty$, $\text{arccsc}(x) \to 0^-$; as $x \to \infty$, $\text{arccsc}(x) \to 0^+$
  - $\text{arccsc}(x) = t$ if and only if $-\frac{\pi}{2} \leq t < 0$ or $0 < t \leq \frac{\pi}{2}$ and $\csc(t) = x$
  - $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ provided $|x| \geq 1$
  - $\csc(\text{arccsc}(x)) = x$ provided $|x| \geq 1$
  - $\text{arccsc}(\csc(x)) = x$ provided $-\frac{\pi}{2} \leq x < 0$ or $0 < x \leq \frac{\pi}{2}$
  - additionally, arccosecant is odd

Example 165 Evaluating the arcsecant and arccosecant functions

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$
   (b) $\text{arccsc}(-2)$
   (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
   (d) $\cot(\text{arccsc}(-3))$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

   (a) $\tan(\text{arcsec}(x))$
   (b) $\cos(\text{arccsc}(4x))$

Solution

1. (a) Using Theorem 78, we have $\text{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.
   (b) Once again, Theorem 78 gives us $\text{arccsc}(-2) = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.
   (c) Since $\frac{5\pi}{4}$ doesn't fall between 0 and $\frac{\pi}{2}$ or $\frac{\pi}{2}$ and π, we cannot use the inverse property stated in Theorem 78. We can, nevertheless, begin by working 'inside out' which yields $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \text{arcsec}(-\sqrt{2}) = \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$.

(d) One way to begin to simplify $\cot(\operatorname{arccsc}(-3))$ is to let $t = \operatorname{arccsc}(-3)$. Then, $\csc(t) = -3$ and, since this is negative, we have that t lies in the interval $\left[-\frac{\pi}{2}, 0\right)$. We are after $\cot(\operatorname{arccsc}(-3)) = \cot(t)$, so we use the Pythagorean Identity $1 + \cot^2(t) = \csc^2(t)$. Substituting, we have $1 + \cot^2(t) = (-3)^2$, or $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since $-\frac{\pi}{2} \leq t < 0$, $\cot(t) < 0$, so we get $\cot(\operatorname{arccsc}(-3)) = -2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\operatorname{arcsec}(x))$ by letting $t = \operatorname{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all t values under consideration, we have no additional restrictions on t. To relate $\sec(t)$ to $\tan(t)$, we use the identity $1 + \tan^2(t) = \sec^2(t)$. This is valid for all values of t under consideration, and when we substitute $\sec(t) = x$, we get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm\sqrt{x^2 - 1}$. If t belongs to $\left[0, \frac{\pi}{2}\right)$ then $\tan(t) \geq 0$; if, on the the other hand, t belongs to $\left(\frac{\pi}{2}, \pi\right]$ then $\tan(t) \leq 0$. As a result, we get a piecewise defined function for $\tan(t)$

$$\tan(t) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } 0 \leq t < \frac{\pi}{2} \\ -\sqrt{x^2 - 1}, & \text{if } \frac{\pi}{2} < t \leq \pi \end{cases}$$

Now we need to determine what these conditions on t mean for x. Since $x = \sec(t)$, when $0 \leq t < \frac{\pi}{2}$, $x \geq 1$, and when $\frac{\pi}{2} < t \leq \pi$, $x \leq -1$. Since we encountered no further restrictions on t, the equivalence below holds for all x in $(-\infty, -1] \cup [1, \infty)$.

$$\tan(\operatorname{arcsec}(x)) = \begin{cases} \sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -\sqrt{x^2 - 1}, & \text{if } x \leq -1 \end{cases}$$

(b) To simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $t = \operatorname{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, and we now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(t)$. Since $\cos(t)$ is defined for all t, we do not encounter any additional restrictions on t. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use if the identity $\cos^2(t) + \sin^2(t) = 1$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$. Solving, we get

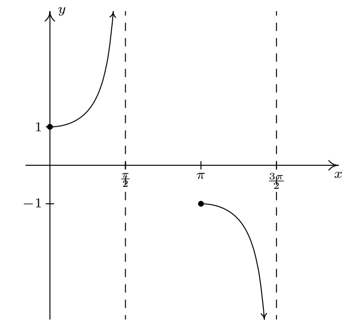
$$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}$$

Since t belongs to $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we know $\cos(t) \geq 0$, so we choose $\cos(t) = \frac{\sqrt{16 - x^2}}{4|x|}$. (The absolute values here are necessary, since x could be negative.) To find the values for which this equivalence is valid, we look back at our original substution, $t = \operatorname{arccsc}(4x)$. Since the domain of $\operatorname{arccsc}(x)$ requires its argument x to satisfy $|x| \geq 1$, the domain of $\operatorname{arccsc}(4x)$ requires $|4x| \geq 1$. Using Theorem 18, we rewrite this inequality and solve to get $x \leq -\frac{1}{4}$ or $x \geq \frac{1}{4}$. Since we had no additional restrictions on t, the equivalence $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2 - 1}}{4|x|}$ holds for all x in $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$.

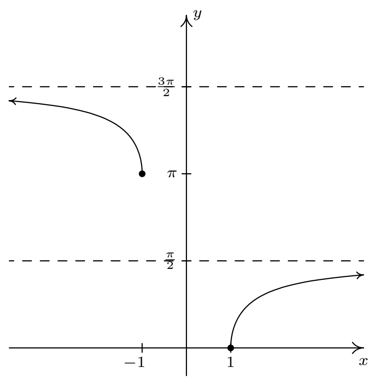
9.1.2 Inverses of Secant and Cosecant: Calculus Friendly Approach

In this subsection, we restrict $f(x) = \sec(x)$ to $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, and we restrict $g(x) = \csc(x)$ to $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

Using these definitions, we get the following result.

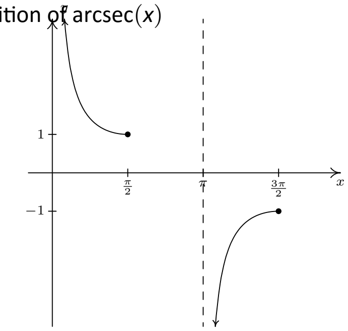


$f(x) = \sec(x)$ on $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

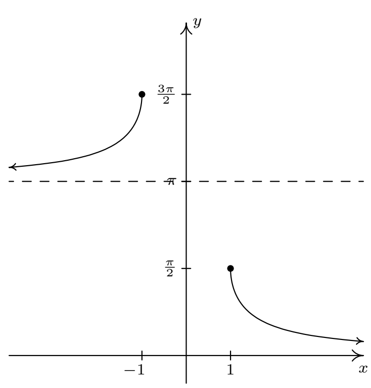


$f^{-1}(x) = \text{arcsec}(x)$

Figure 9.10: The "Calculus Friendly" definition of $\text{arcsec}(x)$



$g(x) = \csc(x)$ on $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$



$g^{-1}(x) = \text{arccsc}(x)$

Figure 9.11: The "Calculus Friendly definition of $\text{arccsc}(x)$

Theorem 79 Properties of the Arcsecant and Arccosecant Functions ("Calculus Friendly" version)

- Properties of $F(x) = \text{arcsec}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
  - as $x \to -\infty$, $\text{arcsec}(x) \to \frac{3\pi}{2}^-$; as $x \to \infty$, $\text{arcsec}(x) \to \frac{\pi}{2}^-$
  - $\text{arcsec}(x) = t$ if and only if $0 \leq t < \frac{\pi}{2}$ or $\pi \leq t < \frac{3\pi}{2}$ and $\sec(t) = x$
  - $\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$ for $x \geq 1$ only (Compare this with the similar result in Theorem 78.)
  - $\sec\left(\text{arcsec}(x)\right) = x$ provided $|x| \geq 1$
  - $\text{arcsec}(\sec(x)) = x$ provided $0 \leq x < \frac{\pi}{2}$ or $\pi \leq x < \frac{3\pi}{2}$
- Properties of $G(x) = \text{arccsc}(x)$
  - Domain: $\{x : |x| \geq 1\} = (-\infty, -1] \cup [1, \infty)$
  - Range: $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$
  - as $x \to -\infty$, $\text{arccsc}(x) \to \pi^+$; as $x \to \infty$, $\text{arccsc}(x) \to 0^+$
  - $\text{arccsc}(x) = t$ if and only if $0 < t \leq \frac{\pi}{2}$ or $\pi < t \leq \frac{3\pi}{2}$ and $\csc(t) = x$
  - $\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$ for $x \geq 1$ only (Compare this with the similar result in Theorem 78.)
  - $\csc\left(\text{arccsc}(x)\right) = x$ provided $|x| \geq 1$
  - $\text{arccsc}(\csc(x)) = x$ provided $0 < x \leq \frac{\pi}{2}$ or $\pi < x \leq \frac{3\pi}{2}$

Our next example is a duplicate of Example 165. The interested reader is invited to compare and contrast the solution to each.

Example 166 Evaluating the arcsecant and arccosecant functions

1. Find the exact values of the following.

   (a) $\text{arcsec}(2)$
   (b) $\text{arccsc}(-2)$
   (c) $\text{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
   (d) $\cot\left(\text{arccsc}\left(-3\right)\right)$

2. Rewrite the following as algebraic expressions of x and state the domain on which the equivalence is valid.

(a) $\tan(\operatorname{arcsec}(x))$ (b) $\cos(\operatorname{arccsc}(4x))$

Solution

1. (a) Since $2 \geq 1$, we can use Theorem 79 to get $\operatorname{arcsec}(2) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

   (b) Unfortunately, -2 is not greater to or equal to 1, so we cannot apply Theorem 79 to $\operatorname{arccsc}(-2)$ and convert this into an arcsine problem. Instead, we appeal to the definition. The real number $t = \operatorname{arccsc}(-2)$ lies in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ and satisfies $\csc(t) = -2$. The t we're after is $t = \frac{7\pi}{6}$, so $\operatorname{arccsc}(-2) = \frac{7\pi}{6}$.

   (c) Since $\frac{5\pi}{4}$ lies between π and $\frac{3\pi}{2}$, we may apply Theorem 79 directly to simplify $\operatorname{arcsec}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$. We encourage the reader to work this through using the definition as we have done in the previous examples to see how it goes.

   (d) To help simplify $\cot\left(\operatorname{arccsc}\left(-3\right)\right)$ we define $t = \operatorname{arccsc}\left(-3\right)$ so that $\cot\left(\operatorname{arccsc}\left(-3\right)\right) = \cot(t)$. We know $\csc(t) = -3$, and since this is negative, t lies in $\left(\pi, \frac{3\pi}{2}\right]$. Using the identity $1 + \cot^2(t) = \csc^2(t)$, we find $1 + \cot^2(t) = (-3)^2$ so that $\cot(t) = \pm\sqrt{8} = \pm 2\sqrt{2}$. Since t is in the interval $\left(\pi, \frac{3\pi}{2}\right]$, we know $\cot(t) > 0$. Our answer is $\cot\left(\operatorname{arccsc}\left(-3\right)\right) = 2\sqrt{2}$.

2. (a) We begin simplifying $\tan(\operatorname{arcsec}(x))$ by letting $t = \operatorname{arcsec}(x)$. Then, $\sec(t) = x$ for t in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, and we seek a formula for $\tan(t)$. Since $\tan(t)$ is defined for all t values under consideration, we have no additional restrictions on t. To relate $\sec(t)$ to $\tan(t)$, we use the identity $1 + \tan^2(t) = \sec^2(t)$. This is valid for all values of t under consideration, and when we substitute $\sec(t) = x$, we get $1 + \tan^2(t) = x^2$. Hence, $\tan(t) = \pm\sqrt{x^2 - 1}$. Since t lies in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$, $\tan(t) \geq 0$, so we choose $\tan(t) = \sqrt{x^2 - 1}$. Since we found no additional restrictions on t, the equivalence $\tan(\operatorname{arcsec}(x)) = \sqrt{x^2 - 1}$ holds for all x in the domain of $t = \operatorname{arcsec}(x)$, namely $(-\infty, -1] \cup [1, \infty)$.

   (b) To simplify $\cos(\operatorname{arccsc}(4x))$, we start by letting $t = \operatorname{arccsc}(4x)$. Then $\csc(t) = 4x$ for t in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, and we now set about finding an expression for $\cos(\operatorname{arccsc}(4x)) = \cos(t)$. Since $\cos(t)$ is defined for all t, we do not encounter any additional restrictions on t. From $\csc(t) = 4x$, we get $\sin(t) = \frac{1}{4x}$, so to find $\cos(t)$, we can make use if the identity $\cos^2(t) + \sin^2(t) = 1$. Substituting $\sin(t) = \frac{1}{4x}$ gives $\cos^2(t) + \left(\frac{1}{4x}\right)^2 = 1$. Solving, we get

   $$\cos(t) = \pm\sqrt{\frac{16x^2 - 1}{16x^2}} = \pm\frac{\sqrt{16x^2 - 1}}{4|x|}$$

   If t lies in $\left(0, \frac{\pi}{2}\right]$, then $\cos(t) \geq 0$, and we choose $\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|}$. Otherwise, t belongs to $\left(\pi, \frac{3\pi}{2}\right]$ in which case $\cos(t) \leq 0$, so, we choose $\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|}$ This leads us to a (momentarily) piecewise defined function for $\cos(t)$

$$\cos(t) = \begin{cases} \dfrac{\sqrt{16x^2-1}}{4|x|}, & \text{if } 0 \leq t \leq \frac{\pi}{2} \\ -\dfrac{\sqrt{16x^2-1}}{4|x|}, & \text{if } \pi < t \leq \frac{3\pi}{2} \end{cases}$$

We now see what these restrictions mean in terms of x. Since $4x = \csc(t)$, we get that for $0 \leq t \leq \frac{\pi}{2}$, $4x \geq 1$, or $x \geq \frac{1}{4}$. In this case, we can simplify $|x| = x$ so

$$\cos(t) = \frac{\sqrt{16x^2-1}}{4|x|} = \frac{\sqrt{16x^2-1}}{4x}$$

Similarly, for $\pi < t \leq \frac{3\pi}{2}$, we get $4x \leq -1$, or $x \leq -\frac{1}{4}$. In this case, $|x| = -x$, so we also get

$$\cos(t) = -\frac{\sqrt{16x^2-1}}{4|x|} = -\frac{\sqrt{16x^2-1}}{4(-x)} = \frac{\sqrt{16x^2-1}}{4x}$$

Hence, in all cases, $\cos(\operatorname{arccsc}(4x)) = \frac{\sqrt{16x^2-1}}{4x}$, and this equivalence is valid for all x in the domain of $t = \operatorname{arccsc}(4x)$, namely $\left(-\infty, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \infty\right)$

9.1.3 Calculators and the Inverse Circular Functions.

In the sections to come, we will have need to approximate the values of the inverse circular functions. On most calculators, only the arcsine, arccosine and arctangent functions are available and they are usually labelled as $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, respectively. If we are asked to approximate these values, it is a simple matter to punch up the appropriate decimal on the calculator. If we are asked for an arccotangent, arcsecant or arccosecant, however, we often need to employ some ingenuity, as our next example illustrates.

Example 167 Inverse trig functions not on the calculator

1. Use a calculator to approximate the following values to four decimal places.

   (a) $\operatorname{arccot}(2)$

   (b) $\operatorname{arcsec}(5)$

   (c) $\operatorname{arccot}(-2)$

   (d) $\operatorname{arccsc}\left(-\frac{3}{2}\right)$

2. Find the domain and range of the following functions. Check your answers using a calculator or computer.

   (a) $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$

   (b) $f(x) = 3\arctan(4x)$.

   (c) $f(x) = \operatorname{arccot}\left(\frac{x}{2}\right) + \pi$

Solution

1. (a) Since $2 > 0$, we can use the property listed in Theorem 77 to rewrite $\text{arccot}(2)$ as $\text{arccot}(2) = \arctan\left(\frac{1}{2}\right)$. In 'radian' mode, we find $\text{arccot}(2) = \arctan\left(\frac{1}{2}\right) \approx 0.4636$.

   (b) Since $5 \geq 1$, we can use the property from either Theorem 78 or Theorem 79 to write $\text{arcsec}(5) = \arccos\left(\frac{1}{5}\right) \approx 1.3694$.

   (c) Since the argument -2 is negative, we cannot directly apply Theorem 77 to help us find $\text{arccot}(-2)$. Let $t = \text{arccot}(-2)$. Then t is a real number such that $0 < t < \pi$ and $\cot(t) = -2$. Moreover, since $\cot(t) < 0$, we know $\frac{\pi}{2} < t < \pi$. Geometrically, this means t corresponds to a Quadrant II angle $\theta = t$ radians. This allows us to proceed using a 'reference angle' approach. Consider α, the reference angle for θ, as pictured in Figure 9.12. By definition, α is an acute angle so $0 < \alpha < \frac{\pi}{2}$, and the Reference Angle Theorem, Theorem 51, tells us that $\cot(\alpha) = 2$. This means $\alpha = \text{arccot}(2)$ radians. Since the argument of arccotangent is now a *positive* 2, we can use Theorem 77 to get $\alpha = \text{arccot}(2) = \arctan\left(\frac{1}{2}\right)$ radians. Since $\theta = \pi - \alpha = \pi - \arctan\left(\frac{1}{2}\right) \approx 2.6779$ radians, we get $\text{arccot}(-2) \approx 2.6779$.

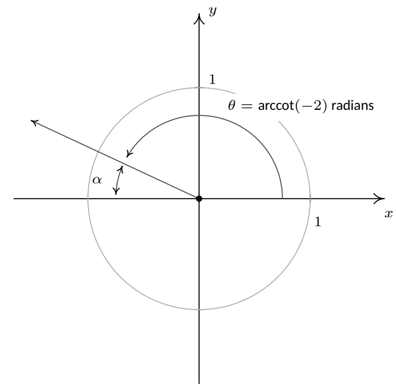


Figure 9.12: Evaluating arccot(−2)

   Another way to attack the problem is to use $\arctan\left(-\frac{1}{2}\right)$. By definition, the real number $t = \arctan\left(-\frac{1}{2}\right)$ satisfies $\tan(t) = -\frac{1}{2}$ with $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Since $\tan(t) < 0$, we know more specifically that $-\frac{\pi}{2} < t < 0$, so t corresponds to an angle β in Quadrant IV. To find the value of $\text{arccot}(-2)$, we once again visualize the angle $\theta = \text{arccot}(-2)$ radians and note that it is a Quadrant II angle with $\tan(\theta) = -\frac{1}{2}$. (See Figure 9.13.) This means it is exactly π units away from β, and we get $\theta = \pi + \beta = \pi + \arctan\left(-\frac{1}{2}\right) \approx 2.6779$ radians. Hence, as before, $\text{arccot}(-2) \approx 2.6779$.

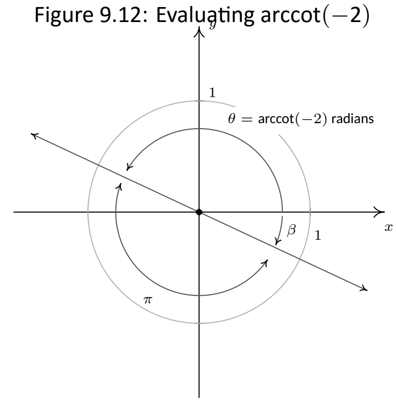


Figure 9.13: Evaluating arccot(−2)

   (d) If the range of arccosecant is taken to be $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, we can use Theorem 78 to get $\text{arccsc}\left(-\frac{3}{2}\right) = \arcsin\left(-\frac{2}{3}\right) \approx -0.7297$. If, on the other hand, the range of arccosecant is taken to be $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$, then we proceed as in the previous problem by letting $t = \text{arccsc}\left(-\frac{3}{2}\right)$. Then t is a real number with $\csc(t) = -\frac{3}{2}$. Since $\csc(t) < 0$, we have that $\pi < \theta \leq \frac{3\pi}{2}$, so t corresponds to a Quadrant III angle, θ, as pictured in Figure 9.14. As above, we let α be the reference angle for θ. Then $0 < \alpha < \frac{\pi}{2}$ and $\csc(\alpha) = \frac{3}{2}$, which means $\alpha = \text{arccsc}\left(\frac{3}{2}\right)$ radians. Since the argument of arccosecant is now positive, we may use Theorem 79 to get $\alpha = \text{arccsc}\left(\frac{3}{2}\right) = \arcsin\left(\frac{2}{3}\right)$ radians. Since $\theta = \pi + \alpha = \pi + \arcsin\left(\frac{2}{3}\right) \approx 3.8713$ radians, $\text{arccsc}\left(-\frac{3}{2}\right) \approx 3.8713$.

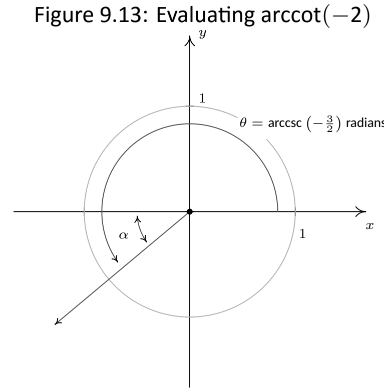


Figure 9.14: Evaluating arccsc $\left(-\frac{3}{2}\right)$

2. (a) Since the domain of $F(x) = \arccos(x)$ is $-1 \leq x \leq 1$, we can find the domain of $f(x) = \frac{\pi}{2} - \arccos\left(\frac{x}{5}\right)$ by setting the argument of the arccosine, in this case $\frac{x}{5}$, between -1 and 1. Solving $-1 \leq \frac{x}{5} \leq 1$ gives $-5 \leq x \leq 5$, so the domain is $[-5, 5]$. To determine the range of f, we take a cue from Section 2.6. Three 'key' points on the graph of $F(x) = \arccos(x)$ are $(-1, \pi)$, $\left(0, \frac{\pi}{2}\right)$ and $(1, 0)$. Following the procedure outlined in Theorem 12, we track these points to $\left(-5, -\frac{\pi}{2}\right)$, $(0, 0)$ and $\left(5, \frac{\pi}{2}\right)$. Plotting these values tells us that the range of f is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. (It also confirms our domain!) The graph in Figure 9.15 confirms our results.

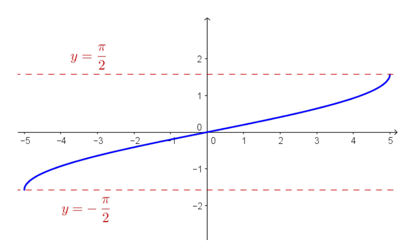


Figure 9.15: $y = f(x) = \dfrac{\pi}{2} - \arccos\left(\dfrac{x}{5}\right)$

   (b) To find the domain and range of $f(x) = 3\arctan(4x)$, we note that since the domain of $F(x) = \arctan(x)$ is all real numbers, the only restrictions, if any, on the domain of $f(x) = 3\arctan(4x)$ come from the argument of the arctangent, in this case, $4x$. Since $4x$ is defined for all real numbers, we have established that the domain of f is all real numbers. To determine the range of f, we can, once again, appeal to Theorem 12. Choosing our 'key' point to be $(0,0)$ and tracking the horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$, we find that the graph of $y = f(x) = 3\arctan(4x)$ differs from the graph of $y = F(x) = \arctan(x)$ by a horizontal compression by a factor of 4 and a vertical stretch by a factor of 3. It is the latter which affects the range, producing a range of $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$. We confirm our findings using GeoGebra in Figure 9.16.

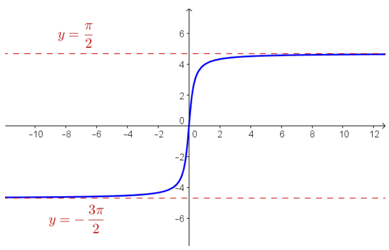


Figure 9.16: $y = f(x) = 3\arctan(4x)$

(c) To find the domain of $g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$, we proceed as above. Since the domain of $G(x) = \text{arccot}(x)$ is $(-\infty, \infty)$, and $\frac{x}{2}$ is defined for all x, we get that the domain of g is $(-\infty, \infty)$ as well. As for the range, we note that the range of $G(x) = \text{arccot}(x)$, like that of $F(x) = \arctan(x)$, is limited by a pair of horizontal asymptotes, in this case $y = 0$ and $y = \pi$. Following Theorem 12, we graph $y = g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$ starting with $y = G(x) = \text{arccot}(x)$ and first performing a horizontal expansion by a factor of 2 and following that with a vertical shift upwards by π. This latter transformation is the one which affects the range, making it now $(\pi, 2\pi)$. To check this graphically, we encounter a bit of a problem, since on many calculators, there is no shortcut button corresponding to the arccotangent function. Taking a cue from number 1c, we attempt to rewrite $g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi$ in terms of the arctangent function. Using Theorem 77, we have that $\text{arccot}\left(\frac{x}{2}\right) = \arctan\left(\frac{2}{x}\right)$ when $\frac{x}{2} > 0$, or, in this case, when $x > 0$. Hence, for $x > 0$, we have $g(x) = \arctan\left(\frac{2}{x}\right) + \pi$. When $\frac{x}{2} < 0$, we can use the same argument in number 1c that gave us $\text{arccot}(-2) = \pi + \arctan\left(-\frac{1}{2}\right)$ to give us $\text{arccot}\left(\frac{x}{2}\right) = \pi + \arctan\left(\frac{2}{x}\right)$. Hence, for $x < 0$, $g(x) = \pi + \arctan\left(\frac{2}{x}\right) + \pi = \arctan\left(\frac{2}{x}\right) + 2\pi$. What about $x = 0$? We know $g(0) = \text{arccot}(0) + \pi = \pi$, and neither of the formulas for g involving arctangent will produce this result. Hence, in order to graph $y = g(x)$ on our computer or calculator, we need to write it as a piecewise defined function:

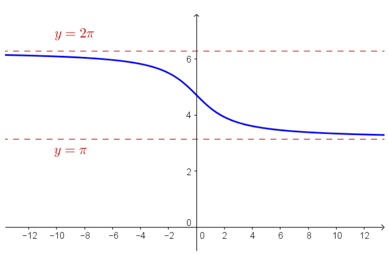


Figure 9.17: $y = g(x) = \text{arccot}\left(\frac{\pi}{2}\right) + \pi$

$$g(x) = \text{arccot}\left(\frac{x}{2}\right) + \pi = \begin{cases} \arctan\left(\frac{2}{x}\right) + 2\pi, & \text{if } x < 0 \\ \pi, & \text{if } x = 0 \\ \arctan\left(\frac{2}{x}\right) + \pi, & \text{if } x > 0 \end{cases}$$

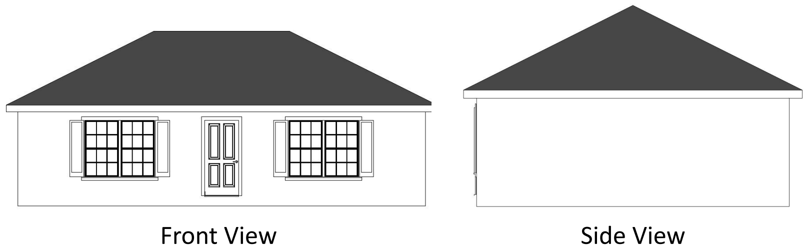
The result is shown in Figure 9.17.

Note: as with a graphing calculator, the GeoGebra software does not have an arccotangent function. To input a piecewise-defined function in GeoGebra, we use the syntax `Function[<function>, <start x value>, <end x value>.]`.

The inverse trigonometric functions are typically found in applications whenever the measure of an angle is required. One such scenario is presented in the following example. (The authors would like to thank Dan Stitz for this problem and associated graphics.)

Example 168 Angle of a pitched roof

The roof on the house below has a '6/12 pitch'. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Express your answer in decimal degrees, rounded to the nearest hundredth of a degree.



SOLUTION If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet. Using Theorem 60, we find the angle of inclination, labelled θ in Figure 9.18, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Since θ is an acute angle, we can use the arctangent function and we find $\theta = \arctan\left(\frac{1}{2}\right)$ radians $\approx 26.56°$.

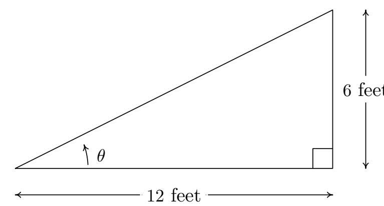


Figure 9.18: Angle of inclination θ for Example 168

9.1.4 Solving Equations Using the Inverse Trigonometric Functions.

In Sections 8.2 and 8.3, we learned how to solve equations like $\sin(\theta) = \frac{1}{2}$ for angles θ and $\tan(t) = -1$ for real numbers t. In each case, we ultimately appealed to the Unit Circle and relied on the fact that the answers corresponded to a set of 'common angles' listed on page 325. If, on the other hand, we had been asked to find all angles with $\sin(\theta) = \frac{1}{3}$ or solve $\tan(t) = -2$ for real numbers t, we would have been hard-pressed to do so. With the introduction of the inverse trigonometric functions, however, we are now in a position to solve these equations. A good parallel to keep in mind is how the square root function can be used to solve certain quadratic equations. The equation $x^2 = 4$ is a lot like $\sin(\theta) = \frac{1}{2}$ in that it has friendly, 'common value' answers $x = \pm 2$. The equation $x^2 = 7$, on the other hand, is a lot like $\sin(\theta) = \frac{1}{3}$. We know there are answers (how do we know this again?), but we can't express them using 'friendly' numbers. (This is all, of course, a matter of opinion. For the record, the authors find $\pm\sqrt{7}$ just as 'nice' as ± 2.) To solve $x^2 = 7$, we make use of the square root function and write $x = \pm\sqrt{7}$. We can certainly *approximate* these answers using a calculator, but as far as exact answers go, we leave them as $x = \pm\sqrt{7}$. In the same way, we will use the arcsine function to solve $\sin(\theta) = \frac{1}{3}$, as seen in the following example.

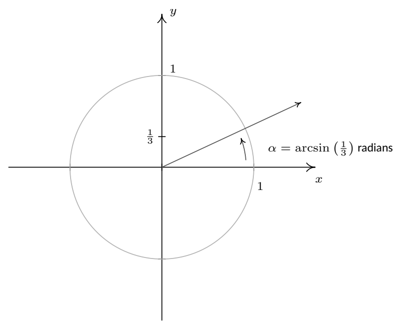
Example 169 Solving trigonometric equations

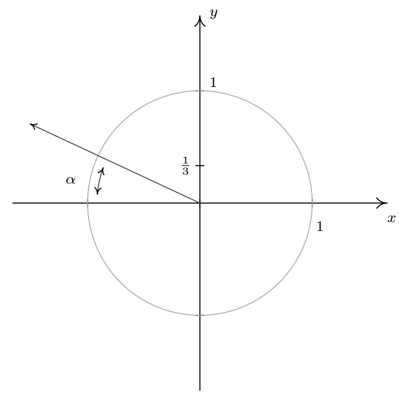
Solve the following equations.

1. Find all angles θ for which $\sin(\theta) = \frac{1}{3}$.
2. Find all real numbers t for which $\tan(t) = -2$
3. Solve $\sec(x) = -\frac{5}{3}$ for x.

SOLUTION

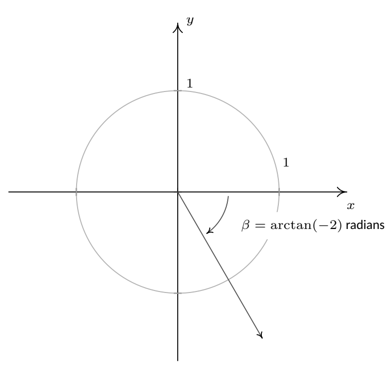
1. If $\sin(\theta) = \frac{1}{3}$, then the terminal side of θ, when plotted in standard position, intersects the Unit Circle at $y = \frac{1}{3}$. Geometrically, we see that this happens at two places: in Quadrant I and Quadrant II. If we let α denote the acute solution to the equation, then all the solutions to this equation in Quadrant I are coterminal with α, and α serves as the reference angle for all of the solutions to this equation in Quadrant II.

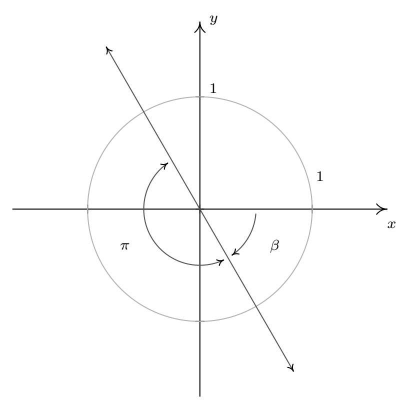




Since $\frac{1}{3}$ isn't the sine of any of the 'common angles' discussed earlier, we use the arcsine functions to express our answers. The real number $t = \arcsin\left(\frac{1}{3}\right)$ is defined so it satisfies $0 < t < \frac{\pi}{2}$ with $\sin(t) = \frac{1}{3}$. Hence, $\alpha = \arcsin\left(\frac{1}{3}\right)$ radians. Since the solutions in Quadrant I are all coterminal with α, we get part of our solution to be $\theta = \alpha + 2\pi k = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k. Turning our attention to Quadrant II, we get one solution to be $\pi - \alpha$. Hence, the Quadrant II solutions are $\theta = \pi - \alpha + 2\pi k = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$, for integers k.

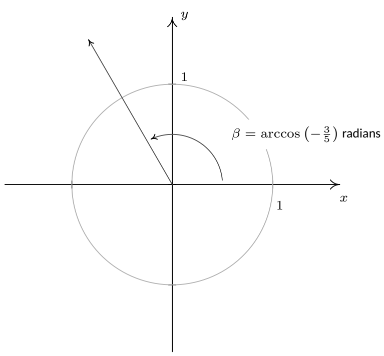
2. We may visualize the solutions to $\tan(t) = -2$ as angles θ with $\tan(\theta) = -2$. Since tangent is negative only in Quadrants II and IV, we focus our efforts there.

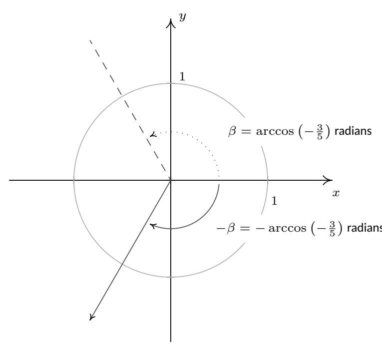




Since -2 isn't the tangent of any of the 'common angles', we need to use the arctangent function to express our answers. The real number $t = \arctan(-2)$ satisfies $\tan(t) = -2$ and $-\frac{\pi}{2} < t < 0$. If we let $\beta = \arctan(-2)$ radians, we see that all of the Quadrant IV solutions to $\tan(\theta) = -2$ are coterminal with β. Moreover, the solutions from Quadrant II differ by exactly π units from the solutions in Quadrant IV, so all the solutions to $\tan(\theta) = -2$ are of the form $\theta = \beta + \pi k = \arctan(-2) + \pi k$ for some integer k. Switching back to the variable t, we record our final answer to $\tan(t) = -2$ as $t = \arctan(-2) + \pi k$ for integers k.

3. The last equation we are asked to solve, $\sec(x) = -\frac{5}{3}$, poses two immediate problems. First, we are not told whether or not x represents an angle or a real number. We assume the latter, but note that we will use angles and the Unit Circle to solve the equation regardless. Second, as we have mentioned, there is no universally accepted range of the arcsecant function. For that reason, we adopt the advice given in Section 8.3 and convert this to the cosine problem $\cos(x) = -\frac{3}{5}$. Adopting an angle approach, we consider the equation $\cos(\theta) = -\frac{3}{5}$ and note that our solutions lie in Quadrants II and III. Since $-\frac{3}{5}$ isn't the cosine of any of the 'common angles', we'll need to express our solutions in terms of the arccosine function. The real number $t = \arccos\left(-\frac{3}{5}\right)$ is defined so that $\frac{\pi}{2} < t < \pi$ with $\cos(t) = -\frac{3}{5}$. If we let $\beta = \arccos\left(-\frac{3}{5}\right)$ radians, we see that β is a Quadrant II angle. To obtain a Quadrant III angle solution, we may simply use $-\beta = -\arccos\left(-\frac{3}{5}\right)$. Since all angle solutions are coterminal with β or $-\beta$, we get our solutions to $\cos(\theta) = -\frac{3}{5}$ to be $\theta = \beta + 2\pi k = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $\theta = -\beta + 2\pi k = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k. Switching back to the variable x, we record our final answer to $\sec(x) = -\frac{5}{3}$ as $x = \arccos\left(-\frac{3}{5}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{3}{5}\right) + 2\pi k$ for integers k.





The reader is encouraged to check the answers found in Example 169 - both analytically and with the calculator (see Section 9.1.3). With practice, the inverse trigonometric functions will become as familiar to you as the square root function. Speaking of practice …

Exercises 9.1

Problems

In Exercises 1 – 40, find the exact value.

1. $\arcsin\left(-1\right)$
2. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
4. $\arcsin\left(-\frac{1}{2}\right)$
5. $\arcsin\left(0\right)$
6. $\arcsin\left(\frac{1}{2}\right)$
7. $\arcsin\left(\frac{\sqrt{2}}{2}\right)$
8. $\arcsin\left(\frac{\sqrt{3}}{2}\right)$
9. $\arcsin\left(1\right)$
10. $\arccos\left(-1\right)$
11. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
12. $\arccos\left(-\frac{\sqrt{2}}{2}\right)$
13. $\arccos\left(-\frac{1}{2}\right)$
14. $\arccos\left(0\right)$
15. $\arccos\left(\frac{1}{2}\right)$
16. $\arccos\left(\frac{\sqrt{2}}{2}\right)$
17. $\arccos\left(\frac{\sqrt{3}}{2}\right)$
18. $\arccos\left(1\right)$
19. $\arctan\left(-\sqrt{3}\right)$
20. $\arctan\left(-1\right)$
21. $\arctan\left(-\frac{\sqrt{3}}{3}\right)$
22. $\arctan\left(0\right)$
23. $\arctan\left(\frac{\sqrt{3}}{3}\right)$
24. $\arctan\left(1\right)$
25. $\arctan\left(\sqrt{3}\right)$
26. $\text{arccot}\left(-\sqrt{3}\right)$
27. $\text{arccot}\left(-1\right)$
28. $\text{arccot}\left(-\frac{\sqrt{3}}{3}\right)$
29. $\text{arccot}\left(0\right)$
30. $\text{arccot}\left(\frac{\sqrt{3}}{3}\right)$
31. $\text{arccot}\left(1\right)$
32. $\text{arccot}\left(\sqrt{3}\right)$
33. $\text{arcsec}\left(2\right)$
34. $\text{arccsc}\left(2\right)$
35. $\text{arcsec}\left(\sqrt{2}\right)$
36. $\text{arccsc}\left(\sqrt{2}\right)$
37. $\text{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
38. $\text{arccsc}\left(\frac{2\sqrt{3}}{3}\right)$
39. $\text{arcsec}\left(1\right)$
40. $\text{arccsc}\left(1\right)$

In Exercises 41 – 48, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value.

41. $\text{arcsec}\left(-2\right)$
42. $\text{arcsec}\left(-\sqrt{2}\right)$
43. $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
44. $\text{arcsec}\left(-1\right)$

45. $\text{arccsc}(-2)$

46. $\text{arccsc}\left(-\sqrt{2}\right)$

47. $\text{arccsc}\left(-\dfrac{2\sqrt{3}}{3}\right)$

48. $\text{arccsc}(-1)$

In Exercises 49 – 56, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value.

49. $\text{arcsec}(-2)$

50. $\text{arcsec}\left(-\sqrt{2}\right)$

51. $\text{arcsec}\left(-\dfrac{2\sqrt{3}}{3}\right)$

52. $\text{arcsec}(-1)$

53. $\text{arccsc}(-2)$

54. $\text{arccsc}\left(-\sqrt{2}\right)$

55. $\text{arccsc}\left(-\dfrac{2\sqrt{3}}{3}\right)$

56. $\text{arccsc}(-1)$

In Exercises 57 – 86, find the exact value or state that it is undefined.

57. $\sin\left(\arcsin\left(\dfrac{1}{2}\right)\right)$

58. $\sin\left(\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)\right)$

59. $\sin\left(\arcsin\left(\dfrac{3}{5}\right)\right)$

60. $\sin(\arcsin(-0.42))$

61. $\sin\left(\arcsin\left(\dfrac{5}{4}\right)\right)$

62. $\cos\left(\arccos\left(\dfrac{\sqrt{2}}{2}\right)\right)$

63. $\cos\left(\arccos\left(-\dfrac{1}{2}\right)\right)$

64. $\cos\left(\arccos\left(\dfrac{5}{13}\right)\right)$

65. $\cos(\arccos(-0.998))$

66. $\cos(\arccos(\pi))$

67. $\tan(\arctan(-1))$

68. $\tan\left(\arctan\left(\sqrt{3}\right)\right)$

69. $\tan\left(\arctan\left(\dfrac{5}{12}\right)\right)$

70. $\tan(\arctan(0.965))$

71. $\tan(\arctan(3\pi))$

72. $\cot(\text{arccot}(1))$

73. $\cot\left(\text{arccot}\left(-\sqrt{3}\right)\right)$

74. $\cot\left(\text{arccot}\left(-\dfrac{7}{24}\right)\right)$

75. $\cot(\text{arccot}(-0.001))$

76. $\cot\left(\text{arccot}\left(\dfrac{17\pi}{4}\right)\right)$

77. $\sec(\text{arcsec}(2))$

78. $\sec(\text{arcsec}(-1))$

79. $\sec\left(\text{arcsec}\left(\dfrac{1}{2}\right)\right)$

80. $\sec(\text{arcsec}(0.75))$

81. $\sec(\text{arcsec}(117\pi))$

82. $\csc\left(\text{arccsc}\left(\sqrt{2}\right)\right)$

83. $\csc\left(\text{arccsc}\left(-\dfrac{2\sqrt{3}}{3}\right)\right)$

84. $\csc\left(\text{arccsc}\left(\dfrac{\sqrt{2}}{2}\right)\right)$

85. $\csc(\text{arccsc}(1.0001))$

86. $\csc\left(\text{arccsc}\left(\dfrac{\pi}{4}\right)\right)$

In Exercises 87 – 106, find the exact value or state that it is undefined.

87. $\arcsin\left(\sin\left(\dfrac{\pi}{6}\right)\right)$

88. $\arcsin\left(\sin\left(-\dfrac{\pi}{3}\right)\right)$

89. $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$

90. $\arcsin\left(\sin\left(\frac{11\pi}{6}\right)\right)$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right)$

92. $\arccos\left(\cos\left(\frac{\pi}{4}\right)\right)$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right)$

94. $\arccos\left(\cos\left(\frac{3\pi}{2}\right)\right)$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

96. $\arccos\left(\cos\left(\frac{5\pi}{4}\right)\right)$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right)$

98. $\arctan\left(\tan\left(-\frac{\pi}{4}\right)\right)$

99. $\arctan\left(\tan\left(\pi\right)\right)$

100. $\arctan\left(\tan\left(\frac{\pi}{2}\right)\right)$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right)$

102. $\text{arccot}\left(\cot\left(\frac{\pi}{3}\right)\right)$

103. $\text{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right)$

104. $\text{arccot}\left(\cot\left(\pi\right)\right)$

105. $\text{arccot}\left(\cot\left(\frac{\pi}{2}\right)\right)$

106. $\text{arccot}\left(\cot\left(\frac{2\pi}{3}\right)\right)$

In Exercises 107 – 118, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ and that the range of arccosecant is $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ when finding the exact value.

107. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

108. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

109. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

110. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

111. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

112. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

113. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

114. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

116. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

118. $\text{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 119 – 130, assume that the range of arcsecant is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and that the range of arccosecant is $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ when finding the exact value.

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right)$

120. $\text{arcsec}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right)$

122. $\text{arcsec}\left(\sec\left(-\frac{\pi}{2}\right)\right)$

123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right)$

124. $\text{arccsc}\left(\csc\left(\frac{\pi}{6}\right)\right)$

125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right)$

126. $\text{arccsc}\left(\csc\left(\frac{2\pi}{3}\right)\right)$

127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right)$

128. $\text{arccsc}\left(\csc\left(\frac{11\pi}{6}\right)\right)$

129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right)$

130. $\operatorname{arccsc}\left(\csc\left(\frac{9\pi}{8}\right)\right)$

In Exercises 131 – 154, find the exact value or state that it is undefined.

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right)$

132. $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$

133. $\sin\left(\arctan\left(-2\right)\right)$

134. $\sin\left(\operatorname{arccot}\left(\sqrt{5}\right)\right)$

135. $\sin\left(\operatorname{arccsc}\left(-3\right)\right)$

136. $\cos\left(\arcsin\left(-\frac{5}{13}\right)\right)$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right)$

138. $\cos\left(\operatorname{arccot}\left(3\right)\right)$

139. $\cos\left(\operatorname{arcsec}\left(5\right)\right)$

140. $\tan\left(\arcsin\left(-\frac{2\sqrt{5}}{5}\right)\right)$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right)$

142. $\tan\left(\operatorname{arcsec}\left(\frac{5}{3}\right)\right)$

143. $\tan\left(\operatorname{arccot}\left(12\right)\right)$

144. $\cot\left(\arcsin\left(\frac{12}{13}\right)\right)$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

146. $\cot\left(\operatorname{arccsc}\left(\sqrt{5}\right)\right)$

147. $\cot\left(\arctan\left(0.25\right)\right)$

148. $\sec\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right)$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right)$

150. $\sec\left(\arctan\left(10\right)\right)$

151. $\sec\left(\operatorname{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right)$

152. $\csc\left(\operatorname{arccot}\left(9\right)\right)$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right)$

154. $\csc\left(\arctan\left(-\frac{2}{3}\right)\right)$

In Exercises 155 – 164, find the exact value or state that it is undefined.

155. $\sin\left(\arcsin\left(\frac{5}{13}\right)+\frac{\pi}{4}\right)$

156. $\cos\left(\operatorname{arcsec}(3)+\arctan(2)\right)$

157. $\tan\left(\arctan(3)+\arccos\left(-\frac{3}{5}\right)\right)$

158. $\sin\left(2\arcsin\left(-\frac{4}{5}\right)\right)$

159. $\sin\left(2\operatorname{arccsc}\left(\frac{13}{5}\right)\right)$

160. $\sin\left(2\arctan\left(2\right)\right)$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right)$

162. $\cos\left(2\operatorname{arcsec}\left(\frac{25}{7}\right)\right)$

163. $\cos\left(2\operatorname{arccot}\left(-\sqrt{5}\right)\right)$

164. $\sin\left(\frac{\arctan(2)}{2}\right)$

In Exercises 165 – 184, rewrite the quantity as algebraic expressions of x and state the domain on which the equivalence is valid.

165. $\sin\left(\arccos\left(x\right)\right)$

166. $\cos\left(\arctan\left(x\right)\right)$

167. $\tan\left(\arcsin\left(x\right)\right)$

168. $\sec\left(\arctan\left(x\right)\right)$

169. $\csc\left(\arccos\left(x\right)\right)$

170. $\sin\left(2\arctan\left(x\right)\right)$

171. $\sin\left(2\arccos\left(x\right)\right)$

172. $\cos\left(2\arctan\left(x\right)\right)$

173. $\sin(\arccos(2x))$

174. $\sin\left(\arccos\left(\frac{x}{5}\right)\right)$

175. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right)$

176. $\cos(\arctan(3x))$

177. $\sin(2\arcsin(7x))$

178. $\sin\left(2\arcsin\left(\frac{x\sqrt{3}}{3}\right)\right)$

179. $\cos(2\arcsin(4x))$

180. $\sec(\arctan(2x))\tan(\arctan(2x))$

181. $\sin(\arcsin(x) + \arccos(x))$

182. $\cos(\arcsin(x) + \arctan(x))$

183. $\tan(2\arcsin(x))$

184. $\sin\left(\frac{1}{2}\arctan(x)\right)$

185. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\theta + \sin(2\theta)$ in terms of x.

186. If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\frac{1}{2}\theta - \frac{1}{2}\sin(2\theta)$ in terms of x.

187. If $\sec(\theta) = \frac{x}{4}$ for $0 < \theta < \frac{\pi}{2}$, find an expression for $4\tan(\theta) - 4\theta$ in terms of x.

In Exercises 188 – 207, solve the equation using the techniques discussed in Example 169 then approximate the solutions which lie in the interval $[0, 2\pi)$ to four decimal places.

188. $\sin(x) = \frac{7}{11}$

189. $\cos(x) = -\frac{2}{9}$

190. $\sin(x) = -0.569$

191. $\cos(x) = 0.117$

192. $\sin(x) = 0.008$

193. $\cos(x) = \frac{359}{360}$

194. $\tan(x) = 117$

195. $\cot(x) = -12$

196. $\sec(x) = \frac{3}{2}$

197. $\csc(x) = -\frac{90}{17}$

198. $\tan(x) = -\sqrt{10}$

199. $\sin(x) = \frac{3}{8}$

200. $\cos(x) = -\frac{7}{16}$

201. $\tan(x) = 0.03$

202. $\sin(x) = 0.3502$

203. $\sin(x) = -0.721$

204. $\cos(x) = 0.9824$

205. $\cos(x) = -0.5637$

206. $\cot(x) = \frac{1}{117}$

207. $\tan(x) = -0.6109$

In Exercises 208 – 210, find the two acute angles in the right triangle whose sides have the given lengths. Express your answers using degree measure rounded to two decimal places.

208. 3, 4 and 5

209. 5, 12 and 13

210. 336, 527 and 625

211. A guy wire 1000 feet long is attached to the top of a tower. When pulled taut it touches level ground 360 feet from the base of the tower. What angle does the wire make with the ground? Express your answer using degree measure rounded to one decimal place.

212. At Cliffs of Insanity Point, The Great Sasquatch Canyon is 7117 feet deep. From that point, a fire is seen at a location known to be 10 miles away from the base of the sheer canyon wall. What angle of depression is made by the line of sight from the canyon edge to the fire? Express your answer using degree measure rounded to one decimal place.

213. Shelving is being built at the Utility Muffin Research Library which is to be 14 inches deep. An 18-inch rod will be attached to the wall and the underside of the shelf at its edge away from the wall, forming a right triangle under the shelf to support it. What angle, to the nearest degree, will the rod make with the wall?

214. A parasailor is being pulled by a boat on Lake Ippizuti. The cable is 300 feet long and the parasailor is 100 feet above the surface of the water. What is the angle of elevation from the boat to the parasailor? Express your answer using degree measure rounded to one decimal place.

215. A tag-and-release program to study the Sasquatch population of the eponymous Sasquatch National Park is begun. From a 200 foot tall tower, a ranger spots a Sasquatch lumbering through the wilderness directly towards the tower. Let θ denote the angle of depression from the top of the tower to a point on the ground. If the range of the rifle with a tranquillizer dart is 300 feet, find the smallest value of θ for which the corresponding point on the ground is in range of the rifle. Round your answer to the nearest hundredth of a degree.

In Exercises 216 – 221, rewrite the given function as a sinusoid of the form $S(x) = A\sin(\omega x + \phi)$ using Exercises 35 and 36 in Section 8.5 for reference. Approximate the value of ϕ (which is in radians, of course) to four decimal places.

216. $f(x) = 5\sin(3x) + 12\cos(3x)$

217. $f(x) = 3\cos(2x) + 4\sin(2x)$

218. $f(x) = \cos(x) - 3\sin(x)$

219. $f(x) = 7\sin(10x) - 24\cos(10x)$

220. $f(x) = -\cos(x) - 2\sqrt{2}\sin(x)$

221. $f(x) = 2\sin(x) - \cos(x)$

In Exercises 222 – 233, find the domain of the given function. Write your answers in interval notation.

222. $f(x) = \arcsin(5x)$

223. $f(x) = \arccos\left(\dfrac{3x-1}{2}\right)$

224. $f(x) = \arcsin\left(2x^2\right)$

225. $f(x) = \arccos\left(\dfrac{1}{x^2-4}\right)$

226. $f(x) = \arctan(4x)$

227. $f(x) = \operatorname{arccot}\left(\dfrac{2x}{x^2-9}\right)$

228. $f(x) = \arctan(\ln(2x-1))$

229. $f(x) = \operatorname{arccot}(\sqrt{2x-1})$

230. $f(x) = \operatorname{arcsec}(12x)$

231. $f(x) = \operatorname{arccsc}(x+5)$

232. $f(x) = \operatorname{arcsec}\left(\dfrac{x^3}{8}\right)$

233. $f(x) = \operatorname{arccsc}\left(e^{2x}\right)$

234. Show that $\operatorname{arcsec}(x) = \arccos\left(\dfrac{1}{x}\right)$ for $|x| \geq 1$ as long as we use $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$ as the range of $f(x) = \operatorname{arcsec}(x)$.

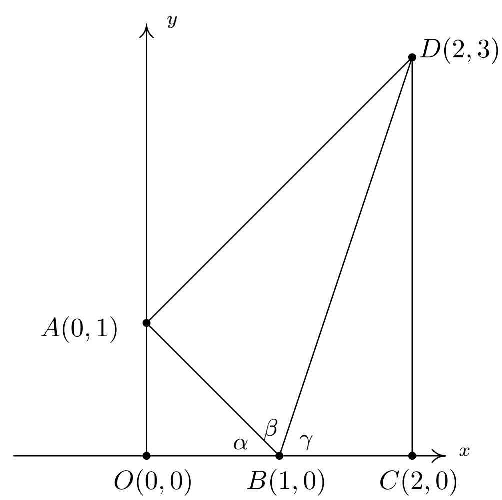
235. Show that $\operatorname{arccsc}(x) = \arcsin\left(\dfrac{1}{x}\right)$ for $|x| \geq 1$ as long as we use $\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$ as the range of $f(x) = \operatorname{arccsc}(x)$.

236. Show that $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$ for $-1 \leq x \leq 1$.

237. Discuss with your classmates why $\arcsin\left(\dfrac{1}{2}\right) \neq 30^\circ$.

238. Use the following picture and series of exercises to show that

$$\arctan(1) + \arctan(2) + \arctan(3) = \pi$$



(a) Clearly $\triangle AOB$ and $\triangle BCD$ are right triangles because the line through O and A and the line through C and D are perpendicular to the x-axis. Use the distance formula to show that $\triangle BAD$ is also a right triangle (with $\angle BAD$ being the right angle) by showing that the sides of the triangle satisfy the Pythagorean Theorem.

(b) Use $\triangle AOB$ to show that $\alpha = \arctan(1)$

(c) Use $\triangle BAD$ to show that $\beta = \arctan(2)$

(d) Use $\triangle BCD$ to show that $\gamma = \arctan(3)$

(e) Use the fact that O, B and C all lie on the x-axis to conclude that $\alpha + \beta + \gamma = \pi$. Thus $\arctan(1) + \arctan(2) + \arctan(3) = \pi$.

10: LIMITS

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17^{th} century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance = rate $\times$ time." But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

The foundation of "the calculus" is the *limit.* It is a tool to describe a particular behaviour of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

10.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \dfrac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value 1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 10.1, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 10.2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \rightarrow \frac{\sin 0}{0} \rightarrow \text{“}\,\frac{0}{0}\,\text{”}.$$

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives

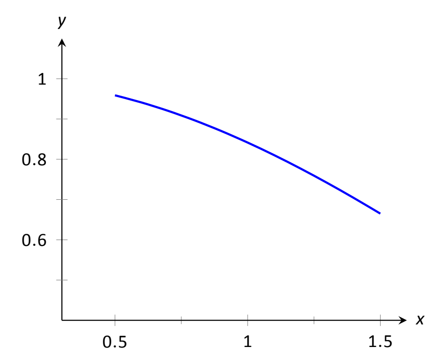


Figure 10.1: $\sin(x)/x$ near $x = 1$.

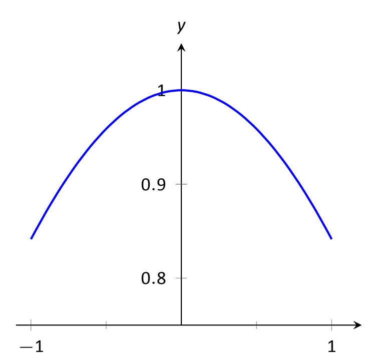


Figure 10.2: $\sin(x)/x$ near $x = 0$.

no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 10.3.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Figure 10.3: Values of $\sin(x)/x$ with x near 1.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 10.2. The table in Figure 10.4 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behaviour of the function *near* 0.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 10.4: Values of $\sin(x)/x$ with x near 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \sin(x)/x$; that is,

$$\lim_{x\to 0} \sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 170 Approximating the value of a limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

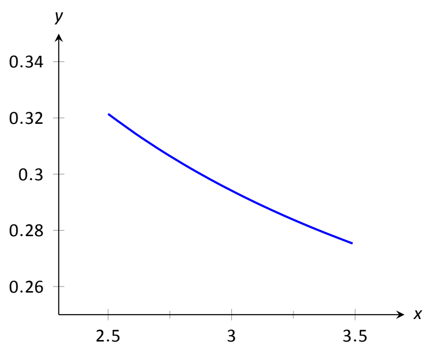


Figure 10.5: Graphically approximating a limit in Example 170.

Solution To graphically approximate the limit, graph

$$y = (x^2 - x - 6)/(6x^2 - 19x + 3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figures 10.5 and 10.6, respectively.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 10.6: Numerically approximating a limit in Example 170.

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?
2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Graphs are useful since they give a visual understanding concerning the behaviour of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 170, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

Example 171 Approximating the value of a limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x < 0 \\ -x^2+1 & x > 0 \end{cases}.$$

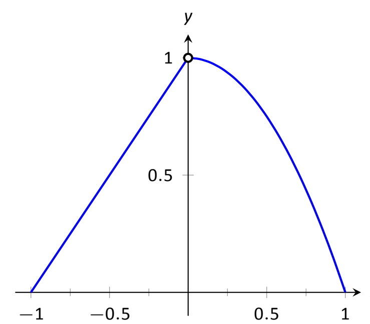


Figure 10.7: Graphically approximating a limit in Example 171.

Solution Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 10.7 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

The table shown in Figure 10.8 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 10.8: Numerically approximating a limit in Example 171.

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.

2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c.

We'll explore each of these in turn.

Example 172 Different Values Approached From Left and Right

Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1 \end{cases}.$$

Solution A graph of $f(x)$ around $x = 1$ and a table are given Figures 10.10 and 10.11, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behaviour is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one value as x approaches 1.

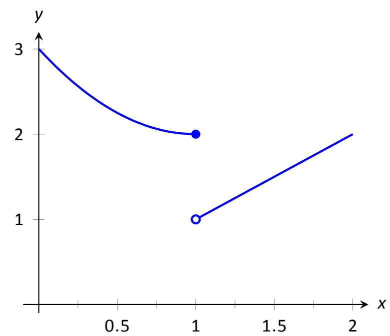


Figure 10.10: Observing no limit as $x \to 1$ in Example 172.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 10.11: Values of $f(x)$ near $x = 1$ in Example 172.

Example 173 The Function Grows Without Bound

Explore why $\lim_{x\to 1} 1/(x-1)^2$ does not exist.

Solution A graph and table of $f(x) = 1/(x-1)^2$ are given in Figures 10.12 and 10.13, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number}.$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist.

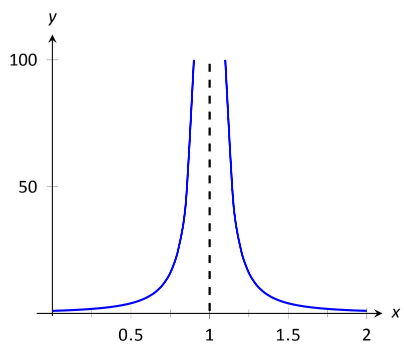


Figure 10.12: Observing no limit as $x \to 1$ in Example 173.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Figure 10.13: Values of $f(x)$ near $x = 1$ in Example 173.

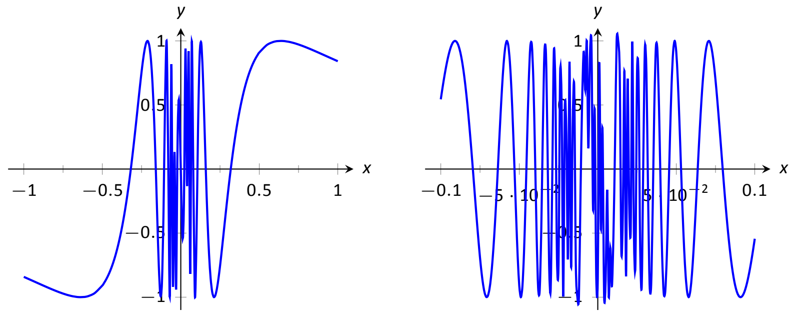
Example 174 The Function Oscillates

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

Solution Two graphs of $f(x) = \sin(1/x)$ are given in Figures 10.9. Figure 10.9(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 10.9(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 10.9(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation,

$\lim_{x\to 0} \sin(1/x)$ does not exist.



(a) (b)

x	$\sin(1/x)$
0.1	−0.544021
0.01	−0.506366
0.001	0.82688
0.0001	−0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	−0.349994
$1. \times 10^{-7}$	0.420548

(c)

Figure 10.9: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 174.

Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 10.14.

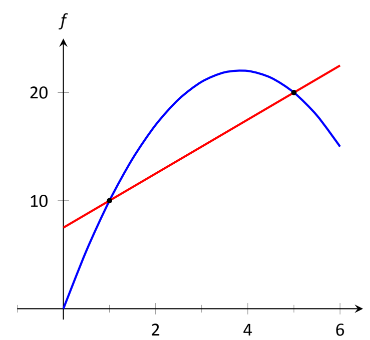


Figure 10.14: Interpreting a difference quotient as the slope of a secant line.

Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See

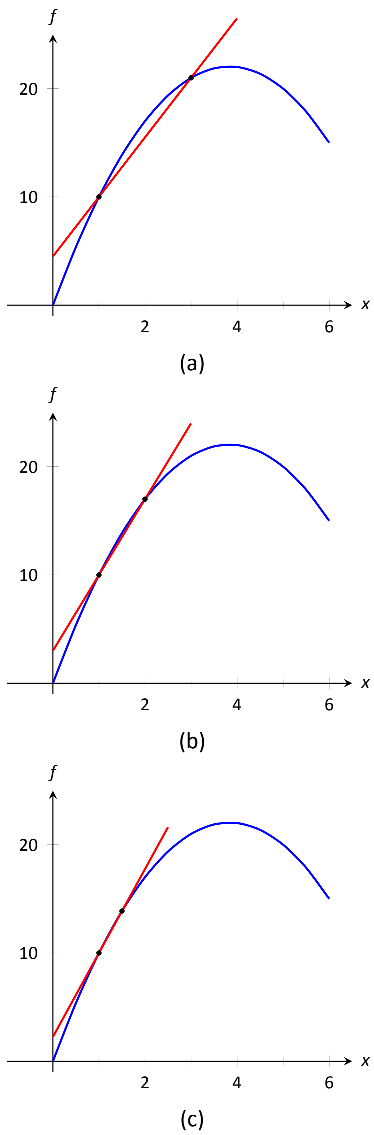


Figure 10.15: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 10.16: The difference quotient evaluated at values of h near 0.

Figure 10.15. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 10.16. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

Unfortunately, the precise definition of the limit, and most of the applications mentioned in the paragraph above, are beyond what we can cover in this course. Instead, we will settle for the following imprecise definition:

Definition 56 Informal Definition of the Limit

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. We say that the **limit of $f(x)$, as x approaches c, is L**, and write

$$\lim_{x \to c} f(x) = L,$$

if we can make the value of $f(x)$ arbitrarily close to L by choosing $x \neq c$ sufficiently close to c.

The formal definition of the limit, which we will not discuss, makes precise the meaning of the phrases "arbitrarily close" and "sufficiently close". The problem with the definition we have given is that, while it gives an intuitive understanding of the meaning of the limit, it's of no use for *proving* theorems about limits. In the next section we will state (but not prove) several theorems about limits which will allow use to compute their values analytically, without recourse to tables of values.

Exercises 10.1

Terms and Concepts

1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

2. An expression of the form $\frac{0}{0}$ is called ____.

3. T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.

4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

5. In your own words, what is a difference quotient?

Problems

In Exercises 6 – 16, approximate the given limits both numerically and graphically.

6. $\lim\limits_{x\to 1} x^2 + 3x - 5$

7. $\lim\limits_{x\to 0} x^3 - 3x^2 + x - 5$

8. $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

9. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

10. $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

11. $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

12. $\lim\limits_{x\to 2} f(x)$, where
$$f(x) = \begin{cases} x+2 & x \leq 2 \\ 3x-5 & x > 2 \end{cases}.$$

13. $\lim\limits_{x\to 3} f(x)$, where
$$f(x) = \begin{cases} x^2-x+1 & x \leq 3 \\ 2x+1 & x > 3 \end{cases}.$$

14. $\lim\limits_{x\to 0} f(x)$, where
$$f(x) = \begin{cases} \cos x & x \leq 0 \\ x^2+3x+1 & x > 0 \end{cases}.$$

15. $\lim\limits_{x\to \pi/2} f(x)$, where
$$f(x) = \begin{cases} \sin x & x \leq \pi/2 \\ \cos x & x > \pi/2 \end{cases}.$$

In Exercises 16 – 24, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

16. $f(x) = -7x + 2, \quad a = 3$

17. $f(x) = 9x + 0.06, \quad a = -1$

18. $f(x) = x^2 + 3x - 7, \quad a = 1$

19. $f(x) = \dfrac{1}{x+1}, \quad a = 2$

20. $f(x) = -4x^2 + 5x - 1, \quad a = -3$

21. $f(x) = \ln x, \quad a = 5$

22. $f(x) = \sin x, \quad a = \pi$

23. $f(x) = \cos x, \quad a = \pi$

10.2 Finding Limits Analytically

In Section 10.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. Proving that these approximations are correct requires a rigorous definition of limits, which is beyond the scope of this course.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x) + g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

The rigorous definition of limits is often known as the " $-\delta$ definition of the limit. You might have a few brief encounters with this definition as you make your way through the calculus sequence, but a careful treatment of limits is usually not encountered until Math 3500.

Theorem 80 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{x\to c} f(x)^n = L^n$
8. Roots: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. Compositions: Adjust our previously given limit situation to:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to L} g(x) = K.$$

Then $\lim_{x\to c} g(f(x)) = K$.

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 175 Using basic limit properties

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

SOLUTION

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2}\big(f(x)+g(x)\big) = 2+3 = 5$.

2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim_{x\to 2}\big(5f(x)+g(x)^2\big) = 5\cdot 2 + 3^2 = 19$.

3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7)\\ &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\ &= 3\cdot 2^2 - 5\cdot 2 + 7\\ &= 9\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 80. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem 81 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$

2. $\lim_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 176 Finding a limit of a rational function

Using Theorem 81, find

$$\lim_{x\to -1} \frac{3x^2 - 5x + 1}{x^4 - x^2 + 3}.$$

Solution Using Theorem 81, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2 - 5x + 1}{x^4 - x^2 + 3} &= \frac{3(-1)^2 - 5(-1) + 1}{(-1)^4 - (-1)^2 + 3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

Using approximations (or worse – the rigorous definition) to deal with limits such as

$$\lim_{x\to 2} x^2 = 4$$

can be annoying, since the result seems fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 176.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behaviour is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 82 Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim_{x\to c} \sin x = \sin c$
2. $\lim_{x\to c} \cos x = \cos c$
3. $\lim_{x\to c} \tan x = \tan c$
4. $\lim_{x\to c} \csc x = \csc c$
5. $\lim_{x\to c} \sec x = \sec c$
6. $\lim_{x\to c} \cot x = \cot c$
7. $\lim_{x\to c} a^x = a^c \ (a > 0)$
8. $\lim_{x\to c} \ln x = \ln c$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Example 177 Evaluating limits analytically

Evaluate the following limits.

1. $\lim_{x\to \pi} \cos x$
2. $\lim_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim_{x\to \pi/2} \cos x \sin x$
4. $\lim_{x\to 1} e^{\ln x}$
5. $\lim_{x\to 0} \frac{\sin x}{x}$

Solution

1. This is a straightforward application of Theorem 82. $\lim_{x\to\pi} \cos x = \cos \pi = -1$.
2. We can approach this in at least two ways. First, by directly applying Theorem 82, we have:
$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$
Using the Pythagorean Theorem, this last expression is 1; therefore
$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = 1.$$
We can also use the Pythagorean Theorem from the start.
$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$
using the Constant limit rule. Either way, we find the limit is 1.
3. Applying the Product limit rule of Theorem 80 and Theorem 82 gives
$$\lim_{x\to \pi/2} \cos x \sin x = \cos(\pi/2)\sin(\pi/2) = 0 \cdot 1 = 0.$$
4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\lim_{x\to 1} e^{\ln x} = \lim_{x\to 1} x = 1$.
We can also use the Composition limit rule of Theorem 80. Using Theorem 82, we have $\lim_{x\to 1} \ln x = \ln 1 = 0$. Applying the Composition rule,
$$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$
Both approaches are valid, giving the same result.

5. We encountered this limit in Section 10.1. Applying our theorems, we attempt to find the limit as

$$\lim_{x\to 0}\frac{\sin x}{x} \rightarrow \frac{\sin 0}{0} \rightarrow \text{“}\frac{0}{0}\text{”}.$$

This, of course, violates a condition of Theorem 80, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 83 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to "squeeze" the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the "evaluating the limit part" very simple.

We use the Squeeze Theorem in the following example to finally prove that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Example 178 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 10.17. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

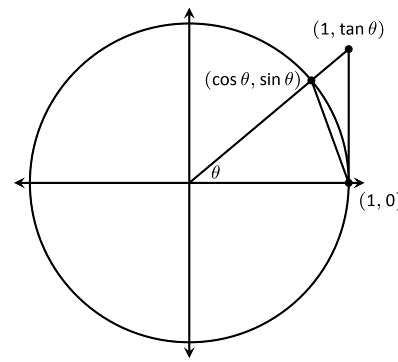
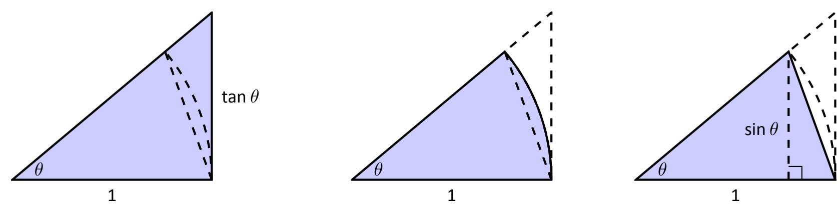


Figure 10.17: The unit circle and related triangles.

Figure 10.17 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0}\cos\theta \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$\cos 0 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 84 Special Limits

1. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$
4. $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equalling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Our final theorem for this section will be motivated by the following example.

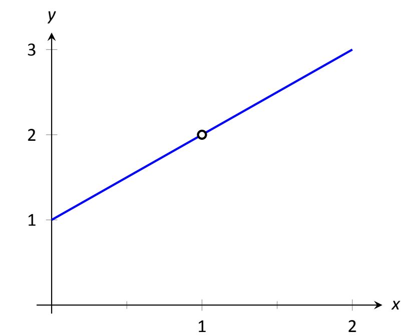


Figure 10.18: Graphing f in Example 179 to understand a limit.

Example 179 Using algebra to evaluate a limit

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution We begin by attempting to apply Theorem 82 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \text{“}\frac{0}{0}\text{”},$$

and indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 10.18, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1.$$

Clearly $\lim\limits_{x\to 1} x+1 = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

The key to the above example is that the functions $y = (x^2 - 1)/(x - 1)$ and $y = x + 1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 85 Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim_{x\to c} \frac{g(x)}{f(x)}$ returns "0/0", then $(x - c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 85. We demonstrate this once more.

Example 180 Evaluating a limit using Theorem 85

Evaluate $\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15}$.

Solution We begin by applying Theorem 82 and substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} = \frac{(x-3)(x^2 + x - 2)}{(x-3)(2x^2 + 9x - 5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 85 we conclude:

$$\begin{aligned}\lim_{x\to 3} \frac{x^3 - 2x^2 - 5x + 6}{2x^3 + 3x^2 - 32x + 15} &= \lim_{x\to 3} \frac{(x-3)(x^2 + x - 2)}{(x-3)(2x^2 + 9x - 5)}\\ &= \lim_{x\to 3} \frac{(x^2 + x - 2)}{(2x^2 + 9x - 5)}\\ &= \frac{10}{40} = \frac{1}{4}.\end{aligned}$$

We end this section by revisiting a limit first seen in Section 10.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Example 181 Evaluating the limit of a difference quotient

Let $f(x) = -1.5x^2 + 11.5x$; find $\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem 82 and substitute 0 for h. However, we see that this gives us

"0/0." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{-1.5(1+h)^2+11.5(1+h)-\big(-1.5(1)^2+11.5(1)\big)}{h}\\
&= \lim_{h\to 0}\frac{-1.5(1+2h+h^2)+11.5+11.5h-10}{h}\\
&= \lim_{h\to 0}\frac{-1.5h^2+8.5h}{h}\\
&= \lim_{h\to 0}\frac{h(-1.5h+8.5)}{h}\\
&= \lim_{h\to 0}(-1.5h+8.5) \quad \big(\text{using Theorem 85, as } h\neq 0\big)\\
&= 8.5 \quad \big(\text{using Theorem 82}\big)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 82; it states that many functions that we use regularly behave in a very nice, predictable way. In the next section we give a name to this nice behaviour; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Exercises 10.2

Terms and Concepts

1. Explain in your own words why $\lim\limits_{x\to c} b = b$.

2. Explain in your own words why $\lim\limits_{x\to c} x = c$.

3. What does the text mean when it says that certain functions' "behaviour is 'nice' in terms of limits"? What, in particular, is "nice"?

4. Sketch a graph that visually demonstrates the Squeeze Theorem.

5. You are given the following information:

   (a) $\lim\limits_{x\to 1} f(x) = 0$

   (b) $\lim\limits_{x\to 1} g(x) = 0$

   (c) $\lim\limits_{x\to 1} f(x)/g(x) = 2$

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

Problems

Using:

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3$$

evaluate the limits given in Exercises 6 – 13, where possible. If it is not possible to know, state so.

6. $\lim\limits_{x\to 9}(f(x) + g(x))$

7. $\lim\limits_{x\to 9}(3f(x)/g(x))$

8. $\lim\limits_{x\to 9}\left(\dfrac{f(x) - 2g(x)}{g(x)}\right)$

9. $\lim\limits_{x\to 6}\left(\dfrac{f(x)}{3 - g(x)}\right)$

10. $\lim\limits_{x\to 9} g\big(f(x)\big)$

11. $\lim\limits_{x\to 6} f\big(g(x)\big)$

12. $\lim\limits_{x\to 6} g\big(f(f(x))\big)$

13. $\lim\limits_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

Using:

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi$$

evaluate the limits given in Exercises 14 – 17, where possible. If it is not possible to know, state so.

14. $\lim\limits_{x\to 1} f(x)^{g(x)}$

15. $\lim\limits_{x\to 10} \cos\big(g(x)\big)$

16. $\lim\limits_{x\to 1} f(x)g(x)$

17. $\lim\limits_{x\to 1} g\big(5f(x)\big)$

In Exercises 18 – 32, evaluate the given limit.

18. $\lim\limits_{x\to 3} x^2 - 3x + 7$

19. $\lim\limits_{x\to \pi}\left(\dfrac{x-3}{x-5}\right)^7$

20. $\lim\limits_{x\to \pi/4} \cos x \sin x$

21. $\lim\limits_{x\to 0} \ln x$

22. $\lim\limits_{x\to 3} 4^{x^3 - 8x}$

23. $\lim\limits_{x\to \pi/6} \csc x$

24. $\lim\limits_{x\to 0} \ln(1 + x)$

25. $\lim\limits_{x\to \pi} \dfrac{x^2 + 3x + 5}{5x^2 - 2x - 3}$

26. $\lim\limits_{x\to \pi} \dfrac{3x + 1}{1 - x}$

27. $\lim\limits_{x\to 6} \dfrac{x^2 - 4x - 12}{x^2 - 13x + 42}$

28. $\lim\limits_{x\to 0} \dfrac{x^2 + 2x}{x^2 - 2x}$

29. $\lim\limits_{x\to 2} \dfrac{x^2 + 6x - 16}{x^2 - 3x + 2}$

30. $\lim\limits_{x\to 2} \dfrac{x^2 - 10x + 16}{x^2 - x - 2}$

31. $\lim\limits_{x\to -2} \dfrac{x^2 - 5x - 14}{x^2 + 10x + 16}$

32. $\lim\limits_{x\to -1} \dfrac{x^2 + 9x + 8}{x^2 - 6x - 7}$

Use the Squeeze Theorem in Exercises 33 – 36, where appropriate, to evaluate the given limit.

33. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

34. $\lim_{x\to 0} \sin x \cos\left(\frac{1}{x^2}\right)$

35. $\lim_{x\to 1} f(x)$, where $3x - 2 \leq f(x) \leq x^3$.

36. $\lim_{x\to 3^+} f(x)$, where $6x - 9 \leq f(x) \leq x^2$ on $[0, 3]$.

Exercises 37 – 40 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.

37. $\lim_{x\to 0} \frac{\sin 3x}{x}$

38. $\lim_{x\to 0} \frac{\sin 5x}{8x}$

39. $\lim_{x\to 0} \frac{\ln(1 + x)}{x}$

40. $\lim_{x\to 0} \frac{\sin x}{x}$, where x is measured in degrees, not radians.

10.3 One Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. The previous section gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 10.1 we explored the three ways in which limits of functions failed to exist:

1. The function approached different values from the left and right,
2. The function grows without bound, and
3. The function oscillates.

In this section we explore in depth the concepts behind #1 by introducing the *one-sided limit*. We begin with definitions that are very similar to the definition of the limit given at the end of Section 10.1, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

Definition 57 One Sided Limits

Left-Hand Limit
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. We say that **limit of $f(x)$, as x approaches c from the left, is L**, or, **the left–hand limit of f at c is L**, and write

$$\lim_{x\to c^-} f(x) = L,$$

if we can make the value of $f(x)$ arbitrarily close to L by choosing $x < c$ sufficiently close to c.

Right-Hand Limit
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. We say that the **limit of $f(x)$, as x approaches c from the right, is L**, or, **the right–hand limit of f at c is L**, and write

$$\lim_{x\to c^+} f(x) = L,$$

if we can make the value of $f(x)$ sufficiently close to L by choosing $x > c$ sufficiently close to c.

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

We practice evaluating left and right-hand limits through a series of examples.

Example 182 **Evaluating one sided limits**

Let $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3-x & 1 < x < 2 \end{cases}$, as shown in Figure 10.19. Find each of the following:

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

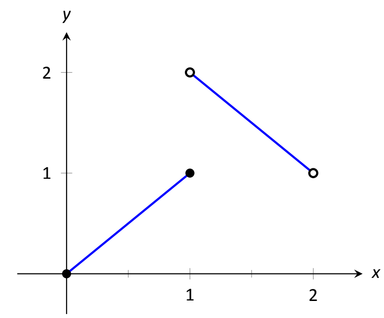


Figure 10.19: A graph of f in Example 182.

SOLUTION For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim\limits_{x\to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist, as discussed in the first section. The function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim\limits_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.
6. Using the definition and the graph, $f(0) = 0$.
7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 2^-} f(x) = 1$.
8. The graph and the definition of the function show that $f(2)$ is not defined.

Note how the left and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left and right-hand limits are equal.

Theorem 86 **Limits and One Sided Limits**

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

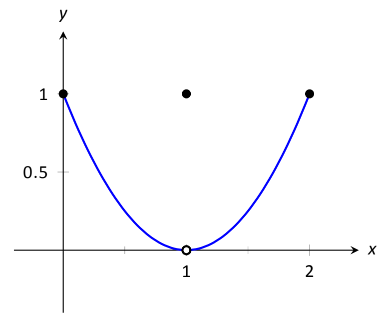


Figure 10.21: Graphing f in Example 184

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 182 – 185 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 183 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} 2-x & 0 < x < 1 \\ (x-2)^2 & 1 < x < 2 \end{cases}$, as shown in Figure 10.20. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

Solution Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim\limits_{x\to 1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim\limits_{x\to 0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim\limits_{x\to 2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

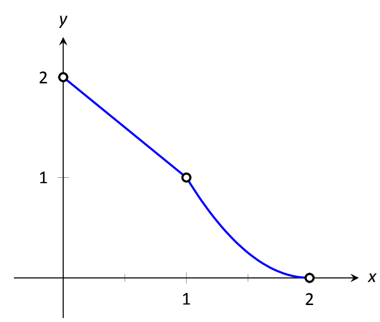


Figure 10.20: A graph of f from Example 183

Example 184 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} (x-1)^2 & 0 \le x \le 2, x \neq 1 \\ 1 & x = 1 \end{cases}$, as shown in Figure 10.21. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear by looking at the graph that both the left and right-hand limits of f, as x approaches 1, is 0. Thus it is also clear that *the* limit is 0; i.e., $\lim_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Example 185 **Evaluating limits of a piecewise–defined function**

Let $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \end{cases}$, as shown in Figure 10.22. Evaluate the following.

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

In Examples 182 – 185 we were asked to find both $\lim_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim_{x\to 1} f(x)$	$f(1)$
Example 182	does not exist	1
Example 183	1	not defined
Example 184	0	1
Example 185	1	1

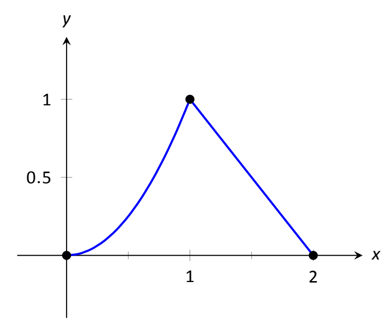


Figure 10.22: Graphing f in Example 185

Only in Example 185 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in the next section, entitled "Continuity." In short, a *continuous function* is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Exercises 10.3

Terms and Concepts

1. What are the three ways in which a limit may fail to exist?

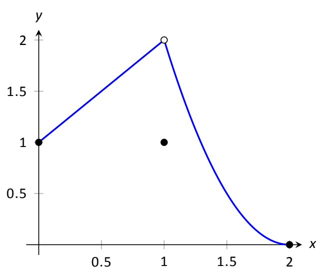
2. T/F: If $\lim\limits_{x\to1^-} f(x) = 5$, then $\lim\limits_{x\to1} f(x) = 5$

3. T/F: If $\lim\limits_{x\to1^-} f(x) = 5$, then $\lim\limits_{x\to1^+} f(x) = 5$

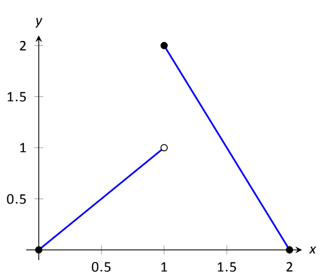
4. T/F: If $\lim\limits_{x\to1} f(x) = 5$, then $\lim\limits_{x\to1^-} f(x) = 5$

Problems

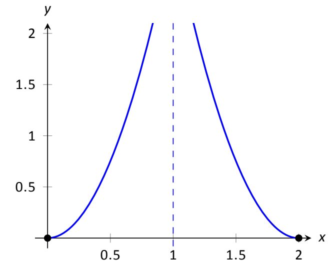
In Exercises 5 – 12, evaluate each expression using the given graph of $f(x)$.

5. 

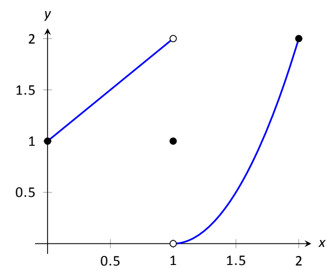
(a) $\lim\limits_{x\to1^-} f(x)$
(b) $\lim\limits_{x\to1^+} f(x)$
(c) $\lim\limits_{x\to1} f(x)$
(d) $f(1)$
(e) $\lim\limits_{x\to0^-} f(x)$
(f) $\lim\limits_{x\to0^+} f(x)$

6. 

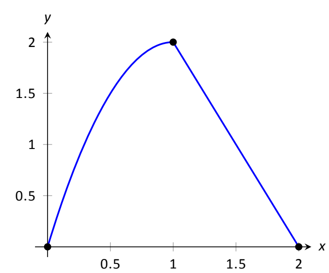
(a) $\lim\limits_{x\to1^-} f(x)$
(b) $\lim\limits_{x\to1^+} f(x)$
(c) $\lim\limits_{x\to1} f(x)$
(d) $f(1)$
(e) $\lim\limits_{x\to2^-} f(x)$
(f) $\lim\limits_{x\to2^+} f(x)$

7. 

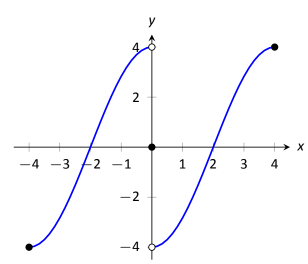
(a) $\lim\limits_{x\to1^-} f(x)$
(b) $\lim\limits_{x\to1^+} f(x)$
(c) $\lim\limits_{x\to1} f(x)$
(d) $f(1)$
(e) $\lim\limits_{x\to2^-} f(x)$
(f) $\lim\limits_{x\to0^+} f(x)$

8. 

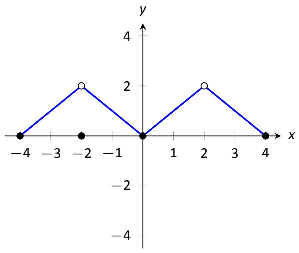
(a) $\lim\limits_{x\to1^-} f(x)$
(b) $\lim\limits_{x\to1^+} f(x)$
(c) $\lim\limits_{x\to1} f(x)$
(d) $f(1)$

9. 

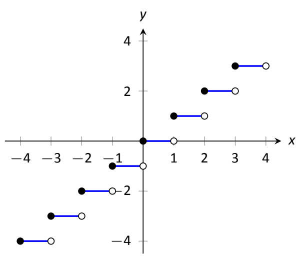
(a) $\lim\limits_{x\to1^-} f(x)$
(b) $\lim\limits_{x\to1^+} f(x)$
(c) $\lim\limits_{x\to1} f(x)$
(d) $f(1)$

10. 

(a) $\lim\limits_{x\to0^-} f(x)$
(b) $\lim\limits_{x\to0^+} f(x)$
(c) $\lim\limits_{x\to0} f(x)$
(d) $f(0)$

11. 

(a) $\lim_{x\to -2^-} f(x)$
(b) $\lim_{x\to -2^+} f(x)$
(c) $\lim_{x\to -2} f(x)$
(d) $f(-2)$
(e) $\lim_{x\to 2^-} f(x)$
(f) $\lim_{x\to 2^+} f(x)$
(g) $\lim_{x\to 2} f(x)$
(h) $f(2)$

12. 

Let $-3 \leq a \leq 3$ be an integer.

(a) $\lim_{x\to a^-} f(x)$
(b) $\lim_{x\to a^+} f(x)$
(c) $\lim_{x\to a} f(x)$
(d) $f(a)$

In Exercises 13 – 21, evaluate the given limits of the piecewise defined functions *f*.

13. $f(x) = \begin{cases} x+1 & x \leq 1 \\ x^2-5 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$
(b) $\lim_{x\to 1^+} f(x)$
(c) $\lim_{x\to 1} f(x)$
(d) $f(1)$

14. $f(x) = \begin{cases} 2x^2+5x-1 & x < 0 \\ \sin x & x \geq 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$
(b) $\lim_{x\to 0^+} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $f(0)$

15. $f(x) = \begin{cases} x^2-1 & x < -1 \\ x^3+1 & -1 \leq x \leq 1 \\ x^2+1 & x > 1 \end{cases}$

(a) $\lim_{x\to -1^-} f(x)$
(b) $\lim_{x\to -1^+} f(x)$
(c) $\lim_{x\to -1} f(x)$
(d) $f(-1)$
(e) $\lim_{x\to 1^-} f(x)$
(f) $\lim_{x\to 1^+} f(x)$
(g) $\lim_{x\to 1} f(x)$
(h) $f(1)$

16. $f(x) = \begin{cases} \cos x & x < \pi \\ \sin x & x \geq \pi \end{cases}$

(a) $\lim_{x\to \pi^-} f(x)$
(b) $\lim_{x\to \pi^+} f(x)$
(c) $\lim_{x\to \pi} f(x)$
(d) $f(\pi)$

17. $f(x) = \begin{cases} 1-\cos^2 x & x < a \\ \sin^2 x & x \geq a \end{cases}$,
where a is a real number.

(a) $\lim_{x\to a^-} f(x)$
(b) $\lim_{x\to a^+} f(x)$
(c) $\lim_{x\to a} f(x)$
(d) $f(a)$

18. $f(x) = \begin{cases} x+1 & x < 1 \\ 1 & x = 1 \\ x-1 & x > 1 \end{cases}$

(a) $\lim_{x\to 1^-} f(x)$
(b) $\lim_{x\to 1^+} f(x)$
(c) $\lim_{x\to 1} f(x)$
(d) $f(1)$

19. $f(x) = \begin{cases} x^2 & x < 2 \\ x+1 & x = 2 \\ -x^2+2x+4 & x > 2 \end{cases}$

(a) $\lim_{x\to 2^-} f(x)$
(b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to 2} f(x)$
(d) $f(2)$

20. $f(x) = \begin{cases} a(x-b)^2+c & x < b \\ a(x-b)+c & x \geq b \end{cases}$,
where a, b and c are real numbers.

(a) $\lim_{x\to b^-} f(x)$
(b) $\lim_{x\to b^+} f(x)$
(c) $\lim_{x\to b} f(x)$
(d) $f(b)$

21. $f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim_{x\to 0^-} f(x)$
(b) $\lim_{x\to 0^+} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $f(0)$

Review

22. Evaluate the limit: $\lim_{x\to -1} \dfrac{x^2+5x+4}{x^2-3x-4}$.

23. Evaluate the limit: $\lim_{x\to -4} \dfrac{x^2-16}{x^2-4x-32}$.

24. Evaluate the limit: $\lim_{x\to -6} \dfrac{x^2-15x+54}{x^2-6x}$.

25. Approximate the limit numerically: $\lim_{x\to 0.4} \dfrac{x^2-4.4x+1.6}{x^2-0.4x}$.

26. Approximate the limit numerically: $\lim_{x\to 0.2} \dfrac{x^2+5.8x-1.2}{x^2-4.2x+0.8}$.

10.4 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim_{x\to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually this. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behaviour.

Definition 58 Continuous Function

Let f be a function defined on an open interval I containing c.

1. f is **continuous at** c if $\lim_{x\to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim_{x\to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim_{x\to c} f(x) = f(c)$.

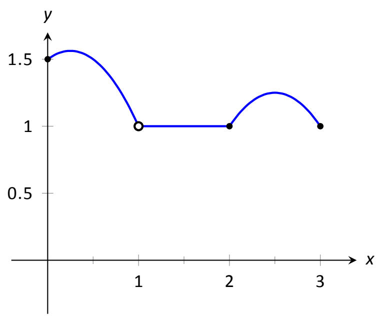


Figure 10.23: A graph of f in Example 186.

Example 186 Finding intervals of continuity

Let f be defined as shown in Figure 10.23. Give the interval(s) on which f is continuous.

Solution We proceed by examining the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ exists for all c between 0 and 3.
2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.
3. The limit $\lim_{x\to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1) \cup (1, 3)$.

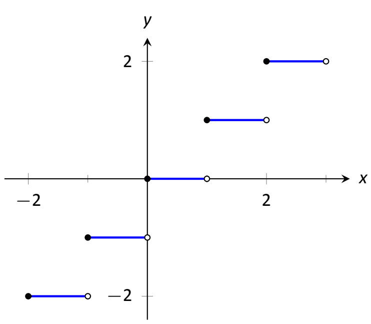


Figure 10.24: A graph of the step function in Example 187.

Example 187 Finding intervals of continuity

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 10.24 demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

Solution We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.

2. The function is defined for all values of c.
3. The limit $\lim_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots.$$

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 59 Continuity on Closed Intervals

Let f be defined on the closed interval $[a,b]$ for some real numbers a, b. f is **continuous on** $[a,b]$ if:

1. f is continuous on (a,b),
2. $\lim_{x\to a^+} f(x) = f(a)$ and
3. $\lim_{x\to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half–open intervals such as $[a,b)$ or $(a,b]$ if necessary.

Example 188 Determining intervals on which a function is continuous
For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1-x^2}$
5. $f(x) = |x|$

Solution We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty,0)\cup(0,\infty)$. As it is a rational function, we apply Theorem 81 to recognize that f is continuous on all of its domain.
2. The domain of $f(x) = \sin x$ is all real numbers, or $(-\infty,\infty)$. Applying Theorem 82 shows that $\sin x$ is continuous everywhere.
3. The domain of $f(x) = \sqrt{x}$ is $[0,\infty)$. Applying Theorem 82 shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0,\infty)$.
4. The domain of $f(x) = \sqrt{1-x^2}$ is $[-1,1]$. Applying Theorems 80 and 82 shows that f is continuous on all of its domain, $[-1,1]$.
5. The domain of $f(x) = |x|$ is $(-\infty,\infty)$. We can define the absolute value function as $f(x) = \begin{cases} -x & x<0 \\ x & x\geq 0 \end{cases}$. Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on

$(-\infty, 0)$ and $[0, \infty)$. We cannot assume this implies that f is continuous on $(-\infty, \infty)$; we need to check that $\lim_{x \to 0} f(x) = f(0)$, as $x = 0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems 80 and 81 apply to continuity as well. Further, now knowing the definition of continuity we can re–read Theorem 82 as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Theorem 87 Properties of Continuous Functions

Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on I)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on I; if n is odd, then true for all values of f on I.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I.

Theorem 88 Continuous Functions

The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \cot x$
5. $f(x) = \sec x$
6. $f(x) = \csc x$
7. $f(x) = \ln x$
8. $f(x) = \sqrt[n]{x}$, (where n is a positive integer)
9. $f(x) = a^x$ ($a > 0$)

We apply these theorems in the following Example.

Example 189 Determining intervals on which a function is continuous
State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x\sin x$
3. $f(x) = \tan x$
4. $f(x) = \sqrt{\ln x}$

Solution We examine each in turn, applying Theorems 87 and 88 as appropriate.

1. The square–root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 10.25.
2. The functions $y = x$ and $y = \sin x$ are each continuous everywhere, hence their product is, too.
3. Theorem 88 states that $f(x) = \tan x$ is continuous "on its domain." Its domain includes all real numbers except odd multiples of $\pi/2$. Thus $f(x) = \tan x$ is continuous on
$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,$$
or, equivalently, on $D = \{x \in \mathbb{R} \mid x \neq n \cdot \frac{\pi}{2}$, n is an odd integer$\}$.
4. The domain of $y = \sqrt{x}$ is $[0, \infty)$. The range of $y = \ln x$ is $(-\infty, \infty)$, but if we restrict its domain to $[1, \infty)$ its range is $[0, \infty)$. So restricting $y = \ln x$ to the domain of $[1, \infty)$ restricts its output is $[0, \infty)$, on which $y = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln x}$ is $[1, \infty)$.

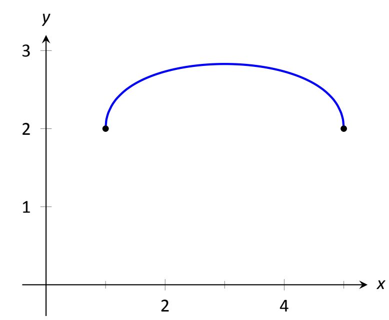


Figure 10.25: A graph of f in Example 189(a).

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. While beyond the scope of this text, this pseudo–definition glosses over some of the finer points of continuity. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

This intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$ and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on $[1, 2]$ f must be equal to -9, and -8, and -7, -6, $\ldots$, 0, $1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem 89 Intermediate Value Theorem

Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is a value c in $[a, b]$ such that $f(c) = y$.

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is a c in $[a, b]$ such that $f(c) = 0$. The theorem does not give us any clue as to where that value is in the interval $[a, b]$, just that it exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$ – we got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$ Then we know there is a root of f on the interval $[d, b]$ – we have halved the size of our interval, hence are closer to a good approximation of the root.

3. $f(d) > 0$ Then we know there is a root of f on the interval $[a, d]$ – again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

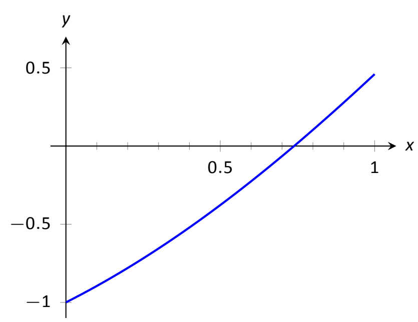


Figure 10.26: Graphing a root of $f(x) = x - \cos x$.

Example 190 Using the Bisection Method

Approximate the root of $f(x) = x - \cos x$, accurate to three places after the decimal.

Solution Consider the graph of $f(x) = x - \cos x$, shown in Figure 10.26. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

Iteration 1: $f(0.7) < 0$, $f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0$, $f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Table 10.27.

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Figure 10.27: Iterations of the Bisection Method of Root Finding

Notice that in the 12th iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, \quad f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Intermediate Value Theorem states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f, accurate to 10 places after the decimal, to be 0.7390851332. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

In the next section we examine one more aspect of limits: limits that involve infinity.

Exercises 10.4

Terms and Concepts

1. In your own words, describe what it means for a function to be continuous.

2. In your own words, describe what the Intermediate Value Theorem states.

3. What is a "root" of a function?

4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

5. T/F: If f is defined on an open interval containing c, and $\lim_{x\to c} f(x)$ exists, then f is continuous at c.

6. T/F: If f is continuous at c, then $\lim_{x\to c} f(x)$ exists.

7. T/F: If f is continuous at c, then $\lim_{x\to c^+} f(x) = f(c)$.

8. T/F: If f is continuous on $[a, b]$, then $\lim_{x\to a^-} f(x) = f(a)$.

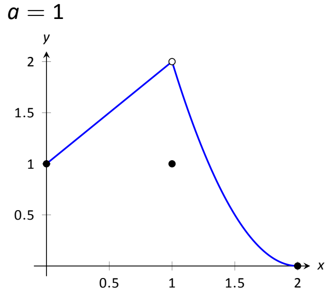
9. T/F: If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.

10. T/F: The sum of continuous functions is also continuous.

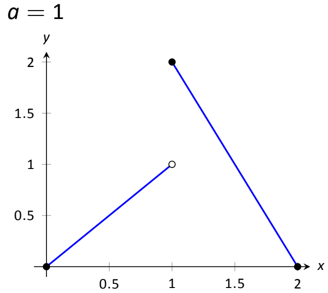
Problems

In Exercises 11 – 17, a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.

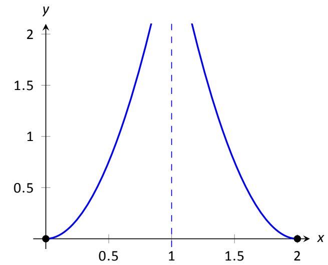
11. $a = 1$



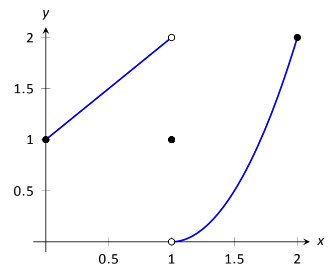
12. $a = 1$



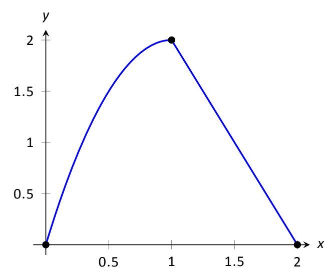
13. $a = 1$



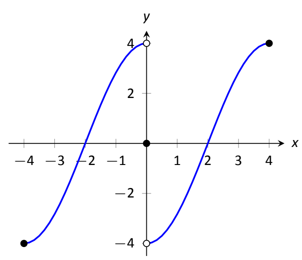
14. $a = 0$



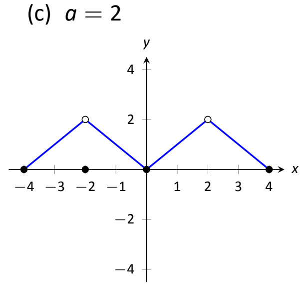
15. $a = 1$



16. $a = 4$



17. (a) $a = -2$
    (b) $a = 0$
    (c) $a = 2$



In Exercises 18 – 21, determine if f is continuous at the indicated values. If not, explain why.

18. $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x > 0 \end{cases}$
    (a) $x = 0$
    (b) $x = \pi$

19. $f(x) = \begin{cases} x^3 - x & x < 1 \\ x - 2 & x \geq 1 \end{cases}$
    (a) $x = 0$
    (b) $x = 1$

20. $f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq -1 \\ 3 & x = -1 \end{cases}$
    (a) $x = -1$
    (b) $x = 10$

21. $f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$
    (a) $x = 0$
    (b) $x = 8$

In Exercises 22 – 32, give the intervals on which the given function is continuous.

22. $f(x) = x^2 - 3x + 9$

23. $g(x) = \sqrt{x^2 - 4}$

24. $h(k) = \sqrt{1-k} + \sqrt{k+1}$

25. $f(t) = \sqrt{5t^2 - 30}$

26. $g(t) = \dfrac{1}{\sqrt{1-t^2}}$

27. $g(x) = \dfrac{1}{1+x^2}$

28. $f(x) = e^x$

29. $g(s) = \ln s$

30. $h(t) = \cos t$

31. $f(k) = \sqrt{1 - e^k}$

32. $f(x) = \sin(e^x + x^2)$

33. Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

34. Let g be continuous on $[-3, 7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?

35. Let f be continuous on $[-1, 1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?

36. Let h be a function on $[-1, 1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?

In Exercises 37 – 40, use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.

37. $f(x) = x^2 + 2x - 4$ on $[1, 1.5]$.

38. $f(x) = \sin x - 1/2$ on $[0.5, 0.55]$

39. $f(x) = e^x - 2$ on $[0.65, 0.7]$.

40. $f(x) = \cos x - \sin x$ on $[0.7, 0.8]$.

Review

41. Let $f(x) = \begin{cases} x^2 - 5 & x < 5 \\ 5x & x \geq 5 \end{cases}$.
    (a) $\lim_{x \to 5^-} f(x)$
    (b) $\lim_{x \to 5^+} f(x)$
    (c) $\lim_{x \to 5} f(x)$
    (d) $f(5)$

42. Numerically approximate the following limits:
    (a) $\lim_{x \to -4/5^+} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$
    (b) $\lim_{x \to -4/5^-} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

43. Give an example of function $f(x)$ for which $\lim_{x \to 0} f(x)$ does not exist.

10.5 Limits Involving Infinity

In Definition 56 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 10.28. Note how, as x approaches 0, $f(x)$ grows very, very large. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

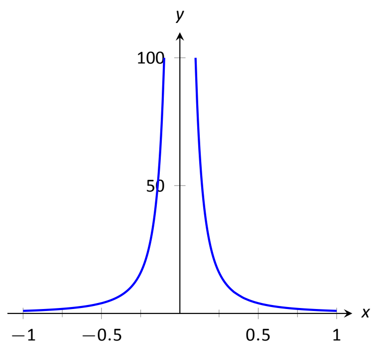


Figure 10.28: Graphing $f(x) = 1/x^2$ for values of x near 0.

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to \infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

Definition 60 Limit of Infinity, ∞

We say $\lim_{x\to c} f(x) = \infty$ if we can make the value of $f(x)$ arbitrarily large by choosing $x \neq c$ sufficiently close to c.

This is once again an informal definition, like Definition 56: we say that if we get close enough to c, then we can make $f(x)$ as large as we want, without giving precise answers to the questions "How close?" or "How large?" We can define limits equal to $-\infty$ in a similar way by requiring $f(x)$ to be large (in absolute value) but negative.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

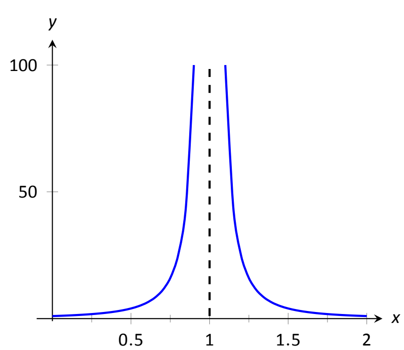


Figure 10.29: Observing infinite limit as $x \to 1$ in Example 191.

Example 191 Evaluating limits involving infinity

Find $\lim_{x\to 1} \dfrac{1}{(x-1)^2}$ as shown in Figure 10.29.

Solution In Example 173 of Section 10.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, we can see that as the difference $|x-1|$ gets smaller, the value of $f(x)$ gets larger and larger, so we may say $\lim_{x\to 1} 1/(x-1)^2 = \infty$.

Example 192 Evaluating limits involving infinity

Find $\lim_{x\to 0} \frac{1}{x}$, as shown in Figure 10.30.

Solution It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behaviour is not consistent, we cannot say that $\lim_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one–sided limits. We can state that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

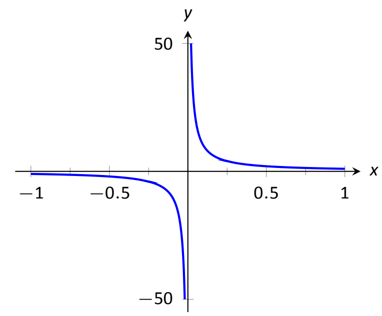


Figure 10.30: Evaluating $\lim_{x\to 0} \frac{1}{x}$.

Vertical asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a **vertical asymptote** at c.

Example 193 Finding vertical asymptotes

Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

Solution Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 10.31. Thus the vertical asymptotes are at $x = \pm 2$.

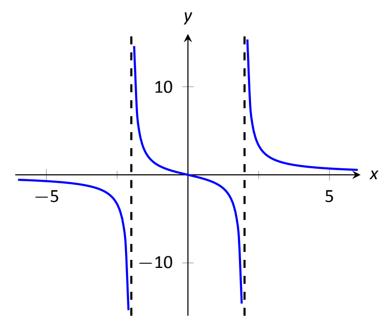


Figure 10.31: Graphing $f(x) = \frac{3x}{x^2-4}$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2-1)/(x-1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 10.32. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Cancelling the common term, we get that $f(x) = x+1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

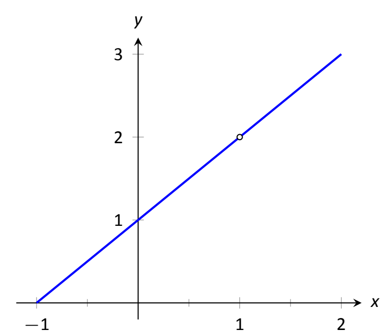


Figure 10.32: Graphically showing that $f(x) = \frac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim_{x\to 0} \frac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 10.2 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2-1}{x-1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, 0/0 is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up 0/0 expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return 0/0. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and cancelling) or it may require a tool such as the Squeeze Theorem. In later courses you may encounter a technique called l'Hospital's Rule that provides another way to handle indeterminate forms using derivatives.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that 1/0 and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behaviour of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 61 Limits at Infinity and Horizontal Asymptote

1. We say $\lim\limits_{x\to\infty} f(x) = L$ if we can make $f(x)$ sufficiently close to L by choosing a sufficiently large (and positive) value for x.

2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if we can make $f(x)$ sufficiently close to L by choosing a sufficiently large (and negative) value for x.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 60.

Example 194 Approximating horizontal asymptotes

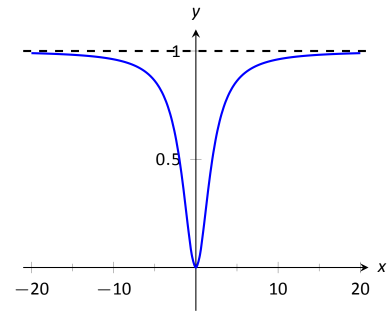
Approximate the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{x^2+4}$.

Solution We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2}{x^2+4}.$$

Figure 10.34(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.

Later, we will show how to determine this analytically.



(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
−10	0.9615
−100	0.9996
−10000	0.999996

(b)

Figure 10.34: Using a graph and a table to approximate a horizontal asymptote in Example 194.

Horizontal asymptotes can take on a variety of forms. Figure 10.33(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 10.33(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 10.33(c) shows that $f(x) = (\sin x)/x$ has even more interesting behaviour than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.

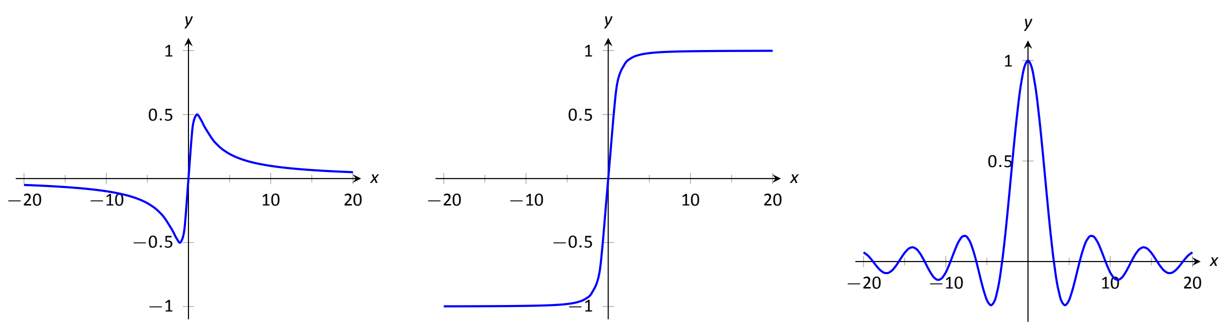


Figure 10.33: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} 1/x$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large enough value of x.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3+2x+1}{4x^3-2x^2+9} &= \lim_{x\to\infty}\frac{1/x^3}{1/x^3}\cdot\frac{x^3+2x+1}{4x^3-2x^2+9}\\ &= \lim_{x\to\infty}\frac{x^3/x^3+2x/x^3+1/x^3}{4x^3/x^3-2x^2/x^3+9/x^3}\\ &= \lim_{x\to\infty}\frac{1+2/x^2+1/x^3}{4-2/x+9/x^3}.\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0}=\frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 90 Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0},$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n=m$, then $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to-\infty}f(x)=\dfrac{a_n}{b_m}$.
2. If $n<m$, then $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to-\infty}f(x)=0$.
3. If $n>m$, then $\lim\limits_{x\to\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$ are both infinite.

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n=m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for a_nx^n/x^n and b_mx^m/x^n. Since $n=m$, this will leave us with the limit a_n/b_m. If $n<m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n>m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to a_nx^n, and likewise all the terms in the denominator are small compared to b_nx^m. If $n=m$, looking only at these two important terms, we have $(a_nx^n)/(b_nx^m)$. This reduces to a_n/b_m. If $n<m$, the function behaves like $a_n/(b_mx^{m-n})$, which tends toward 0. If $n>m$, the function behaves like a_nx^{n-m}/b_m, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty}f(x)$ or $\lim_{x\to-\infty}f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again

$\lim\limits_{x\to\pm\infty} \dfrac{x}{\sqrt{x^2+1}}$, graphed in Figure 10.33(b). When x is very large, $x^2+1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y=1$ and $y=-1$, respectively.

Example 195 Finding a limit of a rational function

Confirm analytically that $y=1$ is the horizontal asymptote of $f(x) = \dfrac{x^2}{x^2+4}$, as approximated in Example 194.

Solution Before using Theorem 90, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 90 directly; in this case $n=m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 196 Finding limits of rational functions

Use Theorem 90 to evaluate each of the following limits.

1. $\lim\limits_{x\to-\infty}\dfrac{x^2+2x-1}{x^3+1}$
2. $\lim\limits_{x\to\infty}\dfrac{x^2+2x-1}{1-x-3x^2}$
3. $\lim\limits_{x\to\infty}\dfrac{x^2-1}{3-x}$

Solution

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 10.36(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 10.36(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 10.36(c).

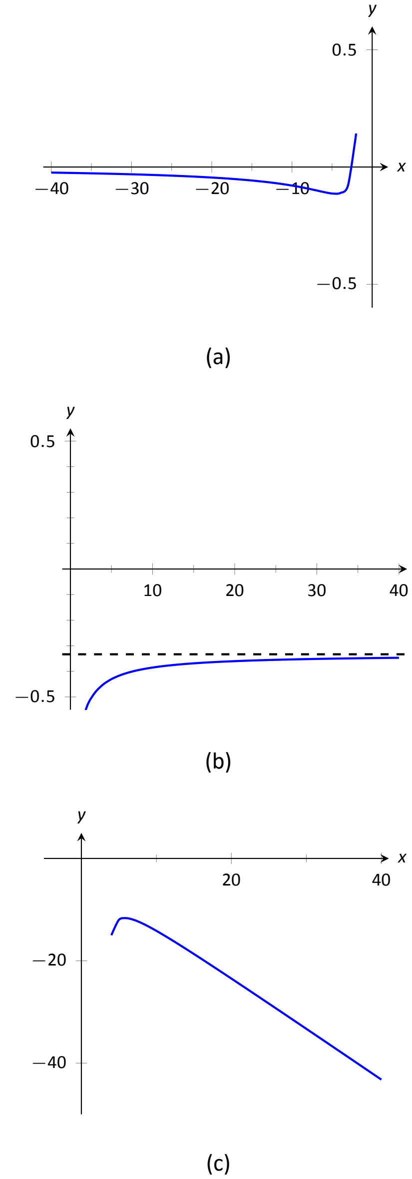


Figure 10.36: Visualizing the functions in Example 196.

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Exercises 10.5

Terms and Concepts

1. T/F: If $\lim_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

2. T/F: If $\lim_{x\to \infty} f(x) = 5$, then we are implicitly stating that the limit exists.

3. T/F: If $\lim_{x\to 1^-} f(x) = -\infty$, then $\lim_{x\to 1^+} f(x) = \infty$

4. T/F: If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

5. T/F: $\infty/0$ is not an indeterminate form.

6. List 5 indeterminate forms.

7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is/is not continuous at $x = 7$.

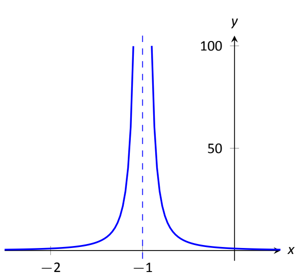
Problems

In Exercises 9 – 14, evaluate the given limits using the graph of the function.

9. $f(x) = \dfrac{1}{(x+1)^2}$

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$



10. $f(x) = \dfrac{1}{(x-3)(x-5)^2}$.

(a) $\lim_{x\to 3^-} f(x)$

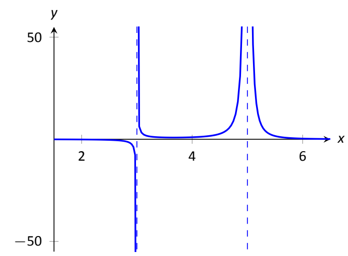
(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $\lim_{x\to 5^-} f(x)$

(e) $\lim_{x\to 5^+} f(x)$

(f) $\lim_{x\to 5} f(x)$



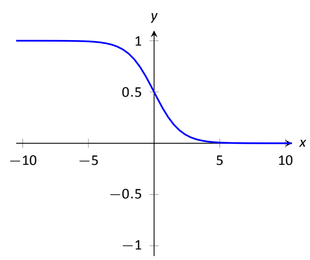
11. $f(x) = \dfrac{1}{e^x + 1}$

(a) $\lim_{x\to -\infty} f(x)$

(b) $\lim_{x\to \infty} f(x)$

(c) $\lim_{x\to 0^-} f(x)$

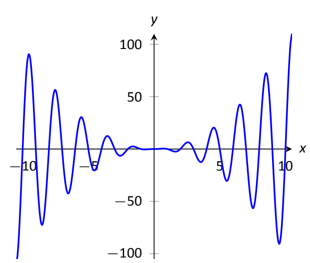
(d) $\lim_{x\to 0^+} f(x)$



12. $f(x) = x^2 \sin(\pi x)$

(a) $\lim_{x\to -\infty} f(x)$

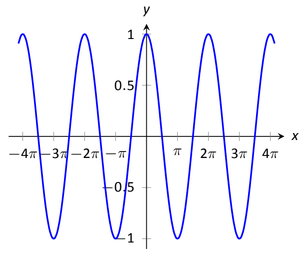
(b) $\lim_{x\to \infty} f(x)$



13. $f(x) = \cos(x)$

(a) $\lim\limits_{x\to-\infty} f(x)$

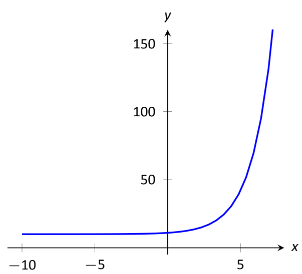
(b) $\lim\limits_{x\to\infty} f(x)$



14. $f(x) = 2^x + 10$

(a) $\lim\limits_{x\to-\infty} f(x)$

(b) $\lim\limits_{x\to\infty} f(x)$



In Exercises 15 – 18, numerically approximate the following limits:

(a) $\lim\limits_{x\to3^-} f(x)$

(b) $\lim\limits_{x\to3^+} f(x)$

(c) $\lim\limits_{x\to3} f(x)$

15. $f(x) = \dfrac{x^2-1}{x^2-x-6}$

16. $f(x) = \dfrac{x^2+5x-36}{x^3-5x^2+3x+9}$

17. $f(x) = \dfrac{x^2-11x+30}{x^3-4x^2-3x+18}$

18. $f(x) = \dfrac{x^2-9x+18}{x^2-x-6}$

In Exercises 19 – 24, identify the horizontal and vertical asymptotes, if any, of the given function.

19. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-20}$

20. $f(x) = \dfrac{-3x^2-9x-6}{5x^2-10x-15}$

21. $f(x) = \dfrac{x^2+x-12}{7x^3-14x^2-21x}$

22. $f(x) = \dfrac{x^2-9}{9x-9}$

23. $f(x) = \dfrac{x^2-9}{9x+27}$

24. $f(x) = \dfrac{x^2-1}{-x^2-1}$

In Exercises 25 – 28, evaluate the given limit.

25. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{x-5}$

26. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{5-x}$

27. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{x^2-5}$

28. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{5-x^2}$

Review

29. Let $\lim\limits_{x\to2} f(x) = 3$ and $\lim\limits_{x\to2} g(x) = -1$. Evaluate the following limits.

(a) $\lim\limits_{x\to2}(f+g)(x)$

(b) $\lim\limits_{x\to2}(fg)(x)$

(c) $\lim\limits_{x\to2}(f/g)(x)$

(d) $\lim\limits_{x\to2} f(x)^{g(x)}$

30. Let $f(x) = \begin{cases} x^2-1 & x<3 \\ x+5 & x\geq 3 \end{cases}$.
Is f continuous everywhere?

31. Evaluate the limit: $\lim\limits_{x\to e} \ln x$.

11: Derivatives

The previous chapter introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe *where* a function is going; derivatives describe *how fast* the function is going.

11.1 Instantaneous Rates of Change: The Derivative

A common amusement park ride lifts riders to a height then allows them to freefall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Student of physics may recall that the height (in feet) of the riders, t seconds after freefall (and ignoring air resistance, etc.) can be accurately modelled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of 86 ft.). How fast will the riders be traveling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an *instantaneous velocity*. We do not currently know how to calculate this.

However, we do know from common experience how to calculate an *average velocity*. (If we travel 60 miles in 2 hours, we know we had an average velocity of 30 mph.) We looked at this concept in Section 10.1 when we introduced the difference quotient. We have

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\text{“ rise ”}}{\text{run}} = \text{average velocity.}$$

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{f(3) - f(2)}{1} = -80 \text{ ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{f(2.5) - f(2)}{0.5} = -72 \text{ ft/s}.$$

We can do this for smaller and smaller intervals of time. For instance, over a time span of $1/10^{\text{th}}$ of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{f(2.1) - f(2)}{0.1} = -65.6 \text{ ft/s}.$$

Over a time span of 1/100th of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \frac{f(2.01) - f(2)}{0.01} = -64.16 \text{ ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2+h) - f(2)}{h}$$

where h is small.

What we really want is for $h = 0$, but this, of course, returns the familiar "0/0" indeterminate form. So we employ a limit, as we did in Section 10.1.

h	Average Velocity ft/s
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Figure 11.2: Approximating the instantaneous velocity with average velocities over a small time period h.

We can approximate the value of this limit numerically with small values of h as seen in Figure 11.2. It looks as though the velocity is approaching -64 ft/s. Computing the limit directly gives

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{-16(2+h)^2 + 150 - (-16(2)^2 + 150)}{h} \\
&= \lim_{h\to 0} \frac{-64h - 16h^2}{h} \\
&= \lim_{h\to 0} -64 - 16h \\
&= -64.
\end{aligned}$$

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2+h, f(2+h))$. In Figure 11.1, the secant line corresponding to $h = 1$ is shown in three contexts. Figure 11.1(a) shows a "zoomed out" version of f with its secant line. In (b), we zoom in around the points of intersection between f and the secant line. Notice how well this secant line approximates f between those two points – it is a common practice to approximate functions with straight lines.

As $h \to 0$, these secant lines approach the *tangent line*, a line that goes through the point $(2, f(2))$ with the special slope of -64. In parts (c) and (d) of Figure 11.1, we zoom in around the point $(2, 86)$. In (c) we see the secant line, which approximates f well, but not as well the tangent line shown in (d).

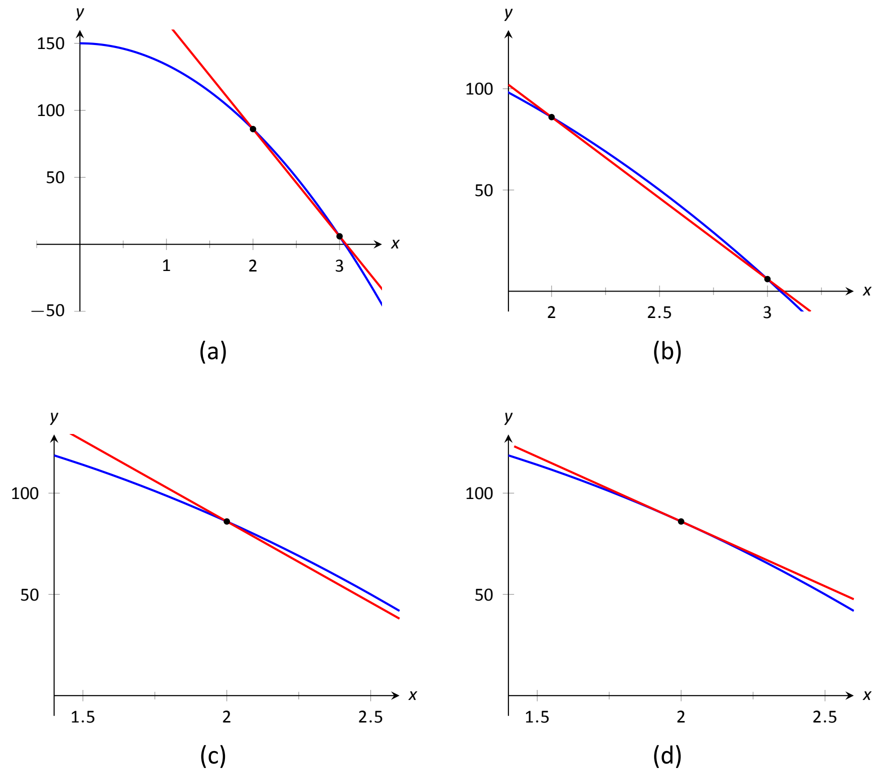


Figure 11.1: Parts (a), (b) and (c) show the secant line to $f(x)$ with $h = 1$, zoomed in different amounts. Part (d) shows the tangent line to f at $x = 2$.

We have just introduced a number of important concepts that we will flesh out more within this section. First, we formally define two of them.

Definition 62 Derivative at a Point

Let f be a continuous function on an open interval I and let c be in I. The **derivative of f at c**, denoted $f'(c)$, is

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that **f is differentiable at c**; if the limit does not exist, then **f is not differentiable at c**. If f is differentiable at every point in I, then **f is differentiable on I**.

Definition 63 Tangent Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The line with equation $\ell(x) = f'(c)(x-c)+f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

Some examples will help us understand these definitions.

Example 197 Finding derivatives and tangent lines

Let $f(x) = 3x^2 + 5x - 7$. Find:

1. $f'(1)$
2. The equation of the tangent line to the graph of f at $x = 1$.
3. $f'(3)$
4. The equation of the tangent line to the graph f at $x = 3$.

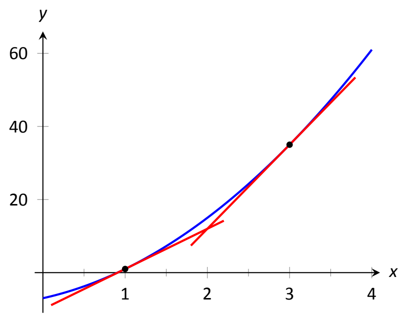


Figure 11.3: A graph of $f(x) = 3x^2+5x-7$ and its tangent lines at $x = 1$ and $x = 3$.

Solution

1. We compute this directly using Definition 62.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2 + 5(1+h) - 7 - (3(1)^2 + 5(1) - 7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2 + 11h}{h} \\ &= \lim_{h\to 0} 3h + 11 = 11. \end{aligned}$$

2. The tangent line at $x = 1$ has slope $f'(1)$ and goes through the point $(1, f(1)) = (1, 1)$. Thus the tangent line has equation, in point-slope form, $y = 11(x - 1) + 1$. In slope-intercept form we have $y = 11x - 10$.

3. Again, using the definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2 + 5(3+h) - 7 - (3(3)^2 + 5(3) - 7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2 + 23h}{h} \\ &= \lim_{h\to 0} 3h + 23 \\ &= 23. \end{aligned}$$

4. The tangent line at $x = 3$ has slope 23 and goes through the point $(3, f(3)) = (3, 35)$. Thus the tangent line has equation $y = 23(x-3)+35 = 23x-34$.

A graph of f is given in Figure 11.3 along with the tangent lines at $x = 1$ and $x = 3$.

Another important line that can be created using information from the derivative is the **normal line.** It is perpendicular to the tangent line, hence its slope is the opposite–reciprocal of the tangent line's slope.

Definition 64 Normal Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x - c) + f(c),$$

where $f'(c) \neq 0$. When $f'(c) = 0$, the normal line is the vertical line through $\big(c, f(c)\big)$; that is, $x = c$.

Example 198 Finding equations of normal lines

Let $f(x) = 3x^2 + 5x - 7$, as in Example 197. Find the equations of the normal lines to the graph of f at $x = 1$ and $x = 3$.

Solution In Example 197, we found that $f'(1) = 11$. Hence at $x = 1$, the normal line will have slope $-1/11$. An equation for the normal line is

$$n(x) = \frac{-1}{11}(x - 1) + 1.$$

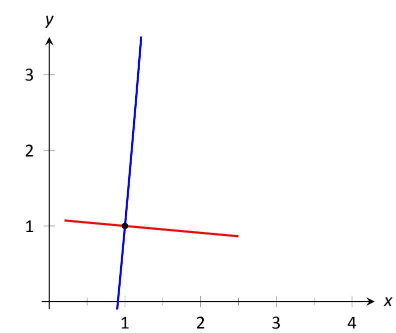


Figure 11.4: A graph of $f(x) = 3x^2 + 5x - 7$, along with its normal line at $x = 1$.

The normal line is plotted with $y = f(x)$ in Figure 11.4. Note how the line looks perpendicular to f. (A key word here is "looks." Mathematically, we say that the normal line *is* perpendicular to f at $x = 1$ as the slope of the normal line is the opposite–reciprocal of the slope of the tangent line. However, normal lines may not always *look* perpendicular. The aspect ratio of the picture of the graph plays a big role in this.)

We also found that $f'(3) = 23$, so the normal line to the graph of f at $x = 3$ will have slope $-1/23$. An equation for the normal line is

$$n(x) = \frac{-1}{23}(x - 3) + 35.$$

Linear functions are easy to work with; many functions that arise in the course of solving real problems are not easy to work with. A common practice in mathematical problem solving is to approximate difficult functions with not–so–difficult functions. Lines are a common choice. It turns out that at any given point on the graph of a differentiable function f, the best linear approximation to f is its tangent line. That is one reason we'll spend considerable time finding tangent lines to functions.

One type of function that does not benefit from a tangent–line approximation is a line; it is rather simple to recognize that the tangent line to a line is the line itself. We look at this in the following example.

Example 199 Finding the Derivative of a Linear Function

Consider $f(x) = 3x + 5$. Find the equation of the tangent line to f at $x = 1$ and $x = 7$.

Solution We find the slope of the tangent line by using Definition 62.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)+5-(3+5)}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

We just found that $f'(1) = 3$. That is, we found the *instantaneous rate of change* of $f(x) = 3x + 5$ is 3. This is not surprising; lines are characterized by being the *only* functions with a *constant rate of change.* That rate of change is called the *slope* of the line. Since their rates of change are constant, their *instantaneous* rates of change are always the same; they are all the slope.

So given a line $f(x) = ax + b$, the derivative at any point x will be a; that is, $f'(x) = a$.

It is now easy to see that the tangent line to the graph of f at $x = 1$ is just f, with the same being true for $x = 7$.

We often desire to find the tangent line to the graph of a function without knowing the actual derivative of the function. In these cases, the best we may be able to do is approximate the tangent line. We demonstrate this in the next example.

Example 200 Numerical Approximation of the Tangent Line

Approximate the equation of the tangent line to the graph of $f(x) = \sin x$ at $x = 0$.

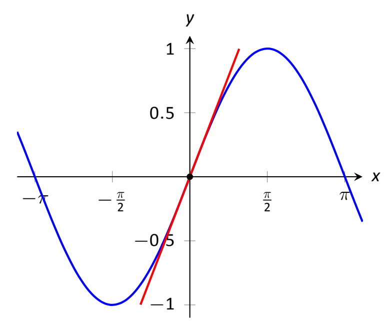


Figure 11.5: $f(x) = \sin x$ graphed with an approximation to its tangent line at $x = 0$.

Solution In order to find the equation of the tangent line, we need a slope and a point. The point is given to us: $(0, \sin 0) = (0, 0)$. To compute the slope, we need the derivative. This is where we will make an approximation. Recall that

$$f'(0) \approx \frac{\sin(0+h)-\sin 0}{h}$$

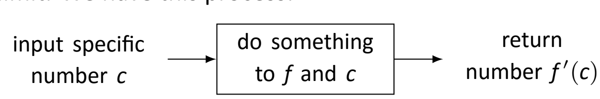
for a small value of h. We choose (somewhat arbitrarily) to let $h = 0.1$. Thus

$$f'(0) \approx \frac{\sin(0.1)-\sin 0}{0.1} \approx 0.9983.$$

Thus our approximation of the equation of the tangent line is $y = 0.9983(x - 0) + 0 = 0.9983x$; it is graphed in Figure 11.5. The graph seems to imply the approximation is rather good.

Recall from Section 10.2 that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, meaning for values of x near 0, $\sin x \approx x$. Since the slope of the line $y = x$ is 1 at $x = 0$, it should seem reasonable that "the slope of $f(x) = \sin x$" is near 1 at $x = 0$. In fact, since we *approximated* the value of the slope to be 0.9983, we might guess the *actual value* is 1. We'll come back to this later.

Consider again Example 197. To find the derivative of f at $x = 1$, we needed to evaluate a limit. To find the derivative of f at $x = 3$, we needed to again evaluate a limit. We have this process:



This process describes a *function*; given one input (the value of c), we return exactly one output (the value of $f'(c)$). The "do something" box is where the tedious work (taking limits) of this function occurs.

Instead of applying this function repeatedly for different values of c, let us apply it just once to the variable x. We then take a limit just once. The process now looks like:

input variable x → [do something to f and x] → return function $f'(x)$

The output is the "derivative function," $f'(x)$. The $f'(x)$ function will take a number c as input and return the derivative of f at c. This calls for a definition.

Definition 65 Derivative Function

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is **the derivative of** f.

Notation:
Let $y = f(x)$. The following notations all represent the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y).$$

Important: The notation $\frac{dy}{dx}$ is one symbol; it is **not** the fraction "dy/dx". The notation, while somewhat confusing at first, was chosen with care. A fraction–looking symbol was chosen because the derivative has many fraction–like properties. Among other places, we see these properties at work when we talk about the units of the derivative, when we discuss the Chain Rule, and when we learn about integration (topics that appear in later sections and chapters).

Examples will help us understand this definition.

Example 201 Finding the derivative of a function
Let $f(x) = 3x^2 + 5x - 7$ as in Example 197. Find $f'(x)$.

Solution We apply Definition 65.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{x\to 0} \frac{3(x+h)^2 + 5(x+h) - 7 - (3x^2 + 5x - 7)}{h} \\ &= \lim_{x\to 0} \frac{3h^2 + 6xh + 5h}{h} \\ &= \lim_{x\to 0} 3h + 6x + 5 \\ &= 6x + 5 \end{aligned}$$

So $f'(x) = 6x + 5$. Recall earlier we found that $f'(1) = 11$ and $f'(3) = 23$. Note our new computation of $f'(x)$ affirm these facts.

Example 202 Finding the derivative of a function

Let $f(x) = \dfrac{1}{x+1}$. Find $f'(x)$.

Solution We apply Definition 65.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \end{aligned}$$

Now find common denominator then subtract; pull $1/h$ out front to facilitate reading.

$$\begin{aligned} &= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{x+1}{(x+1)(x+h+1)} - \frac{x+h+1}{(x+1)(x+h+1)} \right) \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)} \right) \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{-h}{(x+1)(x+h+1)} \right) \\ &= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)(x+1)} \\ &= \frac{-1}{(x+1)^2} \end{aligned}$$

So $f'(x) = \dfrac{-1}{(x+1)^2}$. To practice using our notation, we could also state

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}.$$

Example 203 Finding the derivative of a function

Find the derivative of $f(x) = \sin x$.

Solution Before applying Definition 65, note that once this is found, we can find the actual tangent line to $f(x) = \sin x$ at $x = 0$, whereas we settled for an approximation in Example 200.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \left(\begin{array}{c}\text{Use trig identity} \\ \sin(x+h) = \sin x \cos h + \cos x \sin h\end{array}\right) \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{(regroup)} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} && \text{(split into two fractions)} \\ &= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x \sin h}{h} \right) && \left(\text{use } \lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \text{ and } \lim_{h\to 0} \frac{\sin h}{h} = 1\right) \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x\,! \end{aligned}$$

We have found that when $f(x) = \sin x$, $f'(x) = \cos x$. This should be somewhat surprising; the result of a tedious limit process and the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic – it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Thinking back to Example 200, we can find the slope of the tangent line to $f(x) = \sin x$ at $x = 0$ using our derivative. We approximated the slope as 0.9983; we now know the slope is *exactly* $\cos 0 = 1$.

Example 204 Finding the derivative of a piecewise defined function

Find the derivative of the absolute value function,

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

See Figure 11.6.

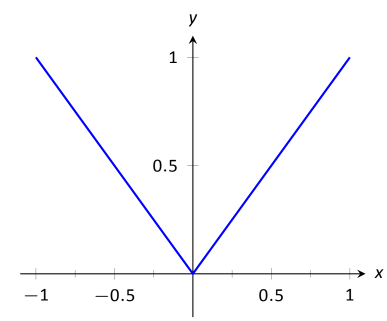


Figure 11.6: The absolute value function, $f(x) = |x|$. Notice how the slope of the lines (and hence the tangent lines) abruptly changes at $x = 0$.

Solution We need to evaluate $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. As f is piecewise–defined, we need to consider separately the limits when $x < 0$ and when $x > 0$.

When $x < 0$:

$$\begin{aligned} \frac{d}{dx}\big(-x\big) &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} -1 \\ &= -1. \end{aligned}$$

When $x > 0$, a similar computation shows that $\frac{d}{dx}\big(x\big) = 1$.

We need to also find the derivative at $x = 0$. By the definition of the derivative at a point, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}.$$

Since $x = 0$ is the point where our function's definition switches from one piece to other, we need to consider left and right-hand limits. Consider the following, where we compute the left and right hand limits side by side.

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \quad \lim_{h\to 0^-} \frac{-h-0}{h} = \quad \lim_{h\to 0^-} -1 = -1$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \quad \lim_{h\to 0^+} \frac{h-0}{h} = \quad \lim_{h\to 0^+} 1 = 1$$

The last lines of each column tell the story: the left and right hand limits are not equal. Therefore the limit does not exist at 0, and f is not differentiable at 0. So we have

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

At $x = 0$, $f'(x)$ does not exist; there is a jump discontinuity at 0; see Figure 11.7. So $f(x) = |x|$ is differentiable everywhere except at 0.

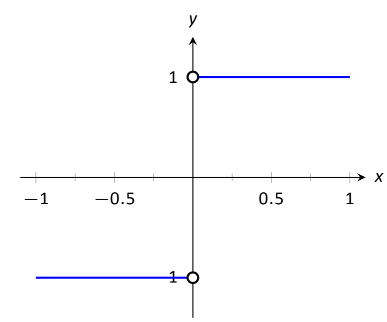


Figure 11.7: A graph of the derivative of $f(x) = |x|$.

The point of non-differentiability came where the piecewise defined function switched from one piece to the other. Our next example shows that this

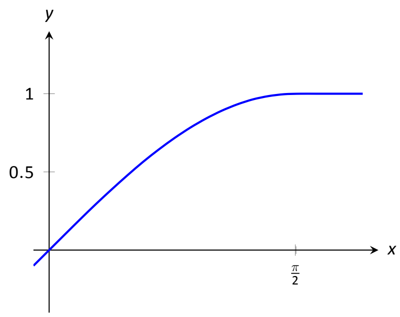


Figure 11.8: A graph of $f(x)$ as defined in Example 205.

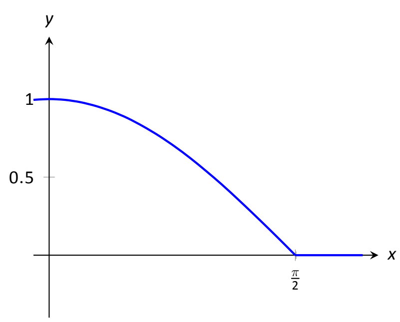


Figure 11.9: A graph of $f'(x)$ in Example 205.

does not always cause trouble.

Example 205 Finding the derivative of a piecewise defined function

Find the derivative of $f(x)$, where $f(x) = \begin{cases} \sin x & x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$. See Figure 11.8.

Solution Using Example 203, we know that when $x < \pi/2$, $f'(x) = \cos x$. It is easy to verify that when $x > \pi/2$, $f'(x) = 0$; consider:

$$\lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{1-1}{h} = \lim_{h \to 0} 0 = 0.$$

So far we have

$$f'(x) = \begin{cases} \cos x & x < \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

We still need to find $f'(\pi/2)$. Notice at $x = \pi/2$ that both pieces of f' are 0, meaning we can state that $f'(\pi/2) = 0$.

Being more rigorous, we can again evaluate the difference quotient limit at $x = \pi/2$, utilizing again left and right–hand limits:

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h \to 0^-} \frac{\sin(\pi/2+h) - \sin(\pi/2)}{h} &= \\ \lim_{h \to 0^-} \frac{\sin(\frac{\pi}{2})\cos(h) + \sin(h)\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})}{h} &= \\ \lim_{h \to 0^-} \frac{1 \cdot \cos(h) + \sin(h) \cdot 0 - 1}{h} &= \\ 0 & \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h \to 0^+} \frac{1-1}{h} &= \\ \lim_{h \to 0^+} \frac{0}{h} &= \\ 0 & \end{aligned}$$

Since both the left and right hand limits are 0 at $x = \pi/2$, the limit exists and $f'(\pi/2)$ exists (and is 0). Therefore we can fully write f' as

$$f'(x) = \begin{cases} \cos x & x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

See Figure 11.9 for a graph of this function.

Recall we pseudo–defined a continuous function as one in which we could sketch its graph without lifting our pencil. We can give a pseudo–definition for differentiability as well: it is a continuous function that does not have any "sharp corners." One such sharp corner is shown in Figure 11.6. Even though the function f in Example 205 is piecewise–defined, the transition is "smooth" hence it is differentiable. Note how in the graph of f in Figure 11.8 it is difficult to tell when f switches from one piece to the other; there is no "corner."

This section defined the derivative; in some sense, it answers the question of "What *is* the derivative?" The next section addresses the question "What does the derivative *mean*?"

Exercises 11.1

Terms and Concepts

1. T/F: Let f be a position function. The average rate of change on $[a, b]$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

2. T/F: The definition of the derivative of a function at a point involves taking a limit.

3. In your own words, explain the difference between the average rate of change and instantaneous rate of change.

4. In your own words, explain the difference between Definitions 62 and 65.

5. Let $y = f(x)$. Give three different notations equivalent to "$f'(x)$."

Problems

In Exercises 6 – 12, use the definition of the derivative to compute the derivative of the given function.

6. $f(x) = 6$

7. $f(x) = 2x$

8. $f(t) = 4 - 3t$

9. $g(x) = x^2$

10. $f(x) = 3x^2 - x + 4$

11. $r(x) = \dfrac{1}{x}$

12. $r(s) = \dfrac{1}{s-2}$

In Exercises 13 – 19, a function and an x–value c are given. (Note: these functions are the same as those given in Exercises 6 through 12.)

(a) Find the tangent line to the graph of the function at c.

(b) Find the normal line to the graph of the function at c.

13. $f(x) = 6$, at $x = -2$.

14. $f(x) = 2x$, at $x = 3$.

15. $f(x) = 4 - 3x$, at $x = 7$.

16. $g(x) = x^2$, at $x = 2$.

17. $f(x) = 3x^2 - x + 4$, at $x = -1$.

18. $r(x) = \dfrac{1}{x}$, at $x = -2$.

19. $r(x) = \dfrac{1}{x-2}$, at $x = 3$.

In Exercises 20 – 23, a function f and an x–value a are given. Approximate the equation of the tangent line to the graph of f at $x = a$ by numerically approximating $f'(a)$, using $h = 0.1$.

20. $f(x) = x^2 + 2x + 1$, $x = 3$

21. $f(x) = \dfrac{10}{x+1}$, $x = 9$

22. $f(x) = e^x$, $x = 2$

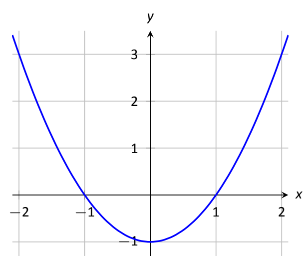
23. $f(x) = \cos x$, $x = 0$

24. The graph of $f(x) = x^2 - 1$ is shown.

   (a) Use the graph to approximate the slope of the tangent line to f at the following points: $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

   (b) Using the definition, find $f'(x)$.

   (c) Find the slope of the tangent line at the points $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

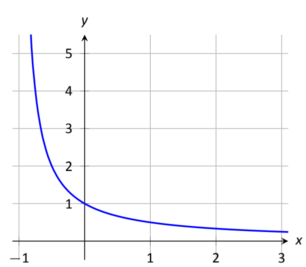


25. The graph of $f(x) = \dfrac{1}{x+1}$ is shown.

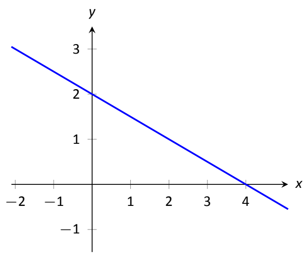
   (a) Use the graph to approximate the slope of the tangent line to f at the following points: $(0, 1)$ and $(1, 0.5)$.

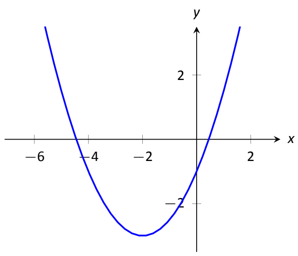
   (b) Using the definition, find $f'(x)$.

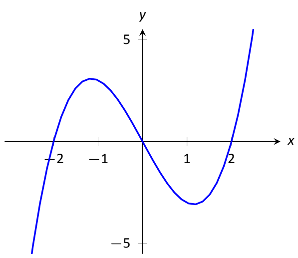
   (c) Find the slope of the tangent line at the points $(0, 1)$ and $(1, 0.5)$.

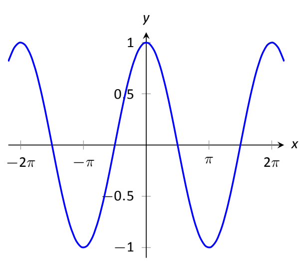


In Exercises 26 – 29, a graph of a function $f(x)$ is given. Using the graph, sketch $f'(x)$.

26. 

27. 

28. 

29. 

30. Using the graph of $g(x)$ below, answer the following questions.

(a) Where is $g(x) > 0$?

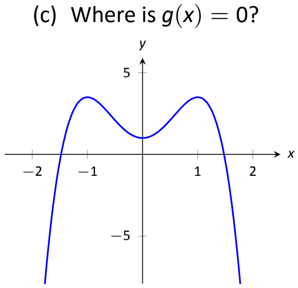
(b) Where is $g(x) < 0$?

(c) Where is $g(x) = 0$?

(c) Where is $g'(x) < 0$?

(d) Where is $g'(x) > 0$?

(e) Where is $g'(x) = 0$?



Review

31. Approximate $\lim_{x\to 5} \dfrac{x^2+2x-35}{x^2-10.5x+27.5}$.

32. Use the Bisection Method to approximate, accurate to two decimal places, the root of $g(x) = x^3 + x^2 + x - 1$ on $[0.5, 0.6]$.

33. Give intervals on which each of the following functions are continuous.

(a) $\dfrac{1}{e^x+1}$

(b) $\dfrac{1}{x^2-1}$

(c) $\sqrt{5-x}$

(d) $\sqrt{5-x^2}$

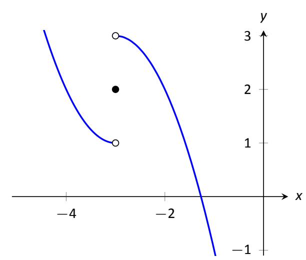
34. Use the graph of $f(x)$ provided to answer the following.

(a) $\lim_{x\to -3^-} f(x) = ?$

(b) $\lim_{x\to -3^+} f(x) = ?$

(c) $\lim_{x\to -3} f(x) = ?$

(d) Where is f continuous?



11.2 Interpretations of the Derivative

The previous section defined the derivative of a function and gave examples of how to compute it using its definition (i.e., using limits). The section also started with a brief motivation for this definition, that is, finding the instantaneous velocity of a falling object given its position function. The next section will give us more accessible tools for computing the derivative, tools that are easier to use than repeated use of limits.

This section falls in between the "What is the definition of the derivative?" and "How do I compute the derivative?" sections. Here we are concerned with "What does the derivative mean?", or perhaps, when read with the right emphasis, "What *is* the derivative?" We offer two interconnected interpretations of the derivative, hopefully explaining why we care about it and why it is worthy of study.

Interpretation of the Derivative #1: Instantaneous Rate of Change

The previous section started with an example of using the position of an object (in this case, a falling amusement–park rider) to find the object's velocity. This type of example is often used when introducing the derivative because we tend to readily recognize that velocity is the *instantaneous rate of change of position.* In general, if f is a function of x, then $f'(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Thinking back to the amusement–park ride, we asked "When time changed, at what rate did the height change?" and found the answer to be "By -64 feet per second."

Now imagine driving a car and looking at the speedometer, which reads "60 mph." Five minutes later, you wonder how far you have travelled. Certainly, lots of things could have happened in those 5 minutes; you could have intentionally sped up significantly, you might have come to a complete stop, you might have slowed to 20 mph as you passed through construction. But suppose that you know, as the driver, none of these things happened. You know you maintained a fairly consistent speed over those 5 minutes. What is a good approximation of the distance travelled?

One could argue the *only* good approximation, given the information provided, would be based on "distance = rate $\times$ time." In this case, we assume a constant rate of 60 mph with a time of $5/60$ hours. Hence we would approximate the distance travelled as 5 miles.

Referring back to the falling amusement–park ride, knowing that at $t = 2$ the velocity was -64 ft/s, we could reasonably assume that 1 second later the riders' height would have dropped by about 64 feet. Knowing that the riders were *accelerating* as they fell would inform us that this is an *under–approximation*. If all we knew was that $f(2) = 86$ and $f'(2) = -64$, we'd know that we'd have to stop the riders quickly otherwise they would hit the ground!

Units of the Derivative

It is useful to recognize the *units* of the derivative function. If y is a function of x, i.e., $y = f(x)$ for some function f, and y is measured in feet and x in seconds, then the units of $y' = f'$ are "feet per second," commonly written as "ft/s." In general, if y is measured in units P and x is measured in units Q, then y' will be measured in units "P per Q", or "P/Q." Here we see the fraction–like behaviour of the derivative in the notation:

$$\text{the units of } \frac{dy}{dx} \text{ are } \frac{\text{units of } y}{\text{units of } x}.$$

Example 206 The meaning of the derivative: World Population

Let $P(t)$ represent the world population t minutes after 12:00 a.m., January 1, 2012. It is fairly accurate to say that $P(0) = 7,028,734,178$ (`www.prb.org`). It is also fairly accurate to state that $P'(0) = 156$; that is, at midnight on January 1, 2012, the population of the world was growing by about 156 *people per minute* (note the units). Twenty days later (or, 28,800 minutes later) we could reasonably assume the population grew by about $28,800 \cdot 156 = 4,492,800$ people.

Example 207 The meaning of the derivative: Manufacturing

The term *widget* is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start–up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$.

Solution The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500. We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at rate of \$0.25 per widget (the units are "dollars per widget.") Since we have no other information to use, our best approximation for $P(1100)$ is:

$$P(1100) \approx P(1000) + P'(1000) \times 100 = \$500 + 100 \cdot 0.25 = \$525.$$

We approximate that selling 1,100 widgets returns a profit of \$525.

The previous examples made use of an important approximation tool that we first used in our previous "driving a car at 60 mph" example at the beginning of this section. Five minutes after looking at the speedometer, our best approximation for distance travelled assumed the rate of change was constant. In Examples 206 and 207 we made similar approximations. We were given rate of change information which we used to approximate total change. Notationally, we would say that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

This approximation is best when h is "small." "Small" is a relative term; when dealing with the world population, $h = 22$ days = 28,800 minutes is small in comparison to years. When manufacturing widgets, 100 widgets is small when one plans to manufacture thousands.

The Derivative and Motion

One of the most fundamental applications of the derivative is the study of motion. Let $s(t)$ be a position function, where t is time and $s(t)$ is distance. For instance, s could measure the height of a projectile or the distance an object has travelled.

Let's let $s(t)$ measure the distance travelled, in feet, of an object after t seconds of travel. Then $s'(t)$ has units "feet per second," and $s'(t)$ measures the *instantaneous rate of distance change* – it measures **velocity**.

Now consider $v(t)$, a velocity function. That is, at time t, $v(t)$ gives the velocity of an object. The derivative of v, $v'(t)$, gives the *instantaneous rate of* *velocity change* – **acceleration**. (We often think of acceleration in terms of cars: a car may "go from 0 to 60 in 4.8 seconds." This is an *average* acceleration, a measurement of how quickly the velocity changed.) If velocity is measured in feet per second, and time is measured in seconds, then the units of acceleration (i.e., the units of $v'(t)$) are "feet per second per second," or (ft/s)/s. We often shorten this to "feet per second squared," or ft/s^2, but this tends to obscure the meaning of the units.

Perhaps the most well known acceleration is that of gravity. In this text, we use $g = 32\text{ft/s}^2$ or $g = 9.8\text{m/s}^2$. What do these numbers mean?

A constant acceleration of 32(ft/s)/s means that the velocity changes by 32ft/s each second. For instance, let $v(t)$ measures the velocity of a ball thrown straight up into the air, where v has units ft/s and t is measured in seconds. The ball will have a positive velocity while travelling upwards and a negative velocity while falling down. The acceleration is thus -32ft/s^2. If $v(1) = 20\text{ft/s}$, then when $t = 2$, the velocity will have decreased by 32ft/s; that is, $v(2) = -12\text{ft/s}$. We can continue: $v(3) = -44\text{ft/s}$, and we can also figure that $v(0) = 42\text{ft/s}$.

These ideas are so important we write them out as a Key Idea.

Key Idea 39 The Derivative and Motion

1. Let $s(t)$ be the position function of an object. Then $s'(t)$ is the velocity function of the object.
2. Let $v(t)$ be the velocity function of an object. Then $v'(t)$ is the acceleration function of the object.

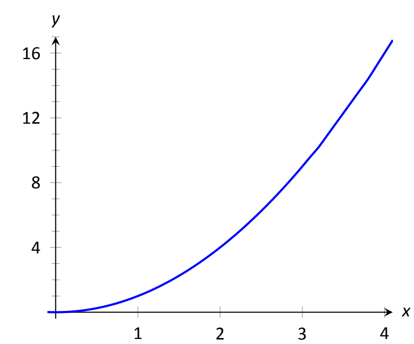


Figure 11.10: A graph of $f(x) = x^2$.

We now consider the second interpretation of the derivative given in this section. This interpretation is not independent from the first by any means; many of the same concepts will be stressed, just from a slightly different perspective.

Interpretation of the Derivative #2: The Slope of the Tangent Line

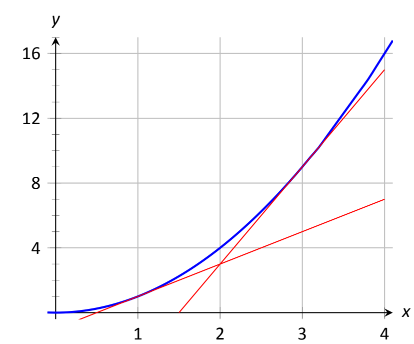


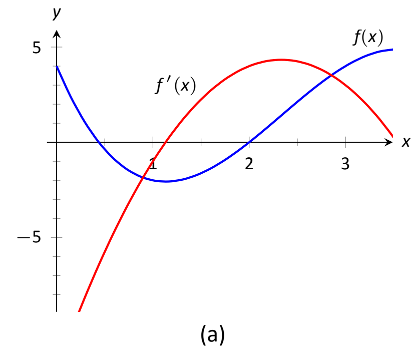
Figure 11.11: A graph of $f(x) = x^2$ and tangent lines.

Given a function $y = f(x)$, the difference quotient $\dfrac{f(c+h) - f(c)}{h}$ gives a change in y values divided by a change in x values; i.e., it is a measure of the "rise over run," or "slope," of the line that goes through two points on the graph of f: $\big(c, f(c)\big)$ and $\big(c+h, f(c+h)\big)$. As h shrinks to 0, these two points come close together; in the limit we find $f'(c)$, the slope of a special line called the tangent line that intersects f only once near $x = c$.

Lines have a constant rate of change, their slope. Nonlinear functions do not have a constant rate of change, but we can measure their *instantaneous rate of change* at a given x value c by computing $f'(c)$. We can get an idea of how f is behaving by looking at the slopes of its tangent lines. We explore this idea in the following example.

Example 208 Understanding the derivative: the rate of change

Consider $f(x) = x^2$ as shown in Figure 11.10. It is clear that at $x = 3$ the function is growing faster than at $x = 1$, as it is steeper at $x = 3$. How much faster is it growing?



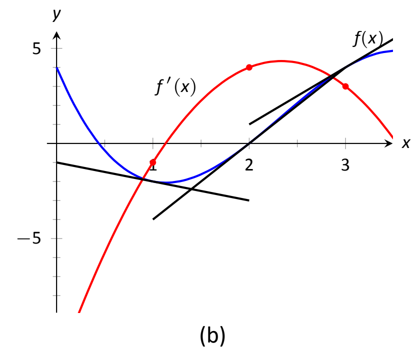


Figure 11.12: Graphs of f and f' in Example 209, along with tangent lines in (b).

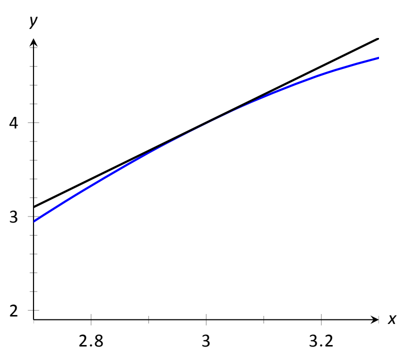


Figure 11.13: Zooming in on f at $x = 3$ for the function given in Examples 209 and 210.

Solution We can answer this directly after the following section, where we learn to quickly compute derivatives. For now, we will answer graphically, by considering the slopes of the respective tangent lines.

With practice, one can fairly effectively sketch tangent lines to a curve at a particular point. In Figure 11.11, we have sketched the tangent lines to f at $x = 1$ and $x = 3$, along with a grid to help us measure the slopes of these lines. At $x = 1$, the slope is 2; at $x = 3$, the slope is 6. Thus we can say not only is f growing faster at $x = 3$ than at $x = 1$, it is growing *three times as fast*.

Example 209 Understanding the graph of the derivative

Consider the graph of $f(x)$ and its derivative, $f'(x)$, in Figure 11.12(a). Use these graphs to find the slopes of the tangent lines to the graph of f at $x = 1$, $x = 2$, and $x = 3$.

Solution To find the appropriate slopes of tangent lines to the graph of f, we need to look at the corresponding values of f'.

The slope of the tangent line to f at $x = 1$ is $f'(1)$; this looks to be about -1.

The slope of the tangent line to f at $x = 2$ is $f'(2)$; this looks to be about 4.

The slope of the tangent line to f at $x = 3$ is $f'(3)$; this looks to be about 3.

Using these slopes, the tangent lines to f are sketched in Figure 11.12(b). Included on the graph of f' in this figure are filled circles where $x = 1$, $x = 2$ and $x = 3$ to help better visualize the y value of f' at those points.

Example 210 Approximation with the derivative

Consider again the graph of $f(x)$ and its derivative $f'(x)$ in Example 209. Use the tangent line to f at $x = 3$ to approximate the value of $f(3.1)$.

Solution Figure 11.13 shows the graph of f along with its tangent line, zoomed in at $x = 3$. Notice that near $x = 3$, the tangent line makes an excellent approximation of f. Since lines are easy to deal with, often it works well to approximate a function with its tangent line. (This is especially true when you don't actually know much about the function at hand, as we don't in this example.)

While the tangent line to f was drawn in Example 209, it was not explicitly computed. Recall that the tangent line to f at $x = c$ is $y = f'(c)(x - c) + f(c)$. While f is not explicitly given, by the graph it looks like $f(3) = 4$. Recalling that $f'(3) = 3$, we can compute the tangent line to be approximately $y = 3(x-3)+4$. It is often useful to leave the tangent line in point–slope form.

To use the tangent line to approximate $f(3.1)$, we simply evaluate y at 3.1 instead of f.

$$f(3.1) \approx y(3.1) = 3(3.1 - 3) + 4 = .1 * 3 + 4 = 4.3.$$

We approximate $f(3.1) \approx 4.3$.

To demonstrate the accuracy of the tangent line approximation, we now state that in Example 210, $f(x) = -x^3 + 7x^2 - 12x + 4$. We can evaluate $f(3.1) = 4.279$. Had we known f all along, certainly we could have just made this computation. In reality, we often only know two things:

1. What $f(c)$ is, for some value of c, and
2. what $f'(c)$ is.

For instance, we can easily observe the location of an object and its instantaneous velocity at a particular point in time. We do not have a "function f"

for the location, just an observation. This is enough to create an approximating function for f.

This last example has a direct connection to our approximation method explained above after Example 207. We stated there that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

If we know $f(c)$ and $f'(c)$ for some value $x = c$, then computing the tangent line at $(c, f(c))$ is easy: $y(x) = f'(c)(x-c) + f(c)$. In Example 210, we used the tangent line to approximate a value of f. Let's use the tangent line at $x = c$ to approximate a value of f near $x = c$; i.e., compute $y(c+h)$ to approximate $f(c+h)$, assuming again that h is "small." Note:

$$y(c+h) = f'(c)\big((c+h) - c\big) + f(c) = f'(c) \cdot h + f(c).$$

This is the exact same approximation method used above! Not only does it make intuitive sense, as explained above, it makes analytical sense, as this approximation method is simply using a tangent line to approximate a function's value.

The importance of understanding the derivative cannot be understated. When f is a function of x, $f'(x)$ measures the instantaneous rate of change of f with respect to x and gives the slope of the tangent line to f at x.

Exercises 11.2

Terms and Concepts

1. What is the instantaneous rate of change of position called?

2. Given a function $y = f(x)$, in your own words describe how to find the units of $f'(x)$.

3. What functions have a constant rate of change?

Problems

4. Given $f(5) = 10$ and $f'(5) = 2$, approximate $f(6)$.

5. Given $P(100) = -67$ and $P'(100) = 5$, approximate $P(110)$.

6. Given $z(25) = 187$ and $z'(25) = 17$, approximate $z(20)$.

7. Knowing $f(10) = 25$ and $f'(10) = 5$ and the methods described in this section, which approximation is likely to be most accurate: $f(10.1)$, $f(11)$, or $f(20)$? Explain your reasoning.

8. Given $f(7) = 26$ and $f(8) = 22$, approximate $f'(7)$.

9. Given $H(0) = 17$ and $H(2) = 29$, approximate $H'(2)$.

10. Let $V(x)$ measure the volume, in decibels, measured inside a restaurant with x customers. What are the units of $V'(x)$?

11. Let $v(t)$ measure the velocity, in ft/s, of a car moving in a straight line t seconds after starting. What are the units of $v'(t)$?

12. The height H, in feet, of a river is recorded t hours after midnight, April 1. What are the units of $H'(t)$?

13. P is the profit, in thousands of dollars, of producing and selling c cars.

    (a) What are the units of $P'(c)$?

    (b) What is likely true of $P(0)$?

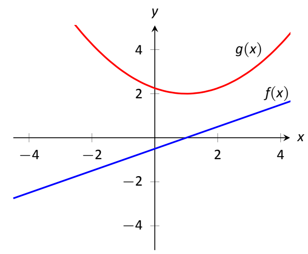
14. T is the temperature in degrees Fahrenheit, h hours after midnight on July 4 in Sidney, NE.

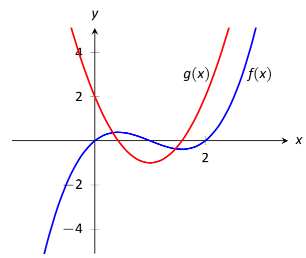
    (a) What are the units of $T'(h)$?

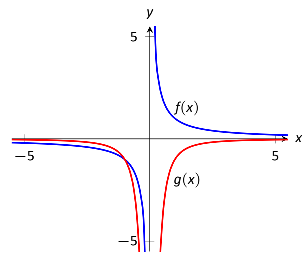
    (b) Is $T'(8)$ likely greater than or less than 0? Why?

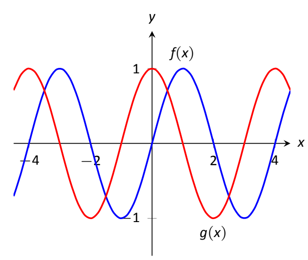
    (c) Is $T(8)$ likely greater than or less than 0? Why?

In Exercises 15 – 18, graphs of functions $f(x)$ and $g(x)$ are given. Identify which function is the derivative of the other.)

15. 

16. 

17. 

18. 

Review

In Exercises 19 – 20, use the definition to compute the derivatives of the following functions.

19. $f(x) = 5x^2$

20. $f(x) = (x - 2)^3$

In Exercises 21 – 22, numerically approximate the value of $f'(x)$ at the indicated x value.

21. $f(x) = \cos x$ at $x = \pi$.

22. $f(x) = \sqrt{x}$ at $x = 9$.

11.3 Basic Differentiation Rules

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction.* For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear function are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h \to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h \to 0} ah + 2ax + b \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem 91 Derivatives of Common Functions

1. **Constant Rule:**
   $\frac{d}{dx}(c) = 0$, where c is a constant.
2. **Power Rule:**
   $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is an integer, $n > 0$.
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(e^x) = e^x$
6. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

This theorem starts by stating an intuitive fact: constant functions have no rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f'(x)$? According to the Power Rule,

$$f'(x) = \frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f *is* x, we are asking "At what rate does x change with respect to x?" The answer is: 1. They change at the same rate.

Let's practice using this theorem.

Example 211 **Using Theorem 91 to find, and use, derivatives**

Let $f(x) = x^3$.

1. Find $f'(x)$.
2. Find the equation of the line tangent to the graph of f at $x = -1$.
3. Use the tangent line to approximate $(-1.1)^3$.
4. Sketch f, f' and the found tangent line on the same axis.

Solution

1. The Power Rule states that if $f(x) = x^3$, then $f'(x) = 3x^2$.
2. To find the equation of the line tangent to the graph of f at $x = -1$, we need a point and the slope. The point is $(-1, f(-1)) = (-1, -1)$. The slope is $f'(-1) = 3$. Thus the tangent line has equation $y = 3(x-(-1)) + (-1) = 3x + 2$.
3. We can use the tangent line to approximate $(-1.1)^3$ as -1.1 is close to -1. We have
$$(-1.1)^3 \approx 3(-1.1) + 2 = -1.3.$$
We can easily find the actual answer; $(-1.1)^3 = -1.331$.
4. See Figure 11.14.

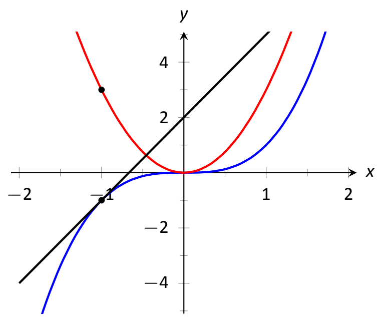


Figure 11.14: A graph of $f(x) = x^3$, along with its derivative $f'(x) = 3x^2$ and its tangent line at $x = -1$.

Theorem 91 gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin x$ nor $y = x^3 \sin x$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem 92 **Properties of the Derivative**

Let f and g be differentiable on an open interval I and let c be a real number. Then:

1. **Sum/Difference Rule:**
$$\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \pm \frac{d}{dx}\Big(g(x)\Big) = f'(x) \pm g'(x)$$
2. **Constant Multiple Rule:**
$$\frac{d}{dx}\Big(c \cdot f(x)\Big) = c \cdot \frac{d}{dx}\Big(f(x)\Big) = c \cdot f'(x).$$

Theorem 92 allows us to find the derivatives of a wide variety of functions. It can be used in conjunction with the Power Rule to find the derivatives of any polynomial. Recall in Example 201 that we found, using the limit definition, the derivative of $f(x) = 3x^2 + 5x - 7$. We can now find its derivative without expressly using limits:

$$\begin{aligned}\frac{d}{dx}\Big(3x^2 + 5x + 7\Big) &= 3\frac{d}{dx}\Big(x^2\Big) + 5\frac{d}{dx}\Big(x\Big) + \frac{d}{dx}\Big(7\Big)\\ &= 3 \cdot 2x + 5 \cdot 1 + 0\\ &= 6x + 5.\end{aligned}$$

We were a bit pedantic here, showing every step. Normally we would do all the arithmetic and steps in our head and readily find $\frac{d}{dx}\Big(3x^2+5x+7\Big) = 6x+5$.

Example 212 Using the tangent line to approximate a function value
Let $f(x) = \sin x + 2x + 1$. Approximate $f(3)$ using an appropriate tangent line.

Solution This problem is intentionally ambiguous; we are to *approximate* using an *appropriate* tangent line. How good of an approximation are we seeking? What does appropriate mean?

In the "real world," people solving problems deal with these issues all time. One must make a judgement using whatever seems reasonable. In this example, the actual answer is $f(3) = \sin 3 + 7$, where the real problem spot is $\sin 3$. What is $\sin 3$?

Since 3 is close to π, we can assume $\sin 3 \approx \sin \pi = 0$. Thus one guess is $f(3) \approx 7$. Can we do better? Let's use a tangent line as instructed and examine the results; it seems best to find the tangent line at $x = \pi$.

Using Theorem 91 we find $f'(x) = \cos x + 2$. The slope of the tangent line is thus $f'(\pi) = \cos \pi + 2 = 1$. Also, $f(\pi) = 2\pi + 1 \approx 7.28$. So the tangent line to the graph of f at $x = \pi$ is $y = 1(x-\pi)+2\pi+1 = x+\pi+1 \approx x+4.14$. Evaluated at $x = 3$, our tangent line gives $y = 3 + 4.14 = 7.14$. Using the tangent line, our final approximation is that $f(3) \approx 7.14$.

Using a calculator, we get an answer accurate to 4 places after the decimal: $f(3) = 7.1411$. Our initial guess was 7; our tangent line approximation was more accurate, at 7.14.

The point is *not* "Here's a cool way to do some math without a calculator." Sure, that might be handy sometime, but your phone could probably give you the answer. Rather, the point is to say that tangent lines are a good way of approximating, and many scientists, engineers and mathematicians often face problems too hard to solve directly. So they approximate.

Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Note: Definition 66 comes with the caveat "Where the corresponding limits exist." With f differentiable on I, it is possible that f' is *not* differentiable on all of I, and so on.

Definition 66 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function on I.

1. The *second derivative* of f is:

$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$

2. The *third derivative* of f is:

$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$

3. The n^{th} *derivative* of f is:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$ notation, not $f''''(x)$; after a while, too many ticks is too confusing.

Let's practice using this new concept.

Example 213 Finding higher order derivatives

Find the first four derivatives of the following functions:

1. $f(x) = 4x^2$
2. $f(x) = \sin x$
3. $f(x) = 5e^x$

Solution

1. Using the Power and Constant Multiple Rules, we have: $f'(x) = 8x$. Continuing on, we have

$$f''(x) = \frac{d}{dx}\big(8x\big) = 8; \qquad f'''(x) = 0; \qquad f^{(4)}(x) = 0.$$

   Notice how all successive derivatives will also be 0.

2. We employ Theorem 91 repeatedly.

$$f'(x) = \cos x; \qquad f''(x) = -\sin x; \qquad f'''(x) = -\cos x; \qquad f^{(4)}(x) = \sin x.$$

   Note how we have come right back to $f(x)$ again. (Can you quickly figure what $f^{(23)}(x)$ is?)

3. Employing Theorem 91 and the Constant Multiple Rule, we can see that

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 5e^x.$$

Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

Our first answer is a bit wordy, but is technically correct and beneficial to understand. That is,

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in Key Idea 39, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

We started this chapter with amusement–park riders free–falling with position function $f(t) = -16t^2 + 150$. It is easy to compute $f'(t) = -32t$ ft/s and $f''(t) = -32$ (ft/s)/s. We may recognize this latter constant; it is the acceleration due to gravity. In keeping with the unit notation introduced in the previous section, we say the units are "feet per second per second." This is usually shortened to "feet per second squared," written as "ft/s^2."

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or "impossible") to understand. The mathematical topic of *series* makes extensive use of higher order derivatives.

Exercises 11.3

Terms and Concepts

1. What is the name of the rule which states that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, where $n > 0$ is an integer?

2. What is $\frac{d}{dx}\left(\ln x\right)$?

3. Give an example of a function $f(x)$ where $f'(x) = f(x)$.

4. Give an example of a function $f(x)$ where $f'(x) = 0$.

5. The derivative rules introduced in this section explain how to compute the derivative of which of the following functions?
   - $f(x) = \frac{3}{x^2}$
   - $g(x) = 3x^2 - x + 17$
   - $h(x) = 5\ln x$
   - $j(x) = \sin x \cos x$
   - $k(x) = e^{x^2}$
   - $m(x) = \sqrt{x}$

6. Explain in your own words how to find the third derivative of a function $f(x)$.

7. Give an example of a function where $f'(x) \neq 0$ and $f''(x) = 0$.

8. Explain in your own words what the second derivative "means."

9. If $f(x)$ describes a position function, then $f'(x)$ describes what kind of function? What kind of function is $f''(x)$?

10. Let $f(x)$ be a function measured in pounds, where x is measured in feet. What are the units of $f''(x)$?

Problems

In Exercises 11 – 25, compute the derivative of the given function.

11. $f(x) = 7x^2 - 5x + 7$

12. $g(x) = 14x^3 + 7x^2 + 11x - 29$

13. $m(t) = 9t^5 - \frac{1}{8}t^3 + 3t - 8$

14. $f(\theta) = 9\sin\theta + 10\cos\theta$

15. $f(r) = 6e^r$

16. $g(t) = 10t^4 - \cos t + 7\sin t$

17. $f(x) = 2\ln x - x$

18. $p(s) = \frac{1}{4}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + s + 1$

19. $h(t) = e^t - \sin t - \cos t$

20. $f(x) = \ln(5x^2)$

21. $f(t) = \ln(17) + e^2 + \sin\pi/2$

22. $g(t) = (1 + 3t)^2$

23. $g(x) = (2x - 5)^3$

24. $f(x) = (1 - x)^3$

25. $f(x) = (2 - 3x)^2$

26. A property of logarithms is that $\log_a x = \frac{\log_b x}{\log_b a}$, for all bases $a, b > 0, \neq 1$.
    (a) Rewrite this identity when $b = e$, i.e., using $\log_e x = \ln x$.
    (b) Use part (a) to find the derivative of $y = \log_a x$.
    (c) Give the derivative of $y = \log_{10} x$.

In Exercises 27 – 32, compute the first four derivatives of the given function.

27. $f(x) = x^6$

28. $g(x) = 2\cos x$

29. $h(t) = t^2 - e^t$

30. $p(\theta) = \theta^4 - \theta^3$

31. $f(\theta) = \sin\theta - \cos\theta$

32. $f(x) = 1,100$

In Exercises 33 – 38, find the equations of the tangent and normal lines to the graph of the function at the given point.

33. $f(x) = x^3 - x$ at $x = 1$

34. $f(t) = e^t + 3$ at $t = 0$

35. $g(x) = \ln x$ at $x = 1$

36. $f(x) = 4\sin x$ at $x = \pi/2$

37. $f(x) = -2\cos x$ at $x = \pi/4$

38. $f(x) = 2x + 3$ at $x = 5$

Review

39. Given that $e^0 = 1$, approximate the value of $e^{0.1}$ using the tangent line to $f(x) = e^x$ at $x = 0$.

40. Approximate the value of $(3.01)^4$ using the tangent line to $f(x) = x^4$ at $x = 3$.

11.4 The Product and Quotient Rules

The previous section showed that, in some ways, derivatives behave nicely. The Constant Multiple and Sum/Difference Rules established that the derivative of $f(x) = 5x^2 + \sin x$ was not complicated. We neglected computing the derivative of things like $g(x) = 5x^2 \sin x$ and $h(x) = \frac{5x^2}{\sin x}$ on purpose; their derivatives are *not* as straightforward. (If you had to guess what their respective derivatives are, you would probably guess wrong.) For these, we need the Product and Quotient Rules, respectively, which are defined in this section.

We begin with the Product Rule.

Theorem 93 Product Rule

Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f(x)g'(x) + f'(x)g(x).$$

Important: $\frac{d}{dx}\Big(f(x)g(x)\Big) \neq f'(x)g'(x)$! While this answer is simpler than the Product Rule, it is wrong.

We practice using this new rule in an example, followed by an example that demonstrates why this theorem is true.

Example 214 Using the Product Rule

Use the Product Rule to compute the derivative of $y = 5x^2 \sin x$. Evaluate the derivative at $x = \pi/2$.

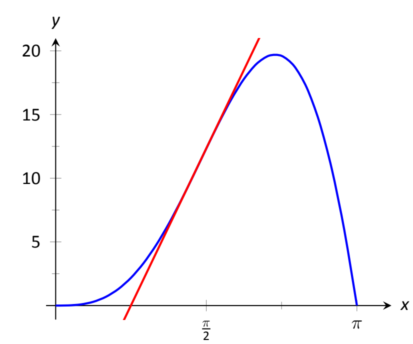


Figure 11.15: A graph of $y = 5x^2 \sin x$ and its tangent line at $x = \pi/2$.

Solution To make our use of the Product Rule explicit, let's set $f(x) = 5x^2$ and $g(x) = \sin x$. We easily compute/recall that $f'(x) = 10x$ and $g'(x) = \cos x$. Employing the rule, we have

$$\frac{d}{dx}\Big(5x^2 \sin x\Big) = 5x^2 \cos x + 10x \sin x.$$

At $x = \pi/2$, we have

$$y'(\pi/2) = 5\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 10\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 5\pi.$$

We graph y and its tangent line at $x = \pi/2$, which has a slope of 5π, in Figure 11.15. While this does not *prove* that the Produce Rule is the correct way to handle derivatives of products, it helps validate its truth.

We now investigate why the Product Rule is true.

Example 215 A proof of the Product Rule

Use the definition of the derivative to prove Theorem 93.

Solution By the limit definition, we have

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We now do something a bit unexpected; add 0 to the numerator (so that nothing is changed) in the form of $-f(x+h)g(x)+f(x+h)g(x)$, then do some regrouping as shown.

$$\begin{aligned}
\frac{d}{dx}\Big(f(x)g(x)\Big) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \qquad \text{(now add 0 to the numerator)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&\hspace{20em}\text{(regroup)}\\
&= \lim_{h\to 0}\frac{\Big(f(x+h)g(x+h)-f(x+h)g(x)\Big)+\Big(f(x+h)g(x)-f(x)g(x)\Big)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h}\\
&\hspace{20em}\text{(factor)}\\
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)\\
&\hspace{20em}\text{(apply limits)}\\
&= f(x)g'(x)+f'(x)g(x)
\end{aligned}$$

It is often true that we can recognize that a theorem is true through its proof yet somehow doubt its applicability to real problems. In the following example, we compute the derivative of a product of functions in two ways to verify that the Product Rule is indeed “right.”

Example 216 Exploring alternate derivative methods

Let $y = (x^2+3x+1)(2x^2-3x+1)$. Find y' two ways: first, by expanding the given product and then taking the derivative, and second, by applying the Product Rule. Verify that both methods give the same answer.

Solution We first expand the expression for y; a little algebra shows that $y = 2x^4+3x^3-6x^2+1$. It is easy to compute y';

$$y' = 8x^3+9x^2-12x.$$

Now apply the Product Rule.

$$\begin{aligned}
y' &= (x^2+3x+1)(4x-3)+(2x+3)(2x^2-3x+1)\\
&= \big(4x^3+9x^2-5x-3\big)+\big(4x^3-7x+3\big)\\
&= 8x^3+9x^2-12x.
\end{aligned}$$

The uninformed usually assume that “the derivative of the product is the product of the derivatives.” Thus we are tempted to say that $y' = (2x+3)(4x-3) = 8x^2+6x-9$. Obviously this is not correct.

Example 217 Using the Product Rule with a product of three functions

Let $y = x^3\ln x\cos x$. Find y'.

Solution We have a product of three functions while the Product Rule only specifies how to handle a product of two functions. Our method of handling this problem is to simply group the latter two functions together, and consider $y = x^3\big(\ln x\cos x\big)$. Following the Product Rule, we have

$$y' = (x^3)\big(\ln x\cos x\big)' + 3x^2\big(\ln x\cos x\big)$$

To evaluate $\big(\ln x \cos x\big)'$, we apply the Product Rule again:

$$\begin{aligned} &= (x^3)\big(\ln x(-\sin x) + \frac{1}{x}\cos x\big) + 3x^2\big(\ln x \cos x\big)\\ &= x^3 \ln x(-\sin x) + x^3\frac{1}{x}\cos x + 3x^2 \ln x \cos x \end{aligned}$$

Recognize the pattern in our answer above: when applying the Product Rule to a product of three functions, there are three terms added together in the final derivative. Each terms contains only one derivative of one of the original functions, and each function's derivative shows up in only one term. It is straightforward to extend this pattern to finding the derivative of a product of 4 or more functions.

We consider one more example before discussing another derivative rule.

Example 218 Using the Product Rule

Find the derivatives of the following functions.

1. $f(x) = x \ln x$
2. $g(x) = x \ln x - x$.

Solution Recalling that the derivative of $\ln x$ is $1/x$, we use the Product Rule to find our answers.

1. $\dfrac{d}{dx}\Big(x \ln x\Big) = x \cdot 1/x + 1 \cdot \ln x = 1 + \ln x.$
2. Using the result from above, we compute

$$\frac{d}{dx}\Big(x \ln x - x\Big) = 1 + \ln x - 1 = \ln x.$$

This seems significant; if the natural log function $\ln x$ is an important function (it is), it seems worthwhile to know a function whose derivative is $\ln x$. We have found one. (We leave it to the reader to find another; a correct answer will be *very* similar to this one.)

We have learned how to compute the derivatives of sums, differences, and products of functions. We now learn how to find the derivative of a quotient of functions.

Theorem 94 Quotient Rule

Let f and g be functions defined on an open interval I, where $g(x) \neq 0$ on I. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Quotient Rule is not hard to use, although it might be a bit tricky to remember. A useful mnemonic works as follows. Consider a fraction's numerator and denominator as "HI" and "LO", respectively. Then

$$\frac{d}{dx}\left(\frac{\text{HI}}{\text{LO}}\right) = \frac{\text{LO}\cdot\,\text{dHI} - \text{HI}\cdot\,\text{dLO}}{\text{LOLO}},$$

read "low dee high minus high dee low, over low low." Said fast, that phrase can roll off the tongue, making it easy to memorize. The "dee high" and "dee low" parts refer to the derivatives of the numerator and denominator, respectively.

Let's practice using the Quotient Rule.

Example 219 Using the Quotient Rule

Let $f(x) = \dfrac{5x^2}{\sin x}$. Find $f'(x)$.

SOLUTION Directly applying the Quotient Rule gives:

$$\begin{aligned}\frac{d}{dx}\left(\frac{5x^2}{\sin x}\right) &= \frac{\sin x \cdot 10x - 5x^2 \cdot \cos x}{\sin^2 x}\\ &= \frac{10x\sin x - 5x^2\cos x}{\sin^2 x}.\end{aligned}$$

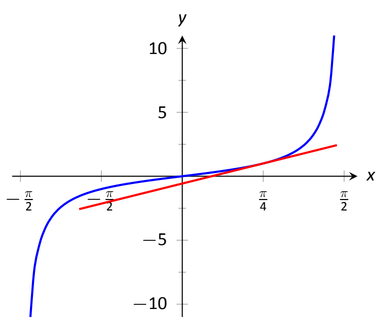


Figure 11.16: A graph of $y = \tan x$ along with its tangent line at $x = \pi/4$.

The Quotient Rule allows us to fill in holes in our understanding of derivatives of the common trigonometric functions. We start with finding the derivative of the tangent function.

Example 220 Using the Quotient Rule to find $\frac{d}{dx}\big(\tan x\big)$.

Find the derivative of $y = \tan x$.

SOLUTION At first, one might feel unequipped to answer this question. But recall that $\tan x = \sin x / \cos x$, so we can apply the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\Big(\tan x\Big) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

This is beautiful result. To confirm its truth, we can find the equation of the tangent line to $y = \tan x$ at $x = \pi/4$. The slope is $\sec^2(\pi/4) = 2$; $y = \tan x$, along with its tangent line, is graphed in Figure 11.16.

We include this result in the following theorem about the derivatives of the trigonometric functions. Recall we found the derivative of $y = \sin x$ in Example 203 and stated the derivative of the cosine function in Theorem 91. The derivatives of the cotangent, cosecant and secant functions can all be computed directly using Theorem 91 and the Quotient Rule.

Theorem 95 Derivatives of Trigonometric Functions

1. $\dfrac{d}{dx}\big(\sin x\big) = \cos x$
2. $\dfrac{d}{dx}\big(\cos x\big) = -\sin x$
3. $\dfrac{d}{dx}\big(\tan x\big) = \sec^2 x$
4. $\dfrac{d}{dx}\big(\cot x\big) = -\csc^2 x$
5. $\dfrac{d}{dx}\big(\sec x\big) = \sec x \tan x$
6. $\dfrac{d}{dx}\big(\csc x\big) = -\csc x \cot x$

To remember the above, it may be helpful to keep in mind that the derivatives of the trigonometric functions that start with "c" have a minus sign in them.

Example 221 Exploring alternate derivative methods

In Example 219 the derivative of $f(x) = \dfrac{5x^2}{\sin x}$ was found using the Quotient Rule. Rewriting f as $f(x) = 5x^2 \csc x$, find f' using Theorem 95 and verify the two answers are the same.

SOLUTION We found in Example 219 that the $f'(x) = \dfrac{10x\sin x - 5x^2\cos x}{\sin^2 x}$. We now find f' using the Product Rule, considering f as $f(x) = 5x^2 \csc x$.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\Big(5x^2 \csc x\Big) \\
&= 5x^2(-\csc x \cot x) + 10x \csc x \qquad \text{(now rewrite trig functions)} \\
&= 5x^2 \cdot \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} + \frac{10x}{\sin x} \\
&= \frac{-5x^2 \cos x}{\sin^2 x} + \frac{10x}{\sin x} \qquad \text{(get common denominator)} \\
&= \frac{10x \sin x - 5x^2 \cos x}{\sin^2 x}
\end{aligned}
$$

Finding f' using either method returned the same result. At first, the answers looked different, but some algebra verified they are the same. In general, there is not one final form that we seek; the immediate result from the Product Rule is fine. Work to "simplify" your results into a form that is most readable and useful to you.

The Quotient Rule gives other useful results, as show in the next example.

Example 222 Using the Quotient Rule to expand the Power Rule

Find the derivatives of the following functions.

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{1}{x^n}$, where $n > 0$ is an integer.

SOLUTION We employ the Quotient Rule.

1. $f'(x) = \dfrac{x \cdot 0 - 1 \cdot 1}{x^2} = -\dfrac{1}{x^2}.$

2. $f'(x) = \dfrac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{(x^n)^2} = -\dfrac{nx^{n-1}}{x^{2n}} = -\dfrac{n}{x^{n+1}}.$

The derivative of $y = \dfrac{1}{x^n}$ turned out to be rather nice. It gets better. Consider:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x^n}\right) &= \frac{d}{dx}\left(x^{-n}\right) && \text{(apply result from Example 222)}\\ &= -\frac{n}{x^{n+1}} && \text{(rewrite algebraically)}\\ &= -nx^{-(n+1)}\\ &= -nx^{-n-1}.\end{aligned}$$

This is reminiscent of the Power Rule: multiply by the power, then subtract 1 from the power. We now add to our previous Power Rule, which had the restriction of $n > 0$.

Theorem 96 Power Rule with Integer Exponents

Let $f(x) = x^n$, where $n \neq 0$ is an integer. Then

$$f'(x) = n \cdot x^{n-1}.$$

Taking the derivative of many functions is relatively straightforward. It is clear (with practice) what rules apply and in what order they should be applied. Other functions present multiple paths; different rules may be applied depending on how the function is treated. One of the beautiful things about calculus is that there is not "the" right way; each path, when applied correctly, leads to the same result, the derivative. We demonstrate this concept in an example.

Example 223 Exploring alternate derivative methods

Let $f(x) = \dfrac{x^2 - 3x + 1}{x}$. Find $f'(x)$ in each of the following ways:

1. By applying the Quotient Rule,
2. by viewing f as $f(x) = \left(x^2 - 3x + 1\right) \cdot x^{-1}$ and applying the Product and Power Rules, and
3. by "simplifying" first through division.

Verify that all three methods give the same result.

Solution

1. Applying the Quotient Rule gives:

$$f'(x) = \frac{x \cdot \left(2x - 3\right) - \left(x^2 - 3x + 1\right) \cdot 1}{x^2} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}.$$

2. By rewriting f, we can apply the Product and Power Rules as follows:

$$\begin{aligned} f'(x) &= \big(x^2-3x+1\big)\cdot(-1)x^{-2} + \big(2x-3\big)\cdot x^{-1} \\ &= -\frac{x^2-3x+1}{x^2} + \frac{2x-3}{x} \\ &= -\frac{x^2-3x+1}{x^2} + \frac{2x^2-3x}{x^2} \\ &= \frac{x^2-1}{x^2} = 1-\frac{1}{x^2}, \end{aligned}$$

   the same result as above.

3. As $x \neq 0$, we can divide through by x first, giving $f(x) = x - 3 + \dfrac{1}{x}$. Now apply the Power Rule.

$$f'(x) = 1 - \frac{1}{x^2},$$

   the same result as before.

Example 223 demonstrates three methods of finding f'. One is hard pressed to argue for a "best method" as all three gave the same result without too much difficulty, although it is clear that using the Product Rule required more steps. Ultimately, the important principle to take away from this is: reduce the answer to a form that seems "simple" and easy to interpret. In that example, we saw different expressions for f', including:

$$1-\frac{1}{x^2} = \frac{x\cdot\big(2x-3\big)-\big(x^2-3x+1\big)\cdot 1}{x^2} = \big(x^2-3x+1\big)\cdot(-1)x^{-2} + \big(2x-3\big)\cdot x^{-1}.$$

They are equal; they are all correct; only the first is "clear." Work to make answers clear.

In the next section we continue to learn rules that allow us to more easily compute derivatives than using the limit definition directly. We have to memorize the derivatives of a certain set of functions, such as "the derivative of $\sin x$ is $\cos x$." The Sum/Difference, Constant Multiple, Power, Product and Quotient Rules show us how to find the derivatives of certain combinations of these functions. The next section shows how to find the derivatives when we *compose* these functions together.

Exercises 11.4

Terms and Concepts

1. T/F: The Product Rule states that $\frac{d}{dx}\big(x^2\sin x\big) = 2x\cos x$.

2. T/F: The Quotient Rule states that $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\cos x}{2x}$.

3. T/F: The derivatives of the trigonometric functions that start with "c" have minus signs in them.

4. What derivative rule is used to extend the Power Rule to include negative integer exponents?

5. T/F: Regardless of the function, there is always exactly one right way of computing its derivative.

6. In your own words, explain what it means to make your answers "clear."

Problems

In Exercises 7 – 10:

(a) Use the Product Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Product Rule.

(c) Show that the answers from (a) and (b) are equivalent.

7. $f(x) = x(x^2 + 3x)$

8. $g(x) = 2x^2(5x^3)$

9. $h(s) = (2s - 1)(s + 4)$

10. $f(x) = (x^2 + 5)(3 - x^3)$

In Exercises 11 – 14:

(a) Use the Quotient Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Quotient Rule.

(c) Show that the answers from (a) and (b) are equivalent.

11. $f(x) = \frac{x^2 + 3}{x}$

12. $g(x) = \frac{x^3 - 2x^2}{2x^2}$

13. $h(s) = \frac{3}{4s^3}$

14. $f(t) = \frac{t^2 - 1}{t + 1}$

In Exercises 15 – 29, compute the derivative of the given function.

15. $f(x) = x\sin x$

16. $f(t) = \frac{1}{t^2}(\csc t - 4)$

17. $g(x) = \frac{x + 7}{x - 5}$

18. $g(t) = \frac{t^5}{\cos t - 2t^2}$

19. $h(x) = \cot x - e^x$

20. $h(t) = 7t^2 + 6t - 2$

21. $f(x) = \frac{x^4 + 2x^3}{x + 2}$

22. $f(x) = (16x^3 + 24x^2 + 3x)\frac{7x - 1}{16x^3 + 24x^2 + 3x}$

23. $f(t) = t^5(\sec t + e^t)$

24. $f(x) = \frac{\sin x}{\cos x + 3}$

25. $g(x) = e^2\big(\sin(\pi/4) - 1\big)$

26. $g(t) = 4t^3e^t - \sin t\cos t$

27. $h(t) = \frac{t^2\sin t + 3}{t^2\cos t + 2}$

28. $f(x) = x^2e^x\tan x$

29. $g(x) = 2x\sin x\sec x$

In Exercises 30 – 33, find the equations of the tangent and normal lines to the graph of g at the indicated point.

30. $g(s) = e^s(s^2 + 2)$ at $(0, 2)$.

31. $g(t) = t\sin t$ at $\left(\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$

32. $g(x) = \frac{x^2}{x - 1}$ at $(2, 4)$

33. $g(\theta) = \frac{\cos\theta - 8\theta}{\theta + 1}$ at $(0, -5)$

In Exercises 34 – 37, find the x–values where the graph of the function has a horizontal tangent line.

34. $f(x) = 6x^2 - 18x - 24$

35. $f(x) = x\sin x$ on $[-1, 1]$

36. $f(x) = \dfrac{x}{x+1}$

37. $f(x) = \dfrac{x^2}{x+1}$

In Exercises 38 – 41, find the requested derivative.

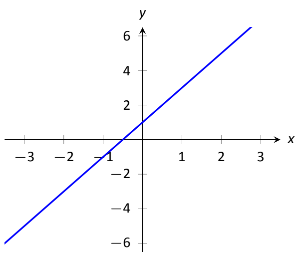
38. $f(x) = x \sin x$; find $f''(x)$.

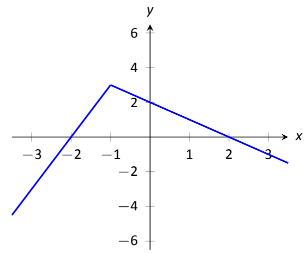
39. $f(x) = x \sin x$; find $f^{(4)}(x)$.

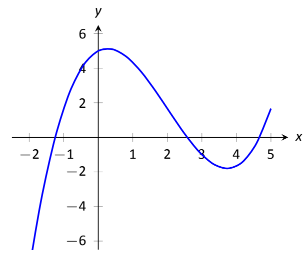
40. $f(x) = \csc x$; find $f''(x)$.

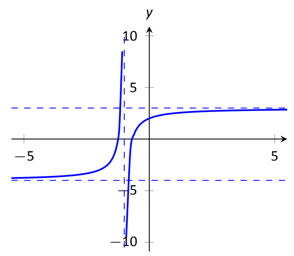
41. $f(x) = (x^3 - 5x + 2)(x^2 + x - 7)$; find $f^{(8)}(x)$.

In Exercises 42 – 45, use the graph of $f(x)$ to sketch $f'(x)$.

42. 

43. 

44. 

45. 

11.5 The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = -\sin(2x)$, where we recognize $-\sin x$ as the derivative of $\cos x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. In Example 227 we'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$, read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$, etc.

To motivate the rule, let's look at three derivatives we can already compute.

Example 224 Exploring similar derivatives

Find the derivatives of $F_1(x) = (1-x)^2$, $F_2(x) = (1-x)^3$, and $F_3(x) = (1-x)^4$. (We'll see later why we are using subscripts for different functions and an uppercase F.)

Solution In order to use the rules we already have, we must first expand each function as $F_1(x) = 1 - 2x + x^2$, $F_2(x) = 1 - 3x + 3x^2 - x^3$ and $F_3(x) = 1 - 4x + 6x^2 - 4x^3 + x^4$.

It is not hard to see that:

$F_1'(x) = -2 + 2x,$

$F_2'(x) = -3 + 6x - 3x^2$ and

$F_3'(x) = -4 + 12x - 12x^2 + 4x^3.$

An interesting fact is that these can be rewritten as

$$F_1'(x) = -2(1-x), \quad F_2'(x) = -3(1-x)^2 \text{ and } F_3'(x) = -4(1-x)^3.$$

A pattern might jump out at you. Recognize that each of these functions is a composition, letting $g(x) = 1 - x$:

$$\begin{aligned} F_1(x) &= f_1(g(x)), \quad \text{where } f_1(x) = x^2, \\ F_2(x) &= f_2(g(x)), \quad \text{where } f_2(x) = x^3, \\ F_3(x) &= f_3(g(x)), \quad \text{where } f_3(x) = x^4. \end{aligned}$$

We'll come back to this example after giving the formal statements of the Chain Rule; for now, we are just illustrating a pattern.

> **Theorem 97 The Chain Rule**
>
> Let $y = f(u)$ be a differentiable function of u and let $u = g(x)$ be a differentiable function of x. Then $y = f(g(x))$ is a differentiable function of x, and
> $$y' = f'(g(x)) \cdot g'(x).$$

To help understand the Chain Rule, we return to Example 224.

Example 225 Using the Chain Rule

Use the Chain Rule to find the derivatives of the following functions, as given in Example 224.

Solution Example 224 ended with the recognition that each of the given functions was actually a composition of functions. To avoid confusion, we ignore most of the subscripts here.

$F_1(x) = (1-x)^2$**:**

We found that

$$y = (1-x)^2 = f(g(x)), \text{ where } f(x) = x^2 \text{ and } g(x) = 1 - x.$$

To find y', we apply the Chain Rule. We need $f'(x) = 2x$ and $g'(x) = -1$.

Part of the Chain Rule uses $f'(g(x))$. This means substitute $g(x)$ for x in the equation for $f'(x)$. That is, $f'(x) = 2(1-x)$. Finishing out the Chain Rule we have

$$y' = f'(g(x)) \cdot g'(x) = 2(1-x) \cdot (-1) = -2(1-x) = 2x - 2.$$

$F_2(x) = (1-x)^3$:

Let $y = (1-x)^3 = f(g(x))$, where $f(x) = x^3$ and $g(x) = (1-x)$. We have $f'(x) = 3x^2$, so $f'(g(x)) = 3(1-x)^2$. The Chain Rule then states

$$y' = f'(g(x)) \cdot g'(x) = 3(1-x)^2 \cdot (-1) = -3(1-x)^2.$$

$F_3(x) = (1-x)^4$:

Finally, when $y = (1-x)^4$, we have $f(x) = x^4$ and $g(x) = (1-x)$. Thus $f'(x) = 4x^3$ and $f'(g(x)) = 4(1-x)^3$. Thus

$$y' = f'(g(x)) \cdot g'(x) = 4(1-x)^3 \cdot (-1) = -4(1-x)^3.$$

Example 225 demonstrated a particular pattern: when $f(x) = x^n$, then $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$. This is called the Generalized Power Rule.

> **Theorem 98 Generalized Power Rule**
>
> Let $g(x)$ be a differentiable function and let $n \neq 0$ be an integer. Then
> $$\frac{d}{dx}\Big(g(x)^n\Big) = n \cdot \big(g(x)\big)^{n-1} \cdot g'(x).$$

This allows us to quickly find the derivative of functions like $y = (3x^2 - 5x + 7 + \sin x)^{20}$. While it may look intimidating, the Generalized Power Rule states that

$$y' = 20(3x^2 - 5x + 7 + \sin x)^{19} \cdot (6x - 5 + \cos x).$$

Treat the derivative–taking process step–by–step. In the example just given, first multiply by 20, the rewrite the inside of the parentheses, raising it all to the 19th power. Then think about the derivative of the expression inside the parentheses, and multiply by that.

We now consider more examples that employ the Chain Rule.

Example 226 **Using the Chain Rule**

Find the derivatives of the following functions:

1. $y = \sin 2x$
2. $y = \ln(4x^3 - 2x^2)$
3. $y = e^{-x^2}$

SOLUTION

1. Consider $y = \sin 2x$. Recognize that this is a composition of functions, where $f(x) = \sin x$ and $g(x) = 2x$. Thus

$$y' = f'(g(x)) \cdot g'(x) = \cos(2x) \cdot 2 = 2\cos 2x.$$

2. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln x$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

This leads us to:

$$y' = \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) = \frac{12x^2 - 4x}{4x^3 - 2x^2} = \frac{4x(3x-1)}{2x(2x^2 - x)} = \frac{2(3x-1)}{2x^2 - x}.$$

3. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$, we have

$$y' = e^{-x^2} \cdot (-2x) = (-2x)e^{-x^2}.$$

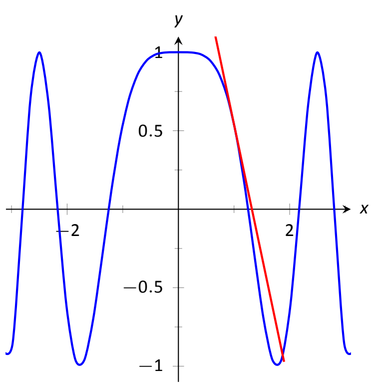


Figure 11.17: $f(x) = \cos x^2$ sketched along with its tangent line at $x = 1$.

Example 227 **Using the Chain Rule to find a tangent line**

Let $f(x) = \cos x^2$. Find the equation of the line tangent to the graph of f at $x = 1$.

SOLUTION The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$ with slope $f'(1)$. To find f', we need the Chain Rule.

$f'(x) = -\sin(x^2) \cdot (2x) = -2x\sin x^2$. Evaluated at $x = 1$, we have $f'(1) = -2\sin 1 \approx -1.68$. Thus the equation of the tangent line is

$$y = -1.68(x - 1) + 0.54.$$

The tangent line is sketched along with f in Figure 11.17.

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}.$$

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15}-\cos x+e^x)\Big)=\frac{45x^{14}+\sin x+e^x}{3x^{15}-\cos x+e^x}.$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\Big(\ln u\Big)=\frac{u'}{u}.$$

The following is a short list of how the Chain Rule can be quickly applied to familiar functions.

1. $\frac{d}{dx}\Big(u^n\Big)=n\cdot u^{n-1}\cdot u'.$
2. $\frac{d}{dx}\Big(e^u\Big)=u'\cdot e^u.$
3. $\frac{d}{dx}\Big(\sin u\Big)=u'\cdot\cos u.$
4. $\frac{d}{dx}\Big(\cos u\Big)=-u'\cdot\sin u.$
5. $\frac{d}{dx}\Big(\tan u\Big)=u'\cdot\sec^2 u.$

Of course, the Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example 228 Using the Product, Quotient and Chain Rules

Find the derivatives of the following functions.

1. $f(x)=x^5\sin 2x^3$ 2. $f(x)=\frac{5x^3}{e^{-x^2}}$.

Solution

1. We must use the Product and Chain Rules. Do not think that you must be able to "see" the whole answer immediately; rather, just proceed step–by–step.

$$f'(x)=x^5\big(6x^2\cos 2x^3\big)+5x^4\big(\sin 2x^3\big)=6x^7\cos 2x^3+5x^4\sin 2x^3.$$

2. We must employ the Quotient Rule along with the Chain Rule. Again, proceed step–by–step.

$$f'(x)=\frac{e^{-x^2}\big(15x^2\big)-5x^3\big((-2x)e^{-x^2}\big)}{\big(e^{-x^2}\big)^2}=\frac{e^{-x^2}\big(10x^4+15x^2\big)}{e^{-2x^2}}$$
$$=e^{x^2}\big(10x^4+15x^2\big).$$

A key to correctly working these problems is to break the problem down into smaller, more manageable pieces. For instance, when using the Product and Chain Rules together, just consider the first part of the Product Rule at first: $f(x)g'(x)$. Just rewrite $f(x)$, then find $g'(x)$. Then move on to the $f'(x)g(x)$ part. Don't attempt to figure out both parts at once.

Likewise, using the Quotient Rule, approach the numerator in two steps and handle the denominator after completing that. Only simplify afterwards.

We can also employ the Chain Rule itself several times, as shown in the next example.

Example 229 Using the Chain Rule multiple times

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

SOLUTION Recognize that we have the $g(x) = \tan(6x^3 - 7x)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \big(\tan(6x^3 - 7x)\big)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step–by–step.

$$y' = 5\big(\tan(6x^3 - 7x)\big)^4 \cdot g'(x).$$

We now find $g'(x)$. We again need the Chain Rule;

$$g'(x) = \sec^2(6x^3 - 7x) \cdot (18x^2 - 7).$$

Combine this with what we found above to give

$$\begin{aligned} y' &= 5\big(\tan(6x^3 - 7x)\big)^4 \cdot \sec^2(6x^3 - 7x) \cdot (18x^2 - 7) \\ &= (90x^2 - 35)\sec^2(6x^3 - 7x)\tan^4(6x^3 - 7x). \end{aligned}$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the derivative can be found. In fact, it is not "hard;" one must take several simple steps and be careful to keep track of how to apply each of these steps.

It is a traditional mathematical exercise to find the derivatives of arbitrarily complicated functions just to demonstrate that it *can be done*. Just break everything down into smaller pieces.

Example 230 Using the Product, Quotient and Chain Rules

Find the derivative of $f(x) = \dfrac{x\cos(x^{-2}) - \sin^2(e^{4x})}{\ln(x^2 + 5x^4)}$.

SOLUTION This function likely has no practical use outside of demonstrating derivative skills. The answer is given below without simplification. It employs the Quotient Rule, the Product Rule, and the Chain Rule three times.

$f'(x) =$

$$\frac{\left(\begin{array}{c} \ln(x^2+5x^4)\cdot \Big[\big(x\cdot(-\sin(x^{-2}))\cdot(-2x^{-3}) + 1\cdot\cos(x^{-2})\big) \\ -2\sin(e^{4x})\cdot\cos(e^{4x})\cdot(4e^{4x})\Big] \\ -\Big(x\cos(x^{-2}) - \sin^2(e^{4x})\Big)\cdot\dfrac{2x+20x^3}{x^2+5x^4} \end{array}\right)}{\big(\ln(x^2+5x^4)\big)^2}.$$

The reader is highly encouraged to look at each term and recognize why it is there. (I.e., the Quotient Rule is used; in the numerator, identify the "LOdHI" term, etc.) This example demonstrates that derivatives can be computed systematically, no matter how arbitrarily complicated the function is.

The Chain Rule also has theoretic value. That is, it can be used to find the derivatives of functions that we have not yet learned as we do in the following example.

Example 231 The Chain Rule and exponential functions
Use the Chain Rule to find the derivative of $y = a^x$ where $a > 0$, $a \neq 1$ is constant.

SOLUTION We only know how to find the derivative of one exponential function: $y = e^x$; this problem is asking us to find the derivative of functions such as $y = 2^x$.

This can be accomplished by rewriting a^x in terms of e. Recalling that e^x and $\ln x$ are inverse functions, we can write

$$a = e^{\ln a} \quad \text{and so} \quad y = a^x = e^{\ln(a^x)}.$$

By the exponent property of logarithms, we can "bring down" the power to get

$$y = a^x = e^{x(\ln a)}.$$

The function is now the composition $y = f(g(x))$, with $f(x) = e^x$ and $g(x) = x(\ln a)$. Since $f'(x) = e^x$ and $g'(x) = \ln a$, the Chain Rule gives

$$y' = e^{x(\ln a)} \cdot \ln a.$$

Recall that the $e^{x(\ln a)}$ term on the right hand side is just a^x, our original function. Thus, the derivative contains the original function itself. We have

$$y' = y \cdot \ln a = a^x \cdot \ln a.$$

The Chain Rule, coupled with the derivative rule of e^x, allows us to find the derivatives of all exponential functions.

The previous example produced a result worthy of its own "box."

Theorem 99 Derivatives of Exponential Functions

Let $f(x) = a^x$, for $a > 0, a \neq 1$. Then f is differentiable for all real numbers and

$$f'(x) = \ln a \cdot a^x.$$

Alternate Chain Rule Notation

It is instructive to understand what the Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u, where $u = g(x)$ is a function of x, as stated in Theorem 97. Then, through the composition $f \circ g$, we can think of y as a function of x, as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ \frac{dy}{dx} &= y'(u) \cdot u'(x) && \text{(since } y = f(u) \text{ and } u = g(x)\text{)} \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{(using "fractional" notation for the derivative)} \end{aligned}$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "du" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

It is important to realize that we *are not* cancelling these terms; the derivative notation of $\frac{dy}{dx}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behaviour. It makes applying the Chain Rule easy with multiple variables. For instance,

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}.$$

where $\bigcirc$ and $\triangle$ are any variables you'd like to use.

One of the most common ways of "visualizing" the Chain Rule is to consider a set of gears, as shown in Figure 11.18. The gears have 36, 18, and 6 teeth, respectively. That means for every revolution of the x gear, the u gear revolves twice. That is, the rate at which the u gear makes a revolution is twice as fast as the rate at which the x gear makes a revolution. Using the terminology of calculus, the rate of u-change, with respect to x, is $\frac{du}{dx} = 2$.

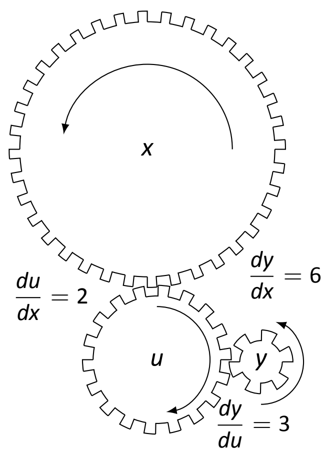


Figure 11.18: A series of gears to demonstrate the Chain Rule. Note how $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Likewise, every revolution of u causes 3 revolutions of y: $\frac{dy}{du} = 3$. How does y change with respect to x? For each revolution of x, y revolves 6 times; that is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 3 = 6.$$

We can then extend the Chain Rule with more variables by adding more gears to the picture.

It is difficult to overstate the importance of the Chain Rule. So often the functions that we deal with are compositions of two or more functions, requiring us to use this rule to compute derivatives. It is often used in practice when actual functions are unknown. Rather, through measurement, we can calculate $\frac{dy}{du}$ and $\frac{du}{dx}$. With our knowledge of the Chain Rule, finding $\frac{dy}{dx}$ is straightforward.

In the next section, we use the Chain Rule to justify another differentiation technique. There are many curves that we can draw in the plane that fail the "vertical line test." For instance, consider $x^2 + y^2 = 1$, which describes the unit circle. We may still be interested in finding slopes of tangent lines to the circle at various points. The next section shows how we can find $\frac{dy}{dx}$ without first "solving for y." While we can in this instance, in many other instances solving for y is impossible. In these situations, *implicit differentiation* is indispensable.

Exercises 11.5

Terms and Concepts

1. T/F: The Chain Rule describes how to evaluate the derivative of a composition of functions.

2. T/F: The Generalized Power Rule states that $\frac{d}{dx}\Big(g(x)^n\Big) = n\big(g(x)\big)^{n-1}$.

3. T/F: $\frac{d}{dx}\big(\ln(x^2)\big) = \frac{1}{x^2}$.

4. T/F: $\frac{d}{dx}\big(3^x\big) \approx 1.1 \cdot 3^x$.

5. T/F: $\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy}$

6. T/F: Taking the derivative of $f(x) = x^2\sin(5x)$ requires the use of both the Product and Chain Rules.

Problems

In Exercises 7 – 28, compute the derivative of the given function.

7. $f(x) = (4x^3 - x)^{10}$

8. $f(t) = (3t - 2)^5$

9. $g(\theta) = (\sin\theta + \cos\theta)^3$

10. $h(t) = e^{3t^2+t-1}$

11. $f(x) = \left(x + \frac{1}{x}\right)^4$

12. $f(x) = \cos(3x)$

13. $g(x) = \tan(5x)$

14. $h(t) = \sin^4(2t)$

15. $p(t) = \cos^3(t^2 + 3t + 1)$

16. $f(x) = \ln(\cos x)$

17. $f(x) = \ln(x^2)$

18. $f(x) = 2\ln(x)$

19. $g(r) = 4^r$

20. $g(t) = 5^{\cos t}$

21. $g(t) = 15^2$

22. $m(w) = \dfrac{3^w}{2^w}$

23. $h(t) = \dfrac{2^t + 3}{3^t + 2}$

24. $m(w) = \dfrac{3^w + 1}{2^w}$

25. $f(x) = \dfrac{3^{x^2} + x}{2^{x^2}}$

26. $f(x) = x^2\sin(5x)$

27. $g(t) = \cos(t^2 + 3t)\sin(5t - 7)$

28. $g(t) = \cos(\frac{1}{t})e^{5t^2}$

In Exercises 29 – 32, find the equations of tangent and normal lines to the graph of the function at the given point. Note: the functions here are the same as in Exercises 7 through 10.

29. $f(x) = (4x^3 - x)^{10}$ at $x = 0$

30. $f(t) = (3t - 2)^5$ at $t = 1$

31. $g(\theta) = (\sin\theta + \cos\theta)^3$ at $\theta = \pi/2$

32. $h(t) = e^{3t^2+t-1}$ at $t = -1$

33. Compute $\frac{d}{dx}\big(\ln(kx)\big)$ two ways:

    (a) Using the Chain Rule, and

    (b) by first using the logarithm rule $\ln(ab) = \ln a + \ln b$, then taking the derivative.

34. Compute $\frac{d}{dx}\big(\ln(x^k)\big)$ two ways:

    (a) Using the Chain Rule, and

    (b) by first using the logarithm rule $\ln(a^p) = p\ln a$, then taking the derivative.

Review

35. The "wind chill factor" is a measurement of how cold it "feels" during cold, windy weather. Let $W(w)$ be the wind chill factor, in degrees Fahrenheit, when it is 25°F outside with a wind of w mph.

    (a) What are the units of $W'(w)$?

    (b) What would you expect the sign of $W'(10)$ to be?

36. Find the derivatives of the following functions.

    (a) $f(x) = x^2e^x\cot x$

    (b) $g(x) = 2^x3^x4^x$

12: The Graphical Behavior of Functions

Our study of limits led to continuous functions, which is a certain class of functions that behave in a particularly nice way. Limits then gave us an even nicer class of functions, functions that are differentiable.

This chapter explores many of the ways we can take advantage of the information that continuous and differentiable functions provide.

12.1 Extreme Values

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object traveled. If a function describes the value of a stock, we might want to know how the highest/lowest values the stock attained over the past year. We call such values *extreme values.*

Definition 67 Extreme Values

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum** (also, **absolute minimum**) of f on I if $f(c) \leq f(x)$ for all x in I.

2. $f(c)$ is the **maximum** (also, **absolute maximum**) of f on I if $f(c) \geq f(x)$ for all x in I.

The maximum and minimum values are the **extreme values**, or **extrema**, of f on I.

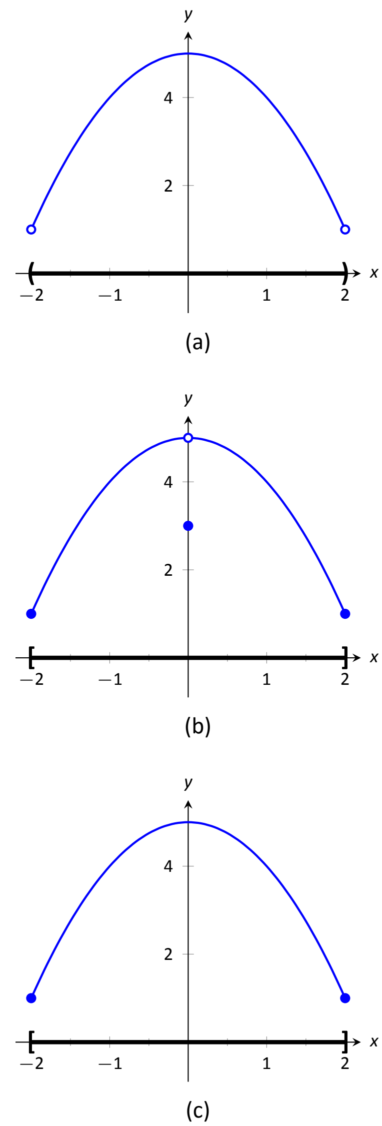


Figure 12.1: Graphs of functions with and without extreme values.

Consider Figure 12.1. The function displayed in (a) has a maximum, but no minimum, as the interval over which the function is defined is open. In (b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in (c) has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.

It is possible for discontinuous functions defined on an open interval to have both a maximum and minimum value, but we have just seen examples where they did not. On the other hand, continuous functions on a closed interval *always* have a maximum and minimum value.

Theorem 100 The Extreme Value Theorem

Let f be a continuous function defined on a closed interval I. Then f has both a maximum and minimum value on I.

Note: The extreme values of a function are "y" values, values the function attains, not the input values.

This theorem states that f has extreme values, but it does not offer any advice about how/where to find these values. The process can seem to be fairly easy, as the next example illustrates. After the example, we will draw on lessons learned to form a more general and powerful method for finding extreme values.

Example 232 Approximating extreme values

Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$, as graphed in Figure 12.2. Approximate the extreme values of f.

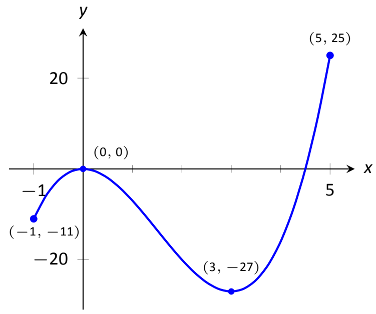


Figure 12.2: A graph of $f(x) = 2x^3 - 9x^2$ as in Example 232.

Solution The graph is drawn in such a way to draw attention to certain points. It certainly seems that the smallest y value is -27, found when $x = 3$. It also seems that the largest y value is 25, found at the endpoint of I, $x = 5$. We use the word *seems*, for by the graph alone we cannot be sure the smallest value is not less than -27. Since the problem asks for an approximation, we approximate the extreme values to be 25 and -27.

Notice how the minimum value came at "the bottom of a hill," and the maximum value came at an endpoint. Also note that while 0 is not an extreme value, it would be if we narrowed our interval to $[-1, 4]$. The idea that the point $(0, 0)$ is the location of an extreme value for some interval is important, leading us to a definition.

Definition 68 Relative Minimum and Relative Maximum

Let f be defined on an interval I containing c.

1. If there is an open interval containing c such that $f(c)$ is the minimum value, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.
2. If there is an open interval containing c such that $f(c)$ is the maximum value, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

The relative maximum and minimum values comprise the **relative extrema** of f.

Note: The terms *local minimum* and *local maximum* are often used as synonyms for *relative minimum* and *relative maximum*.

We briefly practice using these definitions.

Example 233 Approximating relative extrema

Consider $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$, as shown in Figure 12.3. Approximate the relative extrema of f. At each of these points, evaluate f'.

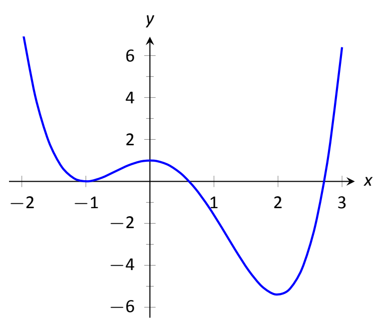


Figure 12.3: A graph of $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$ as in Example 233.

Solution We still do not have the tools to exactly find the relative extrema, but the graph does allow us to make reasonable approximations. It seems f has relative minima at $x = -1$ and $x = 2$, with values of $f(-1) = 0$ and $f(2) = -5.4$. It also seems that f has a relative maximum at the point $(0, 1)$.

We approximate the relative minima to be 0 and -5.4; we approximate the relative maximum to be 1.

It is straightforward to evaluate $f'(x) = \frac{1}{5}(12x^3 - 12x^2 - 24x)$ at $x = 0, 1$ and 2. In each case, $f'(x) = 0$.

Example 234 Approximating relative extrema

Approximate the relative extrema of $f(x) = (x-1)^{2/3} + 2$, shown in Figure 12.4. At each of these points, evaluate f'.

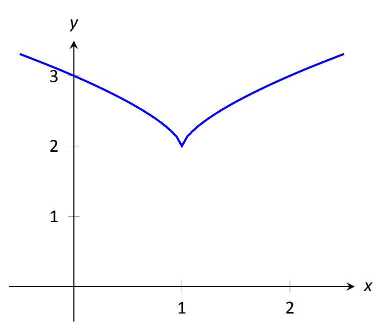


Figure 12.4: A graph of $f(x) = (x-1)^{2/3} + 2$ as in Example 234.

Solution The figure implies that f does not have any relative maxima, but has a relative minimum at $(1, 2)$. In fact, the graph suggests that not only is this point a relative minimum, $y = f(1) = 2$ *the* minimum value of the function.

We compute $f'(x) = \frac{2}{3}(x-1)^{-1/3}$. When $x = 1$, f' is undefined.

What can we learn from the previous two examples? We were able to visually approximate relative extrema, and at each such point, the derivative was either 0 or it was not defined. This observation holds for all functions, leading to a definition and a theorem.

Definition 69 Critical Numbers and Critical Points

Let f be defined at c. The value c is a **critical number** (or **critical value**) of f if $f'(c) = 0$ or $f'(c)$ is not defined.

If c is a critical number of f, then the point $(c, f(c))$ is a **critical point** of f.

Theorem 101 Relative Extrema and Critical Points

Let a function f have a relative extrema at the point $(c, f(c))$. Then c is a critical number of f.

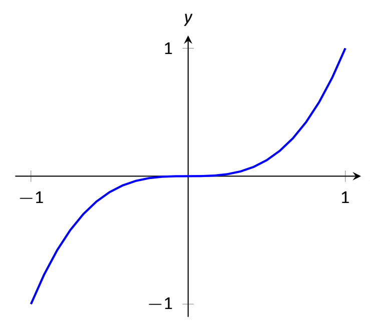


Figure 12.5: A graph of $f(x) = x^3$ which has a critical value of $x = 0$, but no relative extrema.

Be careful to understand that this theorem states "All relative extrema occur at critical points." It does not say "All critical numbers produce relative extrema." For instance, consider $f(x) = x^3$. Since $f'(x) = 3x^2$, it is straightforward to determine that $x = 0$ is a critical number of f. However, f has no relative extrema, as illustrated in Figure 12.5.

Theorem 100 states that a continuous function on a closed interval will have absolute extrema, that is, both an absolute maximum and an absolute minimum. These extrema occur either at the endpoints or at critical values in the interval. We combine these concepts to offer a strategy for finding extrema.

Key Idea 40 Finding Extrema on a Closed Interval

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$:

1. Evaluate f at the endpoints a and b of the interval.
2. Find the critical numbers of f in $[a, b]$.
3. Evaluate f at each critical number.
4. The absolute maximum of f is the largest of these values, and the absolute minimum of f is the least of these values.

We practice these ideas in the next examples.

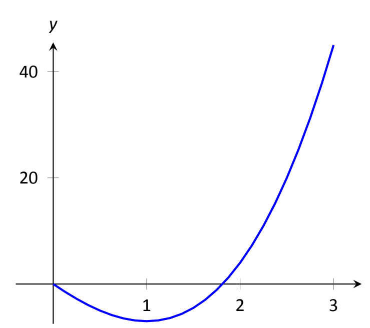


Figure 12.6: A graph of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$ as in Example 235.

Example 235 Finding extreme values

Find the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$, graphed in Figure 12.6.

SOLUTION We follow the steps outlined in Key Idea 40. We first evaluate f at the endpoints:

$$f(0) = 0 \quad \text{and} \quad f(3) = 45.$$

Next, we find the critical values of f on $[0, 3]$. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$; therefore the critical values of f are $x = -2$ and $x = 1$. Since $x = -2$ does not lie in the interval $[0, 3]$, we ignore it. Evaluating f at the only critical number in our interval gives: $f(1) = -7$.

The table in Figure 12.7 gives f evaluated at the "important" x values in $[0, 3]$. We can easily see the maximum and minimum values of f: the maximum value is 45 and the minimum value is -7.

x	$f(x)$
0	0
1	-7
3	45

Figure 12.7: Finding the extreme values of f in Example 235.

Note that all this was done without the aid of a graph; this work followed an analytic algorithm and did not depend on any visualization. Figure 12.6 shows f and we can confirm our answer, but it is important to understand that these answers can be found without graphical assistance.

We practice again.

Example 236 Finding extreme values

Find the maximum and minimum values of f on $[-4, 2]$, where

$$f(x) = \begin{cases} (x-1)^2 & x \leq 0 \\ x + 1 & x > 0 \end{cases}.$$

Solution Here f is piecewise–defined, but we can still apply Key Idea 40. Evaluating f at the endpoints gives:

$$f(-4) = 25 \quad \text{and} \quad f(2) = 3.$$

We now find the critical numbers of f. We have to define f' in a piecewise manner; it is

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that while f is defined for all of $[-4, 2]$, f' is not, as the derivative of f does not exist when $x = 0$. (From the left, the derivative approaches -2; from the right the derivative is 1.) Thus one critical number of f is $x = 0$.

We now set $f'(x) = 0$. When $x > 0$, $f'(x)$ is never 0. When $x < 0$, $f'(x)$ is also never 0. (We may be tempted to say that $f'(x) = 0$ when $x = 1$. However, this is nonsensical, for we only consider $f'(x) = 2(x - 1)$ when $x < 0$, so we will ignore a solution that says $x = 1$.)

So we have three important x values to consider: $x = -4, 2$ and 0. Evaluating f at each gives, respectively, 25, 3 and 1, shown in Figure 12.8. Thus the absolute minimum of f is 1; the absolute maximum of f is 25. Our answer is confirmed by the graph of f in Figure 12.9.

x	$f(x)$
-4	25
0	1
2	3

Figure 12.8: Finding the extreme values of f in Example 236.

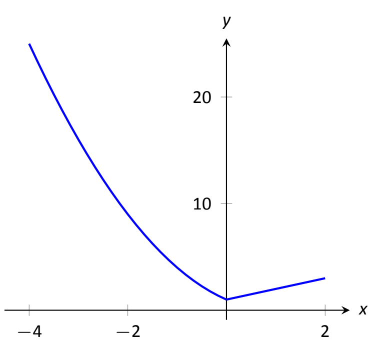


Figure 12.9: A graph of $f(x)$ on $[-4, 2]$ as in Example 236.

Example 237 **Finding extreme values**
Find the extrema of $f(x) = \cos(x^2)$ on $[-2, 2]$.

Solution We again use Key Idea 40. Evaluating f at the endpoints of the interval gives: $f(-2) = f(2) = \cos(4) \approx -0.6536$. We now find the critical values of f.

Applying the Chain Rule, we find $f'(x) = -2x\sin(x^2)$. Set $f'(x) = 0$ and solve for x to find the critical values of f.

We have $f'(x) = 0$ when $x = 0$ and when $\sin(x^2) = 0$. In general, $\sin t = 0$ when $t = \ldots -2\pi, -\pi, 0, \pi, \ldots$ Thus $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$ (x^2 is always positive so we ignore $-\pi$, etc.) So $\sin(x^2) = 0$ when $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots$. The only values to fall in the given interval of $[-2, 2]$ are $-\sqrt{\pi}$ and $\sqrt{\pi}$, approximately ± 1.77.

We again construct a table of important values in Figure 12.10. In this example we have 5 values to consider: $x = 0, \pm 2, \pm\sqrt{\pi}$.

From the table it is clear that the maximum value of f on $[-2, 2]$ is 1; the minimum value is -1. The graph in Figure 12.11 confirms our results.

x	$f(x)$
-2	-0.65
$-\sqrt{\pi}$	-1
0	1
$\sqrt{\pi}$	-1
2	-0.65

Figure 12.10: Finding the extrema of $f(x) = \cos(x^2)$ in Example 237.

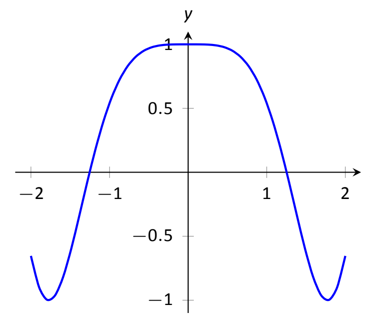


Figure 12.11: A graph of $f(x) = \cos(x^2)$ on $[-2, 2]$ as in Example 237.

We consider one more example.

Example 238 **Finding extreme values**
Find the extreme values of $f(x) = \sqrt{1 - x^2}$.

Solution A closed interval is not given, so we find the extreme values of f on its domain. f is defined whenever $1 - x^2 \geq 0$; thus the domain of f is $[-1, 1]$. Evaluating f at either endpoint returns 0.

Using the Chain Rule, we find $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$. The critical points of f are found when $f'(x) = 0$ or when f' is undefined. It is straightforward to find that $f'(x) = 0$ when $x = 0$, and f' is undefined when $x = \pm 1$, the endpoints of the interval. The table of important values is given in Figure 12.12. The maximum value is 1, and the minimum value is 0.

x	$f(x)$
-1	0
0	1
1	0

Figure 12.12: Finding the extrema of the half–circle in Example 238.

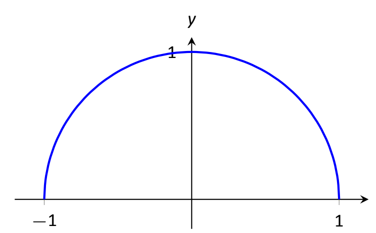


Figure 12.13: A graph of $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$ as in Example 238.

We have seen that continuous functions on closed intervals always have a maximum and minimum value, and we have also developed a technique to find these values. In the next section, we further our study of the information we can glean from "nice" functions with the Mean Value Theorem. On a closed interval, we can find the *average rate of change* of a function (as we did at the beginning of Chapter 2). We will see that differentiable functions always have a point at which their *instantaneous* rate of change is same as the *average* rate of change. This is surprisingly useful, as we'll see.

Exercises 12.1

Terms and Concepts

1. Describe what an "extreme value" of a function is in your own words.

2. Sketch the graph of a function f on $(-1, 1)$ that has both a maximum and minimum value.

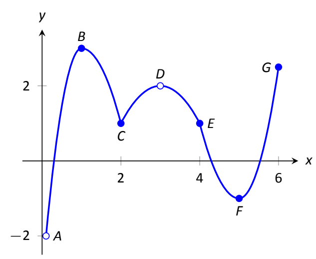
3. Describe the difference between absolute and relative maxima in your own words.

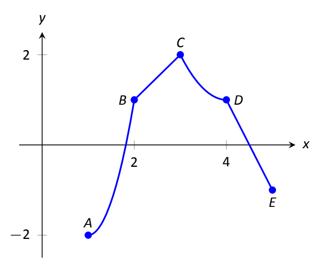
4. Sketch the graph of a function f where f has a relative maximum at $x = 1$ and $f'(1)$ is undefined.

5. T/F: If c is a critical value of a function f, then f has either a relative maximum or relative minimum at $x = c$.

Problems

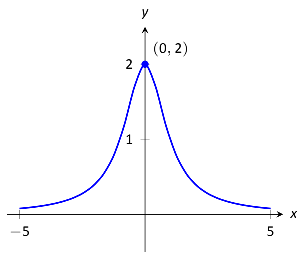
In Exercises 6 – 7, identify each of the marked points as being an absolute maximum or minimum, a relative maximum or minimum, or none of the above.

6. 

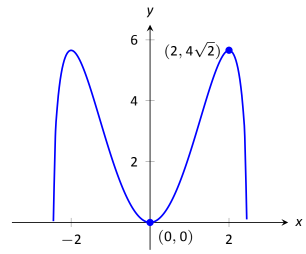
7. 

In Exercises 8 – 14, evaluate $f'(x)$ at the points indicated in the graph.

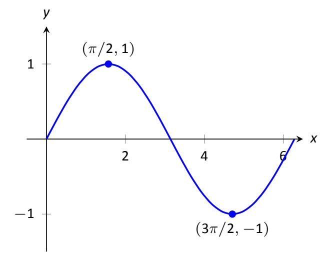
8. $f(x) = \dfrac{2}{x^2 + 1}$



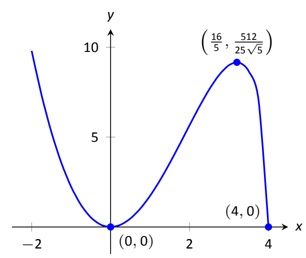
9. $f(x) = x^2\sqrt{6 - x^2}$



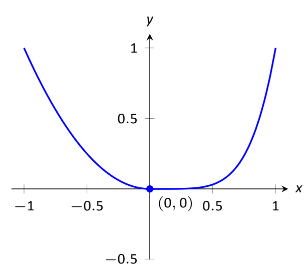
10. $f(x) = \sin x$



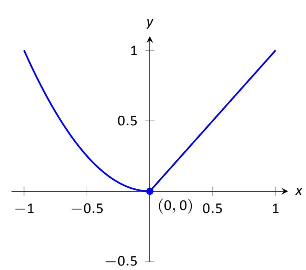
11. $f(x) = x^2\sqrt{4 - x}$



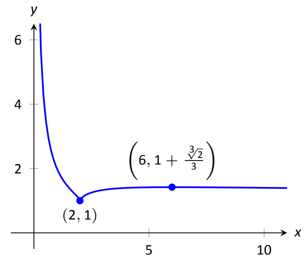
12. $f(x) = \begin{cases} x^2 & x \le 0 \\ x^5 & x > 0 \end{cases}$



13. $f(x) = \begin{cases} x^2 & x \le 0 \\ x & x > 0 \end{cases}$



14. $f(x) = \dfrac{(x-2)^{2/3}}{x}$



In Exercises 15 – 24, find the extreme values of the function on the given interval.

15. $f(x) = x^2 + x + 4$ on $[-1, 2]$.

16. $f(x) = x^3 - \dfrac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.

17. $f(x) = 3\sin x$ on $[\pi/4, 2\pi/3]$.

18. $f(x) = x^2\sqrt{4-x^2}$ on $[-2, 2]$.

19. $f(x) = x + \dfrac{3}{x}$ on $[1, 5]$.

20. $f(x) = \dfrac{x^2}{x^2+5}$ on $[-3, 5]$.

21. $f(x) = e^x \cos x$ on $[0, \pi]$.

22. $f(x) = e^x \sin x$ on $[0, \pi]$.

23. $f(x) = \dfrac{\ln x}{x}$ on $[1, 4]$.

24. $f(x) = x^{2/3} - x$ on $[0, 2]$.

Review

25. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

26. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

27. Let $f(x) = x^3 + x$.

    Evaluate $\displaystyle\lim_{s\to 0} \frac{f(x+s) - f(x)}{s}$.

12.2 Increasing and Decreasing Functions

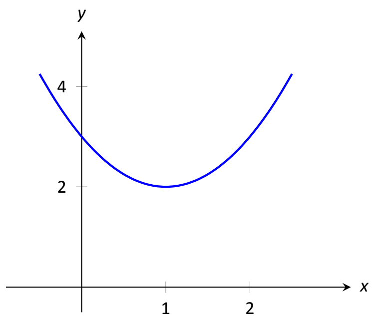


Figure 12.14: A graph of a function f used to illustrate the concepts of *increasing* and *decreasing*.

Our study of "nice" functions f in this chapter has so far focused on individual points: points where f is maximal/minimal, points where $f'(x) = 0$ or f' does not exist, and points c where $f'(c)$ is the average rate of change of f on some interval.

In this section we begin to study how functions behave *between* special points; we begin studying in more detail the shape of their graphs.

We start with an intuitive concept. Given the graph in Figure 12.14, where would you say the function is *increasing*? *Decreasing*? Even though we have not defined these terms mathematically, one likely answered that f is increasing when $x > 1$ and decreasing when $x < 1$. We formally define these terms here.

Definition 70 Increasing and Decreasing Functions

Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $a < b$ in I, $f(a) \leq f(b)$.
2. f is **decreasing** on I if for every $a < b$ in I, $f(a) \geq f(b)$.

A function is **strictly increasing** when $a < b$ in I implies $f(a) < f(b)$, with a similar definition holding for **strictly decreasing**.

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger.

Our interest lies in finding intervals in the domain of f on which f is either increasing or decreasing. Such information should seem useful. For instance, if f describes the speed of an object, we might want to know when the speed was increasing or decreasing (i.e., when the object was accelerating vs. decelerating). If f describes the population of a city, we should be interested in when the population is growing or declining.

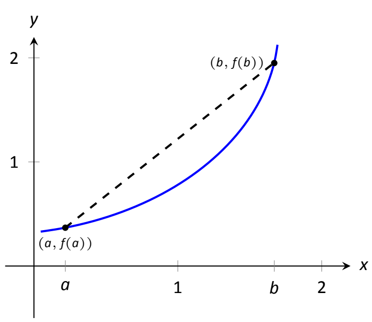


Figure 12.15: Examining the secant line of an increasing function.

To find such intervals, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 12.15, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$ to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$. But note:

$$\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{\text{numerator } > 0}{\text{denominator } > 0} \Rightarrow \begin{array}{c}\text{slope of the}\\ \text{secant line} > 0\end{array} \Rightarrow \begin{array}{c}\text{Average rate of}\\ \text{change of } f \text{ on}\\ [a,b] \text{ is} > 0.\end{array}$$

We have shown mathematically what may have already been obvious: when f is increasing, its secant lines will have a positive slope. Now recall the Mean Value Theorem guarantees that there is a number c, where $a < c < b$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} > 0.$$

By considering all such secant lines in I, we strongly imply that $f'(x) \geq 0$ on I. A similar statement can be made for decreasing functions.

Our above logic can be summarized as "If f is increasing, then f' is probably positive." Theorem 102 below turns this around by stating "If f' is postive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

The **Mean Value Theorem**, which is covered in more advanced courses, like Math 1560, is a remarkably powerful result. It guarantees that if a function f is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$; that is, that at some point the instantaneous rate of change must equal the average rate of change.

Theorem 102 Test For Increasing/Decreasing Functions

Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.

Let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. It follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. This leads us to the following method for finding intervals on which a function is increasing or decreasing.

Note: Theorem 102 also holds if $f'(c) = 0$ for a finite number of values of c in I.

Key Idea 41 Finding Intervals on Which f is Increasing or Decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing and decreasing:

1. Find the critical values of f. That is, find all c in I where $f'(c) = 0$ or f' is not defined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   (a) If $f'(p) > 0$, then f is increasing on that subinterval.
   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

We demonstrate using this process in the following example.

Example 239 Finding intervals of increasing/decreasing
Let $f(x) = x^3 + x^2 - x + 1$. Find intervals on which f is increasing or decreasing.

Solution Using Key Idea 41, we first find the critical values of f. We have $f'(x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1/3$. f' is never undefined.

Since an interval was not specified for us to consider, we consider the entire domain of f which is $(-\infty, \infty)$. We thus break the whole real line into three subintervals based on the two critical values we just found: $(-\infty, -1)$, $(-1, 1/3)$ and $(1/3, \infty)$. This is shown in Figure 12.16.

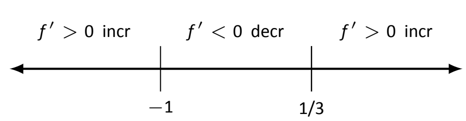


Figure 12.16: Number line for f in Example 239.

We now pick a value p in each subinterval and find the sign of $f'(p)$. All we care about is the sign, so we do not actually have to fully compute $f'(p)$; pick "nice" values that make this simple.

Subinterval 1, $(-\infty, -1)$: We (arbitrarily) pick $p = -2$. We can compute $f'(-2)$ directly: $f'(-2) = 3(-2)^2 + 2(-2) - 1 = 7 > 0$. We conclude that f is increasing on $(-\infty, -1)$.

Note we can arrive at the same conclusion without computation. For instance, we could choose $p = -100$. The first term in $f'(-100)$, i.e., $3(-100)^2$ is clearly positive and very large. The other terms are small in comparison, so we know $f'(-100) > 0$. All we need is the sign.

Subinterval 2, $(-1, 1/3)$: We pick $p = 0$ since that value seems easy to deal with. $f'(0) = -1 < 0$. We conclude f is decreasing on $(-1, 1/3)$.

Subinterval 3, $(1/3, \infty)$: Pick an arbitrarily large value for $p > 1/3$ and note that $f'(p) = 3p^2 + 2p - 1 > 0$. We conclude that f is increasing on $(1/3, \infty)$.

We can verify our calculations by considering Figure 12.17, where f is graphed. The graph also presents f'; note how $f' > 0$ when f is increasing and $f' < 0$ when f is decreasing.

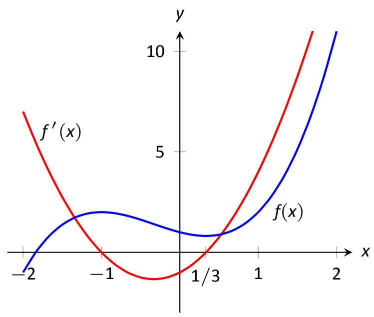


Figure 12.17: A graph of $f(x)$ in Example 239, showing where f is increasing and decreasing.

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant happens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information. We are working to develop mastery.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 12.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice–versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

Theorem 103 First Derivative Test

Let f be differentiable on I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change at c, then $f(c)$ is not a relative extrema of f.

Example 240 Using the First Derivative Test

Find the intervals on which f is increasing and decreasing, and use the First Derivative Test to determine the relative extrema of f, where

$$f(x) = \frac{x^2+3}{x-1}.$$

Solution We start by noting the domain of f: $(-\infty, 1) \cup (1, \infty)$. Key Idea 41 describes how to find intervals where f is increasing and decreasing *when the domain of f is an interval.* Since the domain of f in this example is the union of two intervals, we apply the techniques of Key Idea 41 to both intervals of the domain of f.

Since f is not defined at $x = 1$, the increasing/decreasing nature of f could switch at this value. We do not formally consider $x = 1$ to be a critical value of f, but we will include it in our list of critical values that we find next.

Using the Quotient Rule, we find

$$f'(x) = \frac{x^2-2x-3}{(x-1)^2}.$$

We need to find the critical values of f; we want to know when $f'(x) = 0$ and when f' is not defined. That latter is straightforward: when the denominator of $f'(x)$ is 0, f' is undefined. That occurs when $x = 1$, which we've already recognized as an important value.

$f'(x) = 0$ when the numerator of $f'(x)$ is 0. That occurs when $x^2 - 2x - 3 = (x-3)(x+1) = 0$; i.e., when $x = -1, 3$.

We have found that f has two critical numbers, $x = -1, 3$, and at $x = 1$ something important might also happen. These three numbers divide the real number line into 4 subintervals:

$$(-\infty, -1), \quad (-1, 1), \quad (1, 3) \quad \text{and} \quad (3, \infty).$$

Pick a number p from each subinterval and test the sign of f' at p to determine whether f is increasing or decreasing on that interval. Again, we do well to avoid

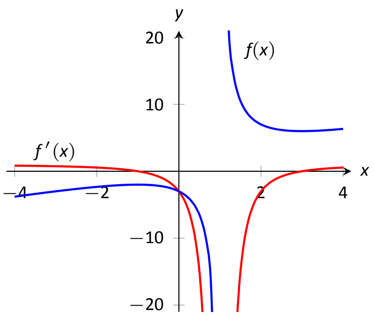


Figure 12.19: A graph of $f(x)$ in Example 240, showing where f is increasing and decreasing.

complicated computations; notice that the denominator of f' is *always* positive so we can ignore it during our work.

Interval 1, $(-\infty, -1)$: Choosing a very small number (i.e., a negative number with a large magnitude) p returns $p^2 - 2p - 3$ in the numerator of f'; that will be positive. Hence f is increasing on $(-\infty, -1)$.

Interval 2, $(-1, 1)$: Choosing 0 seems simple: $f'(0) = -3 < 0$. We conclude f is decreasing on $(-1, 1)$.

Interval 3, $(1, 3)$: Choosing 2 seems simple: $f'(2) = -3 < 0$. Again, f is decreasing.

Interval 4, $(3, \infty)$: Choosing an very large number p from this subinterval will give a positive numerator and (of course) a positive denominator. So f is increasing on $(3, \infty)$.

In summary, f is increasing on the set $(-\infty, -1) \cup (3, \infty)$ and is decreasing on the set $(-1, 1) \cup (1, 3)$. Since at $x = -1$, the sign of f' switched from positive to negative, Theorem 103 states that $f(-1)$ is a relative maximum of f. At $x = 3$, the sign of f' switched from negative to positive, meaning $f(3)$ is a relative minimum. At $x = 1$, f is not defined, so there is no relative extrema at $x = 1$.

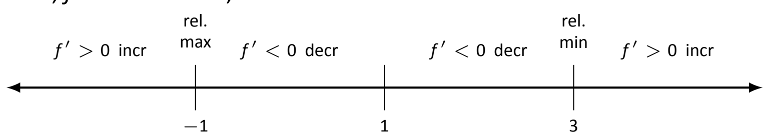


Figure 12.18: Number line for f in Example 240.

This is summarized in the number line shown in Figure 12.18. Also, Figure 12.19 shows a graph of f, confirming our calculations. This figure also shows f', again demonstrating that f is increasing when $f' > 0$ and decreasing when $f' < 0$.

One is often tempted to think that functions always alternate "increasing, decreasing, increasing, decreasing,. . ." around critical values. Our previous example demonstrated that this is not always the case. While $x = 1$ was not technically a critical value, it was an important value we needed to consider. We found that f was decreasing on "both sides of $x = 1$."

We examine one more example.

Example 241 Using the First Derivative Test

Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing and identify the relative extrema.

Solution We again start with taking derivatives. Since we know we want to solve $f'(x) = 0$, we will do some algebra after taking derivatives.

$$\begin{aligned} f(x) &= x^{\frac{8}{3}} - 4x^{\frac{2}{3}} \\ f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^{\frac{6}{3}} - 1\right) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x^2 - 1) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x-1)(x+1). \end{aligned}$$

This derivation of f' shows that $f'(x) = 0$ when $x = \pm 1$ and f' is not defined when $x = 0$. Thus we have 3 critical values, breaking the number line into 4 subintervals as shown in Figure 12.20.

Interval 1, $(\infty, -1)$: We choose $p = -2$; we can easily verify that $f'(-2) < 0$. So f is decreasing on $(-\infty, -1)$.
Interval 2, $(-1, 0)$: Choose $p = -1/2$. Once more we practice finding the sign of $f'(p)$ without computing an actual value. We have $f'(p) = (8/3)p^{-1/3}(p-1)(p+1)$; find the sign of each of the three terms.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{<0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have a "negative $\times$ negative $\times$ positive" giving a positive number; f is increasing on $(-1, 0)$.
Interval 3, $(0, 1)$: We do a similar sign analysis as before, using p in $(0, 1)$.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{>0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have 2 positive factors and one negative factor; $f'(p) < 0$ and so f is decreasing on $(0, 1)$.
Interval 4, $(1, \infty)$: Similar work to that done for the other three intervals shows that $f'(x) > 0$ on $(1, \infty)$, so f is increasing on this interval.

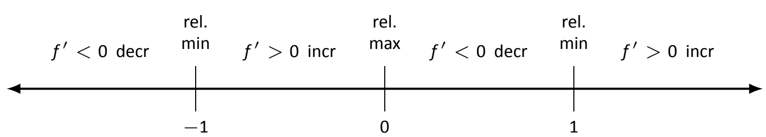


Figure 12.20: Number line for f in Example 241.

We conclude by stating that f is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$. The sign of f' changes from negative to positive around $x = -1$ and $x = 1$, meaning by Theorem 103 that $f(-1)$ and $f(1)$ are relative minima of f. As the sign of f' changes from positive to negative at $x = 0$, we have a relative maximum at $f(0)$. Figure 12.21 shows a graph of f, confirming our result. We also graph f', highlighting once more that f is increasing when $f' > 0$ and is decreasing when $f' < 0$.

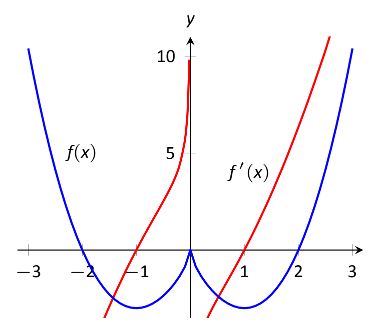


Figure 12.21: A graph of $f(x)$ in Example 241, showing where f is increasing and decreasing.

We have seen how the first derivative of a function helps determine when the function is going "up" or "down." In the next section, we will see how the second derivative helps determine how the graph of a function curves.

Exercises 12.2

Terms and Concepts

1. In your own words describe what it means for a function to be increasing.

2. What does a decreasing function "look like"?

3. Sketch a graph of a function on $[0, 2]$ that is increasing but not strictly increasing.

4. Give an example of a function describing a situation where it is "bad" to be increasing and "good" to be decreasing.

5. A function f has derivative $f'(x) = (\sin x + 2)e^{x^2+1}$, where $f'(x) > 1$ for all x. Is f increasing, decreasing, or can we not tell from the given information?

Problems

In Exercises 6 – 13, a function $f(x)$ is given.

(a) Compute $f'(x)$.

(b) Graph f and f' on the same axes (using technology is permitted) and verify Theorem 102.

6. $f(x) = 2x + 3$

7. $f(x) = x^2 - 3x + 5$

8. $f(x) = \cos x$

9. $f(x) = \tan x$

10. $f(x) = x^3 - 5x^2 + 7x - 1$

11. $f(x) = 2x^3 - x^2 + x - 1$

12. $f(x) = x^4 - 5x^2 + 4$

13. $f(x) = \dfrac{1}{x^2 + 1}$

In Exercises 14 – 23, a function $f(x)$ is given.

(a) Give the domain of f.

(b) Find the critical numbers of f.

(c) Create a number line to determine the intervals on which f is increasing and decreasing.

(d) Use the First Derivative Test to determine whether each critical point is a relative maximum, minimum, or neither.

14. $f(x) = x^2 + 2x - 3$

15. $f(x) = x^3 + 3x^2 + 3$

16. $f(x) = 2x^3 + x^2 - x + 3$

17. $f(x) = x^3 - 3x^2 + 3x - 1$

18. $f(x) = \dfrac{1}{x^2 - 2x + 2}$

19. $f(x) = \dfrac{x^2 - 4}{x^2 - 1}$

20. $f(x) = \dfrac{x}{x^2 - 2x - 8}$

21. $f(x) = \dfrac{(x-2)^{2/3}}{x}$

22. $f(x) = \sin x \cos x$ on $(-\pi, \pi)$.

23. $f(x) = x^5 - 5x$

Review

24. Consider $f(x) = x^2 - 3x + 5$ on $[-1, 2]$; find c guaranteed by the Mean Value Theorem.

25. Consider $f(x) = \sin x$ on $[-\pi/2, \pi/2]$; find c guaranteed by the Mean Value Theorem.

12.3 Concavity and the Second Derivative

Our study of "nice" functions continues. The previous section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. f' has relative maxima and minima where $f'' = 0$ or is undefined.

This section explores how knowing information about f'' gives information about f.

Concavity

We begin with a definition, then explore its meaning.

Definition 71 Concave Up and Concave Down

Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing. The graph of f is **concave down** on I if f' is decreasing. If f' is constant then the graph of f is said to have **no concavity**.

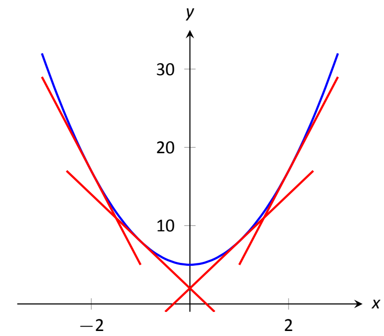


Figure 12.22: A function f with a concave up graph. Notice how the slopes of the tangent lines, when looking from left to right, are increasing.

Note: We often state that "f is concave up" instead of "the graph of f is concave up" for simplicity.

Note: A mnemonic for remembering what concave up/down means is: "Concave up is like a cup; concave down is like a frown." It is admittedly terrible, but it works.

The graph of a function f is *concave up* when f' is *increasing.* That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. Consider Figure 12.22, where a concave up graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, downward, corresponding to a small value of f'. On the right, the tangent line is steep, upward, corresponding to a large value of f'.

If a function is decreasing and concave up, then its rate of decrease is slowing; it is "leveling off." If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

The graph of a function f is *concave down* when f' is *decreasing.* That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. Consider Figure 12.23, where a concave down graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, upward, corresponding to a large value of f'. On the right, the tangent line is steep, downward, corresponding to a small value of f'.

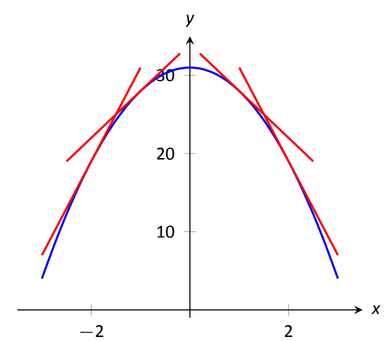


Figure 12.23: A function f with a concave down graph. Notice how the slopes of the tangent lines, when looking from left to right, are decreasing.

If a function is increasing and concave down, then its rate of increase is slowing; it is "leveling off." If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section and to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

Theorem 104 Test for Concavity

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.

Note: Geometrically speaking, a function is concave up if its graph lies above its tangent lines. A function is concave down if its graph lies below its tangent lines.

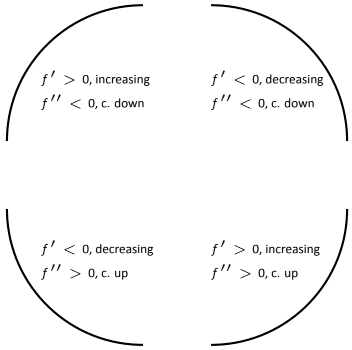


Figure 12.24: Demonstrating the 4 ways that concavity interacts with increasing/decreasing, along with the relationships with the first and second derivatives.

If knowing where a graph is concave up/down is important, it makes sense that the places where the graph changes from one to the other is also important. This leads us to a definition.

Definition 72 Point of Inflection

A **point of inflection** is a point on the graph of f at which the concavity of f changes.

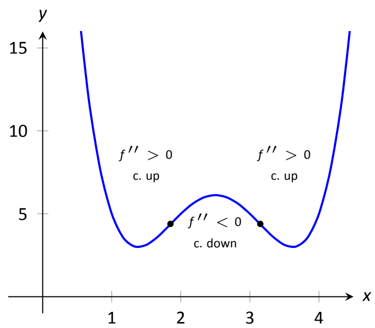


Figure 12.25: A graph of a function with its inflection points marked. The intervals where concave up/down are also indicated.

Figure 12.25 shows a graph of a function with inflection points labeled.

If the concavity of f changes at a point $(c, f(c))$, then f' is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of f'' is changing from positive to negative (or, negative to positive) at $x = c$. This leads to the following theorem.

Theorem 105 Points of Inflection

If $(c, f(c))$ is a point of inflection on the graph of f, then either $f'' = 0$ or f'' is not defined at c.

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following examples.

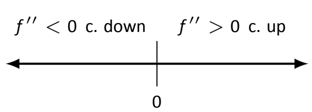


Figure 12.26: A number line determining the concavity of f in Example 242.

Example 242 Finding intervals of concave up/down, inflection points

Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up/down.

SOLUTION We start by finding $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. To find the inflection points, we use Theorem 105 and find where $f''(x) = 0$ or where f'' is undefined. We find f'' is always defined, and is 0 only when $x = 0$. So the point $(0, 1)$ is the only possible point of inflection.

This possible inflection point divides the real line into two intervals, $(-\infty, 0)$ and $(0, \infty)$. We use a process similar to the one used in the previous section to determine increasing/decreasing. Pick any $c < 0$; $f''(c) < 0$ so f is concave down on $(-\infty, 0)$. Pick any $c > 0$; $f''(c) > 0$ so f is concave up on $(0, \infty)$. Since the concavity changes at $x = 0$, the point $(0, 1)$ is an inflection point.

The number line in Figure 12.26 illustrates the process of determining concavity; Figure 12.27 shows a graph of f and f'', confirming our results. Notice how f is concave down precisely when $f''(x) < 0$ and concave up when $f''(x) > 0$.

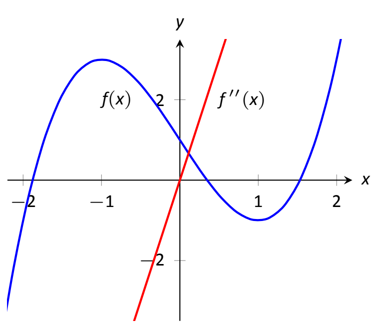


Figure 12.27: A graph of $f(x)$ used in Example 242.

Example 243 Finding intervals of concave up/down, inflection points

Let $f(x) = x/(x^2 - 1)$. Find the inflection points of f and the intervals on which it is concave up/down.

SOLUTION We need to find f' and f''. Using the Quotient Rule and simplifying, we find

$$f'(x) = \frac{-(1 + x^2)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

To find the possible points of inflection, we seek to find where $f''(x) = 0$ and where f'' is not defined. Solving $f''(x) = 0$ reduces to solving $2x(x^2 + 3) = 0$; we find $x = 0$. We find that f'' is not defined when $x = \pm 1$, for then the denominator of f'' is 0. We also note that f itself is not defined at $x = \pm 1$, having a domain of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Since the domain of f is the union of three intervals, it makes sense that the concavity of f could switch across intervals. We technically cannot say that f has a point of inflection at $x = \pm 1$ as they are not part of the domain, but we must still consider these x-values to be important and will include them in our number line.

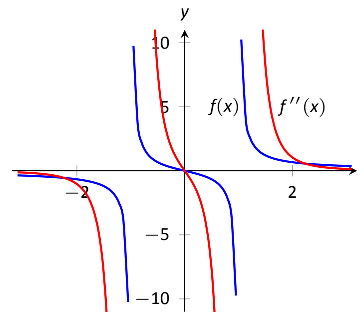


Figure 12.29: A graph of $f(x)$ and $f''(x)$ in Example 243.

The important x-values at which concavity might switch are $x = -1$, $x = 0$ and $x = 1$, which split the number line into four intervals as shown in Figure 12.28. We determine the concavity on each. Keep in mind that all we are concerned with is the *sign* of f'' on the interval.

Interval 1, $(-\infty, -1)$: Select a number c in this interval with a large magnitude (for instance, $c = -100$). The denominator of $f''(x)$ will be positive. In the numerator, the $(c^2 + 3)$ will be positive and the $2c$ term will be negative. Thus the numerator is negative and $f''(c)$ is negative. We conclude f is concave down on $(-\infty, -1)$.

Interval 2, $(-1, 0)$: For any number c in this interval, the term $2c$ in the numerator will be negative, the term $(c^2 + 3)$ in the numerator will be positive, and the term $(c^2 - 1)^3$ in the denominator will be negative. Thus $f''(c) > 0$ and f is concave up on this interval.

Interval 3, $(0, 1)$: Any number c in this interval will be positive and "small." Thus the numerator is positive while the denominator is negative. Thus $f''(c) < 0$ and f is concave down on this interval.

Interval 4, $(1, \infty)$: Choose a large value for c. It is evident that $f''(c) > 0$, so we conclude that f is concave up on $(1, \infty)$.

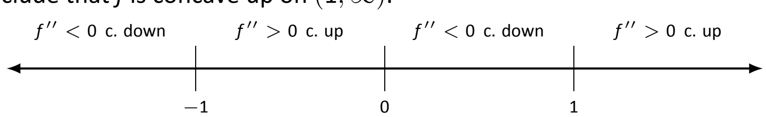


Figure 12.28: Number line for f in Example 243.

We conclude that f is concave up on $(-1, 0) \cup (1, \infty)$ and concave down on $(-\infty, -1) \cup (0, 1)$. There is only one point of inflection, $(0, 0)$, as f is not defined at $x = \pm 1$. Our work is confirmed by the graph of f in Figure 12.29. Notice how f is concave up whenever f'' is positive, and concave down when f'' is negative.

Recall that relative maxima and minima of f are found at critical points of f; that is, they are found when $f'(x) = 0$ or when f' is undefined. Likewise, the relative maxima and minima of f' are found when $f''(x) = 0$ or when f'' is undefined; note that these are the inflection points of f.

What does a "relative maximum of f'" *mean*? The derivative measures the rate of change of f; maximizing f' means finding the where f is increasing the most – where f has the steepest tangent line. A similar statement can be made for minimizing f'; it corresponds to where f has the steepest negatively–sloped tangent line.

We utilize this concept in the next example.

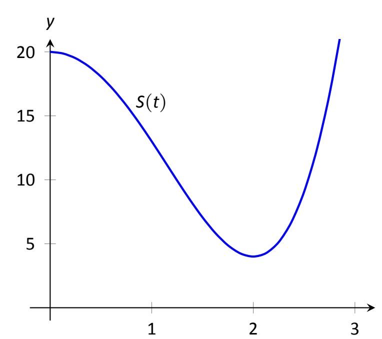


Figure 12.30: A graph of $S(t)$ in Example 244, modeling the sale of a product over time.

Example 244 Understanding inflection points

The sales of a certain product over a three-year span are modeled by $S(t) = t^4 - 8t^2 + 20$, where t is the time in years, shown in Figure 12.30. Over the first two years, sales are decreasing. Find the point at which sales are decreasing at their greatest rate.

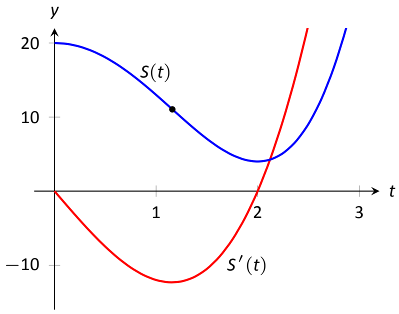


Figure 12.31: A graph of $S(t)$ in Example 244 along with $S'(t)$.

Solution We want to maximize the rate of decrease, which is to say, we want to find where S' has a minimum. To do this, we find where S'' is 0. We find $S'(t) = 4t^3 - 16t$ and $S''(t) = 12t^2 - 16$. Setting $S''(t) = 0$ and solving, we get $t = \sqrt{4/3} \approx 1.16$ (we ignore the negative value of t since it does not lie in the domain of our function S).

This is both the inflection point and the point of maximum decrease. This is the point at which things first start looking up for the company. After the inflection point, it will still take some time before sales start to increase, but at least sales are not decreasing quite as quickly as they had been.

A graph of $S(t)$ and $S'(t)$ is given in Figure 12.31. When $S'(t) < 0$, sales are decreasing; note how at $t \approx 1.16$, $S'(t)$ is minimized. That is, sales are decreasing at the fastest rate at $t \approx 1.16$. On the interval of $(1.16, 2)$, S is decreasing but concave up, so the decline in sales is "leveling off."

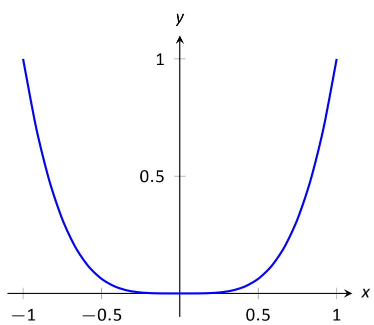


Figure 12.32: A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. It this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

Not every critical point corresponds to a relative extrema; $f(x) = x^3$ has a critical point at $(0, 0)$ but no relative maximum or minimum. Likewise, just because $f''(x) = 0$ we cannot conclude concavity changes at that point. We were careful before to use terminology "*possible* point of inflection" since we needed to check to see if the concavity changed. The canonical example of $f''(x) = 0$ *without* concavity changing is $f(x) = x^4$. At $x = 0$, $f''(x) = 0$ but f is always concave up, as shown in Figure 12.32.

The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 12.33 for a visualization of this.

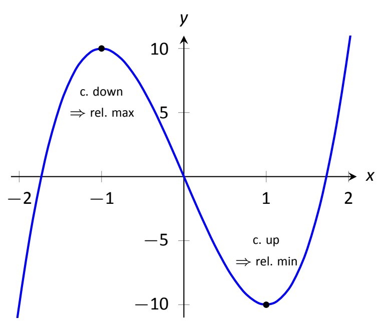


Figure 12.33: Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up.

Theorem 106 The Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

The Second Derivative Test relates to the First Derivative Test in the following way. If $f''(c) > 0$, then the graph is concave up at a critical point c and f' itself is growing. Since $f'(c) = 0$ and f' is growing at c, then it must go from negative to positive at c. This means the function goes from decreasing to increasing, indicating a local minimum at c.

Example 245 Using the Second Derivative Test

Let $f(x) = 100/x + x$. Find the critical points of f and use the Second Derivative Test to label them as relative maxima or minima.

Solution We find $f'(x) = -100/x^2 + 1$ and $f''(x) = 200/x^3$. We set $f'(x) = 0$ and solve for x to find the critical values (note that f' is not defined at $x = 0$, but neither is f so this is not a critical value.) We find the critical values are $x = \pm 10$. Evaluating f'' at $x = 10$ gives $0.1 > 0$, so there is a local minimum at $x = 10$. Evaluating $f''(-10) = -0.1 < 0$, determining a relative maximum at $x = -10$. These results are confirmed in Figure 12.34.

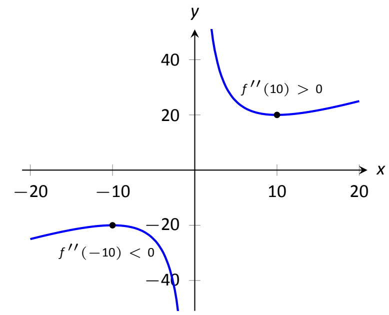


Figure 12.34: A graph of $f(x)$ in Example 245. The second derivative is evaluated at each critical point. When the graph is concave up, the critical point represents a local minimum; when the graph is concave down, the critical point represents a local maximum.

We have been learning how the first and second derivatives of a function relate information about the graph of that function. We have found intervals of increasing and decreasing, intervals where the graph is concave up and down, along with the locations of relative extrema and inflection points. In Chapter 10 we saw how limits explained asymptotic behavior. In the next section we combine all of this information to produce accurate sketches of functions.

Exercises 12.3

Terms and Concepts

1. Sketch a graph of a function $f(x)$ that is concave up on $(0, 1)$ and is concave down on $(1, 2)$.

2. Sketch a graph of a function $f(x)$ that is:

   (a) Increasing, concave up on $(0, 1)$,

   (b) increasing, concave down on $(1, 2)$,

   (c) decreasing, concave down on $(2, 3)$ and

   (d) increasing, concave down on $(3, 4)$.

3. Is is possible for a function to be increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

4. Is is possible for a function to be increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

Problems

In Exercises 5 – 15, a function $f(x)$ is given.

(a) Compute $f''(x)$.

(b) Graph f and f'' on the same axes (using technology is permitted) and verify Theorem 104.

5. $f(x) = -7x + 3$

6. $f(x) = -4x^2 + 3x - 8$

7. $f(x) = 4x^2 + 3x - 8$

8. $f(x) = x^3 - 3x^2 + x - 1$

9. $f(x) = -x^3 + x^2 - 2x + 5$

10. $f(x) = \cos x$

11. $f(x) = \sin x$

12. $f(x) = \tan x$

13. $f(x) = \dfrac{1}{x^2 + 1}$

14. $f(x) = \dfrac{1}{x}$

15. $f(x) = \dfrac{1}{x^2}$

In Exercises 16 – 28, a function $f(x)$ is given.

(a) Find the possible points of inflection of f.

(b) Create a number line to determine the intervals on which f is concave up or concave down.

16. $f(x) = x^2 - 2x + 1$

17. $f(x) = -x^2 - 5x + 7$

18. $f(x) = x^3 - x + 1$

19. $f(x) = 2x^3 - 3x^2 + 9x + 5$

20. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

21. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

22. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

23. $f(x) = \dfrac{1}{x^2 + 1}$

24. $f(x) = \dfrac{x}{x^2 - 1}$

25. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

26. $f(x) = x^2 e^x$

27. $f(x) = x^2 \ln x$

28. $f(x) = e^{-x^2}$

In Exercises 29 – 41, a function $f(x)$ is given. Find the critical points of f and use the Second Derivative Test, when possible, to determine the relative extrema. (Note: these are the same functions as in Exercises 16 – 28.)

29. $f(x) = x^2 - 2x + 1$

30. $f(x) = -x^2 - 5x + 7$

31. $f(x) = x^3 - x + 1$

32. $f(x) = 2x^3 - 3x^2 + 9x + 5$

33. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

34. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

35. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

36. $f(x) = \dfrac{1}{x^2 + 1}$

37. $f(x) = \dfrac{x}{x^2 - 1}$

38. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

39. $f(x) = x^2 e^x$

40. $f(x) = x^2 \ln x$

41. $f(x) = e^{-x^2}$

In Exercises 42 – 54, a function $f(x)$ is given. Find the x values where $f'(x)$ has a relative maximum or minimum. (Note: these are the same functions as in Exercises 16 – 28.)

42. $f(x) = x^2 - 2x + 1$

43. $f(x) = -x^2 - 5x + 7$

44. $f(x) = x^3 - x + 1$

45. $f(x) = 2x^3 - 3x^2 + 9x + 5$

46. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

47. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

48. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

49. $f(x) = \dfrac{1}{x^2 + 1}$

50. $f(x) = \dfrac{x}{x^2 - 1}$

51. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

52. $f(x) = x^2 e^x$

53. $f(x) = x^2 \ln x$

54. $f(x) = e^{-x^2}$

12.4 Curve Sketching

We have been learning how we can understand the behavior of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand–held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast – it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behavior of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behavior we care about is explained by f' and f''. Understanding the interactions between the graph of f and f' and f'' is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands–on" access increases understanding.

The following Key Idea summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Key Idea 42 Curve Sketching

To produce an accurate sketch a given function f, consider the following steps.

1. Find the domain of f. Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.
2. Find the critical values of f.
3. Find the possible points of inflection of f.
4. Find the location of any vertical asymptotes of f (usually done in conjunction with item 1 above).
5. Consider the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ to determine the end behavior of the function.

(continued)

Key Idea 42 Curve Sketching – Continued

6. Create a number line that includes all critical points, possible points of inflection, and locations of vertical asymptotes. For each interval created, determine whether f is increasing or decreasing, concave up or down.

7. Evaluate f at each critical point and possible point of inflection. Plot these points on a set of axes. Connect these points with curves exhibiting the proper concavity. Sketch asymptotes and x and y intercepts where applicable.

Example 246 Curve sketching

Use Key Idea 42 to sketch $f(x) = 3x^3 - 10x^2 + 7x + 5$.

Solution We follow the steps outlined in the Key Idea.

1. The domain of f is the entire real line; there are no values x for which $f(x)$ is not defined.

2. Find the critical values of f. We compute $f'(x) = 9x^2 - 20x + 7$. Use the Quadratic Formula to find the roots of f':
$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(9)(7)}}{2(9)} = \frac{1}{9}\left(10 \pm \sqrt{37}\right) \Rightarrow x \approx 0.435, 1.787.$$

3. Find the possible points of inflection of f. Compute $f''(x) = 18x - 20$. We have
$$f''(x) = 0 \Rightarrow x = 10/9 \approx 1.111.$$

4. There are no vertical asymptotes.

5. We determine the end behavior using limits as x approaches $\pm$infinity.
$$\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to\infty} f(x) = \infty.$$
We do not have any horizontal asymptotes.

6. We place the values $x = (10 \pm \sqrt{37})/9$ and $x = 10/9$ on a number line, as shown in Figure 12.35. We mark each subinterval as increasing or decreasing, concave up or down, using the techniques used in Sections 12.2 and 12.3.

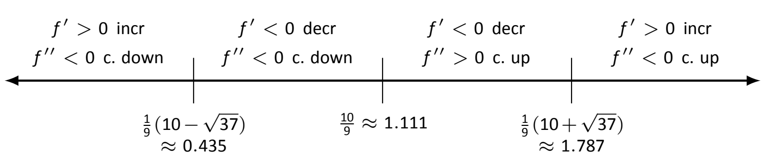


Figure 12.35: Number line for f in Example 246.

7. We plot the appropriate points on axes as shown in Figure 12.36(a) and connect the points with straight lines. In Figure 12.36(b) we adjust these lines to demonstrate the proper concavity. Our curve crosses the y axis at $y = 5$ and crosses the x axis near $x = -0.424$. In Figure 12.36(c) we show a graph of f drawn with a computer program, verifying the accuracy of our sketch.

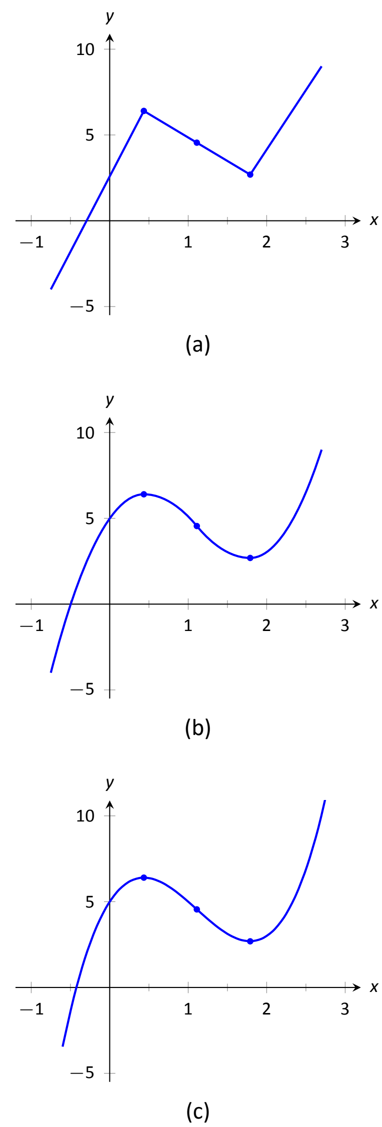


Figure 12.36: Sketching f in Example 246.

Example 247 **Curve sketching**

Sketch $f(x) = \dfrac{x^2 - x - 2}{x^2 - x - 6}$.

Solution We again follow the steps outlined in Key Idea 42.

1. In determining the domain, we assume it is all real numbers and looks for restrictions. We find that at $x = -2$ and $x = 3$, $f(x)$ is not defined. So the domain of f is $D = \{\text{real numbers } x \mid x \neq -2, 3\}$.

2. To find the critical values of f, we first find $f'(x)$. Using the Quotient Rule, we find
$$f'(x) = \frac{-8x + 4}{(x^2 + x - 6)^2} = \frac{-8x + 4}{(x-3)^2(x+2)^2}.$$
$f'(x) = 0$ when $x = 1/2$, and f' is undefined when $x = -2, 3$. Since f' is undefined only when f is, these are not critical values. The only critical value is $x = 1/2$.

3. To find the possible points of inflection, we find $f''(x)$, again employing the Quotient Rule:
$$f''(x) = \frac{24x^2 - 24x + 56}{(x-3)^3(x+2)^3}.$$
We find that $f''(x)$ is never 0 (setting the numerator equal to 0 and solving for x, we find the only roots to this quadratic are imaginary) and f'' is undefined when $x = -2, 3$. Thus concavity will possibly only change at $x = -2$ and $x = 3$.

4. The vertical asymptotes of f are at $x = -2$ and $x = 3$, the places where f is undefined.

5. There is a horizontal asymptote of $y = 1$, as $\lim\limits_{x\to-\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} f(x) = 1$.

6. We place the values $x = 1/2$, $x = -2$ and $x = 3$ on a number line as shown in Figure 12.37. We mark in each interval whether f is increasing or decreasing, concave up or down. We see that f has a relative maximum at $x = 1/2$; concavity changes only at the vertical asymptotes.

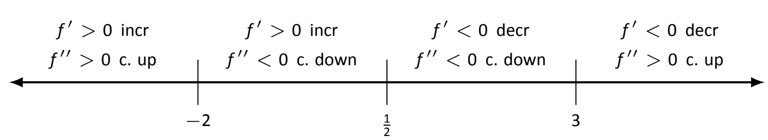


Figure 12.37: Number line for f in Example 247.

7. In Figure 12.38(a), we plot the points from the number line on a set of axes and connect the points with straight lines to get a general idea of what the function looks like (these lines effectively only convey increasing/decreasing information). In Figure 12.38(b), we adjust the graph with the appropriate concavity. We also show f crossing the x axis at $x = -1$ and $x = 2$.

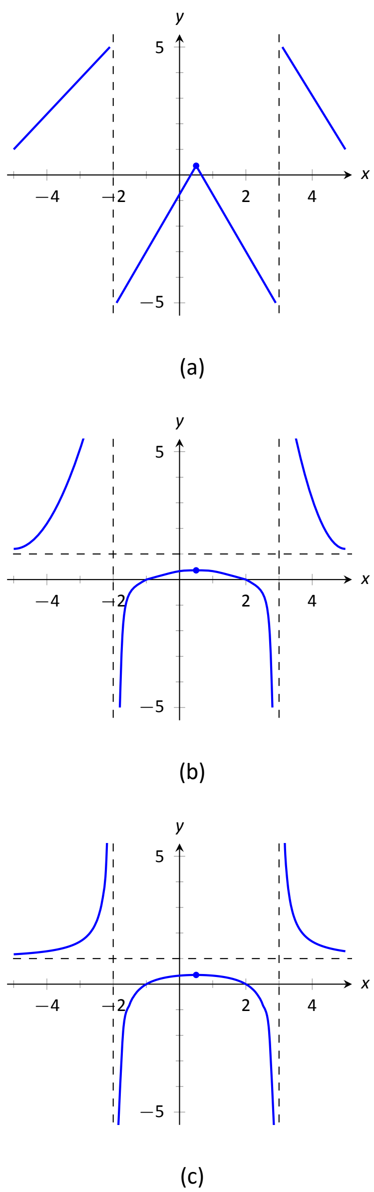


Figure 12.38: Sketching f in Example 247.

Figure 12.38(c) shows a computer generated graph of f, which verifies the accuracy of our sketch.

Example 248 Curve sketching

Sketch $f(x) = \dfrac{5(x-2)(x+1)}{x^2+2x+4}$.

Solution We again follow Key Idea 42.

1. We assume that the domain of f is all real numbers and consider restrictions. The only restrictions come when the denominator is 0, but this never occurs. Therefore the domain of f is all real numbers, $\mathbb{R}$.

2. We find the critical values of f by setting $f'(x) = 0$ and solving for x. We find

$$f'(x) = \frac{15x(x+4)}{(x^2+2x+4)^2} \quad \Rightarrow \quad f'(x) = 0 \text{ when } x = -4, 0.$$

3. We find the possible points of inflection by solving $f''(x) = 0$ for x. We find

$$f''(x) = -\frac{30x^3+180x^2-240}{(x^2+2x+4)^3}.$$

The cubic in the numerator does not factor very "nicely." We instead approximate the roots at $x = -5.759$, $x = -1.305$ and $x = 1.064$.

4. There are no vertical asymptotes.

5. We have a horizontal asymptote of $y = 5$, as $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 5$.

6. We place the critical points and possible points on a number line as shown in Figure 12.39 and mark each interval as increasing/decreasing, concave up/down appropriately.

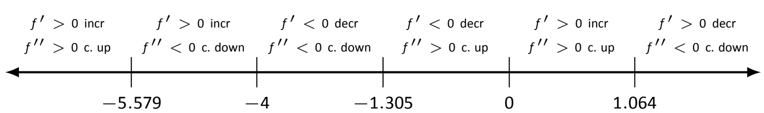
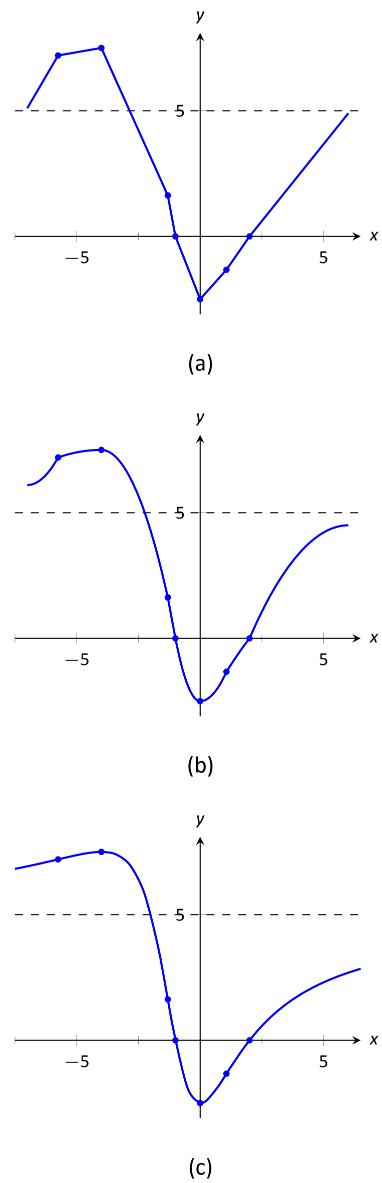


Figure 12.39: Number line for f in Example 248.

7. In Figure 12.40(a) we plot the significant points from the number line as well as the two roots of f, $x = -1$ and $x = 2$, and connect the points with straight lines to get a general impression about the graph. In Figure 12.40(b), we add concavity. Figure 12.40(c) shows a computer generated graph of f, affirming our results.



(a) (b) (c)

Figure 12.40: Sketching f in Example 248.

In each of our examples, we found a few, significant points on the graph of f that corresponded to changes in increasing/decreasing or concavity. We connected these points with straight lines, then adjusted for concavity, and finished by showing a very accurate, computer generated graph.

Why are computer graphics so good? It is not because computers are "smarter" than we are. Rather, it is largely because computers are much faster at computing than we are. In general, computers graph functions much like most students do when first learning to draw graphs: they plot equally spaced points, then connect the dots using lines. By using lots of points, the connecting lines are short and the graph looks smooth.

This does a fine job of graphing in most cases (in fact, this is the method used for many graphs in this text). However, in regions where the graph is very "curvy," this can generate noticeable sharp edges on the graph unless a large number of points are used. High quality computer algebra systems, such as *Mathematica*, use special algorithms to plot lots of points only where the graph is "curvy."

In Figure 12.41, a graph of $y = \sin x$ is given, generated by *Mathematica*. The small points represent each of the places *Mathematica* sampled the function. Notice how at the "bends" of $\sin x$, lots of points are used; where $\sin x$ is relatively straight, fewer points are used. (Many points are also used at the endpoints to ensure the "end behavior" is accurate.)

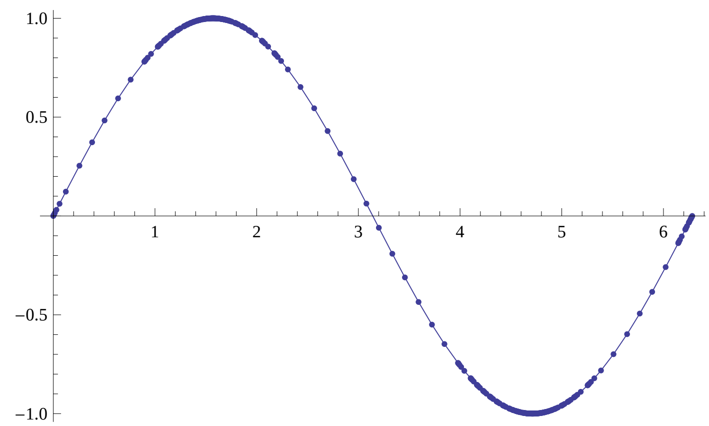


Figure 12.41: A graph of $y = \sin x$ generated by *Mathematica*.

How does *Mathematica* know where the graph is "curvy"? Calculus. When we study *curvature* in a later chapter, we will see how the first and second derivatives of a function work together to provide a measurement of "curviness." *Mathematica* employs algorithms to determine regions of "high curvature" and plots extra points there.

Again, the goal of this section is not "How to graph a function when there is no computer to help." Rather, the goal is "Understand that the shape of the graph of a function is largely determined by understanding the behavior of the function at a few key places." In Example 248, we were able to accurately sketch a complicated graph using only 5 points and knowledge of asymptotes!

There are many applications of our understanding of derivatives beyond curve sketching. The next chapter explores some of these applications, demonstrating just a few kinds of problems that can be solved with a basic knowledge of differentiation.

Exercises 12.4

Terms and Concepts

1. Why is sketching curves by hand beneficial even though technology is ubiquitous?

2. What does "ubiquitous" mean?

3. T/F: When sketching graphs of functions, it is useful to find the critical points.

4. T/F: When sketching graphs of functions, it is useful to find the possible points of inflection.

5. T/F: When sketching graphs of functions, it is useful to find the horizontal and vertical asymptotes.

Problems

In Exercises 6 – 11, practice using Key Idea 42 by applying the principles to the given functions with familiar graphs.

6. $f(x) = 2x + 4$

7. $f(x) = -x^2 + 1$

8. $f(x) = \sin x$

9. $f(x) = e^x$

10. $f(x) = \dfrac{1}{x}$

11. $f(x) = \dfrac{1}{x^2}$

In Exercises 12 – 25, sketch a graph of the given function using Key Idea 42. Show all work; check your answer with technology.

12. $f(x) = x^3 - 2x^2 + 4x + 1$

13. $f(x) = -x^3 + 5x^2 - 3x + 2$

14. $f(x) = x^3 + 3x^2 + 3x + 1$

15. $f(x) = x^3 - x^2 - x + 1$

16. $f(x) = (x - 2)\ln(x - 2)$

17. $f(x) = (x - 2)^2 \ln(x - 2)$

18. $f(x) = \dfrac{x^2 - 4}{x^2}$

19. $f(x) = \dfrac{x^2 - 4x + 3}{x^2 - 6x + 8}$

20. $f(x) = \dfrac{x^2 - 2x + 1}{x^2 - 6x + 8}$

21. $f(x) = x\sqrt{x + 1}$

22. $f(x) = x^2 e^x$

23. $f(x) = \sin x \cos x$ on $[-\pi, \pi]$

24. $f(x) = (x - 3)^{2/3} + 2$

25. $f(x) = \dfrac{(x - 1)^{2/3}}{x}$

In Exercises 26 – 28, a function with the parameters *a* and *b* are given. Describe the critical points and possible points of inflection of *f* in terms of *a* and *b*.

26. $f(x) = \dfrac{a}{x^2 + b^2}$

27. $f(x) = \sin(ax + b)$

28. $f(x) = (x - a)(x - b)$

29. Given $x^2 + y^2 = 1$, use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use this information to justify the sketch of the unit circle.

We have spent considerable time considering the derivatives of a function and their applications. In the section, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. Here, we will only consider very basic examples, and leave most of the heavy lifting to later courses. The importance of antiderivatives becomes apparent in Math 1560, once integration and the Fundamental Theorem of Calculus have been introduced. More advanced techniques for finding antiderivatives are taught in Math 2560.

12.5 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 73 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\,dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 107 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

Given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x)+C$ for some constant C. Using Definition 73, we can say that

$$\int f(x)\,dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

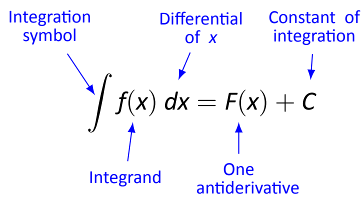


Figure 12.42: Understanding the indefinite integral notation.

Figure 12.42 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 249 Evaluating indefinite integrals

Evaluate $\int \sin x\,dx$.

Solution We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x\,dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x\,dx$$

presents us with a differential, $dy = \sin x\,dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x\,dx$, rewrite

$$\int \sin x\,dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 250 Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5)\ dx$.

SOLUTION We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

For ease of reference, and to stress the relationship between derivatives and antiderivatives, we include below a list of many of the common differentiation rules we have learned, along with the corresponding antidifferentiation rules.

Theorem 108 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\,dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\,dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\,dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\,dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\,dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\,dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\,dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\,dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\,dx = \ln\|x\| + C$

We highlight a few important points from Theorem 108:

- Rule #1 states $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

$$\int 5\cos x\,dx = 5 \cdot \int \cos x\,dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by 5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 250. So:

$$\begin{aligned}\int (3x^2 + 4x + 5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx \\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx \\ &= 3 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 5x + C \\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 11.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinite antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 251 **Solving initial value problems**

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\,dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 252 Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\,dt = \int \cos t\,dt = \sin t + C = f'(t).$$

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\,dt = \int (\sin t + 3)\,dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

If you continue on to Math 1560, you will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function, and how the Fundamental Theorem of Calculus ties together areas and antiderivatives.

Exercises 12.5

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

Problems

In Exercises 8 – 26, evaluate the given indefinite integral.

8. $\int 3x^3\,dx$

9. $\int x^8\,dx$

10. $\int (10x^2 - 2)\,dx$

11. $\int dt$

12. $\int 1\,ds$

13. $\int \frac{1}{3t^2}\,dt$

14. $\int \frac{3}{t^2}\,dt$

15. $\int \frac{1}{\sqrt{x}}\,dx$

16. $\int \sec^2\theta\,d\theta$

17. $\int \sin\theta\,d\theta$

18. $\int (\sec x\tan x + \csc x\cot x)\,dx$

19. $\int 5e^\theta\,d\theta$

20. $\int 3^t\,dt$

21. $\int \frac{5^t}{2}\,dt$

22. $\int (2t+3)^2\,dt$

23. $\int (t^2+3)(t^3-2t)\,dt$

24. $\int x^2x^3\,dx$

25. $\int e^\pi\,dx$

26. $\int a\,dx$

27. This problem investigates why Theorem 108 states that $\int \frac{1}{x}\,dx = \ln|x| + C$.
    (a) What is the domain of $y = \ln x$?
    (b) Find $\frac{d}{dx}\big(\ln x\big)$.
    (c) What is the domain of $y = \ln(-x)$?
    (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.
    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\int \frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

In Exercises 28 – 38, find $f(x)$ described by the given initial value problem.

28. $f'(x) = \sin x$ and $f(0) = 2$

29. $f'(x) = 5e^x$ and $f(0) = 10$

30. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

31. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

32. $f'(x) = 7^x$ and $f(2) = 1$

33. $f''(x) = 5$ and $f'(0) = 7, f(0) = 3$

34. $f''(x) = 7x$ and $f'(1) = -1, f(1) = 10$

35. $f''(x) = 5e^x$ and $f'(0) = 3, f(0) = 5$

36. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2, f(\pi) = 4$

37. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

38. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

39. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

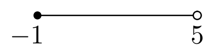
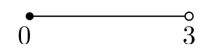
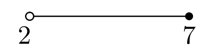
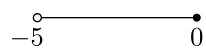
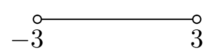
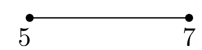
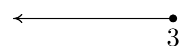
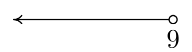
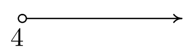
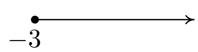
40. Given $y = x^2 e^x \cos x$, find dy.

A: Answers To Selected Problems

Chapter 1

Section 1.1

1.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid -1 \leq x < 5\}$	$[-1, 5)$	
$\{x \mid 0 \leq x < 3\}$	$[0, 3)$	
$\{x \mid 2 < x \leq 7\}$	$(2, 7]$	
$\{x \mid -5 < x \leq 0\}$	$(-5, 0]$	
$\{x \mid -3 < x < 3\}$	$(-3, 3)$	
$\{x \mid 5 \leq x \leq 7\}$	$[5, 7]$	
$\{x \mid x \leq 3\}$	$(-\infty, 3]$	
$\{x \mid x < 9\}$	$(-\infty, 9)$	
$\{x \mid x > 4\}$	$(4, \infty)$	
$\{x \mid x \geq -3\}$	$[-3, \infty)$	

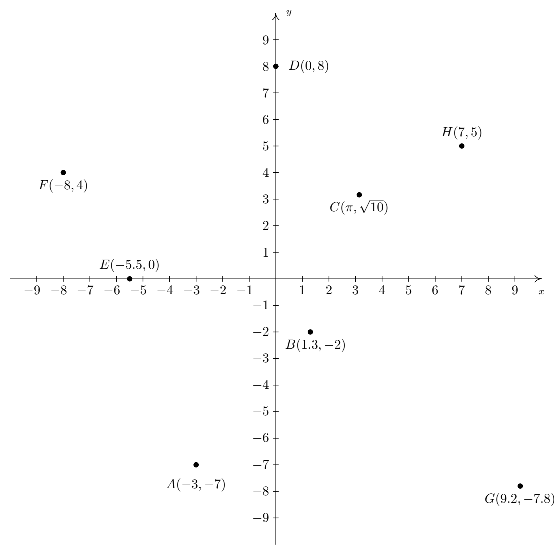
3. $(-1, 1) \cup [0, 6] = (-1, 6]$
5. $(-\infty, 0) \cap [1, 5] = \emptyset$
7. $(-\infty, 5] \cap [5, 8) = \{5\}$
9. $(-\infty, -1) \cup (-1, \infty)$
11. $(-\infty, 0) \cup (0, 2) \cup (2, \infty)$
13. $(-\infty, -4) \cup (-4, 0) \cup (0, 4) \cup (4, \infty)$
15. $(-\infty, \infty)$
17. $(-\infty, 5] \cup \{6\}$
19. $(-3, 3) \cup \{4\}$

Section 1.2

1. 6
3. $\dfrac{2}{21}$
5. $-\dfrac{1}{3}$
7. $\dfrac{3}{5}$
9. $-\dfrac{7}{8}$
11. 0
13. $\dfrac{23}{9}$
15. $-\dfrac{24}{7}$
17. $\dfrac{243}{32}$
19. $\dfrac{9}{22}$
21. 5
23. $\dfrac{107}{27}$
25. $\sqrt{10}$
27. $\sqrt{7}$
29. -1
31. $\dfrac{15}{16}$
33. $-\dfrac{385}{12}$

Section 1.3

1. The required points $A(-3, -7)$, $B(1.3, -2)$, $C(\pi, \sqrt{10})$, $D(0, 8)$, $E(-5.5, 0)$, $F(-8, 4)$, $G(9.2, -7.8)$, and $H(7, 5)$ are plotted in the Cartesian Coordinate Plane below.



3. $d = 5$, $M = \left(-1, \frac{7}{2}\right)$
5. $d = \sqrt{26}$, $M = \left(1, \frac{3}{2}\right)$
7. $d = \sqrt{74}$, $M = \left(\frac{13}{10}, -\frac{13}{10}\right)$
9. $d = \sqrt{83}$, $M = \left(4\sqrt{5}, \frac{5\sqrt{3}}{2}\right)$
11. $(3 + \sqrt{7}, -1)$, $(3 - \sqrt{7}, -1)$
13. $(-1 + \sqrt{3}, 0)$, $(-1 - \sqrt{3}, 0)$
15. $(-3, -4)$, 5 miles, $(4, -4)$
17.
19.
21.

Section 1.4

1. For $z = 2 + 3i$ and $w = 4i$
   - $z + w = 2 + 7i$
   - $zw = -12 + 8i$
   - $z^2 = -5 + 12i$

- $\frac{1}{z} = \frac{2}{13} - \frac{3}{13}i$
- $\frac{z}{w} = \frac{3}{4} - \frac{1}{2}i$
- $\frac{w}{z} = \frac{12}{13} + \frac{8}{13}i$
- $\bar{z} = 2 - 3i$
- $z\bar{z} = 13$
- $(\bar{z})^2 = -5 - 12i$

3. For $z = i$ and $w = -1 + 2i$
   - $z + w = -1 + 3i$
   - $zw = -2 - i$
   - $z^2 = -1$
   - $\frac{1}{z} = -i$
   - $\frac{z}{w} = \frac{2}{5} - \frac{1}{5}i$
   - $\frac{w}{z} = 2 + i$
   - $\bar{z} = -i$
   - $z\bar{z} = 1$
   - $(\bar{z})^2 = -1$

5. For $z = 3 - 5i$ and $w = 2 + 7i$
   - $z + w = 5 + 2i$
   - $zw = 41 + 11i$
   - $z^2 = -16 - 30i$
   - $\frac{1}{z} = \frac{3}{34} + \frac{5}{34}i$
   - $\frac{z}{w} = -\frac{29}{53} - \frac{31}{53}i$
   - $\frac{w}{z} = -\frac{29}{34} + \frac{31}{34}i$
   - $\bar{z} = 3 + 5i$
   - $z\bar{z} = 34$
   - $(\bar{z})^2 = -16 + 30i$

7. For $z = \sqrt{2} - i\sqrt{2}$ and $w = \sqrt{2} + i\sqrt{2}$
   - $z + w = 2\sqrt{2}$
   - $zw = 4$
   - $z^2 = -4i$
   - $\frac{1}{z} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}i$
   - $\frac{z}{w} = -i$
   - $\frac{w}{z} = i$
   - $\bar{z} = \sqrt{2} + i\sqrt{2}$
   - $z\bar{z} = 4$
   - $(\bar{z})^2 = 4i$

9. For $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
   - $z + w = i\sqrt{3}$
   - $zw = -1$
   - $z^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
   - $\frac{1}{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
   - $\frac{z}{w} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
   - $\frac{w}{z} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
   - $\bar{z} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$
   - $z\bar{z} = 1$
   - $(\bar{z})^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

11. $7i$

13. -10

15. -12

17. 3

19. $i^5 = i^4 \cdot i = 1 \cdot i = i$

21. $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

23. $i^{15} = \left(i^4\right)^3 \cdot i^3 = 1 \cdot (-i) = -i$

25. $i^{117} = \left(i^4\right)^{29} \cdot i = 1 \cdot i = i$

27. $x = \dfrac{2 \pm i\sqrt{14}}{3}$

29. $y = \pm 2, \pm i$

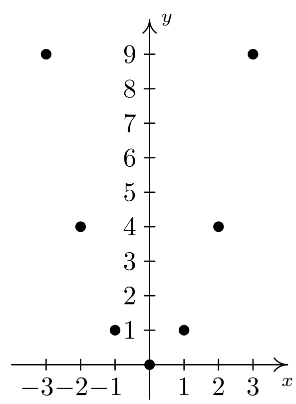
31. $y = \pm \dfrac{3i\sqrt{2}}{2}$

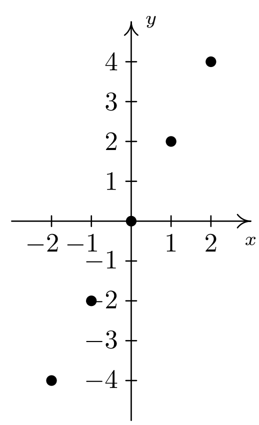
33. $x = \dfrac{\sqrt{5} \pm i\sqrt{3}}{2}$

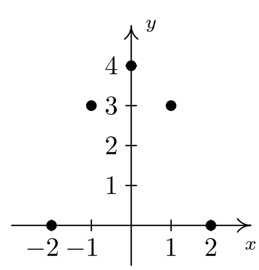
35. $z = \pm 2, \pm 2i$

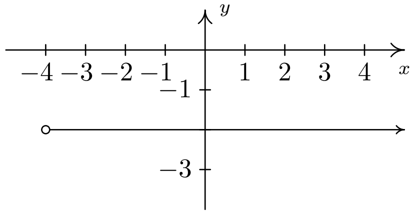
Chapter 2

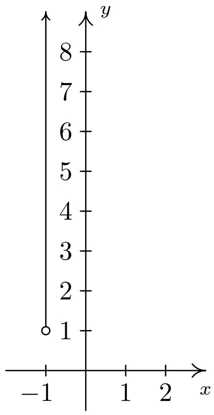
Section 2.1

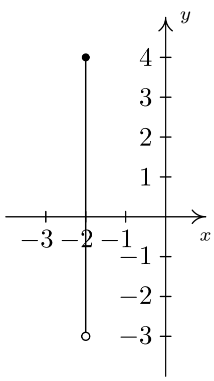
1. 

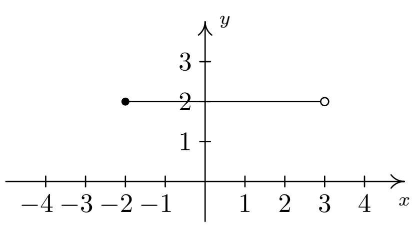
3. 

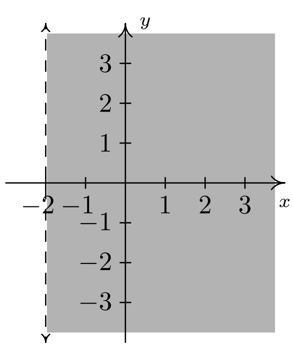
5. 

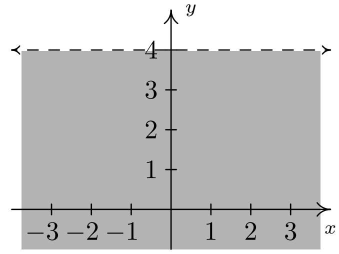
7. 

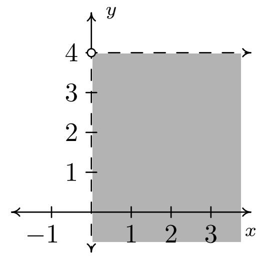
9. 

11. 

13. 

15. 

17. 

19. 

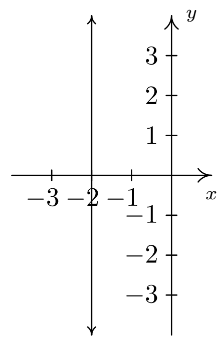
21. $A = \{(-4,-1),(-2,1),(0,3),(1,4)\}$

23. $C = \{(2,y) \mid y > -3\}$

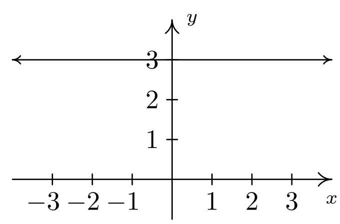
25. $E = \{(x,2) \mid -4 \le x < 3\}$

27. $G = \{(x,y) \mid x > -2\}$

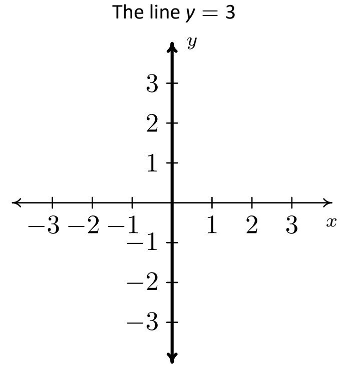
29. $I = \{(x,y) \mid x \ge 0, y \ge 0\}$

31. 

The line $x = -2$

33. 

The line $y = 3$

35. 

The line $x = 0$ is the y-axis

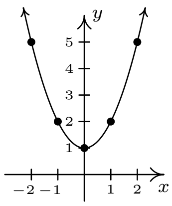
37.

39.

41. The graph has no x-intercepts

y-intercept: $(0, 1)$

x	y	(x, y)
-2	5	$(-2, 5)$
-1	2	$(-1, 2)$
0	1	$(0, 1)$
1	2	$(1, 2)$
2	5	$(2, 5)$



The graph is not symmetric about the x-axis (e.g. $(2, 5)$ is on the graph but $(2, -5)$ is not)

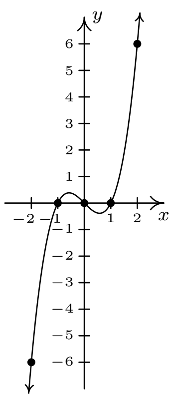
The graph is symmetric about the y-axis

The graph is not symmetric about the origin (e.g. $(2, 5)$ is on the graph but $(-2, -5)$ is not)

43. x-intercepts: $(-1, 0), (0, 0), (1, 0)$

y-intercept: $(0, 0)$

x	y	(x, y)
-2	-6	$(-2, -6)$
-1	0	$(-1, 0)$
0	0	$(0, 0)$
1	0	$(1, 0)$
2	6	$(2, 6)$



The graph is not symmetric about the x-axis. (e.g. $(2, 6)$ is on the graph but $(2, -6)$ is not)

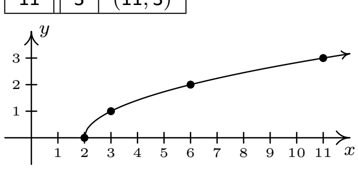
The graph is not symmetric about the y-axis. (e.g. $(2, 6)$ is on the graph but $(-2, 6)$ is not)

The graph is symmetric about the origin.

45. x-intercept: $(2, 0)$

The graph has no y-intercepts

x	y	(x, y)
2	0	$(2, 0)$
3	1	$(3, 1)$
6	2	$(6, 2)$
11	3	$(11, 3)$



The graph is not symmetric about the x-axis (e.g. $(3, 1)$ is on the graph but $(3, -1)$ is not)

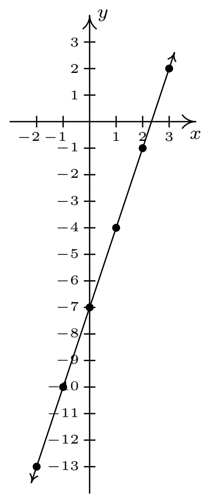
The graph is not symmetric about the y-axis (e.g. $(3, 1)$ is on the graph but $(-3, 1)$ is not)

The graph is not symmetric about the origin (e.g. $(3, 1)$ is on the graph but $(-3, -1)$ is not)

47. x-intercept: $(\frac{7}{3}, 0)$

y-intercept: $(0, -7)$

x	y	(x, y)
-2	-13	$(-2, -13)$
-1	-10	$(-1, -10)$
0	-7	$(0, -7)$
1	-4	$(1, -4)$
2	-1	$(2, -1)$
3	2	$(3, 2)$



The graph is not symmetric about the x-axis (e.g. $(3, 2)$ is on the graph but $(3, -2)$ is not)

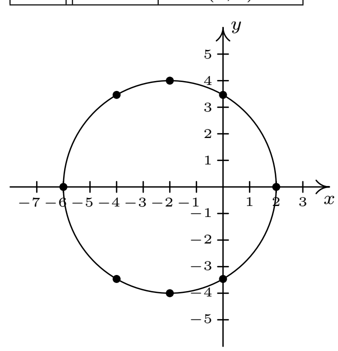
The graph is not symmetric about the y-axis (e.g. $(3, 2)$ is on the graph but $(-3, 2)$ is not)

The graph is not symmetric about the origin (e.g. $(3, 2)$ is on the graph but $(-3, -2)$ is not)

49. x-intercepts: $(-6, 0)$, $(2, 0)$

y-intercepts: $(0, \pm 2\sqrt{3})$

x	y	(x, y)
-6	0	$(-6, 0)$
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-2	± 4	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$(0, \pm 2\sqrt{3})$
2	0	$(2, 0)$



The graph is symmetric about the x-axis

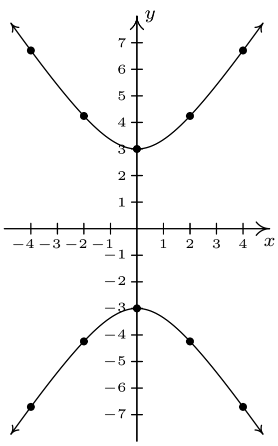
The graph is not symmetric about the y-axis (e.g. $(-6, 0)$ is on the graph but $(6, 0)$ is not)

The graph is not symmetric about the origin (e.g. $(-6, 0)$ is on the graph but $(6, 0)$ is not)

51. $4y^2 - 9x^2 = 36$
Re-write as: $y = \pm\frac{\sqrt{9x^2+36}}{2}$.
The graph has no x-intercepts

y-intercepts: $(0, \pm 3)$

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$(-4, \pm 3\sqrt{5})$
-2	$\pm 3\sqrt{2}$	$(-2, \pm 3\sqrt{2})$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$(2, \pm 3\sqrt{2})$
4	$\pm 3\sqrt{5}$	$(4, \pm 3\sqrt{5})$



The graph is symmetric about the x-axis

The graph is symmetric about the y-axis

The graph is symmetric about the origin

53.

Section 2.2

1. Function
domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
range = $\{0, 1, 4, 9\}$

3. Function
domain = $\{-7, -3, 3, 4, 5, 6\}$
range = $\{0, 4, 5, 6, 9\}$

5. Not a function

7. Function
domain = $\{x | x = 2^n$ for some whole number $n\}$
range = $\{y \mid y \geq 0$ is an integer$\}$

9. Not a function

11. Function
domain = $(-\infty, \infty)$
range = $[0, \infty)$

13. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

15. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

17. Function
domain = $[2, \infty)$
range = $[0, \infty)$

19. Not a function

21. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

23. Function
domain = $[-5, 4)$
range = $[-4, 4)$

25. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

27. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

29. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

31. Not a function

33. Function

35. Function

37. Function

39. Not a function

41. Function

43. Not a function

45. Function

47. Not a function

49.

51.

53.

Section 2.3

1. $f(x) = \frac{2x+3}{4}$
Domain: $(-\infty, \infty)$

3. $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$
Domain: $(-\infty, \infty)$

5. $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$
Domain: $[-3, \infty)$

7. $f(x) = \frac{4}{\sqrt{x}-13}$
Domain: $[0, 169) \cup (169, \infty)$

9. $f(x) = \frac{4}{\sqrt{x}} - 13$
Domain: $(0, \infty)$

11. For $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x - 4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f\left(x^2\right) = 2x^2 + 1$

13. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x - 4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

15. For $f(x) = \frac{x}{x-1}$

- $f(3) = \frac{3}{2}$
- $f(-1) = \frac{1}{2}$
- $f\left(\frac{3}{2}\right) = 3$
- $f(4x) = \frac{4x}{4x-1}$
- $4f(x) = \frac{4x}{x-1}$
- $f(-x) = \frac{x}{x+1}$
- $f(x-4) = \frac{x-4}{x-5}$
- $f(x) - 4 = \frac{x}{x-1} - 4 = \frac{4-3x}{x-1}$
- $f\left(x^2\right) = \frac{x^2}{x^2-1}$

17. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x-4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

19. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 2x^2 - 1$

- $f(2) = 7$
- $f(-2) = 7$
- $f(2a) = 8a^2 - 1$
- $2f(a) = 4a^2 - 2$
- $f(a+2) = 2a^2 + 8a + 7$
- $f(a) + f(2) = 2a^2 + 6$
- $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
- $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
- $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

23. For $f(x) = \sqrt{2x+1}$

- $f(2) = \sqrt{5}$
- $f(-2)$ is not real
- $f(2a) = \sqrt{4a+1}$
- $2f(a) = 2\sqrt{2a+1}$
- $f(a+2) = \sqrt{2a+5}$
- $f(a) + f(2) = \sqrt{2a+1} + \sqrt{5}$
- $f\left(\frac{2}{a}\right) = \sqrt{\frac{4}{a} + 1} = \sqrt{\frac{a+4}{a}}$
- $\frac{f(a)}{2} = \frac{\sqrt{2a+1}}{2}$
- $f(a+h) = \sqrt{2a+2h+1}$

25. For $f(x) = \frac{x}{2}$

- $f(2) = 1$
- $f(-2) = -1$
- $f(2a) = a$
- $2f(a) = a$
- $f(a+2) = \frac{a+2}{2}$
- $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
- $f\left(\frac{2}{a}\right) = \frac{1}{a}$
- $\frac{f(a)}{2} = \frac{a}{4}$
- $f(a+h) = \frac{a+h}{2}$

27. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$

29. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$

31. For $f(x) = \sqrt{x+4}$, $f(0) = 2$ and $f(x) = 0$ when $x = -4$

33. For $f(x) = \frac{3}{4-x}$, $f(0) = \frac{3}{4}$ and $f(x)$ is never equal to 0

35. (a) $f(-4) = 1$
(b) $f(-3) = 2$
(c) $f(3) = 0$
(d) $f(3.001) = 1.999$
(e) $f(-3.001) = 1.999$
(f) $f(2) = \sqrt{5}$

37. $(-\infty, \infty)$

39. $(-\infty, -1) \cup (-1, \infty)$

41. $(-\infty, \infty)$

43. $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$

45. $(-\infty, 3]$

47. $[-3, \infty)$

49. $\left[\frac{1}{3}, \infty\right)$

51. $(-\infty, \infty)$

53. $\left[\frac{1}{3}, 6\right) \cup (6, \infty)$

55. $(-\infty, 8) \cup (8, \infty)$

57. $(8, \infty)$

59. $(-\infty, 8) \cup (8, \infty)$

61. $[0, 5) \cup (5, \infty)$

63. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(x) = 36$ are $x = \pm 6$. Since x is restricted to $x > 0$, we only keep $x = 6$. This means for the area enclosed by the square to be 36 square inches, the length of the side needs to be 6 inches. Since x represents a length, $x > 0$.

65. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(x) = 27$ is $x = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

67. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

69. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

71. $F(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $F(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $F(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

73. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.
    (b) $C(50) = 675$, so it costs \$675 for 50 copies of the book. $C(51) = 612$, so it costs \$612 for 51 copies of the book.
    (c) 56 books.

75. (a) $C(750) = 25$, so it costs \$25 to talk 750 minutes per month with this plan.
    (b) Since 20 hours = 1200 minutes, we substitute $m = 1200$ and get $C(1200) = 45$. It costs \$45 to talk 20 hours per month with this plan.
    (c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

77.

Section 2.4

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$
    - $(f+g)(2) = 9$
    - $(f-g)(-1) = -7$
    - $(g-f)(1) = -1$
    - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
    - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
    - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$

3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
    - $(f+g)(2) = 10$
    - $(f-g)(-1) = -9$
    - $(g-f)(1) = 11$
    - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
    - $\left(\frac{f}{g}\right)(0) = 0$
    - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$

5. For $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$
    - $(f+g)(2) = 3 + \sqrt{5}$
    - $(f-g)(-1) = 3 + \sqrt{2}$
    - $(g-f)(1) = -1$
    - $(fg)\left(\frac{1}{2}\right) = 0$
    - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$
    - $\left(\frac{g}{f}\right)(-2) = -5$

7. For $f(x) = 2x$ and $g(x) = \dfrac{1}{2x+1}$
    - $(f+g)(2) = \frac{21}{5}$
    - $(f-g)(-1) = -1$
    - $(g-f)(1) = -\frac{5}{3}$
    - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
    - $\left(\frac{f}{g}\right)(0) = 0$
    - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

9. For $f(x) = x^2$ and $g(x) = \dfrac{1}{x^2}$
    - $(f+g)(2) = \frac{17}{4}$
    - $(f-g)(-1) = 0$
    - $(g-f)(1) = 0$
    - $(fg)\left(\frac{1}{2}\right) = 1$
    - $\left(\frac{f}{g}\right)(0)$ is undefined.
    - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

11. For $f(x) = 2x + 1$ and $g(x) = x - 2$
    - $(f+g)(x) = 3x - 1$ Domain: $(-\infty, \infty)$
    - $(f-g)(x) = x + 3$ Domain: $(-\infty, \infty)$
    - $(fg)(x) = 2x^2 - 3x - 2$ Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$ Domain: $(-\infty, 2) \cup (2, \infty)$

13. For $f(x) = x^2$ and $g(x) = 3x - 1$
    - $(f+g)(x) = x^2 + 3x - 1$ Domain: $(-\infty, \infty)$
    - $(f-g)(x) = x^2 - 3x + 1$ Domain: $(-\infty, \infty)$
    - $(fg)(x) = 3x^3 - x^2$ Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$ Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

15. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$
    - $(f+g)(x) = x^2 + 3x + 2$ Domain: $(-\infty, \infty)$
    - $(f-g)(x) = x^2 - 3x - 10$ Domain: $(-\infty, \infty)$
    - $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$ Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$ Domain: $(-\infty, -2) \cup (-2, \infty)$

17. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$
    - $(f+g)(x) = \frac{x^2+4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
    - $(f-g)(x) = \frac{x^2-4}{2x}$ Domain: $(-\infty, 0) \cup (0, \infty)$
    - $(fg)(x) = 1$ Domain: $(-\infty, 0) \cup (0, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$ Domain: $(-\infty, 0) \cup (0, \infty)$

19. For $f(x) = x$ and $g(x) = \sqrt{x+1}$
    - $(f+g)(x) = x + \sqrt{x+1}$ Domain: $[-1, \infty)$
    - $(f-g)(x) = x - \sqrt{x+1}$ Domain: $[-1, \infty)$
    - $(fg)(x) = x\sqrt{x+1}$ Domain: $[-1, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$ Domain: $(-1, \infty)$

21. 2

23. 0

25. $-2x - h + 2$

27. $-2x - h + 1$

29. m

31. $\dfrac{-2}{x(x+h)}$

33. $\dfrac{-(2x+h)}{x^2(x+h)^2}$

35. $\dfrac{-4}{(4x-3)(4x+4h-3)}$

37. $\dfrac{-9}{(x-9)(x+h-9)}$

39. $\dfrac{1}{\sqrt{x+h-9} + \sqrt{x-9}}$

41. $\dfrac{-4}{\sqrt{-4x-4h+5} + \sqrt{-4x+5}}$

43. $\dfrac{a}{\sqrt{ax+ah+b} + \sqrt{ax+b}}$

45. $\dfrac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$

47.
    - $C(0) = 100$, so the fixed costs are \$100.
    - $\overline{C}(10) = 20$, so when 10 bottles of tonic are produced, the cost per bottle is \$20.
    - $p(5) = 30$, so to sell 5 bottles of tonic, set the price at \$30 per bottle.

- $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$
- $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$
- $P(x) = 0$ when $x = 5$ and $x = 20$. These are the 'break even' points, so selling 5 bottles of tonic or 20 bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.

49.
- $C(0) = 36$, so the daily fixed costs are \$36.
- $\overline{C}(10) = 6.6$, so when 10 pies are made, the cost per pie is \$6.60.
- $p(5) = 9.5$, so to sell 5 pies a day, set the price at \$9.50 per pie.
- $R(x) = -0.5x^2 + 12x$, $0 \leq x \leq 24$
- $P(x) = -0.5x^2 + 9x - 36$, $0 \leq x \leq 24$
- $P(x) = 0$ when $x = 6$ and $x = 12$. These are the 'break even' points, so selling 6 pies or 12 pies a day will guarantee the revenue earned exactly recoups the cost of production.

51. $(f + g)(-3) = 2$

53. $(fg)(-1) = 0$

55. $(g - f)(3) = 3$

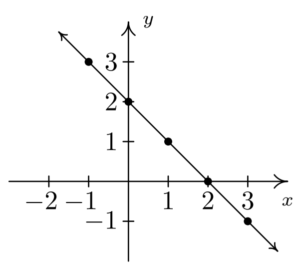
57. $\left(\frac{f}{g}\right)(-2)$ does not exist

59. $\left(\frac{f}{g}\right)(2) = 4$

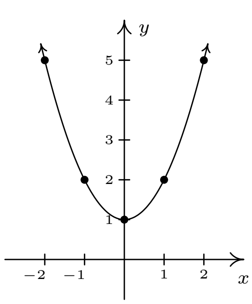
61. $\left(\frac{g}{f}\right)(3) = -2$

Section 2.5

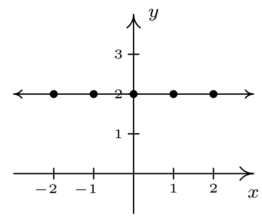
1. $f(x) = 2 - x$
Domain: $(-\infty, \infty)$
x-intercept: $(2, 0)$
y-intercept: $(0, 2)$
No symmetry



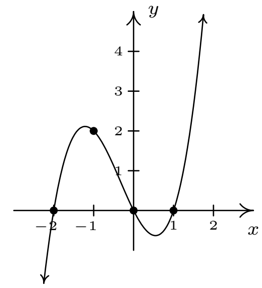
3. $f(x) = x^2 + 1$
Domain: $(-\infty, \infty)$
x-intercept: None
y-intercept: $(0, 1)$
Even



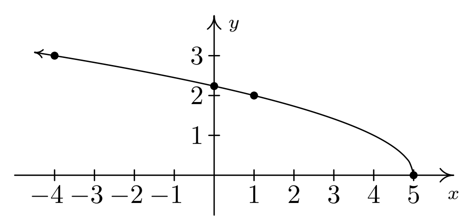
5. $f(x) = 2$
Domain: $(-\infty, \infty)$
x-intercept: None
y-intercept: $(0, 2)$
Even



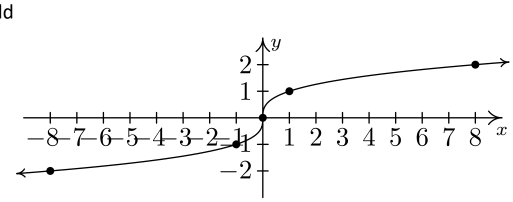
7. $f(x) = x(x - 1)(x + 2)$
Domain: $(-\infty, \infty)$
x-intercepts: $(-2, 0)$, $(0, 0)$, $(1, 0)$
y-intercept: $(0, 0)$
No symmetry



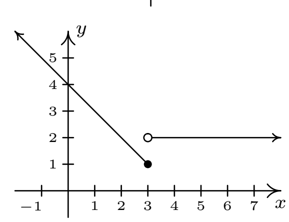
9. $f(x) = \sqrt{5 - x}$
Domain: $(-\infty, 5]$
x-intercept: $(5, 0)$
y-intercept: $(0, \sqrt{5})$
No symmetry

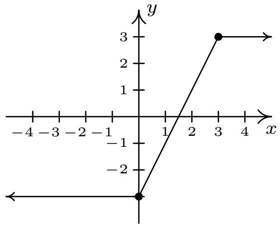


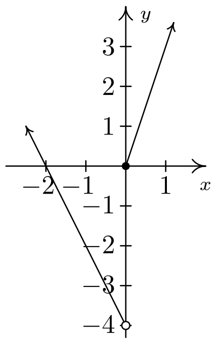
11. $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Odd

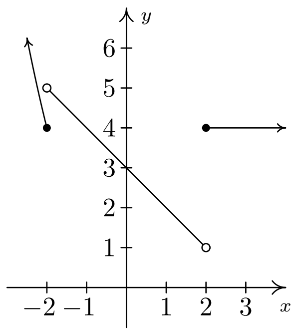


13.



15. 

17. 

19. 

21. odd

23. even

25. even

27. odd

29. even

31. neither

33. even **and** odd

35. even

37. neither

39. odd

41. even

43. $[-5, 4]$

45. $x = -3$

47. $(0, -1)$

49. $[-4, -1] \cup [1, 3]$

51. neither

53. $[-3, 0]$, $[2, 3]$

55. $f(0) = -1$

57. $f(-5) = -5$

59. $[-5, 5)$

61. $x = -2$

63. $(0, 0)$

65. $[-4, 0] \cup \{4\}$

67. neither

69. $[-4, -2]$, $(2, 4]$

71. $f(-2) = -5$, $f(2) = 3$

73. $f(-2) = -5$

75. No absolute maximum
No absolute minimum
Local maximum at $(0, 0)$
Local minimum at $(1.60, -3.28)$
Increasing on $(-\infty, 0]$, $[1.60, \infty)$
Decreasing on $[0, 1.60]$

77. Absolute maximum $f(2.12) \approx 4.50$
Absolute minimum $f(-2.12) \approx -4.50$
Local maximum $(2.12, 4.50)$
Local minimum $(-2.12, -4.50)$
Increasing on $[-2.12, 2.12]$
Decreasing on $[-3, -2.12]$, $[2.12, 3]$

79. $(f + g)(1) = 5$

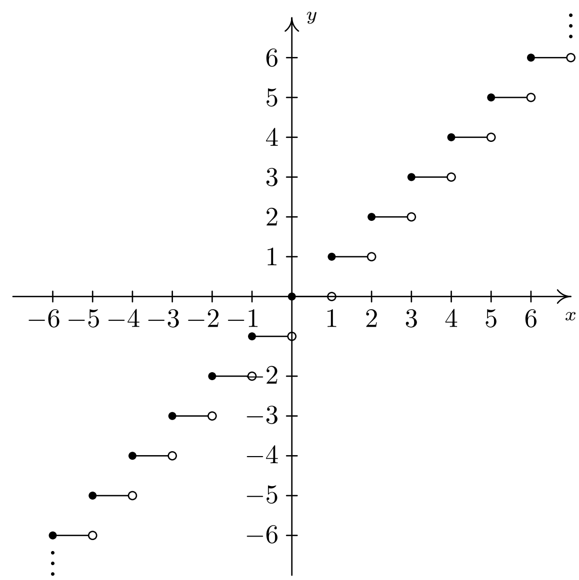
81. $(g - f)(2) = 0$

83. $(fg)(1) = 6$

85. $\left(\frac{g}{f}\right)(2) = 1$

87. $h(15) = 6$, so the Saquatch is 6 feet tall when she is 15 years old.

89. h is constant on $[30, 45]$. This means the Sasquatch's height is constant (at 8 feet) for these years.

91. 

The graph of $f(x) = \lfloor x \rfloor$.

93.

95.

97.

99.

Section 2.6

1. $(2, 0)$

3. $(2, -4)$

5. $(2, -9)$

7. $(2, 3)$

9. $(5, -2)$

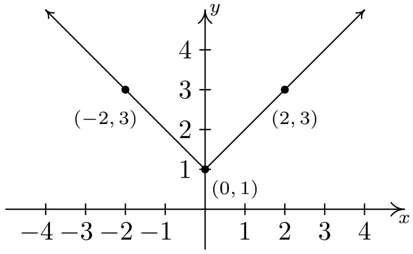
11. $(2, 13)$

13. $\left(2, -\frac{3}{2}\right)$

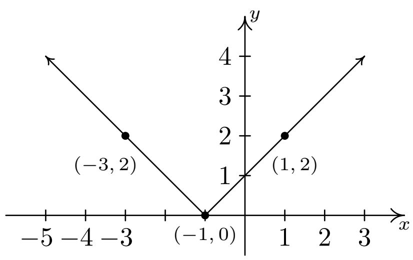
15. $(-1, -7)$

17. $\left(\frac{2}{3}, -2\right)$

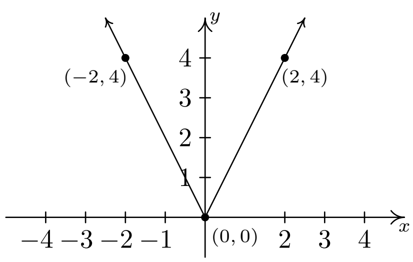
19. $y = f(x) + 1$



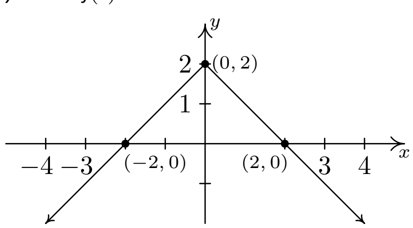
21. $y = f(x + 1)$



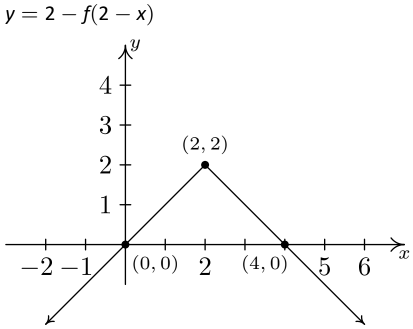
23. $y = 2f(x)$



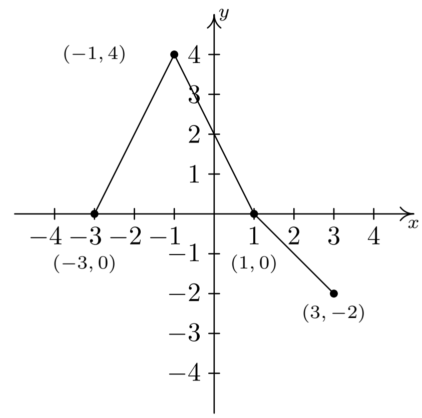
25. $y = 2 - f(x)$



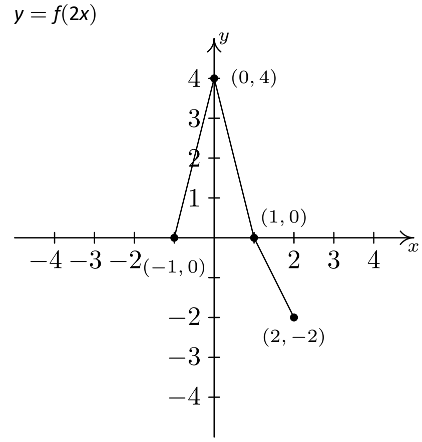
27. $y = 2 - f(2 - x)$



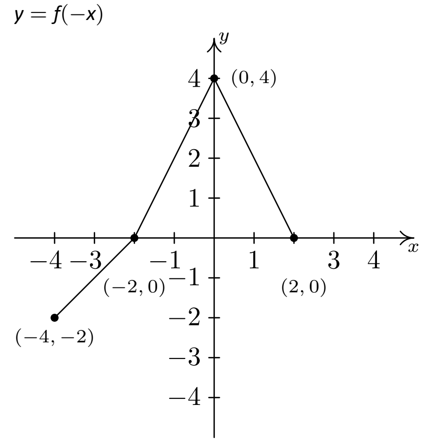
29. $y = f(x + 1)$



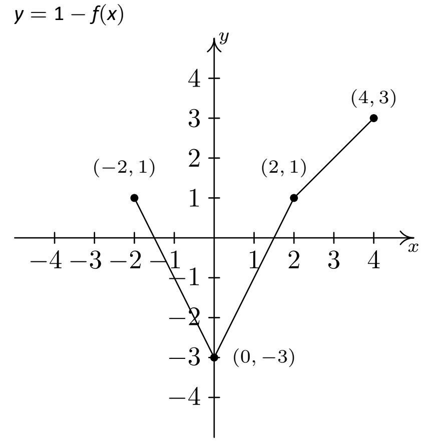
31. $y = f(2x)$



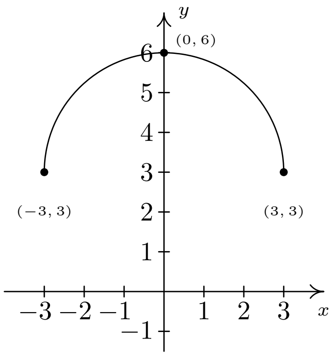
33. $y = f(-x)$



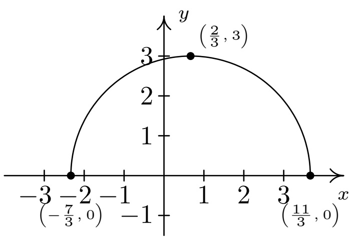
35. $y = 1 - f(x)$



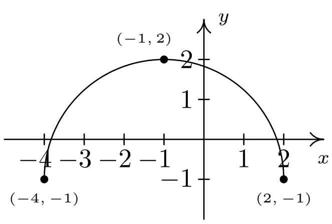
37. $g(x) = f(x) + 3$



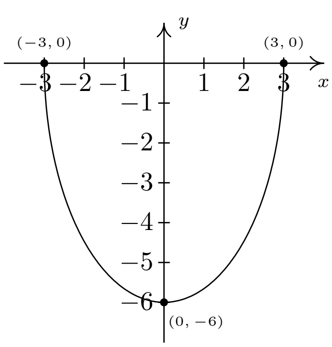
39. $j(x) = f\left(x - \frac{2}{3}\right)$



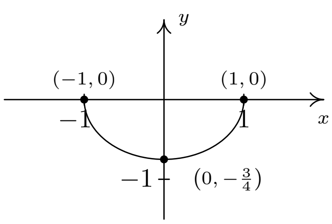
41. $b(x) = f(x + 1) - 1$



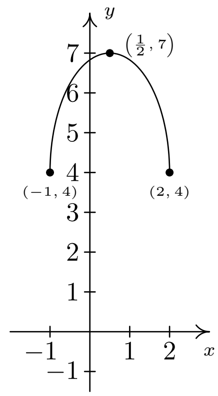
43. $d(x) = -2f(x)$



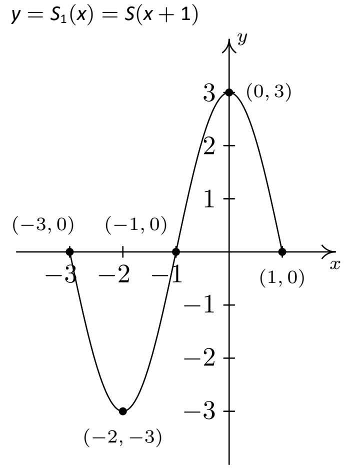
45. $m(x) = -\frac{1}{4}f(3x)$



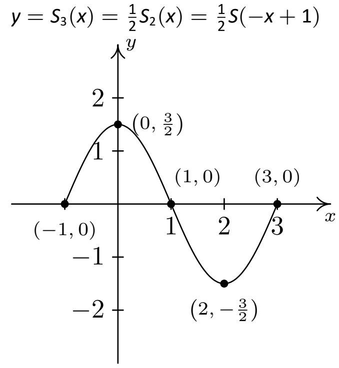
47. $p(x) = 4 + f(1 - 2x)$



49. $y = S_1(x) = S(x + 1)$



51. $y = S_3(x) = \frac{1}{2}S_2(x) = \frac{1}{2}S(-x + 1)$



53. $g(x) = \sqrt{x - 2} - 3$

55. $g(x) = -\sqrt{x} + 1$

57. $g(x) = \sqrt{-(x + 1)} + 2 = \sqrt{-x - 1} + 2$

59. $g(x) = 2\left(\sqrt{x + 3} - 4\right) = 2\sqrt{x + 3} - 8$

61. $g(x) = \sqrt{2(x - 3)} + 1 = \sqrt{2x - 6} + 1$

63.

65.

67. The same thing as reflecting it across the x-axis.

69. The same thing as reflecting it across the y-axis.

71.

Chapter 3

Section 3.1

1. $y + 1 = 3(x - 3)$
$y = 3x - 10$

3. $y + 1 = -(x + 7)$
$y = -x - 8$

5. $y - 4 = -\frac{1}{5}(x - 10)$
$y = -\frac{1}{5}x + 6$

7. $y - 117 = 0$
$y = 117$

9. $y - 2\sqrt{3} = -5(x - \sqrt{3})$
$y = -5x + 7\sqrt{3}$

11. $y = -\frac{5}{3}x$

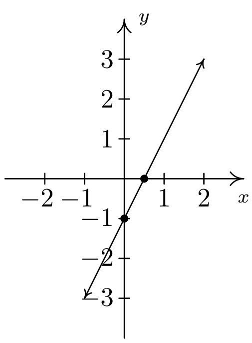
13. $y = \frac{8}{5}x - 8$

15. $y = 5$

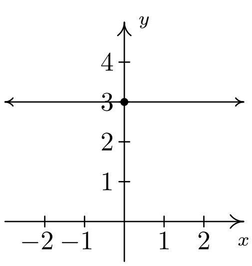
17. $y = -\frac{5}{4}x + \frac{11}{8}$

19. $y = -x$

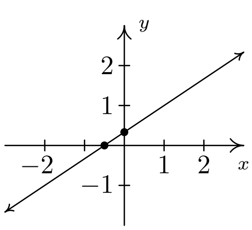
21. $f(x) = 2x - 1$
slope: $m = 2$
y-intercept: $(0, -1)$
x-intercept: $\left(\frac{1}{2}, 0\right)$



23. $f(x) = 3$
slope: $m = 0$
y-intercept: $(0, 3)$
x-intercept: none



25. $f(x) = \frac{2}{3}x + \frac{1}{3}$
slope: $m = \frac{2}{3}$
y-intercept: $\left(0, \frac{1}{3}\right)$
x-intercept: $\left(-\frac{1}{2}, 0\right)$



27. $(-1, -1)$ and $\left(\frac{11}{5}, \frac{27}{5}\right)$

29. $E(t) = 360t,\ t \geq 0$.

31. $C(t) = 80t + 50,\ 0 \leq t \leq 8$

33. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

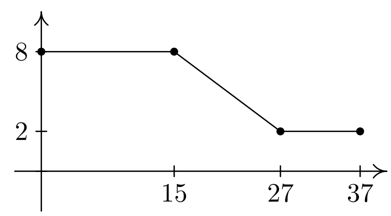
35. (a) $F(C) = \frac{9}{5}C + 32$
(b) $C(F) = \frac{5}{9}(F - 32) = \frac{5}{9}F - \frac{160}{9}$
(c) $F(-40) = -40 = C(-40)$.

37.

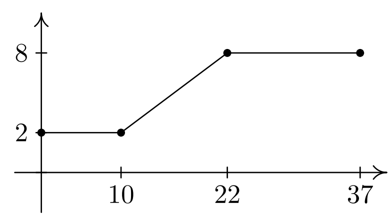
39. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

41. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

43. (a)
$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$
(b)
$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$
(c)



$y = D(d)$



$y = D(s)$

45. $\dfrac{\frac{1}{5} - \frac{1}{1}}{5 - 1} = -\dfrac{1}{5}$

47. $\dfrac{3^2 - (-3)^2}{3 - (-3)} = 0$

49. $\dfrac{(3(2)^2 + 2(2) - 7) - (3(-4)^2 + 2(-4) - 7)}{2 - (-4)} = -4$

51. $\dfrac{-1}{x(x + h)}$

53. $6x + 3h + 2$

55. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F.
$T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F.
$T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.
(b) The average rate of change is $\frac{T(8) - T(4)}{8 - 4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.
(c) The average rate of change is $\frac{T(12) - T(8)}{12 - 8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.

(d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

57.

59. $y = 3x$

61. $y = \frac{2}{3}x - 4$

63. $y = -2$

65. $y = -3x$

67. $y = -\frac{3}{2}x + 9$

69. $x = 3$

71.

73.

Section 3.2

1. $x = -6$ or $x = 6$

3. $x = -3$ or $x = 11$

5. $x = -\frac{1}{2}$ or $x = \frac{1}{10}$

7. $x = -3$ or $x = 3$

9. $x = -\frac{3}{2}$

11. $x = 1$

13. $x = -1$, $x = 0$ or $x = 1$

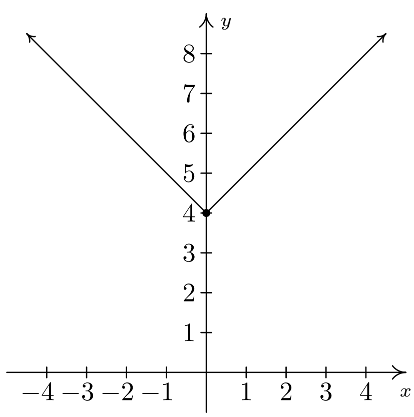
15. $x = -2$ or $x = 2$

17. $x = -\frac{1}{7}$ or $x = 1$

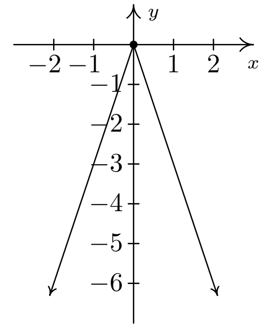
19. $x = 1$

21. $x = \frac{1}{5}$ or $x = 5$

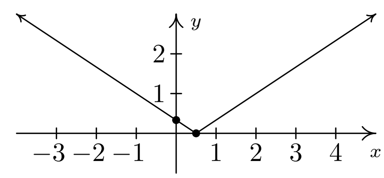
23. $f(x) = |x| + 4$
No zeros
No x-intercepts
y-intercept $(0, 4)$
Domain $(-\infty, \infty)$
Range $[4, \infty)$
Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$
Relative and absolute minimum at $(0, 4)$
No relative or absolute maximum



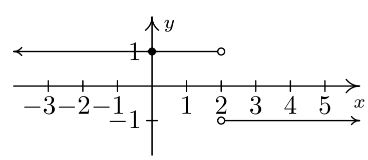
25. $f(x) = -3|x|$
$f(0) = 0$
x-intercept $(0, 0)$
y-intercept $(0, 0)$
Domain $(-\infty, \infty)$
Range $(-\infty, 0]$
Increasing on $(-\infty, 0]$
Decreasing on $[0, \infty)$
Relative and absolute maximum at $(0, 0)$
No relative or absolute minimum



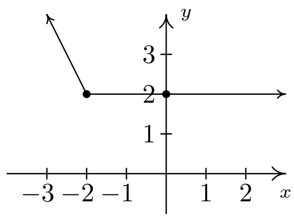
27. $f(x) = \frac{1}{3}|2x - 1|$
$f\left(\frac{1}{2}\right) = 0$
x-intercepts $\left(\frac{1}{2}, 0\right)$
y-intercept $\left(0, \frac{1}{3}\right)$
Domain $(-\infty, \infty)$
Range $[0, \infty)$
Decreasing on $\left(-\infty, \frac{1}{2}\right]$
Increasing on $\left[\frac{1}{2}, \infty\right)$
Relative and absolute min. at $\left(\frac{1}{2}, 0\right)$
No relative or absolute maximum



29. $f(x) = \dfrac{|2 - x|}{2 - x}$
No zeros
No x-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Absolute minimum at every point $(x, -1)$ where $x > 2$
Absolute maximum at every point $(x, 1)$ where $x < 2$
Relative maximum AND minimum at every point on the graph



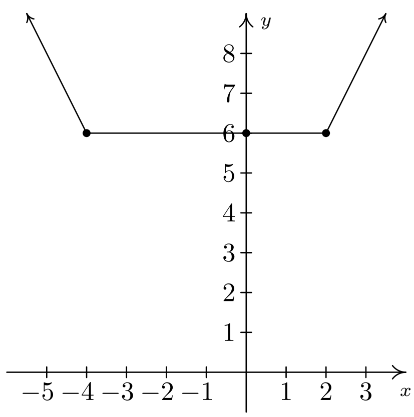
31. Re-write $f(x) = |x + 2| - x$ as
$f(x) = \begin{cases} -2x - 2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$
No zeros
No x-intercepts
y-intercept $(0, 2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Absolute minimum at every point $(x, 2)$ where $x \geq -2$
No absolute maximum
Relative minimum at every point $(x, 2)$ where $x \geq -2$
Relative maximum at every point $(x, 2)$ where $x > -2$



33. Re-write $f(x) = |x+4| + |x-2|$ as

$$f(x) = \begin{cases} -2x-2 & \text{if} \quad x < -4 \\ 6 & \text{if} \quad -4 \le x < 2 \\ 2x+2 & \text{if} \quad x \ge 2 \end{cases}$$

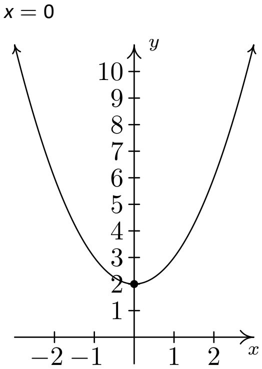
No zeros
No x-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Absolute minimum at every point $(x, 6)$ where $-4 \le x \le 2$
No absolute maximum
Relative minimum at every point $(x, 6)$ where $-4 \le x \le 2$
Relative maximum at every point $(x, 6)$ where $-4 < x < 2$



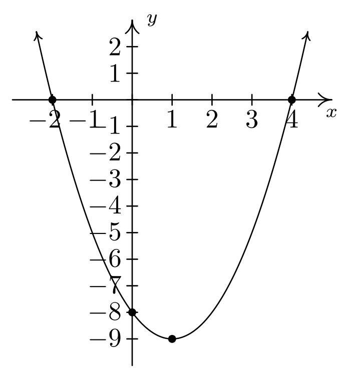
35.

Section 3.3

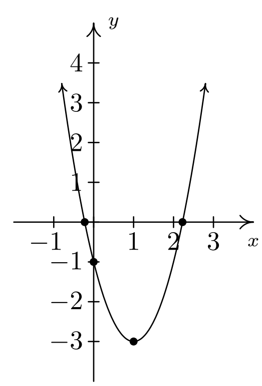
1. $f(x) = x^2 + 2$ (this is both forms!)
No x-intercepts
y-intercept $(0, 2)$
Domain: $(-\infty, \infty)$
Range: $[2, \infty)$
Decreasing on $(-\infty, 0]$
Increasing on $[0, \infty)$
Vertex $(0, 2)$ is a minimum
Axis of symmetry $x = 0$



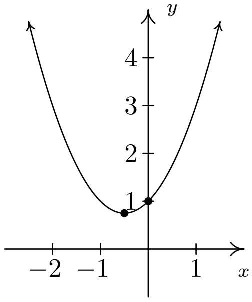
3. $f(x) = x^2 - 2x - 8 = (x-1)^2 - 9$
x-intercepts $(-2, 0)$ and $(4, 0)$
y-intercept $(0, -8)$
Domain: $(-\infty, \infty)$
Range: $[-9, \infty)$
Decreasing on $(-\infty, 1]$
Increasing on $[1, \infty)$
Vertex $(1, -9)$ is a minimum
Axis of symmetry $x = 1$



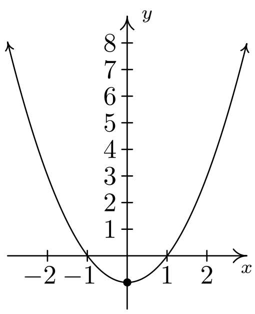
5. $f(x) = 2x^2 - 4x - 1 = 2(x-1)^2 - 3$
x-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$
Increasing on $[1, \infty)$
Decreasing on $(-\infty, 1]$
Vertex $(1, -3)$ is a minimum
Axis of symmetry $x = 1$



7. $f(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$
No x-intercepts
y-intercept $(0, 1)$
Domain: $(-\infty, \infty)$
Range: $\left[\frac{3}{4}, \infty\right)$
Increasing on $\left[-\frac{1}{2}, \infty\right)$
Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
Axis of symmetry $x = -\frac{1}{2}$



9. $f(x) = x^2 - \frac{1}{100}x - 1 = \left(x - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
x-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum
Axis of symmetry $x = \frac{1}{200}$



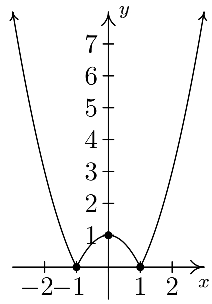
Note: You'll need to plot this on a computer to zoom in far enough to see that the vertex is not the *y*-intercept.

11.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.
- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

13.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

15. 495 cookies
17. 64° at 2 PM (8 hours after 6 AM.)
19. 8 feet by 16 feet; maximum area is 128 square feet.
21. The largest rectangle has area 12.25 square inches.
23. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.
25. (a) The applied domain is $[0, \infty)$.
(d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.
(e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.
(f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

27.



29. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$, D is minimized when $x = -\frac{2}{5}$, so the point on $y = 2x + 1$ closest to $(0, 0)$ is $\left(-\frac{2}{5}, \frac{1}{5}\right)$
31. $x = \pm y\sqrt{10}$
33. $x = \dfrac{m \pm \sqrt{m^2 + 4}}{2}$
35. $y = 2 \pm x$

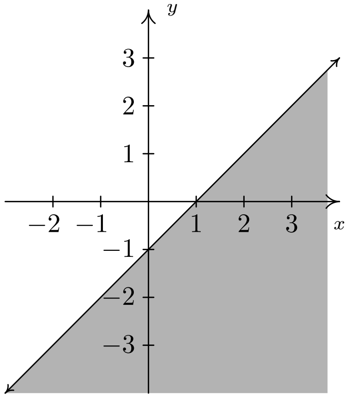
Section 3.4

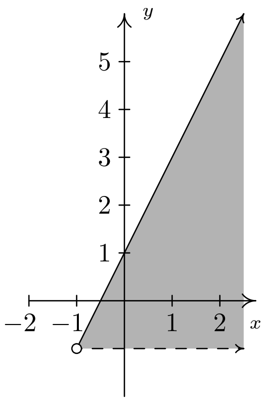
1. $\left[\frac{1}{3}, 3\right]$
3. $(-3, 2)$
5. No solution
7. $(-3, 2] \cup [6, 11)$
9. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
11. $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
13. No Solution
15. $\left(1, \frac{5}{3}\right)$
17. $(-\infty, -3] \cup [1, \infty)$
19. No solution
21. $\{2\}$
23. $\left[-\frac{1}{3}, 4\right]$
25. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
27. $(-3\sqrt{2}, -\sqrt{11}] \cup [-\sqrt{7}, 0) \cup (0, \sqrt{7}] \cup [\sqrt{11}, 3\sqrt{2})$
29. $(-\infty, \infty)$
31. $[-6, -3] \cup [-2, \infty)$
33. $P(x) \geq 50$ on $[10, 15]$. This means anywhere between 10 and 15 bottles of tonic need to be sold to earn at least \$50 in profit.
35. $T(t) > 42$ on $(8 - 2\sqrt{11}, 8 + 2\sqrt{11}) \approx (1.37, 14.63)$, which corresponds to between 7:22 AM (1.37 hours after 6 AM) to 8:38 PM (14.63 hours after 6 AM.) However, since the model is valid only for t, $0 \leq t \leq 12$, we restrict our answer and find it is warmer than 42° Fahrenheit from 7:22 AM to 6 PM.
37. $s(t) = -4.9t^2 + 30t + 2$. $s(t) > 35$ on (approximately) $(1.44, 4.68)$. This means between 1.44 and 4.68 seconds after it is launched into the air, the marble is more than 35 feet off the ground.

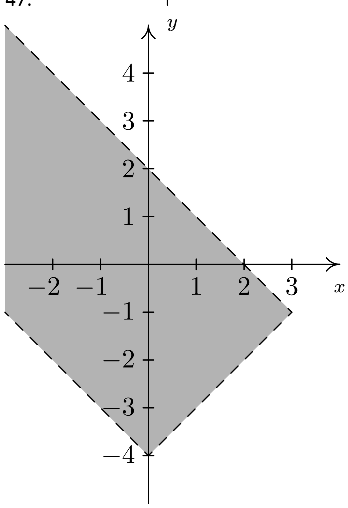
39. $|x-2| \leq 4$, $[-2, 6]$

41. $|x^2-3| \leq 1$, $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

43. Solving $|S(x)-42| \leq 3$, and disregarding the negative solutions yields $\left[\sqrt{\frac{13}{2}}, \sqrt{\frac{15}{2}}\right] \approx [2.550, 2.739]$. The edge length must be within 2.550 and 2.739 centimetres.

45. 

47. 

49. 

Chapter 4

Section 4.1

1. $f(x) = 4 - x - 3x^2$
Degree 2
Leading term $-3x^2$
Leading coefficient -3
Constant term 4
As $x \to -\infty,\ f(x) \to -\infty$
As $x \to \infty,\ f(x) \to -\infty$

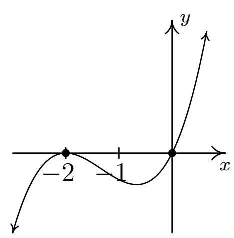
3. $q(r) = 1 - 16r^4$
Degree 4
Leading term $-16r^4$
Leading coefficient -16
Constant term 1
As $r \to -\infty,\ q(r) \to -\infty$
As $r \to \infty,\ q(r) \to -\infty$

5. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
Degree 17
Leading term $\sqrt{3}x^{17}$
Leading coefficient $\sqrt{3}$
Constant term $\frac{1}{3}$
As $x \to -\infty,\ f(x) \to -\infty$
As $x \to \infty,\ f(x) \to \infty$

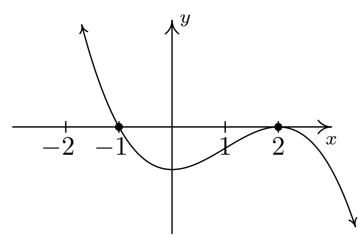
7. $P(x) = (x-1)(x-2)(x-3)(x-4)$
Degree 4
Leading term x^4
Leading coefficient 1
Constant term 24
As $x \to -\infty,\ P(x) \to \infty$
As $x \to \infty,\ P(x) \to \infty$

9. $f(x) = -2x^3(x+1)(x+2)^2$
Degree 6
Leading term $-2x^6$
Leading coefficient -2
Constant term 0
As $x \to -\infty,\ f(x) \to -\infty$
As $x \to \infty,\ f(x) \to -\infty$

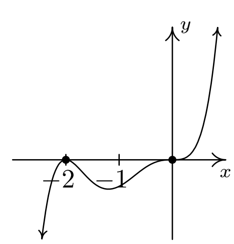
11. $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



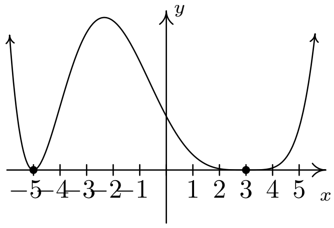
13. $f(x) = -2(x-2)^2(x+1)$
$x = 2$ multiplicity 2
$x = -1$ multiplicity 1



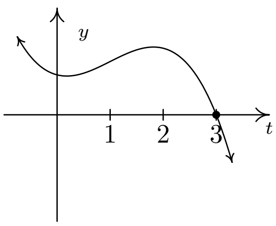
15. $F(x) = x^3(x+2)^2$
$x = 0$ multiplicity 3
$x = -2$ multiplicity 2



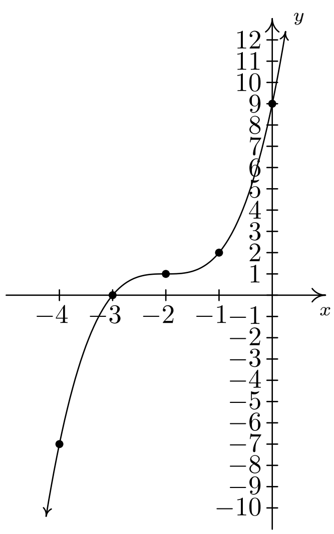
17. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4



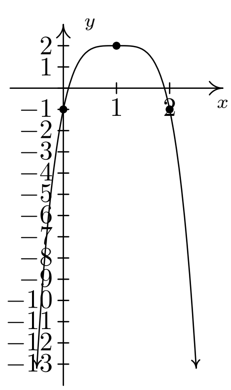
19. $H(t) = (3-t)\left(t^2+1\right)$
$x = 3$ multiplicity 1



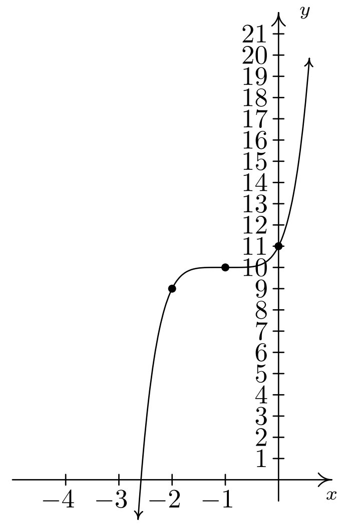
21. $g(x) = (x+2)^3 + 1$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



23. $g(x) = 2 - 3(x-1)^4$
domain: $(-\infty, \infty)$
range: $(-\infty, 2]$



25. $g(x) = (x+1)^5 + 10$
domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$



27. We have
$f(-4) = -23,\ f(-3) = 5,\ f(0) = 5,\ f(1) = -3,\ f(2) = -5$ and $f(3) = 5$ so the Intermediate Value Theorem tells us that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

29. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. Since x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. Since we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is $\$1{,}115{,}416$.

31. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. Since x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. Since we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is $\$35{,}255$.

33. (a) Our ultimate goal is to maximize the volume, so we'll start with the maximum Length + Girth of 130. This means the length is $130 - 4x$. The volume of a rectangular box is always length $\times$ width $\times$ height so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

(b) Graphing $y = V(x)$ on $[0, 33] \times [0, 21000]$ shows a maximum at $(21.67, 20342.59)$ so the dimensions of the box with maximum volume are 21.67in. $\times$ 21.67in. $\times$ 43.32in. for a volume of 20342.59in.^{3}.

(c) If we start with Length + Girth $= 108$ then the length is $108 - 4x$ and the volume is $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ on $[0, 27] \times [0, 11700]$ shows a maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. $\times$ 18.00in. $\times$ 36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all calculator results as approximations and list them as such.)

35.

Section 4.2

1. $4x^2+3x-1=(x-3)(4x+15)+44$
3. $5x^4-3x^3+2x^2-1=\left(x^2+4\right)\left(5x^2-3x-18\right)+(12x+71)$
5. $9x^3+5=(2x-3)\left(\frac{9}{2}x^2+\frac{27}{4}x+\frac{81}{8}\right)+\frac{283}{8}$
7. $\left(3x^2-2x+1\right)=(x-1)(3x+1)+2$
9. $\left(3-4x-2x^2\right)=(x+1)(-2x-2)+5$
11. $\left(x^3+8\right)=(x+2)\left(x^2-2x+4\right)+0$
13. $\left(18x^2-15x-25\right)=\left(x-\frac{5}{3}\right)(18x+15)+0$
15. $\left(2x^3+x^2+2x+1\right)=\left(x+\frac{1}{2}\right)\left(2x^2+2\right)+0$
17. $\left(2x^3-3x+1\right)=\left(x-\frac{1}{2}\right)\left(2x^2+x-\frac{5}{2}\right)-\frac{1}{4}$
19. $\left(x^4-6x^2+9\right)=\left(x-\sqrt{3}\right)\left(x^3+\sqrt{3}x^2-3x-3\sqrt{3}\right)+0$
21. $p(4)=29$
23. $p(-3)=-45$
25. $p(2)=0$, $p(x)=(x-2)\left(3x^2+4\right)$
27. $p\left(\frac{3}{2}\right)=\frac{73}{16}$
29. $p(-\sqrt{7})=0$,
$p(x)=(x+\sqrt{7})\left(x^3+(1-\sqrt{7})x^2+(1-\sqrt{7})x-\sqrt{7}\right)$
31. $x^3-6x^2+11x-6=(x-1)(x-2)(x-3)$
33. $3x^3+4x^2-x-2=3\left(x-\frac{2}{3}\right)(x+1)^2$
35. $x^3+2x^2-3x-6=(x+2)(x+\sqrt{3})(x-\sqrt{3})$
37. $4x^4-28x^3+61x^2-42x+9=4\left(x-\frac{1}{2}\right)^2(x-3)^2$
39. $125x^5-275x^4-2265x^3-3213x^2-1728x-324=$
$125\left(x+\frac{3}{5}\right)^3(x+2)(x-6)$
41. $p(x)=117(x+2)(x-2)(x+1)(x-1)$
43. $p(x)=7(x+3)^2(x-3)(x-6)$
45. $p(x)=a(x+6)^2(x-1)(x-117)$ or
$p(x)=a(x+6)(x-1)(x-117)^2$ where a can be any negative real number

Section 4.3

1. For $f(x)=x^3-2x^2-5x+6$
   - All of the real zeros lie in the interval $[-7,7]$
   - Possible rational zeros are $\pm1,\pm2,\pm3,\pm6$
3. For $f(x)=x^4-9x^2-4x+12$
   - All of the real zeros lie in the interval $[-13,13]$
   - Possible rational zeros are $\pm1,\pm2,\pm3,\pm4,\pm6,\pm12$
5. For $f(x)=x^3-7x^2+x-7$
   - All of the real zeros lie in the interval $[-8,8]$
   - Possible rational zeros are $\pm1,\pm7$
7. For $f(x)=-17x^3+5x^2+34x-10$
   - All of the real zeros lie in the interval $[-3,3]$
   - Possible rational zeros are $\pm\frac{1}{17},\pm\frac{2}{17},\pm\frac{5}{17},\pm\frac{10}{17},\pm1,\pm2,\pm5,\pm10$
9. For $f(x)=3x^3+3x^2-11x-10$
   - All of the real zeros lie in the interval $\left[-\frac{14}{3},\frac{14}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{5}{3},\pm\frac{10}{3},\pm1,\pm2,\pm5,\pm10$
11. $f(x)=x^3-2x^2-5x+6$
$x=-2, x=1, x=3$ (each has mult. 1)
13. $f(x)=x^4-9x^2-4x+12$
$x=-2$ (mult. 2), $x=1$ (mult. 1), $x=3$ (mult. 1)
15. $f(x)=x^3-7x^2+x-7$
$x=7$ (mult. 1)
17. $f(x)=-17x^3+5x^2+34x-10$
$x=\frac{5}{17}$, $x=\pm\sqrt{2}$ (each has mult. 1)
19. $f(x)=3x^3+3x^2-11x-10$
$x=-2$, $x=\frac{3\pm\sqrt{69}}{6}$ (each has mult. 1)
21. $f(x)=9x^3-5x^2-x$
$x=0$, $x=\frac{5\pm\sqrt{61}}{18}$ (each has mult. 1)
23. $f(x)=x^4+2x^2-15$
$x=\pm\sqrt{3}$ (each has mult. 1)
25. $f(x)=3x^4-14x^2-5$
$x=\pm\sqrt{5}$ (each has mult. 1)
27. $f(x)=x^6-3x^3-10$
$x=\sqrt[3]{-2}=-\sqrt[3]{2}$, $x=\sqrt[3]{5}$ (each has mult. 1)
29. $f(x)=x^5-2x^4-4x+8$
$x=2$, $x=\pm\sqrt{2}$ (each has mult. 1)
31. $f(x)=x^5-60x^3-80x^2+960x+2304$
$x=-4$ (mult. 3), $x=6$ (mult. 2)
33. $f(x)=90x^4-399x^3+622x^2-399x+90$
$x=\frac{2}{3}$, $x=\frac{3}{2}$, $x=\frac{5}{3}$, $x=\frac{3}{5}$ (each has mult. 1)
35. $x=0,\ \frac{5\pm\sqrt{61}}{18}$
37. $x=-2,1,3$
39. $x=7$
41. $x=-2,\ \frac{3\pm\sqrt{69}}{6}$
43. $x=\pm\sqrt{5}$
45. $\left(-\infty,\frac{1}{2}\right)\cup(4,5)$
47. $(-\infty,-1]\cup[3,\infty)$
49. $[-2,2]$
51. $(-\infty,-2)\cup\left(-\sqrt{2},\sqrt{2}\right)$
53. $(-\infty,-\sqrt{3})\cup(\sqrt{3},\infty)$
55. $V(x)\geq 80$ on $[1,5-\sqrt{5}]\cup[5+\sqrt{5},\infty)$. Only the portion $[1,5-\sqrt{5}]$ lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than $5-\sqrt{5}\approx 2.76$ inches.
57.

Section 4.4

1. $f(x)=x^2-4x+13=(x-(2+3i))(x-(2-3i))$
Zeros: $x=2\pm3i$
3. $f(x)=3x^2+2x+10=$
$3\left(x-\left(-\frac{1}{3}+\frac{\sqrt{29}}{3}i\right)\right)\left(x-\left(-\frac{1}{3}-\frac{\sqrt{29}}{3}i\right)\right)$
Zeros: $x=-\frac{1}{3}\pm\frac{\sqrt{29}}{3}i$
5. $f(x)=x^3+6x^2+6x+5=(x+5)(x^2+x+1)=$
$(x+5)\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$
Zeros: $x=-5,\ x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
7. $f(x)=x^3+3x^2+4x+12=(x+3)\left(x^2+4\right)=$
$(x+3)(x+2i)(x-2i)$
Zeros: $x=-3,\ \pm2i$

9. $f(x) = x^3 + 7x^2 + 9x - 2 =$
$(x+2)\left(x - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(x - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
Zeros: $x = -2,\ x = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

11. $f(x) = 4x^4 - 4x^3 + 13x^2 - 12x + 3 = \left(x - \frac{1}{2}\right)^2\left(4x^2 + 12\right) =$
$4\left(x - \frac{1}{2}\right)^2 (x + i\sqrt{3})(x - i\sqrt{3})$
Zeros: $x = \frac{1}{2},\ x = \pm\sqrt{3}i$

13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x+2)(x-1)\left(x^2+9\right) =$
$(x+2)(x-1)(x+3i)(x-3i)$
Zeros: $x = -2,\ 1,\ \pm 3i$

15. $f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5 = (x+1)^2\left(-3x^2 - 2x - 5\right)$
$= -3(x+1)^2\left(x - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(x - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$
Zeros: $x = -1,\ x = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

17. $f(x) = x^4 + 9x^2 + 20 = \left(x^2+4\right)\left(x^2+5\right) =$
$(x-2i)(x+2i)\left(x - i\sqrt{5}\right)\left(x + i\sqrt{5}\right)$
Zeros: $x = \pm 2i, \pm i\sqrt{5}$

19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x-1)\left(x^2+3\right)\left(x^2+4\right)$
$= (x-1)(x - i\sqrt{3})(x + i\sqrt{3})(x-2i)(x+2i)$
Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2 - 2x + 26)(x^2+1) =$
$(x - (1+5i))(x - (1-5i))(x+i)(x-i)$
Zeros: $x = 1 \pm 5i,\ x = \pm i$

23. $f(x) = 42(x-1)(x+1)(x-i)(x+i)$

25. $f(x) = -3(x-2)^2(x+2)(x-7i)(x+7i)$

27. $f(x) = -2(x-2i)(x+2i)(x+2)$

Chapter 5

Section 5.1

1. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty, f(x) \to \frac{1}{3}^-$
As $x \to \infty, f(x) \to \frac{1}{3}^+$

3. $f(x) = \dfrac{x}{x^2+x-12} = \dfrac{x}{(x+4)(x-3)}$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4, x = 3$
As $x \to -4^-, f(x) \to -\infty$
As $x \to -4^+, f(x) \to \infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

5. $f(x) = \dfrac{x+7}{(x+3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $x = -3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

7. $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$

9. $f(x) = \dfrac{x^2-x-12}{x^2+x-6} = \dfrac{x-4}{x-2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$

11. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$

13. $f(x) = \dfrac{2x^2+5x-3}{3x+2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $x = -\frac{2}{3}$
As $x \to -\frac{2}{3}^-, f(x) \to \infty$
As $x \to -\frac{2}{3}^+, f(x) \to -\infty$
No holes in the graph
Slant asymptote: $y = \frac{2}{3}x + \frac{11}{9}$
As $x \to -\infty$, the graph is above $y = \frac{2}{3}x + \frac{11}{9}$
As $x \to \infty$, the graph is below $y = \frac{2}{3}x + \frac{11}{9}$

15. $f(x) = \dfrac{-5x^4 - 3x^3 + x^2 - 10}{x^3 - 3x^2 + 3x - 1}$
$= \dfrac{-5x^4 - 3x^3 + x^2 - 10}{(x-1)^3}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $x = 1$
As $x \to 1^-, f(x) \to \infty$
As $x \to 1^+, f(x) \to -\infty$
No holes in the graph
Slant asymptote: $y = -5x - 18$
As $x \to -\infty$, the graph is above $y = -5x - 18$
As $x \to \infty$, the graph is below $y = -5x - 18$

17. $f(x) = \dfrac{18 - 2x^2}{x^2 - 9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
As $x \to \pm\infty, f(x) = -2$

19.
   (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.
   (b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.
   (c) For \$40000 you could remove about 95.76% of the fish.

21.
   (a) $\overline{C}(x) = \frac{100x+2000}{x}, x > 0$.

(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.

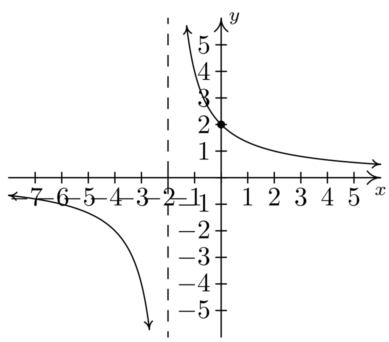
(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.

(d) As $x \to 0^+$, $\overline{C}(x) \to \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.

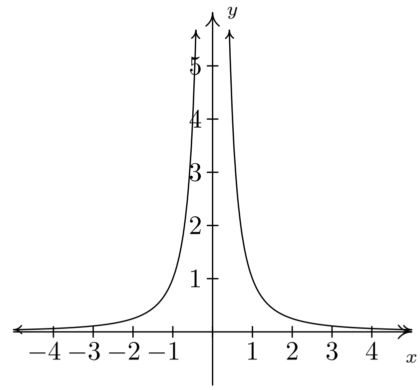
(e) As $x \to \infty$, $\overline{C}(x) \to 100^+$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

Section 5.2

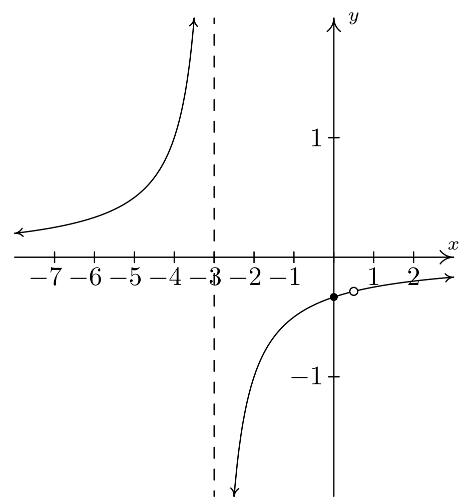
1. $f(x) = \dfrac{4}{x+2}$
Domain: $(-\infty, -2) \cup (-2, \infty)$
No x-intercepts
y-intercept: $(0, 2)$
Vertical asymptote: $x = -2$
As $x \to -2^-$, $f(x) \to -\infty$
As $x \to -2^+$, $f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$



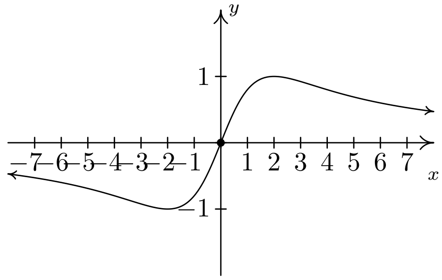
3. $f(x) = \dfrac{1}{x^2}$
Domain: $(-\infty, 0) \cup (0, \infty)$
No x-intercepts
No y-intercepts
Vertical asymptote: $x = 0$
As $x \to 0^-$, $f(x) \to \infty$
As $x \to 0^+$, $f(x) \to \infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^+$



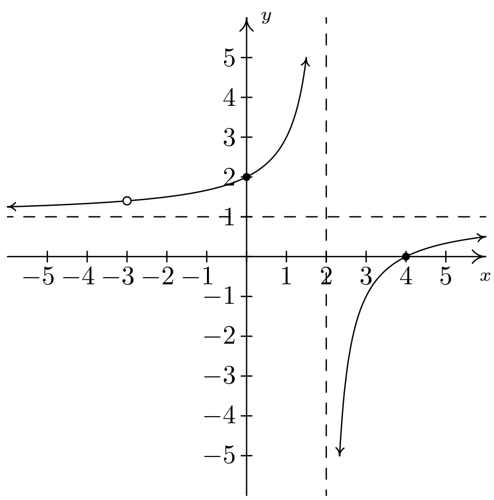
5. $f(x) = \dfrac{2x-1}{-2x^2-5x+3} = -\dfrac{2x-1}{(2x-1)(x+3)}$
Domain: $(-\infty, -3) \cup (-3, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$
No x-intercepts
y-intercept: $(0, -\frac{1}{3})$
$f(x) = \dfrac{-1}{x+3}$, $x \neq \frac{1}{2}$
Hole in the graph at $(\frac{1}{2}, -\frac{2}{7})$
Vertical asymptote: $x = -3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^+$
As $x \to \infty$, $f(x) \to 0^-$



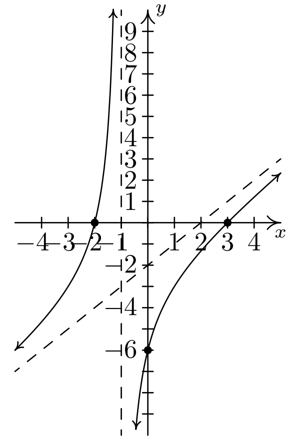
7. $f(x) = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty, f(x) \to 0^-$
As $x \to \infty, f(x) \to 0^+$



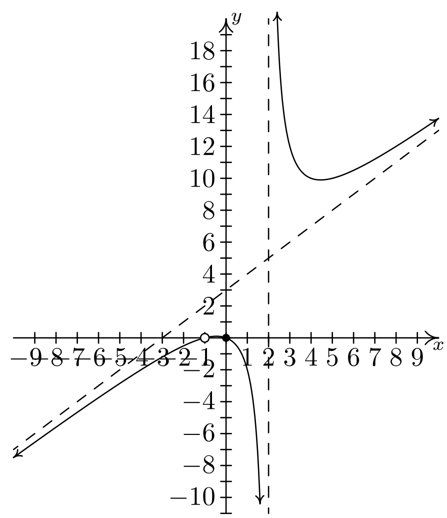
9. $f(x) = \dfrac{x^2 - x - 12}{x^2 + x - 6} = \dfrac{x-4}{x-2}\; x \neq -3$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
x-intercept: $(4, 0)$
y-intercept: $(0, 2)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to \infty$
As $x \to 2^+, f(x) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $x \to -\infty, f(x) \to 1^+$
As $x \to \infty, f(x) \to 1^-$



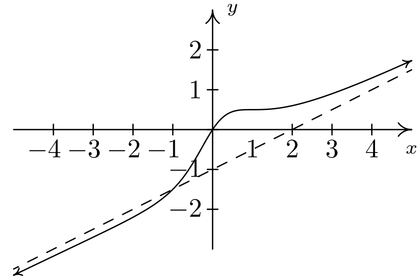
11. $f(x) = \dfrac{x^2 - x - 6}{x + 1} = \dfrac{(x-3)(x+2)}{x+1}$
Domain: $(-\infty, -1) \cup (-1, \infty)$
x-intercepts: $(-2, 0)$, $(3, 0)$
y-intercept: $(0, -6)$
Vertical asymptote: $x = -1$
As $x \to -1^-, f(x) \to \infty$
As $x \to -1^+, f(x) \to -\infty$
Slant asymptote: $y = x - 2$
As $x \to -\infty$, the graph is above $y = x - 2$
As $x \to \infty$, the graph is below $y = x - 2$



13. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}\; x \neq -1$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$

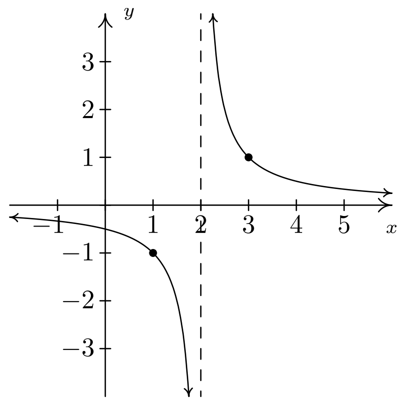


15. $f(x) = \dfrac{x^3 - 2x^2 + 3x}{2x^2 + 2}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Slant asymptote: $y = \frac{1}{2}x - 1$
As $x \to -\infty$, the graph is below $y = \frac{1}{2}x - 1$
As $x \to \infty$, the graph is above $y = \frac{1}{2}x - 1$



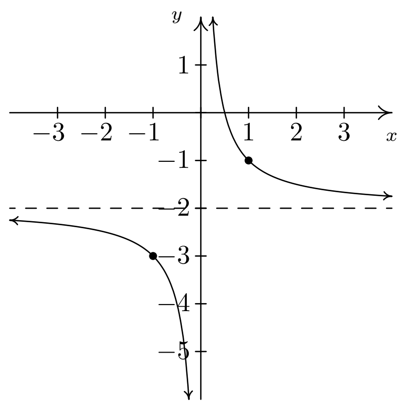
17. $f(x) = \frac{1}{x-2}$

Shift the graph of $y = \frac{1}{x}$ to the right 2 units.



19. $h(x) = \frac{-2x+1}{x} = -2 + \frac{1}{x}$

Shift the graph of $y = \frac{1}{x}$ down 2 units.



21.

23.

25.

27.

Section 5.3

1. $x = -\frac{6}{7}$
3. $x = -1$
5. No solution
7. $(-2, \infty)$
9. $(-1, 0) \cup (1, \infty)$
11. $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
13. $(-1, 0] \cup (2, \infty)$
15. $(-\infty, 1] \cup [2, \infty)$
17. $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$
19. $[-3, 0) \cup (0, 4) \cup [5, \infty)$
21. 4.5 miles per hour
23. 3600 gallons
25. 3 hours
27. The width (and depth) should be 10.00 centimetres, the height should be 5.00 centimetres. The minimum surface area is 300.00 square centimetres.
29. The dimensions are $\approx$ 7 feet by $\approx$ 14 feet; minimum amount of fencing required $\approx$ 28 feet.
31. The radius of the drum should be $\approx$ 1.05 feet and the height of the drum should be $\approx$ 2.12 feet. The minimum surface area of the drum is $\approx$ 20.93 cubic feet.
33. $T = kV$
35. $d = \frac{km}{V}$
37. $D = k\rho\nu^2$
39. Rewriting $f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$ as $f = \frac{\frac{1}{2}\sqrt{T}}{L\sqrt{\mu}}$ we see that the frequency f varies directly with the square root of the tension and varies inversely with the length and the square root of the linear mass.
41.

Chapter 6

Section 6.1

1. For $f(x) = x^2$ and $g(x) = 2x + 1$,
   - $(g \circ f)(0) = 1$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 16$
   - $(g \circ f)(-3) = 19$
   - $(f \circ g)\left(\frac{1}{2}\right) = 4$
   - $(f \circ f)(-2) = 16$
3. For $f(x) = 4 - 3x$ and $g(x) = |x|$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 10$
   - $(g \circ f)(-3) = 13$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
   - $(f \circ f)(-2) = -26$
5. For $f(x) = 4x + 5$ and $g(x) = \sqrt{x}$,
   - $(g \circ f)(0) = \sqrt{5}$
   - $(f \circ g)(-1)$ is not real
   - $(f \circ f)(2) = 57$
   - $(g \circ f)(-3)$ is not real
   - $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
   - $(f \circ f)(-2) = -7$
7. For $f(x) = 6 - x - x^2$ and $g(x) = x\sqrt{x + 10}$,
   - $(g \circ f)(0) = 24$
   - $(f \circ g)(-1) = 0$
   - $(f \circ f)(2) = 6$
   - $(g \circ f)(-3) = 0$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{27 - 2\sqrt{42}}{8}$
   - $(f \circ f)(-2) = -14$
9. For $f(x) = \frac{3}{1-x}$ and $g(x) = \frac{4x}{x^2+1}$,
   - $(g \circ f)(0) = \frac{6}{5}$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = \frac{3}{4}$
   - $(g \circ f)(-3) = \frac{48}{25}$
   - $(f \circ g)\left(\frac{1}{2}\right) = -5$
   - $(f \circ f)(-2)$ is undefined
11. For $f(x) = \frac{2x}{5-x^2}$ and $g(x) = \sqrt{4x + 1}$,
   - $(g \circ f)(0) = 1$
   - $(f \circ g)(-1)$ is not real
   - $(f \circ f)(2) = -\frac{8}{11}$
   - $(g \circ f)(-3) = \sqrt{7}$
   - $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
   - $(f \circ f)(-2) = \frac{8}{11}$
13. For $f(x) = 2x + 3$ and $g(x) = x^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = 2x^2 - 15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

15. For $f(x) = x^2 - 4$ and $g(x) = |x|$

- $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = |x|^2 - 4 = x^2 - 4$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$

17. For $f(x) = |x + 1|$ and $g(x) = \sqrt{x}$

- $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
- $(f \circ g)(x) = |\sqrt{x} + 1| = \sqrt{x} + 1$, domain: $[0, \infty)$
- $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$

19. For $f(x) = |x|$ and $g(x) = \sqrt{4 - x}$

- $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
- $(f \circ g)(x) = |\sqrt{4 - x}| = \sqrt{4 - x}$, domain: $(-\infty, 4]$
- $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

21. For $f(x) = 3x - 1$ and $g(x) = \frac{1}{x+3}$

- $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
- $(f \circ g)(x) = -\frac{x}{x+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
- $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$

23. For $f(x) = \frac{x}{2x+1}$ and $g(x) = \frac{2x+1}{x}$

- $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
- $(f \circ g)(x) = \frac{2x+1}{5x+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
- $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$

25. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$
27. $(g \circ f \circ h)(x) = \sqrt{-2|x|}$, domain: $\{0\}$
29. $(f \circ h \circ g)(x) = -2|\sqrt{x}| = -2\sqrt{x}$, domain: $[0, \infty)$
31. For $f(x) = |x|$ and $g(x) = \sqrt{4 - x}$

- $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
- $(f \circ g)(x) = |\sqrt{4 - x}| = \sqrt{4 - x}$, domain: $(-\infty, 4]$
- $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

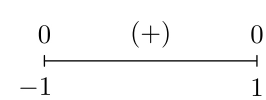
33. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.
35. Let $f(x) = 7 - 3x$ and $g(x) = |x|$, then $H(x) = (g \circ f)(x)$.
37. Let $f(x) = x^2 - 1$ and $g(x) = \frac{7}{x}$, then $R(x) = (g \circ f)(x)$.
39. Let $f(x) = x^3$ and $g(x) = \frac{2x+1}{x-1}$, then $Q(x) = (g \circ f)(x)$.
41. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.
43. $F(x) = 3\sqrt{-x + 2} - 4 = k(j(f(h(g(x)))))$
45. $(f \circ g)(3) = f(g(3)) = f(2) = 4$
47. $(f \circ f)(0) = f(f(0)) = f(1) = 3$
49. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$
51. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$
53. $g(f(g(0))) = g(f(0)) = g(1) = -3$
55. $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$
57. $(g \circ f)(1) = 3$
59. $(g \circ f)(2) = 0$
61. $(f \circ f)(1) = 3$
63. $V(x) = x^3$ so $V(x(t)) = (t + 1)^3$
65.

Section 6.2

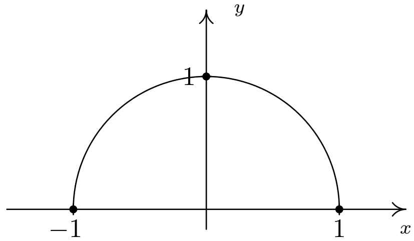
1. $f^{-1}(x) = \dfrac{x + 2}{6}$
3. $f^{-1}(x) = 3x - 10$
5. $f^{-1}(x) = \frac{1}{3}(x - 5)^2 + \frac{1}{3}$, $x \geq 5$
7. $f^{-1}(x) = \frac{1}{9}(x + 4)^2 + 1$, $x \geq -4$
9. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
11. $f^{-1}(x) = 5 + \sqrt{x + 25}$
13. $f^{-1}(x) = 3 - \sqrt{x + 4}$
15. $f^{-1}(x) = \dfrac{4x - 3}{x}$
17. $f^{-1}(x) = \dfrac{4x + 1}{2 - 3x}$
19. $f^{-1}(x) = \dfrac{-3x - 2}{x + 3}$
21.
23.
25. (a) $p^{-1}(x) = \frac{450 - x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$
    (b) $p^{-1}(105) = 23$. This means that if the price is set to $\$105$ then 23 dOpis will be sold.
    (c) $\left(P \circ p^{-1}\right)(x) = -\frac{1}{15}x^2 + \frac{110}{3}x - 5000$, $0 \leq x \leq 450$. The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping $\$41.67$ when the price per dOpi is set to $\$275$. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi at either $\$285$ or $\$270$, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is $\$40$ and it is made by producing and selling 12 dOpis a week at a price of $\$270$ per dOpi.
27. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$
29.
31.
33.

Section 6.3

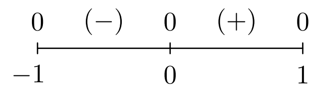
1. $f(x) = \sqrt{1 - x^2}$
   Domain: $[-1, 1]$



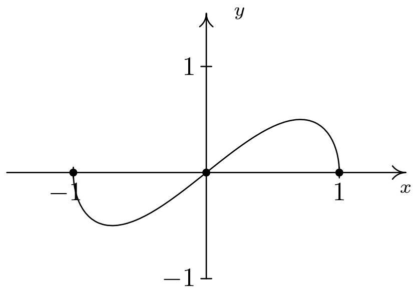
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



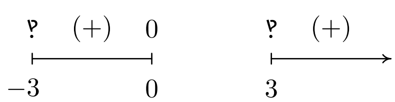
3. $f(x) = x\sqrt{1-x^2}$
Domain: $[-1, 1]$



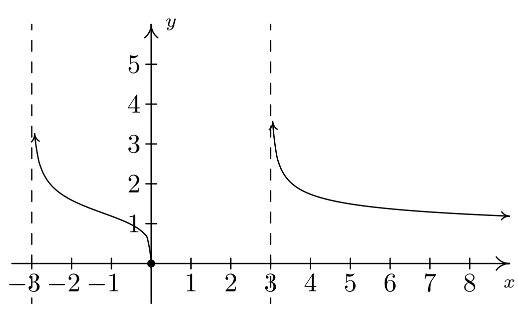
No asymptotes
Unusual steepness at $x = -1$ and $x = 1$
No cusps



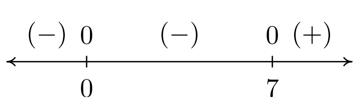
5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2-9}}$
Domain: $(-3, 0] \cup (3, \infty)$



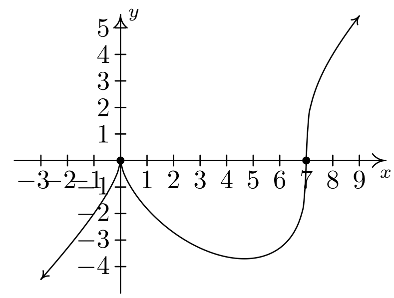
Vertical asymptotes: $x = -3$ and $x = 3$
Horizontal asymptote: $y = 0$
Unusual steepness at $x = 0$
No cusps



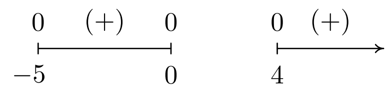
7. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
Domain: $(-\infty, \infty)$



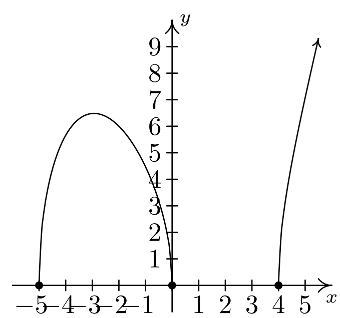
No vertical or horizontal asymptotes[1]
Unusual steepness at $x = 7$
Cusp at $x = 0$



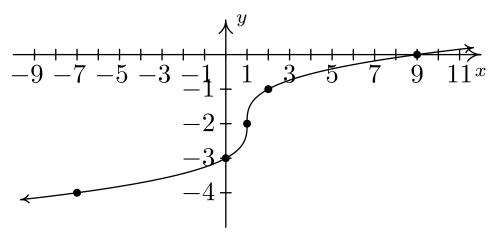
9. $f(x) = \sqrt{x(x+5)(x-4)}$
Domain: $[-5, 0] \cup [4, \infty)$



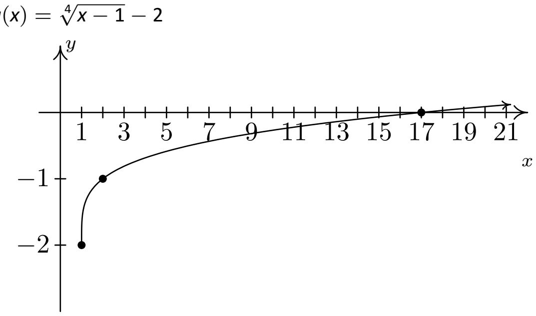
No asymptotes
Unusual steepness at $x = -5$, $x = 0$ and $x = 4$
No cusps



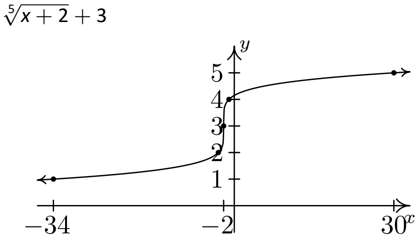
11. $g(x) = \sqrt[3]{x-1} - 2$



13. $g(x) = \sqrt[4]{x-1} - 2$



15. $g(x) = \sqrt[5]{x+2} + 3$



17. $x = 3$

[1]Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.

19. $x = -3$

21. $x = \frac{5+\sqrt{57}}{8}$

23. $x = \pm 8$

25. $x = 4$

27. $[2, \infty)$

29. $(-\infty, 2) \cup (2, 3]$

31. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$

33. $\left(0, \frac{27}{13}\right)$

35. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$

37. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.

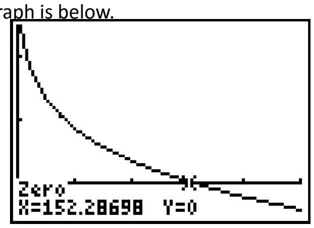
(b) $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$

(c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimetres and the height should be 5.76 centimetres to give a minimum surface area of 90.23 square centimetres.

39. (a) $W(V) = 53.142 - 23.78V^{0.16}$. Since we are told in Exercise 38 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to $V > 3$.

(b) $W(V) = 0$ when $V \approx 152.29$. This means, according to the model, for the wind chill temperature to be 0°F, the wind speed needs to be 152.29 miles per hour.

(c) The graph is below.



41. (a) First rewrite the model as $P = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$. Then $300 = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$ yields $y = \left(\frac{300}{1.23x^{\frac{2}{5}}}\right)^{\frac{5}{3}}$. If $x = 100$ then $y \approx 441.93687$.

43. $k^{-1}(x) = \frac{x}{\sqrt{x^2 - 4}}$

45.

47.

Chapter 7

Section 7.1

1. $\log_2(8) = 3$

3. $\log_4(32) = \frac{5}{2}$

5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$

7. $\ln(1) = 0$

9. $(25)^{\frac{1}{2}} = 5$

11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$

13. $10^{-1} = 0.1$

15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

17. $\log_6(216) = 3$

19. $\log_6\left(\frac{1}{36}\right) = -2$

21. $\log_{36}(216) = \frac{3}{2}$

23. $\log_{\frac{1}{6}}(216) = -3$

25. $\log \frac{1}{1000000} = -6$

27. $\ln\left(e^3\right) = 3$

29. $\log_6(1) = 0$

31. $\log_{36}\left(\sqrt[4]{36}\right) = \frac{1}{4}$

33. $36^{\log_{36}(216)} = 216$

35. $\ln(e^5) = 5$

37. $\log\left(\sqrt[3]{10^5}\right) = \frac{5}{3}$

39. $\log_5\left(3^{\log_3 5}\right) = 1$

41. $\log_2\left(3^{-\log_3(2)}\right) = -1$

43. $(-\infty, \infty)$

45. $(5, \infty)$

47. $(-2, -1) \cup (1, \infty)$

49. $(4, 7)$

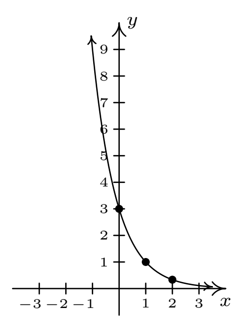
51. $(-\infty, \infty)$

53. $(-\infty, -7) \cup (1, \infty)$

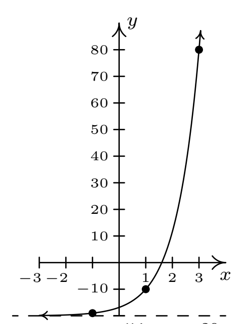
55. $(0, 125) \cup (125, \infty)$

57. $(-\infty, -3) \cup \left(\frac{1}{2}, 2\right)$

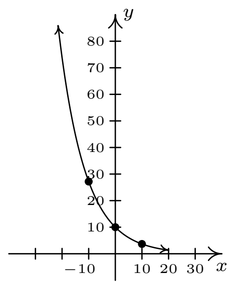
59. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



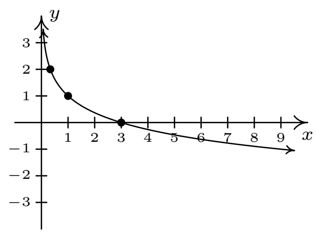
61. Domain of g: $(-\infty, \infty)$
Range of g: $(-20, \infty)$



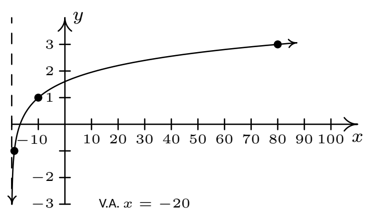
63. Domain of g: $(-\infty, \infty)$
Range of g: $(0, \infty)$



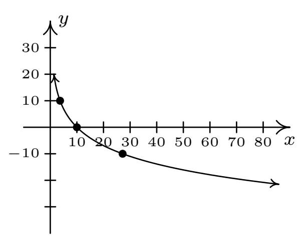
65. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$



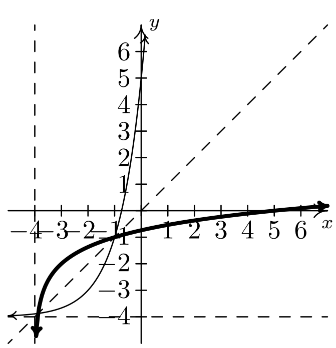
67. Domain of g: $(-20, \infty)$
Range of g: $(-\infty, \infty)$



69. Domain of g: $(0, \infty)$
Range of g: $(-\infty, \infty)$

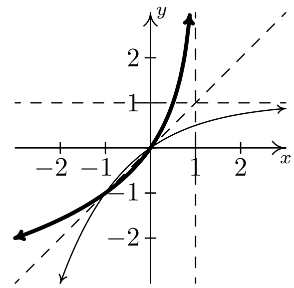


71. $f(x) = 3^{x+2} - 4$
$f^{-1}(x) = \log_3(x+4) - 2$



$y = f(x) = 3^{x+2} - 4$
$\boldsymbol{y = f^{-1}(x) = \log_3(x+4) - 2}$

73. $f(x) = -2^{-x} + 1$
$f^{-1}(x) = -\log_2(1-x)$



$y = f(x) = -2^{-x} + 1$
$\boldsymbol{y = f^{-1}(x) = -\log_2(1-x)}$

75. (a) $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0$.
(b) $M(80,000) = \log\left(\frac{80,000}{0.001}\right) = \log(80,000,000) \approx 7.9$.

77. (a) The pH of pure water is 7.
(b) If $[H^+] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.
(c) $[H^+] = 10^{-0.7} \approx .1995$ moles per liter.

79.

Section 7.2

1. $3\ln(x) + 2\ln(y)$
3. $3\log_5(z) - 6$
5. $\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$
7. $3\log_{\sqrt{2}}(x) + 4$
9. $3 + 3\log(x) + 5\log(y)$
11. $\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$
13. $\frac{5}{3} + \log(x) + \frac{1}{2}\log(y)$
15. $\frac{1}{3}\ln(x) - \ln(10) - \frac{1}{2}\ln(y) - \frac{1}{2}\ln(z)$
17. $\log_2\left(\frac{xy}{z}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2 z}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
27. $\log_2\left(x^{3/2}\right)$
29. $\log_2\left(\frac{x}{x-1}\right)$
31. $\log_3(x+2) = \frac{\log(x+2)}{\log(3)}$
33. $\log(x^2+1) = \frac{\ln(x^2+1)}{\ln(10)}$
35. $\log_5(80) \approx 2.72271$
37. $\log_4\left(\frac{1}{10}\right) \approx -1.66096$
39. $\log_{\frac{2}{3}}(50) \approx -9.64824$
41.
43.
45.

Section 7.3

1. $x = \frac{3}{4}$
3. $x = 2$
5. $x = -\frac{7}{3}$
7. $x = \frac{16}{15}$

9. $x = \frac{\ln(5)}{2\ln(3)}$

11. No solution.

13. $x = \frac{\ln(3)}{12\ln(1.005)}$

15. $t = \frac{\ln(2)}{0.1} = 10\ln(2)$

17. $t = \frac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$

19. $x = \ln(2)$

21. $t = \frac{\ln\left(\frac{1}{29}\right)}{-0.8} = \frac{5}{4}\ln(29)$

23. $x = \ln(2)$

25. $x = \frac{\ln(3)}{\ln(3)-\ln(2)}$

27. $x = \frac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$

29. $x = \ln(2)$

31. $x = \ln(3)$

33. $x = \frac{\ln(5)}{\ln(3)}$

35. $\left[\frac{\ln(3)}{12\ln(1.005)}, \infty\right)$

37. $\left(-\infty, \frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right] = \left(-\infty, \frac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}\right]$

39. $\left[\frac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$

41. $x \approx 0.01866$, $x \approx 1.7115$

43. $(-\infty, 1]$

45. $\approx (2.3217, 4.3717)$

47.

49.

Section 7.4

1. $x = \frac{5}{4}$

3. $x = -2$

5. $x = -1$

7. $x = \pm 10$

9. $x = -\frac{17}{7}$

11. $x = 10^{-5.4}$

13. $x = \frac{25}{2}$

15. $x = 5$

17. $x = 2$

19. $x = 6$

21. $x = 81$

23. $x = 10^{-3}$, 10^5

25. (e, ∞)

27. $[10^{-3}, \infty)$

29. $(10^{-5.4}, 10^{-2.3})$

31. $x \approx 1.3098$

33. $\approx (-\infty, -12.1414) \cup (12.1414, \infty)$

35. $-\frac{1}{2} < x < \frac{e^3-1}{2}$

37. $y = \frac{3}{5e^{2x}+1}$

39. $f^{-1}(x) = \frac{e^{2x}-1}{e^{2x}+1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

41.

Section 7.5

1.
   - $A(t) = 500\left(1 + \frac{0.0075}{12}\right)^{12t}$
   - $A(5) \approx \$519.10$, $A(10) \approx \$538.93$, $A(30) \approx \$626.12$, $A(35) \approx \$650.03$
   - It will take approximately 92 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of $3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of $4.85 per year at this point.

3.
   - $A(t) = 1000\left(1 + \frac{0.0125}{12}\right)^{12t}$
   - $A(5) \approx \$1064.46$, $A(10) \approx \$1133.07$, $A(30) \approx \$1454.71$, $A(35) \approx \$1548.48$
   - It will take approximately 55 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of $13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of $19.23 per year at this point.

5.
   - $A(t) = 5000\left(1 + \frac{0.02125}{12}\right)^{12t}$
   - $A(5) \approx \$5559.98$, $A(10) \approx \$6182.67$, $A(30) \approx \$9453.40$, $A(35) \approx \$10512.13$
   - It will take approximately 33 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of $116.80 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of $220.83 per year at this point.

7.

9. $P = \frac{5000}{\left(1+\frac{0.0225}{12}\right)^{12\cdot 10}} \approx \$3993.42$

11. (a) $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12\cdot 8} \approx \$2394.03$

    (b) $t = \frac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years

    (c) $P = \frac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$

    (d) $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%

13.

15.
   - $k = \frac{\ln(1/2)}{14} \approx -0.0495$
   - $A(t) = 2e^{-0.0495t}$
   - $t = \frac{\ln(0.1)}{-0.0495} \approx 46.52$ days.

17.
   - $k = \frac{\ln(1/2)}{432.7} \approx -0.0016$

- $A(t) = 0.29e^{-0.0016t}$
- $t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11$ years.

19. $t = \frac{\ln(0.1)}{k} = -\frac{\ln(10)}{k}$

21. (a) $G(0) = 9743.77$ This means that the GDP of the US in 2000 was $\$9743.77$ billion dollars.

    (b) $G(7) = 13963.24$ and $G(10) = 16291.25$, so the model predicted a GDP of $\$13,963.24$ billion in 2007 and $\$16,291.25$ billion in 2010.

23. (a) $k = \frac{\ln(2)}{20} \approx 0.0346$

    (b) $N(t) = 1000e^{0.0346t}$

    (c) $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes

25. $N_0 = 52$, $k = \frac{1}{3}\ln\left(\frac{118}{52}\right) \approx 0.2731$, $N(t) = 52e^{0.2731t}$. $N(6) \approx 268$.

27. (a) $P(0) = \frac{120}{4.167} \approx 29$. There are 29 Sasquatch in Bigfoot County in 2010.

    (b) $P(3) = \frac{120}{1+3.167e^{-0.05(3)}} \approx 32$ Sasquatch.

    (c) $t = 20\ln(3.167) \approx 23$ years.

    (d) As $t \to \infty$, $P(t) \to 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.

29. $A(t) = 2.3e^{-0.0138629t}$

31. (a) $T(t) = 75 + 105e^{-0.005005t}$

    (b) The roast would have cooled to 140°F in about 95 minutes.

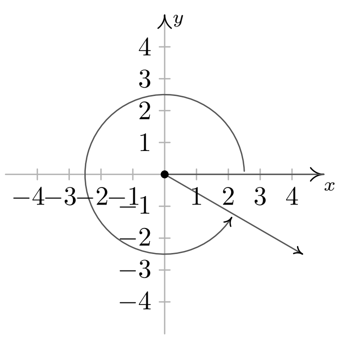
33. The steady state current is 2 amps.

35.

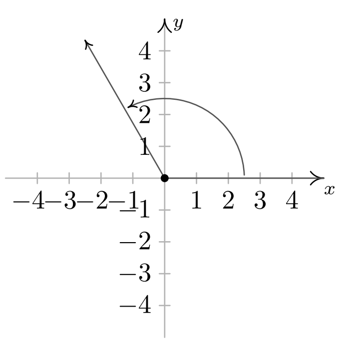
Chapter 8

Section 8.1

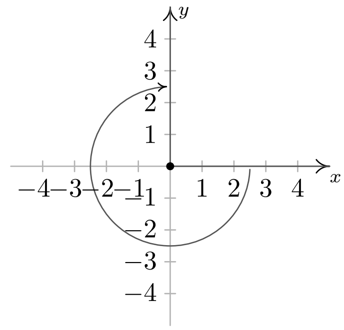
1. 330° is a Quadrant IV angle coterminal with 690° and −30°



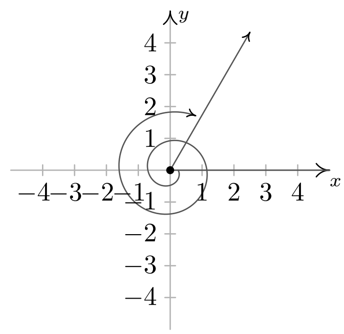
3. 120° is a Quadrant II angle coterminal with 480° and −240°



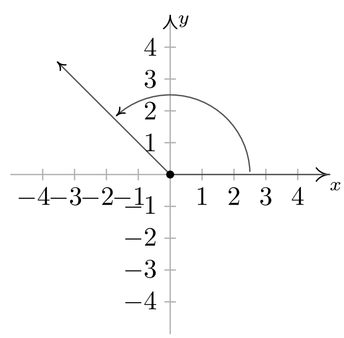
5. −270° lies on the positive y-axis coterminal with 90° and −630°



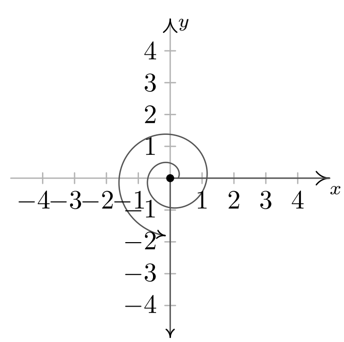
7. $-\frac{11\pi}{3}$ is a Quadrant I angle coterminal with $\frac{\pi}{3}$ and $-\frac{5\pi}{3}$



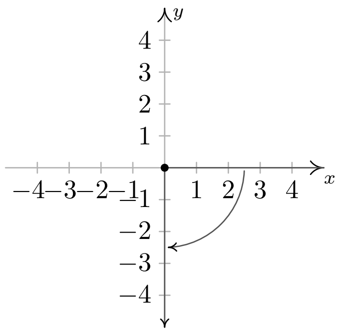
9. $\frac{3\pi}{4}$ is a Quadrant II angle coterminal with $\frac{11\pi}{4}$ and $-\frac{5\pi}{4}$



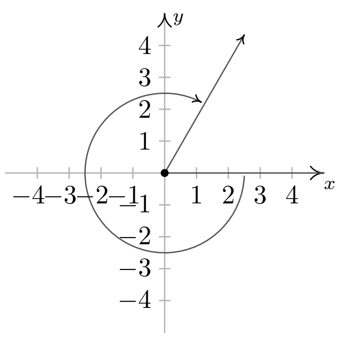
11. $\frac{7\pi}{2}$ lies on the negative y-axis coterminal with $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$



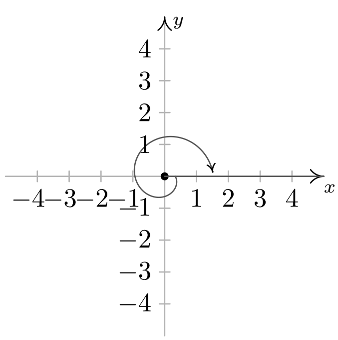
13. $-\dfrac{\pi}{2}$ lies on the negative *y*-axis
coterminal with $\dfrac{3\pi}{2}$ and $-\dfrac{5\pi}{2}$



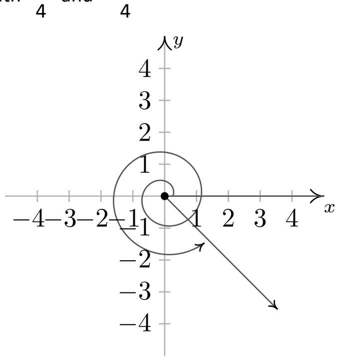
15. $-\dfrac{5\pi}{3}$ is a Quadrant I angle
coterminal with $\dfrac{\pi}{3}$ and $-\dfrac{11\pi}{3}$



17. -2π lies on the positive *x*-axis
coterminal with 2π and -4π



19. $\dfrac{15\pi}{4}$ is a Quadrant IV angle
coterminal with $\dfrac{7\pi}{4}$ and $-\dfrac{\pi}{4}$



21. 0

23. $\dfrac{3\pi}{4}$

25. $-\dfrac{7\pi}{4}$

27. $\dfrac{\pi}{4}$

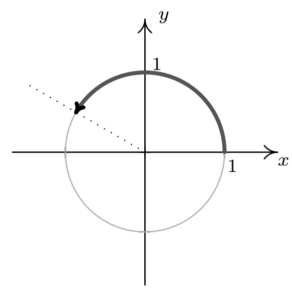
29. 180°

31. 210°

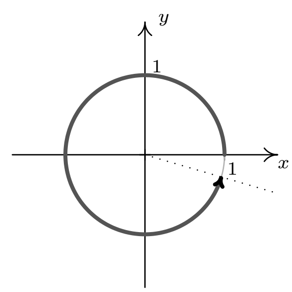
33. 60°

35. -30°

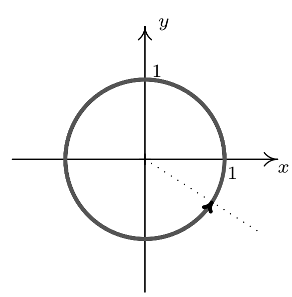
37. $t = \dfrac{5\pi}{6}$



39. $t = 6$



41. $t = 12$ (between 1 and 2 revolutions)



43. About 6274.52 revolutions per minute

45. About 53.55 miles per hour

47. About 4.32 miles per hour

49. 12π square units

51. $79.2825\pi \approx 249.07$ square units

53. $\dfrac{50\pi}{3}$ square units

55.

Section 8.2

1. $\cos(0) = 1,\ \sin(0) = 0$

3. $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2},\ \sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

5. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

7. $\cos(\pi) = -1,\ \sin(\pi) = 0$

9. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

11. $\cos\left(\frac{3\pi}{2}\right) = 0,\ \sin\left(\frac{3\pi}{2}\right) = -1$

13. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

15. $\cos\left(-\frac{13\pi}{2}\right) = 0,\ \sin\left(-\frac{13\pi}{2}\right) = -1$

17. $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

19. $\cos\left(\frac{10\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

21. If $\sin(\theta) = -\frac{7}{25}$ with θ in Quadrant IV, then $\cos(\theta) = \frac{24}{25}$.

23. If $\sin(\theta) = \frac{5}{13}$ with θ in Quadrant II, then $\cos(\theta) = -\frac{12}{13}$.

25. If $\sin(\theta) = -\frac{2}{3}$ with θ in Quadrant III, then $\cos(\theta) = -\frac{\sqrt{5}}{3}$.

27. If $\sin(\theta) = \frac{2\sqrt{5}}{5}$ and $\frac{\pi}{2} < \theta < \pi$, then $\cos(\theta) = -\frac{\sqrt{5}}{5}$.

29. If $\sin(\theta) = -0.42$ and $\pi < \theta < \frac{3\pi}{2}$, then $\cos(\theta) = -\sqrt{0.8236} \approx -0.9075$.

31. $\sin(\theta) = \frac{1}{2}$ when $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for any integer k.

33. $\sin(\theta) = 0$ when $\theta = \pi k$ for any integer k.

35. $\sin(\theta) = \frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{2\pi}{3} + 2\pi k$ for any integer k.

37. $\sin(\theta) = -1$ when $\theta = \frac{3\pi}{2} + 2\pi k$ for any integer k.

39. $\cos(\theta) = -1.001$ never happens

41. $\sin(t) = -\frac{\sqrt{2}}{2}$ when $t = \frac{5\pi}{4} + 2\pi k$ or $t = \frac{7\pi}{4} + 2\pi k$ for any integer k.

43. $\sin(t) = -\frac{1}{2}$ when $t = \frac{7\pi}{6} + 2\pi k$ or $t = \frac{11\pi}{6} + 2\pi k$ for any integer k.

45. $\sin(t) = -2$ never happens

47. $\sin(t) = 1$ when $t = \frac{\pi}{2} + 2\pi k$ for any integer k.

49. $\sin(78.95^\circ) \approx 0.981$

51. $\sin(392.994) \approx -0.291$

53. $\sin(\pi^\circ) \approx 0.055$

55. $\theta = 60^\circ, b = \frac{\sqrt{3}}{3}, c = \frac{2\sqrt{3}}{3}$

57. $\alpha = 57^\circ, a = 8\cos(33^\circ) \approx 6.709, b = 8\sin(33^\circ) \approx 4.357$

59. The hypotenuse has length $\frac{4}{\cos(12^\circ)} \approx 4.089$.

61. The hypotenuse has length $\frac{117.42}{\sin(59^\circ)} \approx 136.99$.

63. The side adjacent to θ has length $10\cos(5^\circ) \approx 9.962$.

65. $\cos(\theta) = -\frac{7}{25},\ \sin(\theta) = \frac{24}{25}$

67. $\cos(\theta) = \frac{5\sqrt{106}}{106},\ \sin(\theta) = -\frac{9\sqrt{106}}{106}$

69. $r = 1.125$ inches, $\omega = 9000\pi\ \frac{\text{radians}}{\text{minute}}$, $x = 1.125\cos(9000\pi\, t)$, $y = 1.125\sin(9000\pi\, t)$. Here x and y are measured in inches and t is measured in minutes.

71. $r = 1.25$ inches, $\omega = 14400\pi\ \frac{\text{radians}}{\text{minute}}$, $x = 1.25\cos(14400\pi\, t)$, $y = 1.25\sin(14400\pi\, t)$. Here x and y are measured in inches and t is measured in minutes.

73. $r = 64$ feet, $\omega = \frac{4\pi}{127}\ \frac{\text{radians}}{\text{second}}$, $x = 64\cos\left(\frac{4\pi}{127}\, t\right)$, $y = 64\sin\left(\frac{4\pi}{127}\, t\right)$. Here x and y are measured in feet and t is measured in seconds.

75.

Section 8.3

1. $\cos(0) = 1,\ \sin(0) = 0$

3. $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

5. $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

7. $\cos(\pi) = -1,\ \sin(\pi) = 0$

9. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

11. $\cos\left(\frac{3\pi}{2}\right) = 0,\ \sin\left(\frac{3\pi}{2}\right) = -1$

13. $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

15. $\cos\left(-\frac{13\pi}{2}\right) = 0,\ \sin\left(-\frac{13\pi}{2}\right) = -1$

17. $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

19. $\cos\left(\frac{10\pi}{3}\right) = -\frac{1}{2},\ \sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

21. $\sin(\theta) = \frac{3}{5}, \cos(\theta) = -\frac{4}{5}, \tan(\theta) = -\frac{3}{4}, \csc(\theta) = \frac{5}{3}, \sec(\theta) = -\frac{5}{4}, \cot(\theta) = -\frac{4}{3}$

23. $\sin(\theta) = \frac{24}{25}, \cos(\theta) = \frac{7}{25}, \tan(\theta) = \frac{24}{7}, \csc(\theta) = \frac{25}{24}, \sec(\theta) = \frac{25}{7}, \cot(\theta) = \frac{7}{24}$

25. $\sin(\theta) = -\frac{\sqrt{91}}{10}, \cos(\theta) = -\frac{3}{10}, \tan(\theta) = \frac{\sqrt{91}}{3}, \csc(\theta) = -\frac{10\sqrt{91}}{91}, \sec(\theta) = -\frac{10}{3}, \cot(\theta) = \frac{3\sqrt{91}}{91}$

27. $\sin(\theta) = -\frac{2\sqrt{5}}{5}, \cos(\theta) = \frac{\sqrt{5}}{5}, \tan(\theta) = -2, \csc(\theta) = -\frac{\sqrt{5}}{2}, \sec(\theta) = \sqrt{5}, \cot(\theta) = -\frac{1}{2}$

29. $\sin(\theta) = -\frac{\sqrt{6}}{6}, \cos(\theta) = -\frac{\sqrt{30}}{6}, \tan(\theta) = \frac{\sqrt{5}}{5}, \csc(\theta) = -\sqrt{6}, \sec(\theta) = -\frac{\sqrt{30}}{5}, \cot(\theta) = \sqrt{5}$

31. $\sin(\theta) = \frac{\sqrt{5}}{5}, \cos(\theta) = \frac{2\sqrt{5}}{5}, \tan(\theta) = \frac{1}{2}, \csc(\theta) = \sqrt{5}, \sec(\theta) = \frac{\sqrt{5}}{2}, \cot(\theta) = 2$

33. $\sin(\theta) = -\frac{\sqrt{110}}{11}, \cos(\theta) = -\frac{\sqrt{11}}{11}, \tan(\theta) = \sqrt{10}, \csc(\theta) = -\frac{\sqrt{110}}{10}, \sec(\theta) = -\sqrt{11}, \cot(\theta) = \frac{\sqrt{10}}{10}$

35. $\csc(78.95^\circ) \approx 1.019$

37. $\cot(392.994) \approx 3.292$

39. $\csc(5.902) \approx -2.688$

41. $\cot(3^\circ) \approx 19.081$

43. $\tan(\theta) = \sqrt{3}$ when $\theta = \frac{\pi}{3} + \pi k$ for any integer k

45. $\csc(\theta) = -1$ when $\theta = \frac{3\pi}{2} + 2\pi k$ for any integer k.

47. $\tan(\theta) = 0$ when $\theta = \pi k$ for any integer k

49. $\csc(\theta) = 2$ when $\theta = \frac{\pi}{6} + 2\pi k$ or $\theta = \frac{5\pi}{6} + 2\pi k$ for any integer k.

51. $\tan(\theta) = -1$ when $\theta = \frac{3\pi}{4} + \pi k$ for any integer k

53. $\csc(\theta) = -\frac{1}{2}$ never happens

55. $\tan(\theta) = -\sqrt{3}$ when $\theta = \frac{2\pi}{3} + \pi k$ for any integer k

57. $\cot(\theta) = -1$ when $\theta = \frac{3\pi}{4} + \pi k$ for any integer k

59. $\tan(t) = \frac{\sqrt{3}}{3}$ when $t = \frac{\pi}{6} + \pi k$ for any integer k

61. $\csc(t) = 0$ never happens

63. $\tan(t) = -\frac{\sqrt{3}}{3}$ when $t = \frac{5\pi}{6} + \pi k$ for any integer k

65. $\csc(t) = \frac{2\sqrt{3}}{3}$ when $t = \frac{\pi}{3} + 2\pi k$ or $t = \frac{2\pi}{3} + 2\pi k$ for any integer k

67. $\alpha = 56^\circ$, $b = 12\tan(34^\circ) = 8.094$, $c = 12\sec(34^\circ) = \frac{12}{\cos(34^\circ)} \approx 14.475$

69. $\beta = 40^\circ$, $b = 2.5\tan(50^\circ) \approx 2.979$, $c = 2.5\sec(50^\circ) = \frac{2.5}{\cos(50^\circ)} \approx 3.889$

71. The side opposite θ has length $10\sin(15^\circ) \approx 2.588$

73. The hypotenuse has length $14\csc(38.2^\circ) = \frac{14}{\sin(38.2^\circ)} \approx 22.639$

75. The side opposite θ has length $31\tan(42^\circ) \approx 27.912$

77. The lights are about 75 feet apart.

79. The tree is about 41 feet tall.

81. The tower is about 682 feet tall. The guy wire hits the ground about 731 feet away from the base of the tower.

83.

85.

87.

89.

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101.

103.

105.

107.

109.

111.

113.

115.

117.

119.

121.

123.

125.

127.

129.

131.

133.

135.

137.

Section 8.4

1.

3.

5.

7. $\cos(75^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$

9. $\sin(105^\circ) = \frac{\sqrt{6} + \sqrt{2}}{4}$

11. $\cot(255^\circ) = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$

13. $\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{6} + \sqrt{2}}{4}$

15. $\tan\left(\frac{13\pi}{12}\right) = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = 2 - \sqrt{3}$

17. $\tan\left(\frac{17\pi}{12}\right) = 2 + \sqrt{3}$

19. $\cot\left(\frac{11\pi}{12}\right) = -(2 + \sqrt{3})$

21. $\sec\left(-\frac{\pi}{12}\right) = \sqrt{6} - \sqrt{2}$

23. (a) $\cos(\alpha + \beta) = -\frac{4 + 7\sqrt{2}}{30}$

(b) $\sin(\alpha + \beta) = \frac{28 - \sqrt{2}}{30}$

(c) $\tan(\alpha + \beta) = \frac{-28 + \sqrt{2}}{4 + 7\sqrt{2}} = \frac{63 - 100\sqrt{2}}{41}$

(d) $\cos(\alpha - \beta) = \frac{-4 + 7\sqrt{2}}{30}$

(e) $\sin(\alpha - \beta) = -\frac{28 + \sqrt{2}}{30}$

(f) $\tan(\alpha - \beta) = \frac{28 + \sqrt{2}}{4 - 7\sqrt{2}} = -\frac{63 + 100\sqrt{2}}{41}$

25. (a) $\csc(\alpha - \beta) = -\frac{5}{4}$

(b) $\sec(\alpha + \beta) = \frac{125}{117}$

(c) $\cot(\alpha + \beta) = \frac{117}{44}$

27.

29.

31.

33.

35.

37.

39. $\cos(75^\circ) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

41. $\cos(67.5^\circ) = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

43. $\tan(112.5^\circ) = -\sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}} = -1-\sqrt{2}$

45. $\sin\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

47. $\sin\left(\dfrac{5\pi}{8}\right) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

49.
- $\sin(2\theta) = -\dfrac{336}{625}$
- $\sin\left(\frac{\theta}{2}\right) = \dfrac{\sqrt{2}}{10}$
- $\cos(2\theta) = \dfrac{527}{625}$
- $\cos\left(\frac{\theta}{2}\right) = -\dfrac{7\sqrt{2}}{10}$
- $\tan(2\theta) = -\dfrac{336}{527}$
- $\tan\left(\frac{\theta}{2}\right) = -\dfrac{1}{7}$

51.
- $\sin(2\theta) = \dfrac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \dfrac{3\sqrt{13}}{13}$
- $\cos(2\theta) = -\dfrac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\dfrac{2\sqrt{13}}{13}$
- $\tan(2\theta) = -\dfrac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\dfrac{3}{2}$

53.
- $\sin(2\theta) = \dfrac{24}{25}$
- $\sin\left(\frac{\theta}{2}\right) = \dfrac{\sqrt{5}}{5}$
- $\cos(2\theta) = -\dfrac{7}{25}$
- $\cos\left(\frac{\theta}{2}\right) = \dfrac{2\sqrt{5}}{5}$
- $\tan(2\theta) = -\dfrac{24}{7}$
- $\tan\left(\frac{\theta}{2}\right) = \dfrac{1}{2}$

55.
- $\sin(2\theta) = -\dfrac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \dfrac{\sqrt{26}}{26}$
- $\cos(2\theta) = \dfrac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\dfrac{5\sqrt{26}}{26}$
- $\tan(2\theta) = -\dfrac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\dfrac{1}{5}$

57.
- $\sin(2\theta) = -\dfrac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \dfrac{\sqrt{50-10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\dfrac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\dfrac{\sqrt{50+10\sqrt{5}}}{10}$
- $\tan(2\theta) = \dfrac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\dfrac{5-\sqrt{5}}{5+\sqrt{5}}} = \dfrac{5-5\sqrt{5}}{10}$

59.

61.

63.

65.

67.

69.

71.

73.

75. $\dfrac{\cos(5\theta)-\cos(9\theta)}{2}$

77. $\dfrac{\cos(4\theta)+\cos(8\theta)}{2}$

79. $\dfrac{\sin(2\theta)+\sin(4\theta)}{2}$

81. $-2\cos\left(\dfrac{9}{2}\theta\right)\sin\left(\dfrac{5}{2}\theta\right)$

83. $2\cos(4\theta)\sin(5\theta)$

85. $-\sqrt{2}\sin\left(\theta-\dfrac{\pi}{4}\right)$

87.

89.

91. $\dfrac{14x}{x^2+49}$

93.

95.

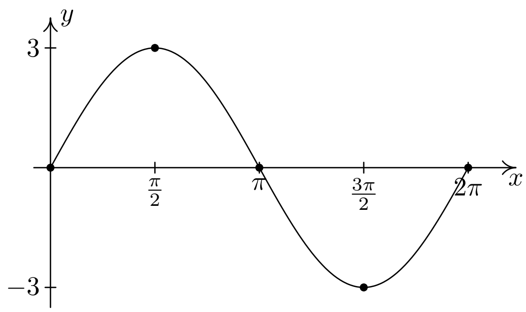
97.

99.

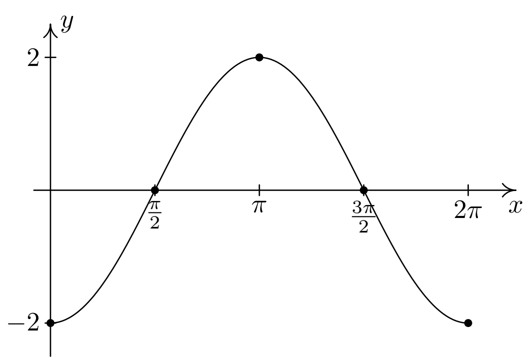
101.

Section 8.5

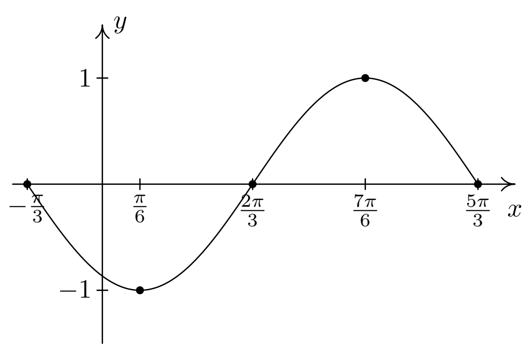
1. $y = 3\sin(x)$
Period: 2π
Amplitude: 3
Phase Shift: 0
Vertical Shift: 0



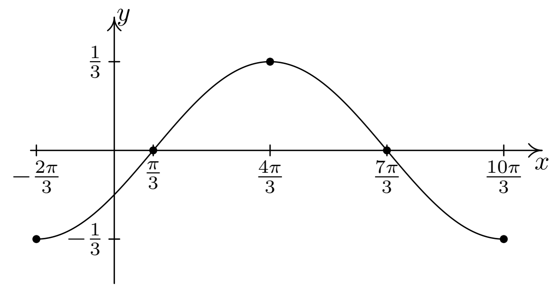
3. $y = -2\cos(x)$
   Period: 2π
   Amplitude: 2
   Phase Shift: 0
   Vertical Shift: 0



5. $y = -\sin\left(x + \dfrac{\pi}{3}\right)$
   Period: 2π
   Amplitude: 1
   Phase Shift: $-\dfrac{\pi}{3}$
   Vertical Shift: 0

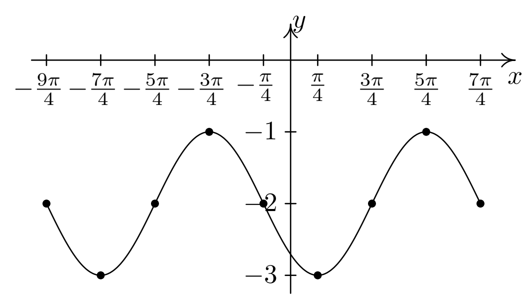


7. $y = -\dfrac{1}{3}\cos\left(\dfrac{1}{2}x + \dfrac{\pi}{3}\right)$
   Period: 4π
   Amplitude: $\dfrac{1}{3}$
   Phase Shift: $-\dfrac{2\pi}{3}$
   Vertical Shift: 0

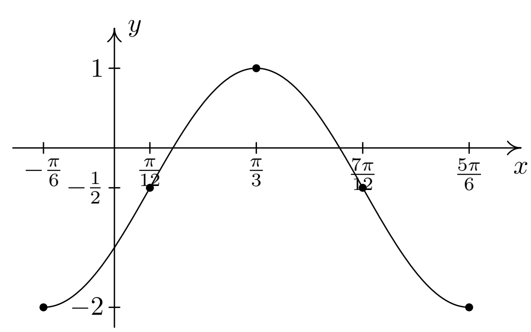


9. $y = \sin\left(-x - \dfrac{\pi}{4}\right) - 2$
   Period: 2π
   Amplitude: 1
   Phase Shift: $-\dfrac{\pi}{4}$ (You need to use

   $y = -\sin\left(x + \dfrac{\pi}{4}\right) - 2$ to find this.)[2]
   Vertical Shift: -2

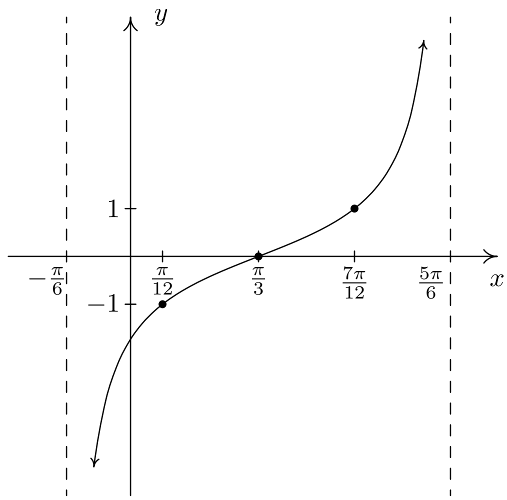


11. $y = -\dfrac{3}{2}\cos\left(2x + \dfrac{\pi}{3}\right) - \dfrac{1}{2}$
    Period: π
    Amplitude: $\dfrac{3}{2}$
    Phase Shift: $-\dfrac{\pi}{6}$
    Vertical Shift: $-\dfrac{1}{2}$

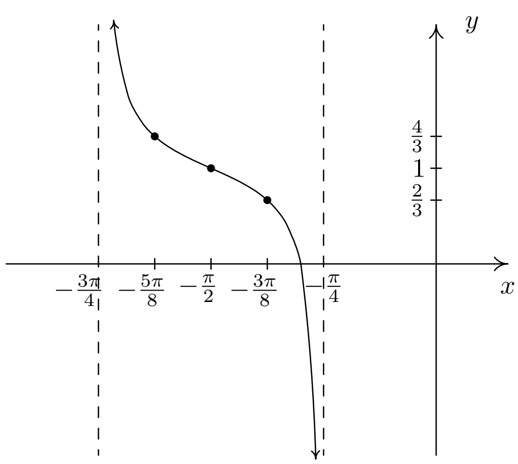


13. $y = \tan\left(x - \dfrac{\pi}{3}\right)$
    Period: π

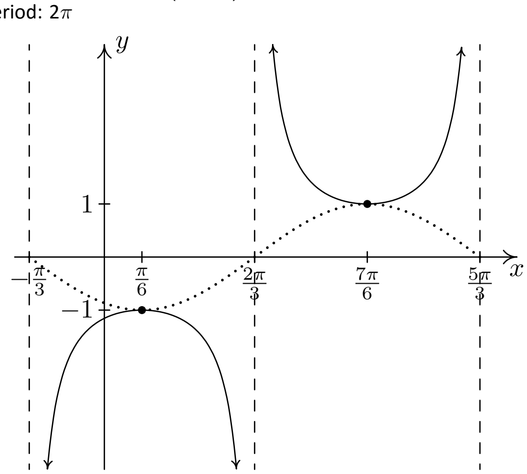
[2]Two cycles of the graph are shown to illustrate the discrepancy discussed on page 375.



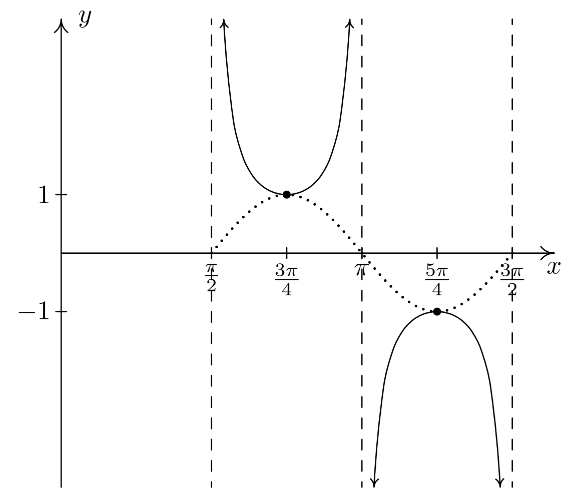
15. $y = \frac{1}{3}\tan(-2x - \pi) + 1$
is equivalent to
$y = -\frac{1}{3}\tan(2x + \pi) + 1$
via the Even / Odd identity for tangent.
Period: $\frac{\pi}{2}$



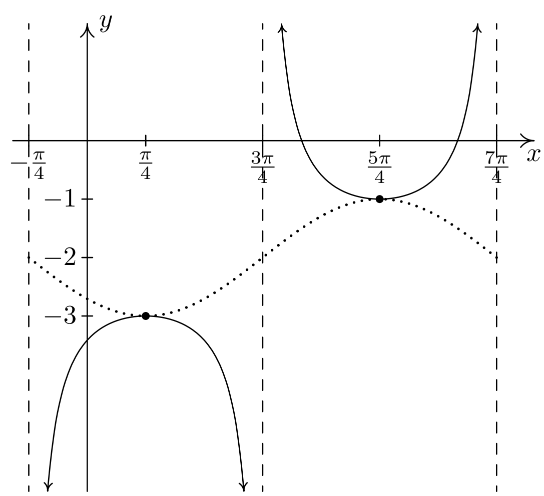
17. $y = -\csc\left(x + \frac{\pi}{3}\right)$
Start with $y = -\sin\left(x + \frac{\pi}{3}\right)$
Period: 2π



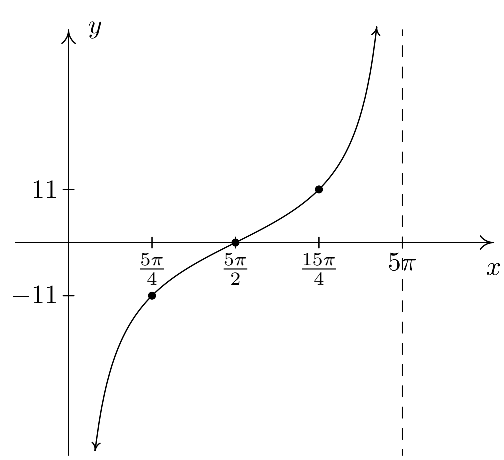
19. $y = \csc(2x - \pi)$
Start with $y = \sin(2x - \pi)$
Period: π



21. $y = \csc\left(-x - \frac{\pi}{4}\right) - 2$
Start with $y = \sin\left(-x - \frac{\pi}{4}\right) - 2$
Period: 2π



23. $y = -11\cot\left(\frac{1}{5}x\right)$
Period: 5π



25. $f(x) = \sqrt{2}\sin(x) + \sqrt{2}\cos(x) + 1 = 2\sin\left(x + \frac{\pi}{4}\right) + 1 = 2\cos\left(x + \frac{7\pi}{4}\right) + 1$

27. $f(x) = -\sin(x) + \cos(x) - 2 = \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right) - 2 = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 2$

29. $f(x) = 2\sqrt{3}\cos(x) - 2\sin(x) = 4\sin\left(x + \frac{2\pi}{3}\right) = 4\cos\left(x + \frac{\pi}{6}\right)$

31. $f(x) = -\frac{1}{2}\cos(5x) - \frac{\sqrt{3}}{2}\sin(5x) = \sin\left(5x + \frac{7\pi}{6}\right) = \cos\left(5x + \frac{2\pi}{3}\right)$

33. $f(x) = \frac{5\sqrt{2}}{2}\sin(x) - \frac{5\sqrt{2}}{2}\cos(x) = 5\sin\left(x + \frac{7\pi}{4}\right) = 5\cos\left(x + \frac{5\pi}{4}\right)$

35.

37.

39.

41.

43.

45.

47.

49.

51.

Chapter 9

Section 9.1

1. $\arcsin(-1) = -\frac{\pi}{2}$

3. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

5. $\arcsin(0) = 0$

7. $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

9. $\arcsin(1) = \frac{\pi}{2}$

11. $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

13. $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

15. $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

17. $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

19. $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$

21. $\arctan\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$

23. $\arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

25. $\arctan(\sqrt{3}) = \frac{\pi}{3}$

27. $\operatorname{arccot}(-1) = \frac{3\pi}{4}$

29. $\operatorname{arccot}(0) = \frac{\pi}{2}$

31. $\operatorname{arccot}(1) = \frac{\pi}{4}$

33. $\operatorname{arcsec}(2) = \frac{\pi}{3}$

35. $\operatorname{arcsec}(\sqrt{2}) = \frac{\pi}{4}$

37. $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right) = \frac{\pi}{6}$

39. $\operatorname{arcsec}(1) = 0$

41. $\operatorname{arcsec}(-2) = \frac{4\pi}{3}$

43. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{7\pi}{6}$

45. $\operatorname{arccsc}(-2) = \frac{7\pi}{6}$

47. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{4\pi}{3}$

49. $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$

51. $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$

53. $\operatorname{arccsc}(-2) = -\frac{\pi}{6}$

55. $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$

57. $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

59. $\sin\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

61. $\sin\left(\arcsin\left(\frac{5}{4}\right)\right)$ is undefined.

63. $\cos\left(\arccos\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$

65. $\cos(\arccos(-0.998)) = -0.998$

67. $\tan(\arctan(-1)) = -1$

69. $\tan\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{12}$

71. $\tan(\arctan(3\pi)) = 3\pi$

73. $\cot(\operatorname{arccot}(-\sqrt{3})) = -\sqrt{3}$

75. $\cot(\operatorname{arccot}(-0.001)) = -0.001$

77. $\sec(\operatorname{arcsec}(2)) = 2$

79. $\sec\left(\operatorname{arcsec}\left(\frac{1}{2}\right)\right)$ is undefined.

81. $\sec(\operatorname{arcsec}(117\pi)) = 117\pi$

83. $\csc\left(\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)\right) = -\frac{2\sqrt{3}}{3}$

85. $\csc(\text{arccsc}(1.0001)) = 1.0001$

87. $\arcsin\left(\sin\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

89. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$

91. $\arcsin\left(\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{\pi}{3}$

93. $\arccos\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

95. $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

97. $\arctan\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

99. $\arctan(\tan(\pi)) = 0$

101. $\arctan\left(\tan\left(\frac{2\pi}{3}\right)\right) = -\frac{\pi}{3}$

103. $\text{arccot}\left(\cot\left(-\frac{\pi}{4}\right)\right) = \frac{3\pi}{4}$

105. $\text{arccot}\left(\cot\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

107. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

109. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{7\pi}{6}$

111. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

113. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = \frac{5\pi}{4}$

115. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = \frac{3\pi}{2}$

117. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{13\pi}{12}$

119. $\text{arcsec}\left(\sec\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

121. $\text{arcsec}\left(\sec\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6}$

123. $\text{arcsec}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \frac{\pi}{3}$

125. $\text{arccsc}\left(\csc\left(\frac{5\pi}{4}\right)\right) = -\frac{\pi}{4}$

127. $\text{arccsc}\left(\csc\left(-\frac{\pi}{2}\right)\right) = -\frac{\pi}{2}$

129. $\text{arcsec}\left(\sec\left(\frac{11\pi}{12}\right)\right) = \frac{11\pi}{12}$

131. $\sin\left(\arccos\left(-\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

133. $\sin(\arctan(-2)) = -\frac{2\sqrt{5}}{5}$

135. $\sin(\text{arccsc}(-3)) = -\frac{1}{3}$

137. $\cos\left(\arctan\left(\sqrt{7}\right)\right) = \frac{\sqrt{2}}{4}$

139. $\cos(\text{arcsec}(5)) = \frac{1}{5}$

141. $\tan\left(\arccos\left(-\frac{1}{2}\right)\right) = -\sqrt{3}$

143. $\tan(\text{arccot}(12)) = \frac{1}{12}$

145. $\cot\left(\arccos\left(\frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}$

147. $\cot(\arctan(0.25)) = 4$

149. $\sec\left(\arcsin\left(-\frac{12}{13}\right)\right) = \frac{13}{5}$

151. $\sec\left(\text{arccot}\left(-\frac{\sqrt{10}}{10}\right)\right) = -\sqrt{11}$

153. $\csc\left(\arcsin\left(\frac{3}{5}\right)\right) = \frac{5}{3}$

155. $\sin\left(\arcsin\left(\frac{5}{13}\right) + \frac{\pi}{4}\right) = \frac{17\sqrt{2}}{26}$

157. $\tan\left(\arctan(3) + \arccos\left(-\frac{3}{5}\right)\right) = \frac{1}{3}$

159. $\sin\left(2\,\text{arccsc}\left(\frac{13}{5}\right)\right) = \frac{120}{169}$

161. $\cos\left(2\arcsin\left(\frac{3}{5}\right)\right) = \frac{7}{25}$

163. $\cos\left(2\,\text{arccot}\left(-\sqrt{5}\right)\right) = \frac{2}{3}$

165. $\sin(\arccos(x)) = \sqrt{1-x^2}$ for $-1 \le x \le 1$

167. $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$ for $-1 < x < 1$

169. $\csc(\arccos(x)) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

171. $\sin(2\arccos(x)) = 2x\sqrt{1-x^2}$ for $-1 \le x \le 1$

173. $\sin(\arccos(2x)) = \sqrt{1-4x^2}$ for $-\frac{1}{2} \le x \le \frac{1}{2}$

175. $\cos\left(\arcsin\left(\frac{x}{2}\right)\right) = \frac{\sqrt{4-x^2}}{2}$ for $-2 \le x \le 2$

177. $\sin(2\arcsin(7x)) = 14x\sqrt{1-49x^2}$ for $-\frac{1}{7} \le x \le \frac{1}{7}$

179. $\cos(2\arcsin(4x)) = 1 - 32x^2$ for $-\frac{1}{4} \le x \le \frac{1}{4}$

181. $\sin(\arcsin(x) + \arccos(x)) = 1$ for $-1 \le x \le 1$

183. $\tan(2\arcsin(x)) = \frac{2x\sqrt{1-x^2}}{1-2x^2}$ for x in $\left(-1, -\frac{\sqrt{2}}{2}\right) \cup \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, 1\right)$[3]

185. If $\sin(\theta) = \frac{x}{2}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\theta + \sin(2\theta) = \arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2}$

187. If $\sec(\theta) = \frac{x}{4}$ for $0 < \theta < \frac{\pi}{2}$, then $4\tan(\theta) - 4\theta = \sqrt{x^2-16} - 4\,\text{arcsec}\left(\frac{x}{4}\right)$

[3] The equivalence for $x = \pm 1$ can be verified independently of the derivation of the formula, but Calculus is required to fully understand what is happening at those x values. You'll see what we mean when you work through the details of the identity for $\tan(2t)$. For now, we exclude $x = \pm 1$ from our answer.

189. $x = \arccos\left(-\frac{2}{9}\right) + 2\pi k$ or $x = -\arccos\left(-\frac{2}{9}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 1.7949,\ 4.4883$

191. $x = \arccos(0.117) + 2\pi k$ or $x = 2\pi - \arccos(0.117) + 2\pi k$, in $[0, 2\pi)$, $x \approx 1.4535,\ 4.8297$

193. $x = \arccos\left(\frac{359}{360}\right) + 2\pi k$ or $x = 2\pi - \arccos\left(\frac{359}{360}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.0746,\ 6.2086$

195. $x = \arctan\left(-\frac{1}{12}\right) + \pi k$, in $[0, 2\pi)$, $x \approx 3.0585,\ 6.2000$

197. $x = \pi + \arcsin\left(\frac{17}{90}\right) + 2\pi k$ or $x = 2\pi - \arcsin\left(\frac{17}{90}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 3.3316,\ 6.0932$

199. $x = \arcsin\left(\frac{3}{8}\right) + 2\pi k$ or $x = \pi - \arcsin\left(\frac{3}{8}\right) + 2\pi k$, in $[0, 2\pi)$, $x \approx 0.3844,\ 2.7572$

201. $x = \arctan(0.03) + \pi k$, in $[0, 2\pi)$, $x \approx 0.0300,\ 3.1716$

203. $x = \pi + \arcsin(0.721) + 2\pi k$ or $x = 2\pi - \arcsin(0.721) + 2\pi k$, in $[0, 2\pi)$, $x \approx 3.9468,\ 5.4780$

205. $x = \arccos(-0.5637) + 2\pi k$ or $x = -\arccos(-0.5637) + 2\pi k$, in $[0, 2\pi)$, $x \approx 2.1697,\ 4.1135$

207. $x = \arctan(-0.6109) + \pi k$, in $[0, 2\pi)$, $x \approx 2.5932,\ 5.7348$

209. 22.62° and 67.38°

211. 68.9°

213. 51°

215. 41.81°

217. $f(x) = 3\cos(2x) + 4\sin(2x) = 5\sin\left(2x + \arcsin\left(\frac{3}{5}\right)\right) \approx 5\sin(2x + 0.6435)$

219. $f(x) = 7\sin(10x) - 24\cos(10x) = 25\sin\left(10x + \arcsin\left(-\frac{24}{25}\right)\right) \approx 25\sin(10x - 1.2870)$

221. $f(x) = 2\sin(x) - \cos(x) = \sqrt{5}\sin\left(x + \arcsin\left(-\frac{\sqrt{5}}{5}\right)\right) \approx \sqrt{5}\sin(x - 0.4636)$

223. $\left[-\frac{1}{3}, 1\right]$

225. $(-\infty, -\sqrt{5}] \cup [-\sqrt{3}, \sqrt{3}] \cup [\sqrt{5}, \infty)$

227. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

229. $\left[\frac{1}{2}, \infty\right)$

231. $(-\infty, -6] \cup [-4, \infty)$

233. $[0, \infty)$

235.

237.

Chapter 10

Section 10.1

1. Answers will vary.
3. F
5. Answers will vary.
7. -5
9. 2
11. Limit does not exist.
13. 7
15. Limit does not exist.

17. The limit seems to be exactly 9.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

19. The limit is approx. -0.11.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

21. The limit is approx. 0.2.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

23. The limit is approx. 0.005.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.0499583
-0.01	-0.00499996
0.01	0.00499996
0.1	0.0499583

Section 10.2

1. Answers will vary.
3. Answers will vary.
5. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (I.e., $f(x) \approx 2g(x)$.)
7. 6
9. Limit does not exist.
11. Not possible to know.
13. -45
15. -1
17. π
19. $-0.000000015 \approx 0$
21. Limit does not exist
23. 2
25. $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$
27. -8
29. 10
31. $-3/2$
33. 0
35. 1
37. 3
39. 1

Section 10.3

1. The function approaches different values from the left and right; the function grows without bound; the function oscillates.
3. F
5. (a) 2
   (b) 2
   (c) 2
   (d) 1

(e) As f is not defined for $x < 0$, this limit is not defined.
(f) 1

7. (a) Does not exist.
(b) Does not exist.
(c) Does not exist.
(d) Not defined.
(e) 0
(f) 0

9. (a) 2
(b) 2
(c) 2
(d) 2

11. (a) 2
(b) 2
(c) 2
(d) 0
(e) 2
(f) 2
(g) 2
(h) Not defined

13. (a) 2
(b) -4
(c) Does not exist.
(d) 2

15. (a) 0
(b) 0
(c) 0
(d) 0
(e) 2
(f) 2
(g) 2
(h) 2

17. (a) $1 - \cos^2 a = \sin^2 a$
(b) $\sin^2 a$
(c) $\sin^2 a$
(d) $\sin^2 a$

19. (a) 4
(b) 4
(c) 4
(d) 3

21. (a) -1
(b) 1
(c) Does not exist
(d) 0

23. $2/3$

25. -9

Section 10.4

1. Answers will vary.

3. A root of a function f is a value c such that $f(c) = 0$.

5. F

7. T

9. F

11. No; $\lim_{x\to 1} f(x) = 2$, while $f(1) = 1$.

13. No; $f(1)$ does not exist.

15. Yes

17. (a) No; $\lim_{x\to -2} f(x) \neq f(-2)$
(b) Yes
(c) No; $f(2)$ is not defined.

19. (a) Yes
(b) No; the left and right hand limits at 1 are not equal.

21. (a) Yes
(b) No. $\lim_{x\to 8} f(x) = 16/5 \neq f(8) = 5$.

23. $(-\infty, -2] \cup [2, \infty)$

25. $(-\infty, -\sqrt{6}] \cup [\sqrt{6}, \infty)$

27. $(-\infty, \infty)$

29. $(0, \infty)$

31. $(-\infty, 0]$

33. Yes, by the Intermediate Value Theorem.

35. We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.

37. Approximate root is $x = 1.23$. The intervals used are:
$[1, 1.5]$ $[1, 1.25]$ $[1.125, 1.25]$
$[1.1875, 1.25]$ $[1.21875, 1.25]$ $[1.234375, 1.25]$
$[1.234375, 1.2421875]$ $[1.234375, 1.2382813]$

39. Approximate root is $x = 0.69$. The intervals used are:
$[0.65, 0.7]$ $[0.675, 0.7]$ $[0.6875, 0.7]$
$[0.6875, 0.69375]$ $[0.690625, 0.69375]$

41. (a) 20
(b) 25
(c) Limit does not exist
(d) 25

43. Answers will vary.

Section 10.5

1. F

3. F

5. T

7. Answers will vary.

9. (a) ∞
(b) ∞

11. (a) 1
(b) 0
(c) $1/2$
(d) $1/2$

13. (a) Limit does not exist
(b) Limit does not exist

15. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

	x	$f(x)$	
(b)	3.1	16.8824	It seems $\lim_{x\to 3^+} f(x) = \infty$.
	3.01	160.88	
	3.001	1600.88	

(c) It seems $\lim_{x\to 3} f(x)$ does not exist.

17. Tables will vary.

	x	$f(x)$	
(a)	2.9	132.857	It seems $\lim_{x\to 3^-} f(x) = \infty$.
	2.99	12124.4	

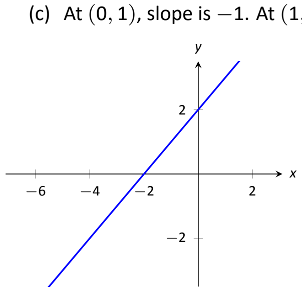
	x	$f(x)$	
(b)	3.1	108.039	It seems $\lim_{x\to 3^+} f(x) = \infty$.
	3.01	11876.4	

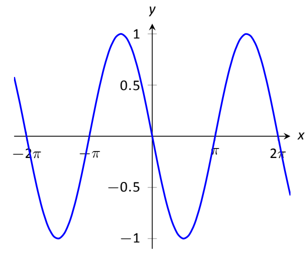
(c) It seems $\lim_{x\to 3} f(x) = \infty$.

19. Horizontal asymptote at $y = 2$; vertical asymptotes at $x = -5, 4$.
21. Horizontal asymptote at $y = 0$; vertical asymptotes at $x = -1, 0$.
23. No horizontal or vertical asymptotes.
25. ∞
27. $-\infty$
29. (a) 2
    (b) -3
    (c) -3
    (d) $1/3$
31. 1

Chapter 11

Section 11.1

1. T
3. Answers will vary.
5. Answers will vary.
7. $f'(x) = 2$
9. $g'(x) = 2x$
11. $r'(x) = \frac{-1}{x^2}$
13. (a) $y = 6$
    (b) $x = -2$
15. (a) $y = -3x + 4$
    (b) $y = 1/3(x - 7) - 17$
17. (a) $y = -7(x + 1) + 8$
    (b) $y = 1/7(x + 1) + 8$
19. (a) $y = -1(x - 3) + 1$
    (b) $y = 1(x - 3) + 1$
21. $y = -0.099(x - 9) + 1$
23. $y = -0.05x + 1$
25. (a) Approximations will vary; they should match (c) closely.
    (b) $f'(x) = -1/(x + 1)^2$
    (c) At $(0, 1)$, slope is -1. At $(1, 0.5)$, slope is $-1/4$.
27. 

29. 

31. Approximately 24.
33. (a) $(-\infty, \infty)$
    (b) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
    (c) $(-\infty, 5]$
    (d) $[-5, 5]$

Section 11.2

1. Velocity
3. Linear functions.
5. -17
7. $f(10.1)$ is likely most accurate, as accuracy is lost the farther from $x = 10$ we go.
9. 6
11. ft/s^2
13. (a) thousands of dollars per car
    (b) It is likely that $P(0) < 0$. That is, negative profit for not producing any cars.
15. $f(x) = g'(x)$
17. Either $g(x) = f'(x)$ or $f(x) = g'(x)$ is acceptable. The actual answer is $g(x) = f'(x)$, but is very hard to show that $f(x) \neq g'(x)$ given the level of detail given in the graph.
19. $f'(x) = 10x$
21. $f'(\pi) \approx 0$.

Section 11.3

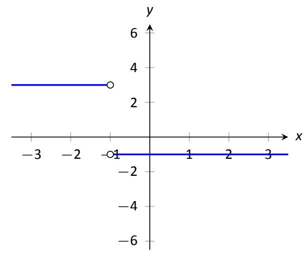
1. Power Rule.
3. One answer is $f(x) = 10e^x$.
5. $g(x)$ and $h(x)$
7. One possible answer is $f(x) = 17x - 205$.
9. $f'(x)$ is a velocity function, and $f''(x)$ is acceleration.
11. $f'(x) = 14x - 5$
13. $m'(t) = 45t^4 - \frac{3}{8}t^2 + 3$
15. $f'(r) = 6e^r$
17. $f'(x) = \frac{2}{x} - 1$
19. $h'(t) = e^t - \cos t + \sin t$
21. $f'(t) = 0$
23. $g'(x) = 24x^2 - 120x + 150$
25. $f'(x) = 18x - 12$
27. $f'(x) = 6x^5$ $f''(x) = 30x^4$ $f'''(x) = 120x^3$ $f^{(4)}(x) = 360x^2$
29. $h'(t) = 2t - e^t$ $h''(t) = 2 - e^t$ $h'''(t) = -e^t$ $h^{(4)}(t) = -e^t$
31. $f'(\theta) = \cos\theta + \sin\theta$ $f''(\theta) = -\sin\theta + \cos\theta$
    $f'''(\theta) = -\cos\theta - \sin\theta$ $f^{(4)}(\theta) = \sin\theta - \cos\theta$

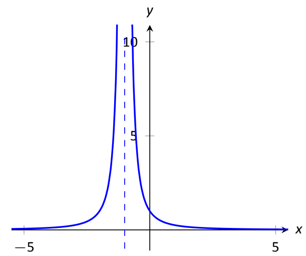
33. Tangent line: $y = 2(x-1)$
Normal line: $y = -1/2(x-1)$

35. Tangent line: $y = x - 1$
Normal line: $y = -x + 1$

37. Tangent line: $y = \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \sqrt{2}$
Normal line: $y = \frac{-2}{\sqrt{2}}(x - \frac{\pi}{4}) - \sqrt{2}$

39. The tangent line to $f(x) = e^x$ at $x = 0$ is $y = x + 1$; thus $e^{0.1} \approx y(0.1) = 1.1$.

43. 

45. 

Section 11.4

1. F
3. T
5. F
7. (a) $f'(x) = (x^2 + 3x) + x(2x + 3)$
   (b) $f'(x) = 3x^2 + 6x$
   (c) They are equal.
9. (a) $h'(s) = 2(s + 4) + (2s - 1)(1)$
   (b) $h'(s) = 4s + 7$
   (c) They are equal.
11. (a) $f'(x) = \frac{x(2x) - (x^2+3)1}{x^2}$
   (b) $f'(x) = 1 - \frac{3}{x^2}$
   (c) They are equal.
13. (a) $h'(s) = \frac{4s^3(0) - 3(12s^2)}{16s^6}$
   (b) $h'(s) = -9/4s^{-4}$
   (c) They are equal.
15. $f'(x) = \sin x + x \cos x$
17. $g'(x) = \frac{-12}{(x-5)^2}$
19. $h'(x) = -\csc^2 x - e^x$
21. (a) $f'(x) = \frac{(x+2)(4x^3+6x^2) - (x^4+2x^3)(1)}{(x+2)^2}$
   (b) $f(x) = x^3$ when $x \neq -2$, so $f'(x) = 3x^2$.
   (c) They are equal.
23. $f'(t) = 5t^4(\sec t + e^t) + t^5(\sec t \tan t + e^t)$
25. $g'(x) = 0$
27. $f'(x) = \frac{(t^2 \cos t + 2)(2t \sin t + t^2 \cos t) - (t^2 \sin t + 3)(2t \cos t - t^2 \sin t)}{(t^2 \cos t + 2)^2}$
29. $g'(x) = 2 \sin x \sec x + 2x \cos x \sec x + 2x \sin x \sec x \tan x = 2 \tan x + 2x + 2x \tan^2 x = 2 \tan x + 2x \sec^2 x$
31. Tangent line: $y = -(x - \frac{3\pi}{2}) - \frac{3\pi}{2} = -x$
   Normal line: $y = (x - \frac{3\pi}{2}) - \frac{3\pi}{2} = x - 3\pi$
33. Tangent line: $y = -9x - 5$
   Normal line: $y = 1/9x - 5$
35. $x = 0$
37. $x = -2, 0$
39. $f^{(4)}(x) = -4 \cos x + x \sin x$
41. $f^{(8)} = 0$

Section 11.5

1. T
3. F
5. T
7. $f'(x) = 10(4x^3 - x)^9 \cdot (12x^2 - 1) = (120x^2 - 10)(4x^3 - x)^9$
9. $g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta)$
11. $f'(x) = 4\left(x + \frac{1}{x}\right)^3\left(1 - \frac{1}{x^2}\right)$
13. $g'(x) = 5\sec^2(5x)$
15. $p'(t) = -3\cos^2(t^2 + 3t + 1)\sin(t^2 + 3t + 1)(2t + 3)$
17. $f'(x) = 2/x$
19. $g'(r) = \ln 4 \cdot 4^r$
21. $g'(t) = 0$
23. $f'(x) = \frac{(3^t+2)\left((\ln 2)2^t\right) - (2^t+3)\left((\ln 3)3^t\right)}{(3^t+2)^2}$
25. $f'(x) = \frac{2^{x^2}(\ln 3 \cdot 3^x x^2 2x + 1) - (3^{x^2} + x)(\ln 2 \cdot 2^{x^2} 2x)}{2^{2x^2}}$
27. $g'(t) = 5\cos(t^2+3t)\cos(5t-7) - (2t+3)\sin(t^2+3t)\sin(5t-7)$
29. Tangent line: $y = 0$
   Normal line: $x = 0$
31. Tangent line: $y = -3(\theta - \pi/2) + 1$
   Normal line: $y = 1/3(\theta - \pi/2) + 1$
33. In both cases the derivative is the same: $1/x$.
35. (a) $^\circ$ F/mph
   (b) The sign would be negative; when the wind is blowing at 10 mph, any increase in wind speed will make it feel colder, i.e., a lower number on the Fahrenheit scale.

Chapter 12

Section 12.1

1. Answers will vary.
3. Answers will vary.
5. F
7. A: abs. min B: none C: abs. max D: none E: none
9. $f'(0) = 0\, f'(2) = 0$

11. $f'(0) = 0 \, f'(3.2) = 0 \, f'(4)$ is undefined

13. $f'(0)$ is not defined

15. min: $(-0.5, 3.75)$
    max: $(2, 10)$

17. min: $(\pi/4, 3\sqrt{2}/2)$
    max: $(\pi/2, 3)$

19. min: $(\sqrt{3}, 2\sqrt{3})$
    max: $(5, 28/5)$

21. min: $(\pi, -e^{\pi})$
    max: $(\pi/4, \frac{\sqrt{2}e^{\pi/4}}{2})$

23. min: $(1, 0)$
    max: $(e, 1/e)$

25. $\frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$

27. $3x^2 + 1$

Section 12.2

1. Answers will vary.
3. Answers will vary.
5. Increasing
7. Graph and verify.
9. Graph and verify.
11. Graph and verify.
13. Graph and verify.
15. domain=$(-\infty, \infty)$
    c.p. at $c = -2, 0$;
    increasing on $(-\infty, -2) \cup (0, \infty)$;
    decreasing on $(-2, 0)$;
    rel. min at $x = 0$;
    rel. max at $x = -2$.
17. domain=$(-\infty, \infty)$
    c.p. at $c = 1$;
    increasing on $(-\infty, \infty)$;
19. domain=$(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
    c.p. at $c = 0$;
    decreasing on $(-\infty, -1) \cup (-1, 0)$;
    increasing on $(0, 1) \cup (1, \infty)$;
    rel. min at $x = 0$;
21. domain=$(-\infty, 0) \cup (0, \infty)$;
    c.p. at $c = 2, 6$;
    decreasing on $(-\infty, 0) \cup (0, 2) \cup (6, \infty)$;
    increasing on $(2, 6)$;
    rel. min at $x = 2$; rel. max at $x = 6$.
23. domain = $(-\infty, \infty)$;
    c.p. at $c = -1, 1$;
    decreasing on $(-1, 1)$;
    increasing on $(-\infty, -1) \cup (1, \infty)$;
    rel. min at $x = 1$;
    rel. max at $x = -1$
25. $c = \pm \cos^{-1}(2/\pi)$

Section 12.3

1. Answers will vary.
3. Yes; Answers will vary.
5. Graph and verify.
7. Graph and verify.
9. Graph and verify.
11. Graph and verify.
13. Graph and verify.
15. Graph and verify.
17. Possible points of inflection: none; concave down on $(-\infty, \infty)$
19. Possible points of inflection: $x = 1/2$; concave down on $(-\infty, 1/2)$; concave up on $(1/2, \infty)$
21. Possible points of inflection: $x = (1/3)(2 \pm \sqrt{7})$; concave up on $((1/3)(2 - \sqrt{7}), (1/3)(2 + \sqrt{7}))$; concave down on $(-\infty, (1/3)(2 - \sqrt{7})) \cup ((1/3)(2 + \sqrt{7}), \infty)$
23. Possible points of inflection: $x = \pm 1/\sqrt{3}$; concave down on $(-1/\sqrt{3}, 1/\sqrt{3})$; concave up on $(-\infty, -1/\sqrt{3}) \cup (1/\sqrt{3}, \infty)$
25. Possible points of inflection: $x = -\pi/4, 3\pi/4$; concave down on $(-\pi/4, 3\pi/4)$ concave up on $(-\pi, -\pi/4) \cup (3\pi/4, \pi)$
27. Possible points of inflection: $x = 1/e^{3/2}$; concave down on $(0, 1/e^{3/2})$ concave up on $(1/e^{3/2}, \infty)$
29. min: $x = 1$
31. max: $x = -1/\sqrt{3}$ min: $x = 1/\sqrt{3}$
33. min: $x = 1$
35. min: $x = 1$
37. critical values: $x = -1, 1$; no max/min
39. max: $x = -2$; min: $x = 0$
41. max: $x = 0$
43. f' has no maximal or minimal value
45. f' has a minimal value at $x = 1/2$
47. f' has a relative max at: $x = (1/3)(2 + \sqrt{7})$ relative min at: $x = (1/3)(2 - \sqrt{7})$
49. f' has a relative max at $x = -1/\sqrt{3}$; relative min at $x = 1/\sqrt{3}$
51. f' has a relative min at $x = 3\pi/4$; relative max at $x = -\pi/4$
53. f' has a relative min at $x = 1/\sqrt{e^3} = e^{-3/2}$

Section 12.4

1. Answers will vary.
3. T
5. T
7. A good sketch will include the x and y intercepts..
9. Use technology to verify sketch.
11. Use technology to verify sketch.
13. Use technology to verify sketch.
15. Use technology to verify sketch.
17. Use technology to verify sketch.
19. Use technology to verify sketch.
21. Use technology to verify sketch.
23. Use technology to verify sketch.
25. Use technology to verify sketch.
27. Critical points: $x = \frac{n\pi/2 - b}{a}$, where n is an odd integer Points of inflection: $(n\pi - b)/a$, where n is an integer.

29. $\frac{dy}{dx} = -x/y$, so the function is increasing in second and fourth quadrants, decreasing in the first and third quadrants.
$\frac{d^2y}{dx^2} = -1/y - x^2/y^3$, which is positive when $y < 0$ and is negative when $y > 0$. Hence the function is concave down in the first and second quadrants and concave up in the third and fourth quadrants.

Section 12.5

1. Answers will vary.
3. Answers will vary.
5. Answers will vary.
7. velocity
9. $1/9x^9 + C$
11. $t + C$
13. $-1/(3t) + C$
15. $2\sqrt{x} + C$
17. $-\cos\theta + C$
19. $5e^\theta + C$
21. $\frac{5^t}{2\ln 5} + C$
23. $t^6/6 + t^4/4 - 3t^2 + C$
25. $e^\pi x + C$
27. (a) $x > 0$
    (b) $1/x$
    (c) $x < 0$
    (d) $1/x$
    (e) $\ln|x| + C$. Explanations will vary.
29. $5e^x + 5$
31. $\tan x + 4$
33. $5/2x^2 + 7x + 3$
35. $5e^x - 2x$
37. $\frac{2x^4\ln^2(2)+2^x+x\ln 2)(\ln 32-1)+\ln^2(2)\cos(x)-1-\ln^2(2)}{\ln^2(2)}$
39. No answer provided.

Index

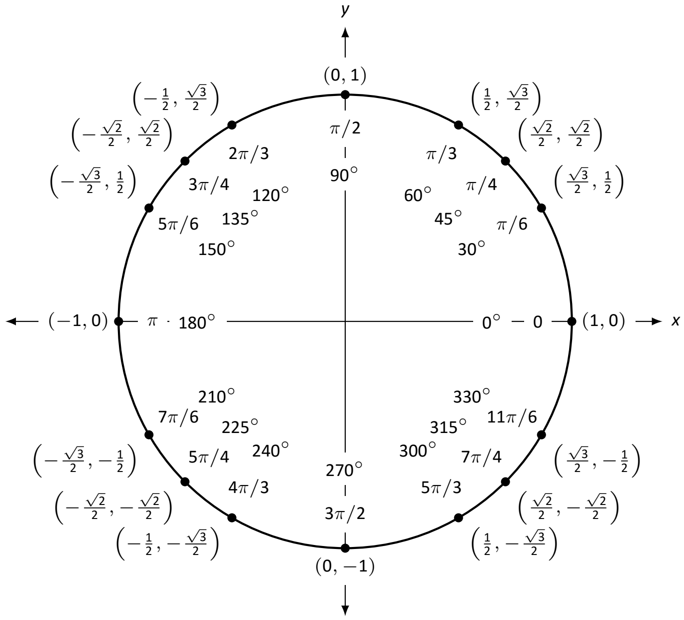
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

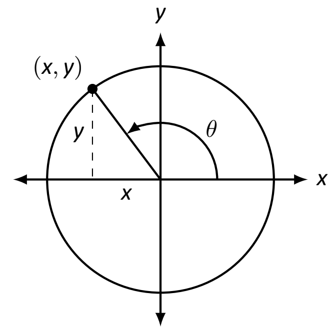
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
   $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln |x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

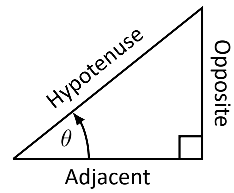


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$

Right Triangle Definition



$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$\sin 2x = 2\sin x\cos x$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Sum to Product Formulas

$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Power–Reducing Formulas

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$

$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$

$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$

Angle Sum/Difference Formulas

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

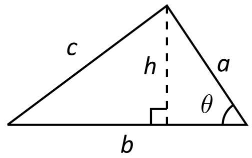
Areas and Volumes

Triangles

$h = a \sin\theta$

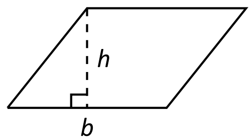
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



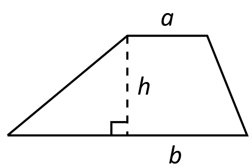
Parallelograms

Area = bh



Trapezoids

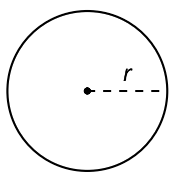
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

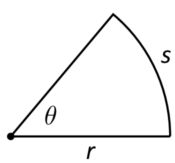


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

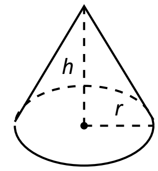
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

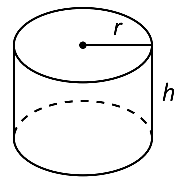
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

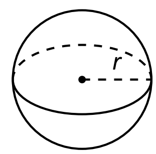
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

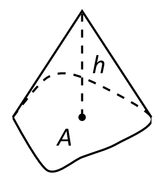
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

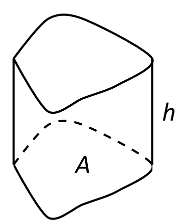
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x-a)(x+a)$ $\quad$ $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$ $\quad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$ $\quad$ $(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\quad$ $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\quad$ $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b+c)$ $\quad$ $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ $\quad$ $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

$\dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\dfrac{a}{b}\right)\left(\dfrac{d}{c}\right) = \dfrac{ad}{bc}$ $\quad$ $\dfrac{\left(\frac{a}{b}\right)}{c} = \dfrac{a}{bc}$ $\quad$ $\dfrac{a}{\left(\frac{b}{c}\right)} = \dfrac{ac}{b}$

$a\left(\dfrac{b}{c}\right) = \dfrac{ab}{c}$ $\quad$ $\dfrac{a-b}{c-d} = \dfrac{b-a}{d-c}$ $\quad$ $\dfrac{ab + ac}{a} = b + c$

Exponents and Radicals

$a^0 = 1, \ a \neq 0$ $\quad$ $(ab)^x = a^x b^x$ $\quad$ $a^x a^y = a^{x+y}$ $\quad$ $\sqrt{a} = a^{1/2}$ $\quad$ $\dfrac{a^x}{a^y} = a^{x-y}$ $\quad$ $\sqrt[n]{a} = a^{1/n}$

$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $\quad$ $\sqrt[n]{a^m} = a^{m/n}$ $\quad$ $a^{-x} = \dfrac{1}{a^x}$ $\quad$ $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\quad$ $(a^x)^y = a^{xy}$ $\quad$ $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\,\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\,\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
*n*th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\text{Sum} = \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		$\text{Sum} = \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive. Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive. Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$