Almost CR quantization via the index of transversally elliptic Dirac operators

by

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Abstract

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Let M be a smooth compact manifold equipped with a co-oriented subbundle $E \subset TM$. We suppose that a compact Lie group G acts on M preserving E, such that the G-orbits are transverse to E. If the fibres of E are equipped with a complex structure then it is possible to construct a G-invariant Dirac operator $\not{D}_b$ in terms of the resulting almost CR structure.

We show that there is a canonical equivariant differential form with generalized coefficients $\mathcal{J}(E, X)$ defined on M that depends only on the G-action and the co-oriented subbundle E. Moreover, the group action is such that $\not{D}_b$ is a G-transversally elliptic operator in the sense of Atiyah [Ati74]. Its index is thus defined as a generalized function on G. Beginning with the equivariant index formula of Paradan and Vergne [PV08b], we obtain an index formula for $\not{D}_b$ computed as an integral over M that is free of choices and growth conditions. This formula necessarily involves equivariant differential forms with generalized coefficients and we show that the only such form required is the canonical form $\mathcal{J}(E, X)$.

In certain cases the index of $\not{D}_b$ can be interpreted in terms of a CR analogue of the space of holomorphic sections, allowing us to view our index formula as a character formula for the G-equivariant quantization of the almost CR manifold (M, E). In particular, we obtain the "almost CR" quantization of a contact manifold, in a manner directly analogous to the almost complex quantization of a symplectic manifold.

Dedication

To Jana.

To my parents, for supporting me in spite of not knowing exactly what it is I'm doing.

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Some of the results in this thesis have been accepted for publication in two journal articles. I have submitted the following articles:

- *An equivariant index formula in contact geometry*, submitted to Mathematics Research Letters, forthcoming.

- *An equivariant index formula for almost CR manifolds*, International Mathematics Research Notices, DOI: 10.1093/imrn/rnp057.

The first of these articles presents some of the results of this thesis as they apply to contact manifolds. (We note that many of the original proofs are now somewhat obsolete). The second article contains the statement and proof of the general index formula presented here in Chapter 8, as well as the construction of the form $\mathcal{J}(E, X)$ given in Chapter 7.

Contents

Chapter 1

Introduction

Transversally elliptic operators were introduced by Atiyah in [Ati74][1]. In the commentary for Volume 3 of his collected works [Ati88], Atiyah remarked that not much had come of this work, and speculated that the likely cause was a lack of interesting examples of transversally elliptic operators.

One goal of this thesis is to show that these remarks by Atiyah were perhaps too pessimistic, by constructing geometric examples of transversally elliptic operators. These will be geometric Dirac operators (in the sense of Nicolaescu [Nic05]) on a manifold M associated to well-studied geometries such as contact and CR structures. Given the prevalence of such structures in the mathematics literature, the hope is that we have succeeded in producing transversally elliptic operators that one might in fact encounter in the wild.

Since transversally elliptic operators are not in general Fredholm, their equivariant index must be defined with care. Atiyah showed that while the resulting G-representations given by the kernel and cokernel of such operators are in general infinite-dimensional, the character of the virtual representation given by the equivariant index is defined as a generalized ($C^{-\infty}$) function on G. Our starting point will be the formula for the equiv-

[1]While Atiyah is the sole author of the book, he notes in the introduction that the material in [Ati74] is the result of joint work with I. M. Singer.

ariant index of transversally elliptic operators given recently by Paradan and Vergne [PV08b]. (One could also start from the Berline-Vergne index formula [BV96a, BV96b]; either approach has certain advantages and disadvantages that we will discuss.) The general index formula is given as an integral over T^*M. Because we choose specific Dirac operators, we are able to explicitly carry out the integral over the fibres of T^*M to obtain an index formula as an integral over M of certain characteristic forms determined by the geometric structure used to define the Dirac operator.

Due to the fact that the equivariant index of a transversally elliptic operator defines a generalized function on G, we cannot expect all of the forms that appear in our index formula to be smooth. We show that the non-smooth part of the index formula is given by a completely canonical equivariant differential form with generalized coefficients denoted by $\mathcal{J}(E,X)$, whose coefficients are given in terms of the Dirac delta distribution and its derivatives.

Summary of results

Let M be compact manifold equipped with an action $\Phi : G \to \mathrm{Diff}(M)$ of a compact Lie group G, and suppose we are given a smooth, G-invariant distribution $E \subset TM$ whose annihilator $E^0 \subset T^*M$ satisfies the following conditions:

(i) E^0 is oriented

(ii) $E^0 \cap T^*_G M = 0$,

where $T^*_G M \subset T^*M$ denotes the space of covectors that annihilate vectors tangent to the G-orbits (4.2). Property (i) is the statement that E is co-oriented. When the pair (E,Φ) satisfies property (ii), we say that the action of G is *transverse* to E.

Given a pair (E,Φ) satisfying properties (i) and (ii), it is possible to define a natural equivariant differential form with generalized coefficients $\mathcal{J}(E,X)$ that depends only on the distribution E and the action Φ, as follows:

Denote by θ the canonical 1-form on T^*M, and let $\imath : E^0 \hookrightarrow T^*M$ and $q : E^0 \to M$ denote inclusion and projection, respectively. We denote by $D\theta(X) = d\theta - \iota(X)\theta$ the equivariant differential of θ. We then define[2]

$$\mathcal{J}(E, X) = (2\pi i)^k q_* \imath^* e^{iD\theta(X)} \quad \text{for any } X \in \mathfrak{g}, \tag{1.1}$$

where $k = \operatorname{rank} E^0$ and q_* denotes integration over the fibres of E^0. The assumption that the action of G on M is transverse to E implies that this fibre integral is well-defined as an oscillatory integral in the sense of Hörmander [Hör83], and determines an equivariant differential form with generalized coefficients on M.

In the case of a contact distribution $E = \ker\alpha$, we obtain the explicit expression

$$\mathcal{J}(E, X) = \alpha \wedge \delta_0(D\alpha(X))$$

in terms of the Dirac delta on $\mathbb{R}$. If one carries out the fibre integral in (1.1) locally, in terms of some frame $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ for E^0, one obtains the similar expression

$$\mathcal{J}(E, X) = \alpha_k \wedge \cdots \wedge \alpha_1 \delta_0(D\boldsymbol{\alpha}(X)), \tag{1.2}$$

where δ_0 now denotes the Dirac delta function on $\mathbb{R}^k$. Using the properties of δ_0, one can show directly that the expression (1.2) is independent of the choice of frame $\boldsymbol{\alpha}$, and that $D\mathcal{J}(E, X) = 0$.

Let π, r and s be the projections given by the following diagram:

$$\begin{tikzcd} T^*M \arrow[r, "r"'] \arrow[d, "\pi"'] & E^* \arrow[dl, "s"] \\ M & \end{tikzcd}$$

Let $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^- \to M$ be a G-equivariant $\mathbb{Z}_2$-graded vector bundle, and suppose $P : \Gamma(M, \mathcal{V}^+) \to \Gamma(M, \mathcal{V}^-)$ is a differential operator with principal symbol

$$\sigma(P) : \pi^*\mathcal{V}^+ \to \pi^*\mathcal{V}^-$$

[2] We will later define $\mathcal{J}(E, X)$ in terms of Paradan's form $P_\theta(X)$, whose construction we will give later, and then show that the two definitions are equivalent.

on T^*M such that $\sigma(P) = r^*\sigma_E$, for some symbol $\sigma_E : s^*\mathcal{V}^+ \to s^*\mathcal{V}^-$ defined on E^*. If σ_E is invertible on $E^* \setminus 0$, then $\mathrm{Supp}(\sigma(P)) \subset E^0$, and the assumption that the action of G is transverse to E implies that $\mathrm{Supp}(\sigma(P)) \cap T^*_G M = 0$, whence P satisfies Atiyah's definition of a G-transversally elliptic operator.

Example. Suppose that M is a Sasakian manifold. Then M is is a Cauchy-Riemann (CR) manifold, and the real distribution E underlying the CR distribution $E_{1,0} \subset TM \otimes \mathbb{C}$ is a contact distribution. The operator $P = \overline{\partial}_b + \overline{\partial}_b^*$, where $\overline{\partial}_b$ is the tangential CR operator, is transversally elliptic provided that the action of G is transverse to E. Since the Reeb vector field of a Sasakian manifold is Killing [Bla76], we can take G to be the one-parameter group of isometries of (M, g) generated by the Reeb field.

Example. Given a principal H-bundle $\pi : M \to B$, we let $E \cong \pi^*TB$ be the horizontal distribution with respect to some choice of connection. Then, if P is an elliptic operator on T^*B, its pullback to T^*M will be transversally elliptic. This example, already present in [Ati74], was studied in detail by Berline and Vergne [BV96b]. Also in [Ati74] is the case of a locally free action, for which a general cohomological formula was given by Vergne in [Ver96]. The formula we give below can be thought of as a further extension of these results to a broader class of group actions, since not all group actions transverse to a distribution are locally free.

We begin with the Paradan-Vergne formula for the equivariant index of transversally elliptic operators and show that for the operators we consider, it is possible to integrate over the fibres of T^*M to obtain a formula given as an integral over M. Specializing to the case of an G-invariant almost CR structure we can simplify further by explicitly constructing a transversally elliptic Dirac operator $\not{D}_b$ as follows:

Given an almost CR structure $E_{1,0} \subset T_{\mathbb{C}}M$, there exists a metric g on M whose restriction to the Levi distribution E is compatible with the complex structure on the fibres of E. We then form the bundle of Clifford algebras $\mathbb{C}l(E)$ whose fibre over $x \in M$ is the complex Clifford algebra of E_x^* with respect to the quadratic form defined by g. The exterior algebra bundle $\mathcal{S} = \bigwedge E^{0,1}$ becomes a spinor module for $\mathbb{C}l(E)$. A compatible connection ∇ on $E_{1,0}$ induces a connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$, and we can define the transversally elliptic operator $\not{D}_b : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ given by

$$\not{D}_b = \mathbf{c} \circ r \circ \nabla^{\mathcal{S}},$$

where $r : T^*M \to E^*$ is orthogonal projection onto E, and $\mathbf{c} : \mathbb{C}l(E) \to \operatorname{End}(\mathcal{S})$ is the Clifford action. Given a G-equivariant Hermitian vector bundle $\mathcal{W} \to M$ with G-invariant connection $\nabla^{\mathcal{W}}$, we can form the twisted Dirac operator $\not{D}_b^{\mathcal{W}} : \Gamma(M, \mathcal{S}^+ \otimes \mathcal{W}) \to \Gamma(M, \mathcal{S}^- \otimes \mathcal{W})$ using the tensor product connection and Clifford action $\mathbf{c} \otimes \mathrm{Id}_{\mathcal{W}}$. The principal symbol $\sigma(\not{D}_b^{\mathcal{W}}) : \pi^*(\mathcal{S}^+ \otimes \mathcal{W}) \to \pi^*(\mathcal{S}^- \otimes \mathcal{W})$ then depends only on E^*, and has support E^0. The integral over the fibres of T^*M of the corresponding Chern character can be computed explicitly in terms of equivariant characteristic classes on M, giving:

Theorem. *Let $\mathcal{W} \to M$ be an G-equivariant Hermitian vector bundle on M with Hermitian connection $\nabla^{\mathcal{W}}$, and let $\not{D}_b{}^{\mathcal{W}}$ denote the corresponding twisted Dirac operator. Then the G-equivariant index of $\not{D}_b{}^{\mathcal{W}}$ is the generalized function on G whose germ at $1 \in G$ is given, for $X \in \mathfrak{g}$ sufficiently small, by*

$$\operatorname{index}^G(\not{D}_b^{\mathcal{W}})(e^X) = \int_M (2\pi i)^{-\operatorname{rank} E/2} \operatorname{Td}(E, X) \widehat{\mathrm{A}}^2(E^0, X) \mathcal{J}(E, X) \operatorname{Ch}(\mathcal{W}, X), \tag{1.3}$$

with similar formulas near other elements $g \in G$.

Applications and further work

The original motivation for this work was the desire to construct a contact analogue of the geometric quantization of a symplectic manifold. We note that if a symplectic manifold (N, ω) is prequantizable, then its prequantum circle bundle $M \to N$ is a contact manifold, and a contact form is given (up to a factor of $\sqrt{-1}$) by a choice of connection on M.

When N is Kähler, the Hamiltonian action of a Lie group G on N induces a virtual representation on the "quantized" space $Q(N) = \sum(-1)^i H^i(N, \mathcal{O}(\mathbb{L}))$, where $\mathcal{O}(\mathbb{L})$ is the sheaf of holomorphic sections of the associated prequantum line bundle $\mathbb{L} \to N$.

In the Kähler case the spaces $H^i(N, \mathcal{O}(\mathbb{L}))$ are isomorphic to the cohomology spaces $H^{0,k}(N, \mathbb{L})$ of the Dolbeault complex, twisted by the line bundle $\mathbb{L}$, and the latter spaces make sense for an almost complex structure that is not integrable. By choosing a compatible almost complex structure, one can perform the "almost complex" quantization of (N, ω), given by $Q(N) = \sum(-1)^i H^{0,i}(M, \mathbb{L})$, and this, in turn, is equal to the index of the "Dolbeault Dirac" operator $\overline{\partial} + \overline{\partial}^*$ [GGK02].

The Dirac operator $\not{D}_b$ on a contact manifold (M, E) is (in a manner that we will explain) the contact analogue of the Dolbeault Dirac operator, and so we view the virtual G-representation whose character is given by $\mathrm{index}^G(\not{D}_b)$ as a quantization of M. We call this the "almost CR" quantization of M, since the construction of our Dirac operator makes use of a "compatible" almost CR structure. That is, we define our almost CR structure using a complex structure on the fibres of E that is compatible with the symplectic structure on the fibres of E. Indeed, the almost CR structure (and a compatible metric) are all we really need to define such a quantization.

In general we do not consider this quantization to be associated to an analogue of the prequantum line bundle. There are, however, certain cases where we can attach additional geometric meaning to the almost CR quantization of an almost CR manifold M. The first is the case of a contact manifold whose compatible almost CR structure is integrable. (The choice of a compatible metric makes M a Sasakian manifold.) In this case we can choose ∇ such that $\not{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*)$, and interpret the index of $\not{D}_b$ in terms of the Kohn-Rossi cohomology groups [KR65] of M.

Another application is to induced representations. Suppose we are given a complex homogeneous space $M = G/H$, and a unitary H-representation $\tau : H \to \mathrm{End}(V_\tau)$. The associated vector bundle $\mathcal{V}_\tau = G \times_H V_\tau$ is holomorphic, and one may define two different induced representations of G: the Frobenius induced representation $\mathrm{ind}_H^G(\tau)$ on the space of L^2-sections of $\mathcal{V}_\tau$, and the holomorphic induced representation hol-ind_H^G on the space of holomorphic sections of $\mathcal{V}_\tau$.

A formula for the character of $\mathrm{ind}_H^G(\tau)$ was given by Berline and Vergne [BV92] as the G-equivariant index of the zero symbol, while the character of hol-$\mathrm{ind}_H^G(\tau)$ is the G-equivariant index of the elliptic Dolbeault-Dirac operator (given by the equivariant Riemann-Roch formula).

We can, by varying the rank of E, view both of these formulas as special cases of our formula (1.3), for $\mathcal{V} = \mathcal{V}_\tau$: When $E = 0$, we have $P = 0$, and obtain the Berline-Vergne formula for the character of $\mathrm{ind}_H^G(\tau)$. When $E = TM$, we can take P to be the Dolbeault-Dirac operator, and $\mathrm{index}^G(P)$ gives the character of hol-$\mathrm{ind}_H^G(\tau)$. Thus we see explicitly the nature of the index theorem for transversally elliptic operators as a marriage between two extreme cases.

One question that we are currently investigating is whether we can give geometric constructions of "prequantization spaces" with respect to which we may view $\mathrm{index}^G(\not{D}_b)$ as the character of a corresponding quantization. Instead of a holomorphic line bundle, the appropriate object would seem to be a *CR holomorphic* vector bundle equipped with a Hermitian connection. In most cases any natural 2-form associated to the geometric structure is exact, so it may be that the most natural object is a trivial line bundle. In the case of a trivial bundle over a Sasakian manifold, we obtain a relationship between the index of our Dirac operator and the Kohn-Rossi cohomology. We are also interested in considering generalizations to higher corank using so-called S-structures (see [LP04].)

Another obvious question is whether or not a "quantization commutes with reduction" result holds in contact geometry. There have been a few results in recent years concerning contact reduction [Wil02, ZZ05], and we expect that our quantization procedure is compatible with contact reduction in a manner similar to the symplectic case.

Basic definitions and notation

For the sake of convenience, we will first list some of the notation we will use throughout this thesis. Let M be a smooth manifold. For any fibre bundle $\mathcal{V} \to M$, we will denote by $\Gamma(\mathcal{V})$ the sections of this bundle. We will use the notation $\mathcal{A}(M)$ for the complex of differential forms on M, $\mathcal{A}^k(M)$ if we wish to denote the forms of degree k. (So that, for example, $\mathcal{A}^1(M) = \Gamma(T^*M)$.) If $\mathcal{W}$ is a vector bundle (or vector space), we will denote by $\mathcal{A}(M, \mathcal{W})$ the space of differential forms on M that take values in $\mathcal{W}$. We will generally denote manifolds by M, N, Lie groups and their elements by $g \in G$, $h \in H$, and the corresponding Lie algebras and their elements by $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$. We will at times also use X, Y to denote vector fields but when this is the case it should be clear by the context. To avoid confusion with Lie group elements, Riemannian metrics will be denoted by the sans serif letters g, h.

Chapter 2

The index of transversally elliptic operators

The equivariant index theorem for transversally elliptic operators is a synthesis of two extreme cases: the equivariant index theorem for elliptic operators [AS68a, AS68b, AS68c], and the Frobenius reciprocity theorem for induced representations [Kna02, BV92]. In the former case, as noted by Atiyah in the introduction to [Ati74], the role of the Lie group G is essentially algebraic, while the latter is an exercise in Fourier analysis on Lie groups. (In the case of induced representations, our results will lead to a way of seeing explicitly the spectrum that exists between these two extremes; see Section 9.5.)

Let M be a smooth compact manifold. Let $\mathcal{V}$, $\mathcal{W}$ be G-equivariant vector bundles over M, and let

$$P : \Gamma_c^\infty(M, \mathcal{V}) \to \Gamma^\infty(M, \mathcal{W})$$

be a differential operator of order m [1]. Let $\pi : T^*M \to M$ denote projection. We define the principal symbol

$$\sigma(P) : \pi^*\mathcal{V} \to \pi^*\mathcal{W}$$

[1] In general, P may be a pseudodifferential operator. However, we will only be interested in differential Dirac operators, so we will omit the details regarding pseudodifferential operators, which can be found in [AS68b, Hör83].

of P pointwise on $T^*M \setminus 0$ by

$$\sigma(P)(x,\xi) = \lim_{t\to\infty} t^{-m}(e^{-itf}Pe^{itf})(x), \tag{2.1}$$

where $f \in C^\infty(M)$ is any smooth function such that $df(x) = \xi$ [Ati74, BGV91]. Recall that P is *elliptic* if $\sigma(P)$ is invertible on $T^*M \setminus 0$. When this is the case, all solutions of P are C^∞, the spaces $\ker P$ and $\operatorname{coker} P$ are finite-dimensional, and we can define

$$\operatorname{index}(P) = \dim\ker P - \dim\operatorname{coker} P. \tag{2.2}$$

A cohomological formula for $\operatorname{index}(P)$ was given in this case by the Atiyah-Singer index theorem [AS68b, AS68c]; this result is sufficiently famous that we need not repeat it here. Besides the papers of Atiyah and Singer, useful references on the subject are (among many others) [BGV91] and [LM89].

Suppose now that a compact Lie group G acts smoothly on M, and that P is a G-invariant differential operator. In this case, the spaces $\ker P$ and $\operatorname{coker} P$ become G-representations, and the equivariant index of P is the virtual character defined by

$$\operatorname{index}^G(P)(g) = \operatorname{Tr}(g|_{\ker P}) - \operatorname{Tr}(g|_{\operatorname{coker} P}). \tag{2.3}$$

In this case we again have a cohomological formula for the index, based on the Lefschetz fixed point formula, and due to Atiyah, Segal and Singer [AS68a, AS68b, AS68c], that we again do not repeat here.

We now come to the case of transversally elliptic operators. We first need a preliminary definition. We denote by $\mathfrak{g}$ the Lie algebra of G, and by $X_M \in \Gamma(TM)$ the vector field generated by the infinitesimal action of $X \in \mathfrak{g}$.

Definition 2.1. *We denote by T^*_GM the space of covectors that vanish on vectors tangent to the G-orbits, given pointwise by*

$$(T^*_GM)_x = \{\xi \in T^*_xM | \xi(X_M) = 0 \text{ for all } X \in \mathfrak{g}\}. \tag{2.4}$$

We note that T_G^*M is not a subbundle of T^*M in general, since it need not have constant rank. We can now give the definition of a transversally elliptic operator from [Ati74].

Definition 2.2. *A G-invariant differential operator*

$$P : \Gamma_c^\infty(M, \mathcal{V}) \to \Gamma^\infty(M, \mathcal{W})$$

*is said to be G-*transversally elliptic *if its principal symbol $\sigma(P) : \pi^*\mathcal{V} \to \pi^*\mathcal{W}$ is invertible on $T_G^*M \setminus 0$.*

Remark 2.3. For any morphism $\sigma : \pi^*\mathcal{V} \to \pi^*\mathcal{W}$, we define the *support* of σ by

$$\mathrm{Supp}(\sigma) = \{(x, \xi) \in T^*M | \sigma(x, \xi) \text{ is not invertible}\}. \tag{2.5}$$

We then have that P is a G-transversally elliptic operator if and only if

$$T_G^*M \cap \mathrm{Supp}(\sigma(P)) = 0. \tag{2.6}$$

Given a transversally elliptic operator P, one still wishes to use (2.3) to define the equivariant index of P. However, such operators are in general not Fredholm; that is, the spaces $\ker P$ and $\operatorname{coker} P$ need not be finite-dimensional. By [Ati74, Theorem 2.2], we can still make sense of $\mathrm{index}^G(P)$ as a generalized function (that is, distribution) on G. In other words, the character of the G-representation $\mathrm{index}^G(P) : G \to \mathcal{L}(\ker P)$ is generally not defined pointwise on G, but for any smooth, compactly supported test function $\varphi \in C_c^\infty(G)$, the operator

$$< \mathrm{index}^G(P), \varphi >= \int_G \varphi(g)\, \mathrm{index}^G(P)(g)$$

is of trace class, and thus we can define $\mathrm{index}^G(P)$ as the distributional character of this representation.

We will not repeat here all of the results in [Ati74] regarding the properties of the index. Those that are most relevant to us are the basic property that $\mathrm{index}^G(P)$ depends only on the homotopy class of $\sigma(P)$ in the space of transversally elliptic symbols[2], and the free action property: if a compact Lie group H acts freely on M, commuting with the G-action, then

$$\mathrm{index}^{G\times H}(\pi^*\sigma(P)) = \sum_{\tau\in\hat{H}} \mathrm{Tr}(\tau)\cdot \mathrm{index}^G(\sigma(P)\otimes \mathrm{Id}_{V_\tau^*}),$$

where $\pi : M \to M/H$ denotes projection. The index also enjoys additive, multiplicative, and excision properties. We also have a localization result: the support of $\mathrm{index}^G(P)$ in G is the set of all $g\in G$ such that the fixed-point set $M(g)$ is non-empty. In particular, if G acts on M freely, then $\mathrm{index}^G(P)$ is supported at the identity in G, which suggests that the generalized function $\mathrm{index}^G(P)$ can be expressed in terms of Dirac delta functions. Indeed, in the case of S^1 acting on itself by multiplication, the zero operator is transversally elliptic, and its index is given by the Dirac delta δ_1 at $1\in S^1$. (This should be contrasted with the case of an elliptic operator, for which $\mathrm{index}^G(P) = 0$ when G acts freely.)

By using the properties mentioned above, the general problem can be reduced to the case of a torus acting on Euclidean space, where the index can be computed explicitly. For general compact Lie groups, a method is given in [Ati74] for reduction to its maximal torus, but explicit cohomological computations are carried out only for the torus case. It was suggested by Atiyah in the introduction that new techniques were required to handle the general case.

In [BV85], Berline and Vergne gave a reformulation of the equivariant index theorem for elliptic operators in terms of equivariant characteristic forms (see also [BGV91, Chapter 8]) using the Kirillov character formula (see [Kir04]). In this form, the equivariant index formula can be more easily generalized to the case of transversally elliptic operators. In [BV96a], Berline and Vergne showed that for a transversally elliptic symbol σ,

[2] In fact, the K-group $K_G(T^*_G M)$ consists entirely of homotopy classes of transversally elliptic symbols. Although most of the results in [Ati74] are given in terms of K-theory, we will restrict our attention to cohomological formulas.

there exists a generalized function on G whose germ at $g \in G$ is given by the expression

$$\chi(ge^X) = \int_{T^*M(g)} (2\pi i)^{-\dim M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)\,\mathrm{Ch}_g(\sigma, X) e^{iD\theta(X)}}{D_g(\mathcal{N}(g), X)}, \tag{2.7}$$

for $X \in \mathfrak{g}(g)$ sufficiently small, and in [BV96b], they showed that when $\sigma = \sigma(P)$, the generalized function thus defined is indeed the equivariant index of P: we have the equality of generalized functions $\chi(ge^X) = \mathrm{index}^G(P)(ge^X)$. We will define the terms appearing in this formula in Chapter 5; let us make a few comments on this formula. The integrand consists of smooth equivariant differential forms, but does not have compact support, and the integral converges only as a generalized function. The Chern character $\mathrm{Ch}_g(\sigma, X)$ is modeled on Quillen's Chern character [Qui85], and is rapidly decreasing in the directions along which σ is invertible. The term $D\theta(X)$ is the equivariant differential of the canonical 1-form on T^*M, and is used to ensure convergence in the remaining directions, once the above expression is integrated against a compactly supported test function on $\mathfrak{g}(g)$. Ultimately, one has to carefully prove estimates ensuring that the integral above converges as a generalized function, and the symbol σ is required to satisfy somewhat subtle growth conditions.

Recently, Paradan and Vergne produced a new version of the equivariant index formula that makes use of equivariant differential forms with generalized coefficients (see Chapter 6). They show that one can replace the term $e^{iD\theta(X)}$ by an equivariant differential form with generalized coefficients $P_\theta(X)$ constructed by Paradan [Par99, Par00] that is supported on $T^*_G M$, so that the directions in which σ is not invertible do not contribute to the integral. One then has the formula

$$\mathrm{index}^G(P)(ge^X) = \int_{T^*M(g)} (2\pi i)^{-\dim M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)\,\mathrm{Ch}_g(\sigma, X) P_\theta(X)}{D_g(\mathcal{N}(g), X)}, \tag{2.8}$$

for $X \in \mathfrak{g}(g)$ sufficiently small. The construction of the form $P_\theta(X)$ makes use of a cutoff function χ supported on a neighbourhood of $T^*_G M$. While the cohomology class of $P_\theta(X)$ is independent of the choice of cutoff function, such a choice must be made in order to compute the above integral. We will show in Chapter 8 that this choice does not affect the integral over the fibres of $T^*M(g)$, and carry out this fibre integral to obtain a formula as an integral over $M(g)$ that requires neither growth conditions nor cutoff functions.

Chapter 3

Geometric structures on manifolds

3.1 Symplectic and (almost) complex structures

We begin by quickly reviewing some basic facts about symplectic and complex structures on a manifold M. Recall that a 2-form $\omega \in \mathcal{A}^2(M)$ is *symplectic* if $d\omega = 0$, and the map $X \mapsto \iota(X)\omega$ determines an isomorphism $\Gamma(TM) \cong \mathcal{A}^1(M)$ (that is, ω is *non-degenerate*). It follows that M must have even dimension $2n$, and that ω^n defines a volume form on M.

The subject of symplectic geometry is sufficiently immense that we do not dare attempt to give a comprehensive list of references on the topic, and instead mention only the book by Ana Cannas da Silva [CdS01] from which the author first learned the basics of symplectic, complex and contact geometry.

We further recall that a section $J \in \mathrm{End}(TM)$ is an *almost complex structure* if $J^2 = -\mathrm{Id}$. An almost complex structure determines a splitting

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

of the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ into the $\pm i$-eigenbundles of J. An almost complex structure is *integrable* if $[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M)$; in this case we say that J is a *complex structure* and call M a complex manifold. As is well-known, the integrability of J is equivalent to the vanishing of the *Nijenhuis tensor* $\mathrm{Nij}(J) \in \mathcal{A}^2(M, TM)$ given by

$$\mathrm{Nij}(J)(X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY].$$

By the Newlander-Nirenberg theorem [NN57], the integrability of J implies the existence of an atlas of complex coordinate charts whose transition functions are holomorphic.

A Riemannian metric g is said to be *compatible* with an almost complex structure J if

$$\mathsf{g}(JX,JY) = \mathsf{g}(X,Y) \quad \text{for all} \quad X,Y \in \Gamma(TM).$$

A compatible metric determines a Hermitian metric h on $T_{\mathbb{C}}M$ whose real part is g. The imaginary part of h is a non-degenerate 2-form ω satisfying

$$\omega(X,Y) = \mathsf{g}(JX,Y) \quad \text{for all} \quad X,Y \in \Gamma(TM).$$

If M is equipped with an almost complex structure J and compatible metric g, we call the triple (M, J, g) an *almost Hermitian* manifold; if $\mathrm{Nij}(J) = 0$, M is then called *Hermitian*. In the case that $d\omega = 0$ (that is, if ω is symplectic), we call $(M, J, \omega, \mathsf{g})$ an *almost Kähler* manifold, and finally, a *Kähler* manifold is a manifold M equipped with a "compatible triple" (J, ω, g) such that J is integrable, ω is symplectic, and

$$\mathsf{g}(X,Y) = \omega(X,JY) \tag{3.1}$$

is a Riemannian metric. It is easy to check that the Riemannian metric defined in this way satisfies $\mathsf{g}(JX,JY) = \mathsf{g}(X,Y)$. To summarize the above discussion, we reproduce here the following hierarchical diagram of these geometric structures, which is due to Blair [Bla76].

$$\begin{tikzcd}
\boxed{\begin{matrix}(M,J)\\ \text{Almost Complex}\end{matrix}} \arrow[r, "\mathsf{g}(JX,JY)=\mathsf{g}(X,Y)"] \arrow[d, "\mathrm{Nij}(J)=0"] & \boxed{\begin{matrix}(M,J,\mathsf{g})\\ \text{Almost Hermitian}\end{matrix}} \arrow[r, "d\omega=0"] \arrow[d, "\mathrm{Nij}(J)=0"] \arrow[dr, "\nabla J=0"] & \boxed{\begin{matrix}(M,J,\mathsf{g},\omega)\\ \text{Almost Kähler}\end{matrix}} \arrow[d, "\mathrm{Nij}(J)=0"] \\
\boxed{\begin{matrix}(M,J)\\ \text{Complex}\end{matrix}} \arrow[r, "\mathsf{g}(JX,JY)=\mathsf{g}(X,Y)"] & \boxed{\begin{matrix}(M,J,\mathsf{g})\\ \text{Hermitian}\end{matrix}} \arrow[r, "d\omega=0"] & \boxed{\begin{matrix}(M,J,\mathsf{g},\omega)\\ \text{Kähler}\end{matrix}}
\end{tikzcd}$$

Remark 3.1. Given a symplectic manifold (M,ω) it is always possible to find a compatible almost complex structure J, in the sense that (3.1) defines a Riemannian metric on M. (See [CdS01, Section 12.3].) Moreover, the space of compatible almost complex structures is contractible, a fact that is useful when one wishes to define an invariant of (M,ω) with the aid of an almost complex structure.

Similarly, one can define a *symplectic vector bundle* $E \to M$ to be a vector bundle equipped with a smoothly-varying symplectic structure ω on its fibres; that is, on each fibre E_x, the map

$$\omega_x : E_x \times E_x \to \mathbb{R}$$

is skew-symmetric, bilinear and non-degenerate. It is then possible to define an almost complex structure J fibrewise on E that is compatible with ω in the sense that $\omega(\cdot, J\cdot)$ determines a Riemannian metric on E.

We will make use of the following result in the next chapter. The proof is essentially the same as that of [GS90, Lemma 27.1], and thus we do not repeat it here.

Proposition 3.2. *Let $E \to M$ be a G-invariant symplectic vector bundle, and suppose that the action of G on E preserves the symplectic structure. Then for each $g \in G$, the set of g-fixed vectors $E(g)$ is a symplectic subbundle of E, and the symplectic structure on $E(g)$ is given by the pullback under inclusion of the symplectic structure on E.*

3.2 Contact structures

We now begin our survey of the geometries with respect to which we will construct our transversally elliptic Dirac operators, beginning with contact manifolds.

Definition 3.3. *Let M be a smooth manifold, and $E \subset TM$ a smooth subbundle of corank one. We say E is a* contact structure*, and call (M, E) a* contact manifold*, if E is given locally as the kernel of a 1-form α such that $d\alpha|_E$ defines a symplectic structure on the fibres of E.*

Equivalently, for each $m \in M$, if α is a non-zero 1-form defining E locally in some neighbourhood U of m, then E is contact if and only if for some k, $\alpha \wedge (d\alpha)^k$ defines a volume form on U. From either definition, we see that the dimension of M is necessarily odd.

Remark 3.4. We will always assume that our contact structures are *co-oriented*, meaning that the annihilator $E^0 \subset T^*M$ of E is an oriented (and hence, trivial) real line bundle. Thus, there exists a global contact form $\alpha \in \Gamma(E^0) \setminus 0$. We say that α determines a *positively-oriented* contact structure if the orientation of M agrees with that defined by the volume form $\alpha \wedge (d\alpha)^n$.

Remark 3.5. If α and α' are two (locally) defining 1-forms for E, we see that $\alpha' = f\alpha$ for some non-vanishing $f \in C^\infty(M)$. If (M, E) is co-oriented, then α and α' define the same co-oriented contact structure if $f > 0$.

Definition 3.6. *Let (M, E) be a co-oriented contact manifold, and let α be a choice of contact form. The* Reeb field *associated to α is the unique vector field ξ such that $\iota(\xi)\alpha = 1$ and $\iota(\xi)d\alpha = 0$. We say that α is a* regular *contact form if its Reeb field ξ is regular; that is, if the integral curves of ξ are embedded submanifolds of M.*

Remark 3.7. We note that regularity is a property of the contact *form*, rather than an intrinsic property of the contact *structure*. As discussed in [Gei08], the dynamics of the

Reeb flow can be dramatically altered if the contact form is rescaled by a non-constant function. (See for example [Gei08, Example 2.2.5].)

Example 3.8. Let (M, ω) be a symplectic manifold such that the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ of ω lies in the image of the coefficient homomorphism from $H^2(M; \mathbb{Z})$. (We call such an ω is an *integral* symplectic form.) It is then possible to define a principal $U(1)$-bundle $\pi : P \to M$ equipped with a connection 1-form $\widetilde{\alpha} \in \mathcal{A}^1(M, i\mathbb{R})$ such that

$$d\widetilde{\alpha} = -i\pi^*\omega. \tag{3.2}$$

Such a bundle (P, α) is referred to as a *prequantum circle bundle* for M and appears in the geometric quantization of (M, ω). (See [GGK02].) The corresponding real form $\alpha = i\widetilde{\alpha}$ is a regular contact form on P corresponding to the contact distribution $E = \pi^*TM$, since ω is non-degenerate.

The following theorem, due to Boothby and Wang (stated here in somewhat updated language), states that every contact manifold equipped with a regular contact form is of the above type, also known as a Boothby-Wang fibration. For example, every odd-dimensional sphere is a contact manifold, and the Boothby-Wang fibration on S^{2n+1} corresponding to the standard contact structure coincides with the Hopf fibration. As observed in [Gei08], the proof in [BW58] is not complete, although correct proofs have since been given by several authors, and appear in both [Bla76] and [Gei08].

Theorem 3.9 ([BW58])**.** *Let (M, α) be a contact manifold, and suppose that α is a regular contact form. Then the action generated by the flow of the corresponding Reeb field is free and effective, and M is a prequantum circle bundle over an symplectic manifold (M, ω). Conversely, any principal $U(1)$-bundle over a symplectic manifold with connection 1-form $\tilde{\alpha}$ satisfying* (3.2) *is a contact manifold.*

Definition 3.10. *Let $E = \ker \alpha$ be a contact structure on M. We say a diffeomorphism $\phi : M \to M$ is a* contact transformation *(or* contactomorphism*) if $\phi_* E = E$. If α is any contact form, this implies that $\phi^*\alpha = f\alpha$ for some non-vanishing $f \in C^\infty(M)$.*

Example 3.11. The prototypical example of a contact manifold is the space $\mathbb{R}^{2n+1}$ with coordinates $z, x_1, y_1, \ldots, x_n, y_n$, equipped with the contact form

$$\alpha = dz - \sum y_i dx_i.$$

The corresponding contact distribution E is spanned by the vector fields $X_i = \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}$ and $Y_i = \frac{\partial}{\partial y_i}$. By the contact version of Darboux's theorem (see [Bla76, CdS01]), every contact manifold is locally isomorphic to $\mathbb{R}^{2n+1}$ equipped with the contact structure $dz - \sum y_i dx_i$ given above. An equivalent definition of a contact manifold requires every coordinate chart to be of this form, such that the transition functions on the overlap of two coordinate neighbourhoods are contact transformations.

Example 3.12. A more exotic example of a contact structure on $\mathbb{R}^3$ (from [Bla76]) is given by the contact form

$$\eta = \cos z\, dx + \sin z\, dy.$$

One easily checks that $\eta \wedge d\eta = -dx \wedge dy \wedge dz$, so that η indeed defines a contact structure. The corresponding contact distribution is spanned by the vectors $X = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}$ and $Y = \frac{\partial}{\partial z}$. Here we see a nice illustration of the non-integrability of the contact distribution, since motion in the z direction results in the rotation of the contact planes.

Since η is invariant under translations of the coordinate functions by 2π, it descends to a contact form on the 3-torus $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. The vector field

$$\xi = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}$$

dual to η, defines the Reeb field on $\mathbb{T}^3$. If we consider the integral curves of ξ through the point $(0, 0, \pi/3)$ we find that $x = t/2$, $y = \sqrt{3}t/2$, and $z = \pi/3$, so that ξ has an irrational flow on the 2-torus defined by $z = \pi/3$. This contact structure is thus an example of a non-regular contact manifold; it is shown in [Bla76] that no regular contact structure on $\mathbb{T}^3$ exists.

For the remainder of this thesis we will focus on co-oriented contact manifolds, for which [Ler03] is a useful reference. Suppose then that (M, E) is a co-oriented contact manifold. A choice of contact form α determines a trivialization $E^0 = M \times \mathbb{R}$ and a splitting $T^*M = E^* \oplus E^0$. The corresponding Reeb field $\xi \subset \Gamma(TM)$ determines a splitting $TM = E \oplus \mathbb{R}\xi$, dual to the splitting $T^*M = E^* \oplus \mathbb{R}\alpha$ given by the choice of contact form.

Remark 3.13. If $E \subset TM$ is a co-oriented contact distribution, then $E^0 \setminus 0$ is a symplectic submanifold of T^*M. We denote by E^0_+ the connected component of $E^0 \setminus 0$ that contains $\alpha(M)$; E^0_+ is known as the *symplectization* of (M, E). If $\imath : E^0_+ \hookrightarrow T^*M$ denotes the inclusion mapping, then the symplectic form on E^0_+ is given by $\omega_0 = \imath^*\omega$, where ω is the canonical symplectic form on T^*M.

The pair (E^0_+, ω_0) is then a *symplectic cone*; that is, there is a free action of $\mathbb{R}^+$ on E^0_+, given by $t \mapsto \rho_t$, such that $\rho_t^*\omega_0 = t\omega_0$. The base of the cone is then $E^0_+/\mathbb{R}^+ = M$.

Conversely, suppose (N, ω) is a symplectic cone, and let $\pi : N \to M = N/\mathbb{R}^+$ denote the quotient mapping. We let $\Xi \in \Gamma(TN)$ denote the infinitesimal generator of the $\mathbb{R}^+$ action, given by

$$\Xi = \frac{d}{dt}\rho_{\exp t}\Big|_{t=0}. \tag{3.3}$$

It is then easy to check that $\mathcal{L}_\Xi\omega_0 = \omega_0$, and that $\omega_0 = -d\xi$, where $\xi = -\iota(\Xi)\omega_0$. Thus, $\ker\xi$ is the symplectic orthogonal of $\mathbb{R}\Xi \subset TN$. It follows that $E = \pi_* \ker\xi$ is a contact distribution on M.

Remark 3.14. The choice of a contact form α makes $E \to M$ into a symplectic vector bundle, with symplectic structure given by $d\alpha|_E$, and hence, as noted in Remark 3.1, it is possible to choose a complex structure on the fibres of E compatible with $d\alpha|_E$, in the sense that $\mathsf{g}(X, Y) = d\alpha(X, JY)$, for $X, Y \in \Gamma(E)$, is a positive-definite symmetric form on E. This compatibility depends only on the contact structure E and its co-orientation, since if $\beta = e^f\alpha$ is another choice of contact form, we have $d\beta = e^f df \wedge \alpha + e^f d\alpha$, whence

$d\beta|_E = e^f d\alpha|_E$. By declaring the Reeb vector field to be orthogonal to E and of unit length, we can extend g to a Riemannian metric g_α on TM. We will see below in the context of CR geometry that this metric is an example of what is known as a Webster metric. This metric depends on the choice of contact form, but unlike the CR case described below, it is always an honest (that is, positive-definite) Riemannian metric.

3.3 Almost contact and Sasakian structures

An almost contact structure is a generalization of a co-oriented contact manifold: we still assume that a global, non-vanishing 1-form η exists, but drop the non-degeneracy condition in the definition of a contact structure. On the other hand, an almost contact manifold has added structure, since we assume the a priori existence of a complex structure on the fibres of $\ker \eta$.

Definition 3.15. *An* almost contact structure *on an odd-dimensional manifold M is a triple (ϕ, ξ, η), where $\phi \in \mathcal{A}(M, TM)$ is an endomorphism of TM, $\xi \in \Gamma(TM)$ is a vector field, and $\eta \in \mathcal{A}^1(M)$ is a 1-form, such that*

$$\phi^2 = -\operatorname{Id} + \eta \otimes \xi \qquad \text{and} \qquad \eta(\xi) = 1. \tag{3.4}$$

It follows from (3.4) that $\eta \circ \phi = \phi(\xi) = 0$. We see that η plays the role of the contact form, while ξ is the analogue of the Reeb field. The endomorphism ϕ has rank $\dim M - 1$, and defines a complex structure on the fibres of $E = \ker(\eta)$. From [Bla76], we have the following proposition:

Proposition 3.16. *Let M be equipped with an almost contact structure (ϕ, ξ, η). Then M admits a Riemannian metric g such that*

$$\mathsf{g}(\phi X, \phi Y) = \mathsf{g}(X, Y) - \eta(X)\eta(Y) \qquad \text{and} \qquad \eta(X) = \mathsf{g}(X, \xi), \tag{3.5}$$

for all vector fields $X, Y \in \Gamma(TM)$. (The data (ϕ, ξ, η, g) is called an almost contact metric structure.*)*

Remark 3.17. An almost contact structure defines a contact structure $E = \ker(\eta)$ if and only if $\eta \wedge (d\eta)^n$ defines a volume form on M. An alternative definition of an almost contact metric structure specifies the 1-form η and a 2-form Φ such that $\eta \wedge \Phi^n$ is a volume form. An almost contact metric structure is therefore contact when $\Phi = \pm d\eta$. The two definitions are equivalent; for example, given ϕ and g, we can define $\Phi \in \mathcal{A}^2(M)$ by $\Phi(X,Y) = \mathsf{g}(X, \phi Y)$.

We like to think of the data (Φ, ϕ, g) as defining a "compatible triple" analogous to the triple (ω, J, g) of the previous section. The Nijenhuis tensor of $\mathrm{Nij}(\phi) \in \mathcal{A}^2(M, TM)$ of ϕ is again defined by

$$\mathrm{Nij}(\phi)(X,Y) = [\phi,\phi](X,Y) = \phi^2[X,Y] - [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]. \tag{3.6}$$

We say that the almost contact structure (ϕ, ξ, η) is *normal* if

$$\mathrm{Nij}(\phi) + 2d\eta \otimes \xi = 0.$$

A normal almost contact structure is the odd-dimensional equivalent of an integrable almost-complex structure (see Proposition 3.28) . When $\Phi = d\eta$, we obtain the contact analogue of a Kähler manifold:

Definition 3.18. *We say that a manifold M equipped with a normal contact metric structure $(\phi, \xi, \eta, \mathsf{g})$ is a* Sasakian manifold.

Indeed, if M is Sasakian, then its symplectization is a Kähler cone. Additionally, let ∇ be the Levi-Civita connection for the metric g. Then the almost contact metric structure $(\phi, \xi, \eta, \mathsf{g})$ is Sasakian if and only if $(\nabla_X \phi)Y = \mathsf{g}(X,Y)\xi - \eta(Y)X$ for all vector fields X and Y. It follows that on a Sasakian manifold, the Reeb field ξ is Killing. As in the previous section, we can give a diagram outlining the hierarchy of the various structures we have described; this diagram is once again borrowed from [Bla76].

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}(M,\phi,\xi,\eta)\\ \text{Almost Contact}\end{array}} \arrow[r, "{\mathsf{g}(\phi X,\phi Y)=}", "{\mathsf{g}(X,Y)-\eta(X)\eta(Y)}"'] \arrow[d, "\mathrm{Nij}(\phi)=0"] & \boxed{\begin{array}{c}(M,\phi,\xi,\eta,\mathsf{g})\\ \text{Almost Contact}\\ \text{Metric}\end{array}} \arrow[r, "\Phi=d\eta"] \arrow[d, "\mathrm{Nij}(\phi)=0"] \arrow[dr, "{(\nabla_X\phi)Y=\mathsf{g}(X,Y)\xi-\eta(Y)X}"] & \boxed{\begin{array}{c}(M,\phi,\xi,\eta,\mathsf{g})\\ \text{Contact Metric}\end{array}} \arrow[d, "\mathrm{Nij}(\phi)=0"] \\
\boxed{\begin{array}{c}(M,\phi,\xi,\eta)\\ \text{Normal Almost}\\ \text{Contact}\end{array}} \arrow[r, "{\mathsf{g}(\phi X,\phi Y)=}", "{\mathsf{g}(X,Y)-\eta(X)\eta(Y)}"'] & \boxed{\begin{array}{c}(M,\phi,\xi,\eta,\mathsf{g})\\ \text{Normal Almost}\\ \text{Contact Metric}\end{array}} \arrow[r, "\Phi=d\eta"] & \boxed{\begin{array}{c}(M,\phi,\xi,\eta,\mathsf{g})\\ \text{Sasakian}\end{array}}
\end{tikzcd}
$$

3.4 CR and almost CR manifolds

A natural way of generalizing the case of contact manifolds to distributions of higher corank is to consider almost CR (Cauchy-Riemann) structures: while we lose the contact form and the additional structure it provides, we keep the existence of a complex structure on the fibres of the distribution.

Definition 3.19. *An* almost CR *structure on a manifold M is a subbundle $E_{1,0}(M) \subset TM \otimes \mathbb{C}$ of the complexified tangent bundle such that $E_{1,0}(M) \cap E_{0,1}(M) = 0$, where $E_{0,1}(M) = \overline{E_{1,0}(M)}$. An almost CR structure defines a* CR structure *if it is integrable; that is, if the space of sections of $E_{1,0}(M)$ is closed under the Lie bracket.*

We denote the corresponding dual spaces in $T^*M \otimes \mathbb{C}$ by $E^{1,0}(M) = E_{1,0}(M)^*$ and $E^{0,1}(M) = E_{0,1}(M)^*$.

Remark 3.20. When only one manifold M is being discussed, we may omit the dependence on M in the above notation, and simply write $E_{1,0}$ for $E_{1,0}(M)$. The complex rank n of $E_{1,0}(M)$ is called the *CR dimension* of $(M, E_{1,0}(M))$, and $k = \dim M - 2n$ is called the *CR codimension* of $(M, E_{1,0}(M))$. The pair (n, k) is called the *type* of the CR structure. We note that an almost CR manifold of type $(n, 0)$ is simply an almost complex manifold. Almost CR structures of type $(n, 1)$ are closely related to almost contact manifolds; we will discuss this relationship below. Many (almost) CR manifolds arise as submanifolds of (almost) complex manifolds. Indeed, from [DT06] we have the following proposition:

Proposition 3.21. *If M is a real, codimension one submanifold of a complex manifold N, then*

$$E_{1,0}(M) = T^{1,0}N \cap T_{\mathbb{C}}M \tag{3.7}$$

defines a CR structure on M.

In general (i.e for higher codimensions) $E_{1,0}$ need not have constant rank; in [Bog91] the definition of an imbedded CR manifold is given by the requirement that $E_{1,0}(M)$ have constant rank.

Example 3.22. Let (M, E) be a co-oriented contact manifold, and let α be a choice of contact form. Then a choice of complex structure J on the fibres of E compatible with the symplectic structure determined by $d\alpha$ on E determines an almost CR structure of type $(n, 1)$ on M via the decomposition $E \otimes \mathbb{C} = E_{1,0} \oplus E_{0,1}$, where $E_{1,0}$ and $E_{0,1}$ are the $+i$ and $-i$ eigenbundles of J, respectively. More generally, we may apply the same argument in the case of an almost contact manifold (M, ϕ, ξ, η): the restriction of ϕ to $E = \ker \eta$ determines a decomposition into $\pm i$-eigenbundles of $\phi|_E$.

Definition 3.23. *The* Levi distribution *of an (almost) CR manifold $(M, E_{1,0}(M))$ of type (n, k) is the real subbundle $E(M) \subset TM$ of rank $2n$ given by*

$$E(M) = \mathrm{Re}\{E_{1,0}(M) \oplus E_{0,1}(M)\}.$$

The distribution $E(M)$ is equipped with a fibrewise complex structure $J_b : E(M) \to E(M)$ given by

$$J_b(X + \overline{X}) = i(X - \overline{X}),$$

for each $X + \overline{X} \in E(M)$.

Remark 3.24. We will sometimes refer to the Levi distribution as the *underlying real distribution* of the almost CR structure. As in Example 3.22, an almost CR structure may be given by specifying $E(M)$ and an endomorphism $J_b \in \mathrm{End}(E(M))$ such that $J_b^2 = -\mathrm{Id}$; the space $E_{1,0}(M)$ is then the $+i$-eigenbundle of J_b. It follows (see [DT06]) that an almost CR structure $E_{1,0}(M)$ is integrable if and only if for any local sections X, Y of $E(M)$, the vector field

$$W = [J_b X, Y] + [X, J_b Y]$$

is also a local section of $E(M)$, and the Nijenhuis tensor of J_b vanishes. The complex structure J_b on $E(M)$ induces a complex structure on $E(M)^*$ (which we will also denote by J_b), given by $J_b(\eta)(\xi) = \eta(J_b(\xi))$ for all vector fields $\xi \in \Gamma(E(M))$ and 1-forms $\eta \in \Gamma(E(M)^*)$. We note the isomorphism $\psi : E(M)^* \xrightarrow{\simeq} E^{0,1}(M)$ given by

$$\psi(\eta) = \frac{1}{2}(\eta + iJ_b(\eta)). \tag{3.8}$$

Definition 3.25. *Let $(M, E_{1,0})$ be a Cauchy-Riemann manifold. We define the* Levi form

$$\mathcal{L}_p : (E_{1,0})_p \times (E_{1,0})_p \to T_p,$$

where $T = T_{\mathbb{C}}M/(E_{1,0} \oplus E_{0,1})$, by

$$\mathcal{L}_p(Z_p, W_p) = \frac{i}{2}\pi_p([Z, \overline{W}])_p), \tag{3.9}$$

where $Z, W \in E_{1,0}$ are any vector fields equal to Z_p and W_p at $p \in M$, and $\pi_p : T_pM \otimes \mathbb{C} \to T_p$ is the natural projection.

We call $(M, E_{1,0})$ a *non-degenerate* CR manifold if the Levi form $\mathcal{L}$ is non-degenerate. The Levi form is of particular interest in the case of CR codimension one, as explained in [DT06]: If $(M, E_{1,0})$ is an oriented CR manifold of type $(n, 1)$, we let E^0 denote the annihilator of the Levi distribution E of $E_{1,0}$. We see that E^0 is a subbundle of T^*M isomorphic to TM/E. Since M is oriented and $E_{1,0}$ is oriented by the complex structure J_b, it follows that E^0 is orientable, and hence trivial, whence it admits a nowhere-vanishing section η.

Definition 3.26. *Let* $(M, E_{1,0})$ *be a non-degenerate CR manifold. A nowhere-vanishing global section* $\eta \in \Gamma(E^0)$ *is called a* pseudo-Hermitian structure *on* M*. Associated to a pseudo-Hermitian structure* η *is the* Levi form $\mathcal{L}_\eta$ *defined for any* $Z, W \in E_{1,0}$ *by*

$$\mathcal{L}_\eta(Z, W) = -id\eta(Z, \overline{W}). \tag{3.10}$$

Since the form (3.10) is scalar-valued, it is possible to make the following definition:

Definition 3.27. *We say that a CR manifold* $(M, E_{1,0})$ *is* strongly pseudoconvex *if for some pseudo-Hermitian structure* η *the associated Levi form* $\mathcal{L}_\eta$ *is positive-definite.*

Let us now consider the relationship between CR and contact geometry. Suppose that $(M, E_{1,0})$ is a CR manifold of type $(n, 1)$ with pseudo-Hermitian structure η. It can be shown (see [DT06]) that there exists a unique vector field $\xi \in TM$ such that $\eta(\xi) = 1$ and $\iota(\xi)d\eta = 0$. The vector field ξ determines a splitting

$$TM = E \oplus \mathbb{R}\xi,$$

where E is the Levi distribution of $E_{1,0}$. We can then define an endomorphism field ϕ on M by setting $\phi(X) = J_b X$ for all $X \in E$, and $\phi T = 0$, and it is easy to see that (ϕ, ξ, η) defines an almost contact structure. Moreover, if the Levi form $\mathcal{L}_\eta$ is non-degenerate, it follows that (M, η) is a contact manifold, since the restriction of $d\eta$ to $E = \ker \eta$ is non-degenerate. We define a symmetric bilinear form G_η on $E \subset TM$ by setting

$$\mathsf{G}_\eta(X, Y) = d\eta(X, J_b Y).$$

When M is CR, the vanishing of $\mathrm{Nij}(J_b)$ implies that $\mathsf{G}_\eta(J_b X, J_b Y) = \mathsf{G}_\eta(X, Y)$. Note, however, that in general $d\eta$ need not be non-degenerate on E. We can use the splitting $TM = E \oplus \mathbb{R}\xi$ to extend G_η to a semi-Riemannian metric on M by declaring ξ to be the unit normal vector to E. That is, we set

$$\mathsf{g}_\eta(X, Y) = \mathsf{G}_\eta(X, Y), \quad \mathsf{g}_\eta(X, \xi) = 0, \quad \mathsf{g}_\eta(\xi, \xi) = 1.$$

The resulting symmetric form g_η is called the *Webster metric* of (M, η). It is a Riemannian metric if and only if $\mathcal{L}_\eta$ is positive definite; that is, if M is strongly pseudoconvex. It follows that when M is strongly pseudoconvex, the data $(\phi, \xi, \eta, \mathsf{g})$ determine a contact metric structure on M.

As mentioned at the beginning of this section, if (M, ϕ, ξ, η) is an almost contact manifold, then the $+i$-eigenbundle of $\phi|_E$ determines an almost CR structure on M. The question of whether or not this structure is integrable is answered by the following proposition, due to S. Ianuş [Ian72]:

Proposition 3.28. *If an almost contact structure (ϕ, ξ, η) on M is normal, then the almost CR structure defined by the $+i$-eigenbundle $E_{1,0} = \{X - i\phi X | X \in E = \ker \eta\}$ of $\phi|_E$ is a CR structure.*[1]

3.4.1 The tangential Cauchy-Riemann complex

Definition 3.29. *Let $(M, E_{1,0})$ be a CR manifold of type (n, k). We define the operator*

$$\overline{\partial}_b : \Gamma\left(\bigwedge\nolimits^{\bullet} E^{0,1}\right) \to \Gamma\left(\bigwedge\nolimits^{\bullet+1} E^{0,1}\right) \tag{3.11}$$

as follows: for each smooth function $f : M \to \mathbb{C}$ and $Z \in \Gamma(E_{1,0})$, we set

$$\iota(\overline{Z})(\overline{\partial}_b f) = \overline{Z} f,$$

and for any k-form $\psi \in \Gamma(\bigwedge^k E^{0,1})$ and $Z_1, \ldots, Z_{k+1} \in \Gamma(E_{1,0})$, we have

$$\begin{aligned}(\overline{\partial}_b \psi)(\overline{Z}_1, \ldots, \overline{Z}_{k+1}) = \frac{1}{k+1} \Bigg\{ &\sum_{i=1}^{k+1} (-1)^{i+1} \overline{Z}_i (\psi(\overline{Z}_1, \ldots, \widehat{\overline{Z}}_i, \ldots, \overline{Z}_{k+1})) \\ &+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \psi([\overline{Z}_i, \overline{Z}_j], \overline{Z}_1, \ldots, \widehat{\overline{Z}}_i, \ldots, \widehat{\overline{Z}}_j, \ldots, \overline{Z}_{k+1}) \Bigg\}. \end{aligned} \tag{3.12}$$

We note that this operator does not make sense for an almost CR structure that is not integrable, since the Lie bracket $[\overline{Z}_i, \overline{Z}_j]$ may not again be a section of $E_{1,0}$. Let

[1] In particular, every Sasakian manifold admits a CR structure. We note that the converse of the above theorem does not hold in general, according to [DT06].

$(M, E_{1,0})$ be a CR manifold of type $(n, 1)$. There is then an alternative construction of the $\overline{\partial}_b$ operator in terms of the usual de Rham differential d. Since the annihilator E^0 is trivial, we may choose a pseudo-Hermitian form η, allowing us to identify elements of $\Gamma(M, \bigwedge E^{0,1})$ with differential forms on M, and giving a decomposition of the complexified cotangent bundle as

$$T^*_C M = E^{1,0} \oplus E^0_{\mathbb{C}} \oplus E^{0,1},$$

where $E^0_{\mathbb{C}} \cong \mathbb{C}\eta$. Let us denote by $\widetilde{E}^{1,0}(M)$ the subbundle

$$\widetilde{E}^{1,0}(M) = E^{1,0}(M) \oplus E^0_{\mathbb{C}}.$$

We then define the space of (p, q)-forms $\mathcal{A}^{(p,q)}(M)$ as the space of smooth sections of

$$\bigwedge^{(p,q)}(M) = \bigwedge^p \widetilde{E}^{1,0}(M) \wedge \bigwedge^q E^{0,1}(M), \tag{3.13}$$

and for $r = p + q$ we let $\pi^{p,q} : \mathcal{A}^r(M) \to \mathcal{A}^{(p,q)}(M)$ denote the natural projection. Following [Bog91] we define an operator

$$\pi^{p,q+1} \circ d : \mathcal{A}^{(p,q)}(M) \to \mathcal{A}^{(p,q+1)}(M), \tag{3.14}$$

where $d : \mathcal{A}^r(M) \to \mathcal{A}^{r+1}(M)$ is the usual de Rham differential.

Proposition 3.30 ([DT06])**.** *The tangential Cauchy-Riemann operator $\overline{\partial}_b$ on a CR manifold $(M, E_{1,0})$ of type $(n, 1)$ agrees with the operator*

$$\pi^{0,q+1} \circ d : \mathcal{A}^{(0,q)}(M) \to \mathcal{A}^{(0,q+1)}(M).$$

Remark 3.31. For a CR structure of type (n, k), where $k > 1$, one must choose a splitting $T^*M = E^* \oplus E^0$ of the cotangent bundle in order to identify $E^{0,1}$ with a subbundle of $T^*_{\mathbb{C}}M$ and define the projection $\pi^{p,q}$. It is possible to show that

$$\overline{\partial}_b \circ \overline{\partial}_b = 0$$

using either definition of $\overline{\partial}_b$ but the proof is much simpler for the second. (Compare [Bog91, Section 8.2] to [DT06, Section 1.7].) Much study has been devoted to the resulting *tangential Cauchy-Riemann complex* of a CR manifold and its associated Kohn-Rossi cohomology [KR65]; see [Bog91] and [DT06] and the references therein for details. We note that the definition $\overline{\partial}_b = \pi^{0,q} \circ d$ does not a priori require that the distribution $E_{1,0}$ be integrable, but the proof that $\overline{\partial}_b^2 = 0$ depends very much on this fact. Thus while it may be possible to define an operator $\overline{\partial}_b$ on an almost CR manifold, we do not obtain a complex unless we have a CR structure.

We conclude this section with a brief mention of CR-holomorphic vector bundles, which are presented in [DT06] as the CR analogue of holomorphic vector bundles on complex manifolds.

Definition 3.32. *We say that a function $f \in C^\infty(M, \mathbb{C})$ is* CR-holomorphic *if it is a solution to the tangential CR equations*

$$\overline{\partial}_b f = 0; \tag{3.15}$$

that is, if $\overline{Z} f = 0$ for each $Z \in \Gamma(E_{1,0})$.

Remark 3.33. In the case that $(M, E_{1,0})$ is embedded as a submanifold of a complex manifold N, then any holomorphic function on N restricts to a CR-holomorphic function on M. However, not all CR-holomorphic functions are obtained in this way; in particular, there are often smooth functions on M that are CR-holomorphic but not real analytic, and hence do not come from a holomorphic function on M. The problem of determining which CR-holomorphic functions can be extended to a holomorphic function on the ambient complex space is discussed in [Bog91].

Suppose now that $\mathcal{V} \to M$ is a complex vector bundle over a CR manifold $(M, E_{1,0})$. We would like to extend the notion of CR-holomorphic functions to CR-holomorphic sections of $\mathcal{V}$.

Definition 3.34. *We say that $\mathcal{V} \to M$ is a* CR-holomorphic vector bundle *if $\mathcal{V}$ is equipped with a first-order differential operator*

$$\overline{\partial}_{\mathcal{V}} : \Gamma(\mathcal{V}) \to \Gamma(E^{0,1} \otimes \mathcal{V})$$

such that

$$\overline{\partial}_{\mathcal{V}}(fs) = f\overline{\partial}_{\mathcal{V}}s + (\overline{\partial}_b f) \otimes s, \tag{3.16}$$

$$[\overline{Z}, \overline{W}]s = \overline{Z}\,\overline{W}s - \overline{W}\,\overline{Z}s, \tag{3.17}$$

for any $f \in C^\infty(M, \mathbb{C})$*,* $s \in \Gamma(\mathcal{V})$ *and* $Z, W \in \Gamma(E_{1,0}(M))$*, where we have used the notation* $\overline{Z}s = \iota(\overline{Z})\overline{\partial}_{\mathcal{V}}s$*. We say that a section* $s \in \Gamma(\mathcal{V})$ *is a* CR-holomorphic section *if*

$$\overline{\partial}_{\mathcal{V}}s = 0. \tag{3.18}$$

3.5 f-Structures

We end our discussion of geometric structures on manifolds with a brief mention of f-structures, as they provide a general framework with respect to which many of the structures discussed above may be viewed as special cases. A good introduction to f-structures can be found in the books [KY83, KY84].

Definition 3.35. *An* f-structure *on a smooth manifold* M *is a smooth non-null section* $f \in \mathrm{End}(TM)$ *such that*

$$f^3 + f = 0.$$

Given such an f, we define operators $l = -f^2$ and $m = f^2 + \mathrm{Id}$. It follows immediately that l and m are complementary projections on TM, since $l + m = \mathrm{Id}$, $l^2 = l$, $m^2 = m$, and $ml = lm = 0$. If f has constant rank we then have the distributions $E = l(TM)$ and $T = m(TM)$. Indeed, we have $T = \ker(f)$, $E = \mathrm{ran}(f)$, and a splitting [LP04]

$$TM = \ker(f) \oplus \mathrm{ran}(f). \tag{3.19}$$

Example 3.36. If $\mathrm{rank}\, f = n = \dim M$, then $f^2 + \mathrm{Id} = 0$, and (M, f) is an almost complex manifold. If M is orientable and $\mathrm{rank}\, f = n - 1$, then f determines an almost contact structure on M.

Any f-structure of rank $2n$ on a manifold M of dimension $2n+k$ determines an almost CR structure of type (n,k). The following proposition is easy to prove, and shows that an almost CR structure is actually the most general geometry that we consider.

Proposition 3.37. *Let $(M, E_{1,0})$ be an almost CR manifold, let E denote the Levi distribution and J_b the corresponding almost complex structure on E. If we choose a splitting $E \oplus E^{\perp} = TM$ then the endomorphism $f = J_b \oplus 0$ is an f-structure on M.*

Thus, an f-structure is simply an almost CR structure together with a choice of complement to the Levi distribution E. However, an f structure allows us to give extensions to TM of objects previously defined only on E. For example, we have (see [Soa97]):

Lemma 3.38. *There exists a Riemannian metric g on M such that*

$$\mathsf{g}(fX, Y) = -\mathsf{g}(X, fY). \tag{3.20}$$

Proof. Given any Riemannian metric $\tilde{\mathsf{g}}$ define

$$\mathsf{g}(X, Y) = (\tilde{\mathsf{g}}(fX, fY) + \tilde{\mathsf{g}}(f^2X, f^2Y))/2. \qquad \square$$

Using f-structures, we can extend the notion of an almost contact structure to the case $k > 1$.

Definition 3.39 ([BLY73, Sae81])**.** *We say an f-structure of rank $2n$ on a manifold M of dimension $2n+k$ is a* framed *f-structure if there exist vector fields $\xi_1, \ldots \xi_k$ on M, and corresponding 1-forms $\eta_1, \ldots, \eta_k$ such that*

$$f\xi_i = \eta_i \circ f = 0, \quad \eta_i(\xi_j) = \delta_{ij}, \quad \textit{and} \quad f^2 = -I + \sum \eta_i \otimes \xi_i. \tag{3.21}$$

In other words, a framed f-structure is one whose kernel $T = (f^2+I)(TM)$ is a trivial bundle, and the 1-forms $\eta_1, \ldots, \eta_k$ are a parallelizing frame for E^0. We can continue the analogy with almost contact geometry further. We say that a framed f-structure is *regular* if each of the ξ_i are regular vector fields, and *normal* if

$$\mathrm{Nij}(f) + 2\sum d\eta_i \otimes \xi_i = 0.$$

Given any framed f-structure, there also exists a Riemannian metric adapted to the structure [BLY73], in the sense that

$$\mathsf{g}(fX, fY) = \mathsf{g}(X, Y) - \sum \eta_i(X)\eta_i(Y). \tag{3.22}$$

If we equip M with such a metric, we obtain a framed metric f-structure. If the 2-form $\Phi(X, Y) = \mathsf{g}(X, fY)$ associated to a normal f-structure is closed, f is called a K-structure, because each of the vector fields ξ_i are then Killing vector fields. Finally, a K-structure is called an S-structure if each 2-form $d\eta_i$ is a multiple of Φ. In terms of this language, we have the following generalization of the Boothby-Wang theorem:

Theorem 3.40 ([BLY73])**.** *Let M be a compact connected manifold of dimension $2n+k$ equipped with a regular normal framed f-structure of rank $2n$. Then M is the total space of a principal torus bundle over a complex manifold N of dimension $2n$. Moreover, if f is a K-structure, then N is Kähler.*

In the case that each of the $d\eta_i$ are equal, it is shown in addition in [Sae81] that N is in fact a *Hodge manifold*; that is, a Kähler manifold whose symplectic form is integral, thus giving an extension of Theorem 3.9 to higher corank.

Chapter 4

Group actions and differential operators on (M, E)

4.1 Group actions transverse to a distribution

Let G be a compact Lie group, and let M be a G-manifold. We will make use of the following notation throughout this thesis:

Definition 4.1. *Let $\eta \in \Omega^1(M)$ be a G-invariant 1-form on M. We define the* η-moment map *to be the map $f_\eta : M \to \mathfrak{g}^*$ given by the pairing*

$$< f_\eta(m), X >= -\eta_m(X_M(m)), \tag{4.1}$$

for any $X \in \mathfrak{g}$, where X_M is the vector field on M generated by X via the infinitesimal action of $\mathfrak{g}$ on M.

We denote by C_η the zero-level set $f_\eta^{-1}(0) \subset M$ of the η-moment map.

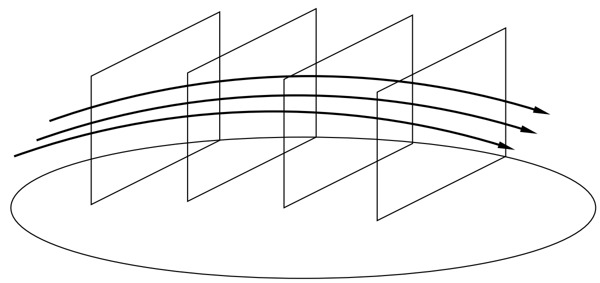
Remark 4.2. Let θ be the canonical 1-form on T^*M, and consider the lift of the action of G on M to T^*M. This action is Hamiltonian, and $f_\theta : T^*M \to \mathfrak{g}^*$ is the corresponding moment map. Recall that the space $T^*_G M$ defined by (2.4) is the set of all covectors that annihilate vectors tangent to the G-orbits. In terms of the θ-moment map, we have the convenient description of the space $T^*_G M$ given by

$$T^*_G M = C_\theta. \tag{4.2}$$

Now, suppose $E \subset TM$ is a G-invariant distribution. We wish to consider those actions of G on M such that the G-orbits are transverse to E in the following sense:

Definition 4.3. *We say that the action of G on M is* transverse *to E if $E + \mathfrak{g}_M = TM$. Equivalently, the action of G on M is transverse to E if and only if we have*

$$T^*_G M \cap E^0 = \{0\}. \tag{4.3}$$



Remark 4.4. The assumption of transversality implies that $\operatorname{rank} E^0 \leq \dim G$. In the case that $\operatorname{rank} E^0 = \dim G$, the action of G on M is locally free. More precisely, at any $x \in M$, one has $\operatorname{rank} E^0 \leq \dim G - \dim G_x$, whence $\operatorname{rank} E^0 = \dim G$ implies that $\dim G_x = 0$. A subbundle E that is transverse to the G-orbits is then the space of horizontal vectors with respect to some choice of connection, and the annihilator $E^0 = M \times \mathfrak{g}^*$ is trivial. We are then in the same setting considered in [BV96b] in the case of a free action, or [Ver96], in the orbifold case. Our results follow a similar approach, but allow for a broader class of group actions, since any locally free action will be transverse to a horizontal distribution, but not all actions satisfying (4.3) are locally free.

For any G-set V, we will denote by $V(g)$ the subset of V fixed by the action of an element $g \in G$. For example, $G(g)$ denotes the centralizer of g in G, and $\mathfrak{g}(g)$ denotes its Lie algebra, the set of points fixed by g under the adjoint action. Given the action of G on M, we have the decomposition

$$TM|_{M(g)} = TM(g) \oplus \mathcal{N}(g), \tag{4.4}$$

where $TM(g) = \ker(g - \mathrm{Id})$ denotes the points in TM fixed by the action of g, and $\mathcal{N}(g) = \mathrm{ran}(g - \mathrm{Id})$ denotes the normal bundle. From the corresponding action on T^*M, we have the canonical identification $T^*(M(g)) \cong (T^*M)(g)$. With respect to the action of $G(g)$ on $T^*M(g)$, we note the following lemmas:

Lemma 4.5. [BV96a, Lemma 19]

1. *The canonical 1-form θ^g on $T^*M(g)$ is the pullback under inclusion of the canonical 1-form θ on T^*M.*

2. *The corresponding moment map $f_{\theta^g} : T^*M(g) \to \mathfrak{g}^*(g)$ is given by the restriction of f_θ to $T^*M(g)$.*

3. *$T^*_{G(g)}M(g) = C_{\theta^g} = (T^*_G M)(g)$.*

Lemma 4.6. *For any $g \in G$, we have the identification*

$$\mathfrak{g}_M(g) = \mathfrak{g}(g)_M. \tag{4.5}$$

Proof. At any $x \in M$ we have that $\mathfrak{g}_M|_x \cong \mathfrak{g}/\mathfrak{g}_x$. Choose an G_x-equivariant splitting $s : \mathfrak{g}/\mathfrak{g}_x \to \mathfrak{g}$ of the exact sequence

$$0 \to \mathfrak{g}_x \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_x \to 0.$$

By the equivariance of s, we thus have $s\left((\mathfrak{g}/\mathfrak{g}_x)(g)\right) \subset \mathfrak{g}(g)$, whence $\mathfrak{g}_M(g) \subset \mathfrak{g}(g)_M$. The opposite inclusion is clear, and thus the result follows. □

Proposition 4.7. *If the action of G on M is transverse to $E \subset TM$, then for any $g \in G$, the action of $G(g)$ on $M(g)$ is transverse to $E(g) \subset TM(g)$.*

Proof. If G acts on M transverse to E, then we have

$$TM(g) = (E + \mathfrak{g}_M)(g) = E(g) + \mathfrak{g}_M(g) = E(g) + \mathfrak{g}(g)_M,$$

by averaging with respect to the subgroup generated by g, and then using Lemma 4.6. □

Remark 4.8. When (M, E) is a co-oriented contact manifold, there is a nice characterization of group actions transverse to E as follows: Let E^0_+ denote the symplectization of E, equipped with symplectic form ω_0. If G acts on M by contact transformations, then the cotangent lift of this action restricts to a Hamiltonian action on E^0_+, with moment map $f_\xi : M \to \mathfrak{g}^*$ given for each $X \in \mathfrak{g}$ by the pairing of X_M against $\xi = -\iota(\Xi)\omega_0$, where Ξ is the vector field (3.3). The action of G is then transverse to E if and only if $C_\xi = \emptyset$.

Proposition 4.9. *Let (M, E) be a co-oriented contact manifold, and suppose G is a compact Lie group acting on M by co-orientation preserving contact transformations. For any $g \in G$, let $i : M(g) \hookrightarrow M$ denote inclusion of the g-fixed points. If the action of G is transverse to $E = \ker \alpha$, then we have:*

1. *The submanifold $M(g) \subset M$ is a contact manifold, and if α is a contact form on M, then $\alpha^g = i^*\alpha$ is a contact form on $M(g)$.*

2. *The action of the centralizer $G(g)$ of g in G on $M(g)$ is transverse to $E(g)$.*

Proof. The second point is simply a restatement of Proposition 4.7. Let us prove the first. If α is a choice of contact form on M, which we can assume is G-invariant, then

$$\ker(i^*\alpha) = \ker \alpha \cap TM(g) = E \cap TM(g) = E(g).$$

Let $j : E(g) \hookrightarrow E$ denote the inclusion of the G-fixed points in E. Since α is G-invariant, $d\alpha|_E$ is a G-invariant symplectic structure on E, and thus $j^*(d\alpha|_E) = d\alpha^g|_{E(g)}$ is a $G(g)$-invariant symplectic structure on $E(g)$ by Proposition 3.2, whence α^g is a contact form on $M(g)$. □

4.2 The Dolbeault-Dirac operator

The main goal of this chapter is to construct a class of differential operators adapted to a subbundle $E \subset TM$. These operators will be modelled on the Dolbeault-Dirac operator defined on an almost Hermitian manifold, so let us begin by quickly reviewing the construction of this operator. Further details may be found in [BGV91] and [GGK02]; see also [Nic05] and the references therein for an explicit construction of this operator.

Recall that if a manifold M of even dimension is equipped with a Riemannian metric g, then one can form the *Clifford bundle* $\mathbb{C}l(M) \to M$ whose fibre at $x \in M$ is the complexified Clifford algebra of T_x^*M with respect to the quadratic form induced by g_x. A *Clifford module* is a $\mathbb{Z}_2$-graded complex vector bundle $\mathcal{V} \to M$ equipped with an action $\mathbf{c} : \mathbb{C}l(M) \to \mathrm{End}(\mathcal{V})$. Given any Clifford module $\mathcal{V}$ and a complex vector bundle $\mathcal{W}$, we obtain a new Clifford module $\mathcal{V} \otimes \mathcal{W}$ equipped with the action $\nu \in \mathbb{C}l(M) \mapsto \mathbf{c}(\nu) \otimes \mathrm{Id}$. The Clifford bundle $\mathbb{C}l(M)$ is equipped with a canonical anti-automorphism $*$ given by

$$* (\nu_1 \cdots \nu_k) = \overline{\nu_k} \cdots \overline{\nu_1}, \tag{4.6}$$

where $\overline{\nu_i}$ denotes the complex conjugate in $T_{\mathbb{C}}^*M$.

Definition 4.10. *An Hermitian vector bundle $\mathcal{S}$ is a* spinor module *for $\mathbb{C}l(M)$ if there is a $*$-isomorphism*

$$\mathbf{c} : \mathbb{C}l(M) \to \mathrm{End}(\mathcal{S}). \tag{4.7}$$

Most of the spinor modules in which we will be interested arise from almost complex structures. Following [BGV91], we will call a subbundle $\mathcal{P} \subset T_{\mathbb{C}}M$ a *polarization* if $\mathcal{P}$ is isotropic with respect to the metric g (extended to $T_{\mathbb{C}}M$ by complex linearity), and $T_{\mathbb{C}}M = \mathcal{P} \oplus \overline{\mathcal{P}}$. A Clifford module $\mathcal{S}$ of the form

$$\mathcal{S}^{\pm} = \bigwedge^{\pm} \mathcal{P}^*, \tag{4.8}$$

for some polarization $\mathcal{P}$, is an example of a spinor module. The action of the Clifford bundle is given as follows. For any $\mu \in T^*M$, write $\mu = \nu + \overline{\nu}$, where $\nu \in \mathcal{P}^*$. We then set, for any $\gamma \in \bigwedge \mathcal{P}^*$,

$$\mathbf{c}(\mu)\gamma = \sqrt{2}(\varepsilon(\nu)\gamma - \iota(\overline{\nu})\gamma), \tag{4.9}$$

where $\varepsilon(\nu)\gamma = \nu \wedge \gamma$, and we define the contraction $\iota(\overline{\nu})$ by identifying $\overline{\mathcal{P}^*}$ with $\mathcal{P}$ using the metric g. Note that we have

$$\mathbf{c}(\mu)^2\gamma = -2(\varepsilon(\nu)\iota(\overline{\nu}) + \iota(\overline{\nu})\varepsilon(\nu)) = -2\mathsf{g}(\nu, \overline{\nu}) = -2\mathsf{g}(\mu, \mu),$$

from which it follows (after a dimension count) that $\mathcal{S}$ is indeed a spinor module.

Remark 4.11. The existence of a polarization is equivalent to an almost Hermitian structure on M. In general a spinor module may be defined pointwise in this fashion, but not all spinor modules are globally defined in terms of polarizations. In Section 4.3 below, we consider a variation on the theme of polarization, replacing the condition $\mathcal{P} \oplus \overline{\mathcal{P}} = T_{\mathbb{C}}M$ by $\mathcal{P} \cap \overline{\mathcal{P}} = 0$.

Suppose now that (M, J, g) is an almost Hermitian manifold, let h be the Hermitian metric on $T_{\mathbb{C}}M$ induced by g, and let $\widehat{\nabla}$ be a Hermitian connection on M (that is, $\widehat{\nabla} J = \widehat{\nabla}\mathsf{h} = 0$). Let $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ denote the usual decomposition of the complexified tangent bundle into the $\pm i$-eigenbundles of J, and $T_{\mathbb{C}}^*M = T^{1,0}M^* \oplus T^{0,1}M^*$ the corresponding dual splitting. Since this splitting defines a polarization on M (with respect to h), the vector bundle $\mathcal{S} = \bigwedge T^{0,1}M^* \to M$ becomes a spinor module for $\mathbb{C}l(M)$, with the Clifford action of $\mu \in TM$ given by (4.9), where $\nu = \frac{1}{2}(\mu + iJ\mu)$.

Remark 4.12. We have the usual decomposition $\mathsf{h} = \mathsf{g} + i\omega$ of the Hermitian metric h, where $\mathsf{g} = \frac{1}{2}(\mathsf{h} + \overline{\mathsf{h}})$, and $\omega = \frac{i}{2}(\mathsf{h} - \overline{\mathsf{h}})$. We have that $\mathsf{g}(JX, Y) = \omega(X, Y)$, and $\mathsf{g}(X, Y) = \omega(X, JY)$. With regard to the discussion in Section 3.1, we note that M is a Hermitian manifold if the almost complex structure is integrable, almost Kähler if $d\omega = 0$, and Kähler if both of these conditions hold. In general the Levi-Civita connection ∇^{g} need not preserve $T^{1,0}M$ and $T^{0,1}M$; however, we have $\nabla^{\mathsf{g}}J = 0$ if and only if M is Kähler, in which case ∇^{g} is in fact a Hermitian connection. Unlike the spin case (see [BGV91]), we will usually not have a canonical choice of connection, and our connections will not usually be torsion free; see the discussion in [Nic05].

The Hermitian connection $\widehat{\nabla}$ on M induces a canonical[1] connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$ compatible with the Clifford action on $\mathcal{S}$ in the sense that [BGV91, Nic05]

$$[\nabla^{\mathcal{S}}_X, \mathbf{c}(\nu)]\gamma = \mathbf{c}(\widehat{\nabla}_X \nu)\gamma.$$

If $\mathcal{W} \to M$ is a complex vector bundle equipped with a Hermitian connection $\nabla^{\mathcal{W}}$, we may form the tensor product connection ∇, acting on $s \otimes w \in \mathcal{S} \otimes \mathcal{W}$ by

$$\nabla(s \otimes w) = \nabla^{\mathcal{S}} s \otimes w + s \otimes \nabla^{\mathcal{W}} w. \tag{4.10}$$

It is then possible to define a Dirac operator $\not{D}$ on sections of $\mathcal{S} \otimes \mathcal{W}$ by the composition

$$\Gamma(M, \mathcal{S} \otimes \mathcal{W}) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes (\mathcal{S} \otimes \mathcal{W})) \xrightarrow{\mathbf{c} \otimes \mathrm{Id}_{\mathcal{W}}} \Gamma(M, \mathcal{S} \otimes \mathcal{W}). \tag{4.11}$$

The operator $\not{D}$ is then a *geometric Dirac operator* in the sense of [Nic05]. By [BGV91, Proposition 3.67], if M is Kähler and $\mathcal{W}$ is a holomorphic vector bundle equipped with its canonical holomorphic connection, then the operator so defined coincides with an operator given in terms of the holomorphic $\overline{\partial}_{\mathcal{W}}$ operator:

$$\not{D} = (\mathbf{c} \otimes \mathrm{Id}_{\mathcal{W}}) \circ \nabla = \sqrt{2}(\overline{\partial}_{\mathcal{W}} + \overline{\partial}^*_{\mathcal{W}}).$$

4.3 Differential operators on (M, E)

We now present a construction of a differential operator associated to a subbundle $E \subset TM$ analogous to the construction of the Dolbeault-Dirac operator on an almost Hermitian manifold. In fact, if M is symplectic and $E = TM$, this is precisely the operator that we obtain. We proceed as above, with one modification: when M is odd-dimensional, we must allow polarizations that are not of maximal rank. Thus, we will declare a subbundle $\mathcal{P} \subset T_{\mathbb{C}}M$ to be a polarization if $\mathcal{P}$ is isotropic with respect some

[1] By [Nic05, Proposition 1.3], any other connection compatible with both $\hat{\nabla}$ and the Clifford action differs from $\nabla^{\mathcal{S}}$ by the action of an imaginary 1-form.

Riemannian metric g on M, and $\mathcal{P}\cap\overline{\mathcal{P}}=0$. We note that this definition of polarization agrees with the definition in [Wei97] of a complex polarization on a contact manifold.

By Definition 3.19, such a polarization $\mathcal{P}$ defines an almost CR structure on M. Thus, we suppose M is a smooth manifold equipped with an almost CR structure $E\otimes\mathbb{C}=E_{1,0}(M)\oplus E_{0,1}(M)$, where $E\subset TM$ is a smooth subbundle of even rank. Let $J_b\in\mathrm{End}(E)$ denote the corresponding complex structure on E, and let g be a Riemannian metric on M such that

$$\mathsf{g}(J_bX,J_bY)=\mathsf{g}(X,Y)\quad\text{for all}\quad X,Y\in\Gamma(E).$$

The metric g determines a Hermitian metric h on $E\otimes\mathbb{C}$ with respect to which $E_{1,0}$ is isotropic. We suppose that a compact Lie group G acts on M preserving E; by averaging, we can assume that J_b and g are G-invariant. Let $E^*\otimes\mathbb{C}=E^{1,0}(M)\oplus E^{0,1}(M)$ denote the dual splitting, and equip E with a G-invariant Hermitian connection ∇ (that is, $\nabla\mathsf{h}=\nabla J_b=0$).

Let $\imath:E^0\hookrightarrow T^*M$ and $r:T^*M\to E^*$ denote the inclusion and projection of the exact sequence

$$0\longrightarrow E^0\overset{\imath}{\hookrightarrow}T^*M\overset{r}{\twoheadrightarrow}E^*\longrightarrow 0. \tag{4.12}$$

Let $\mathbb{C}l(E)\to M$ be the bundle whose fibre over $x\in M$ is the Clifford algebra of E_x^* with respect to the restriction of g to E. For any $\mu\in E_x^*$, write $\mu=\nu+\overline{\nu}$, where $\overline{\nu}=\psi(\mu)\in E^{0,1}$ is given by the isomorphism (3.8). The bundle

$$\mathcal{S}=\bigwedge E^{0,1} \tag{4.13}$$

is then a spinor module for $\mathbb{C}l(E)$ with respect to the Clifford multiplication

$$\mathbf{c}(\mu)=\sqrt{2}(\varepsilon(\overline{\nu})-\iota(\nu)), \tag{4.14}$$

where $\epsilon(\overline{\nu})$ denotes exterior multiplication by $\overline{\nu}$, and $\iota(\nu)$ denotes contraction with respect to the Hermitian metric h on $E_{\mathbb{C}}$. Using $\mathbf{c}$ and the connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$ induced by ∇, we define a G-invariant differential operator $\not{D}_b:\Gamma(\mathcal{S})\to\Gamma(\mathcal{S})$ by the composition

$$\not{D}_b:\Gamma(\mathcal{S})\xrightarrow{\nabla^{\mathcal{S}}}\Gamma(T^*M\otimes\mathcal{S})\xrightarrow{r}\Gamma(E^*\otimes\mathcal{S})\xrightarrow{\mathbf{c}}\Gamma(\mathcal{S}). \tag{4.15}$$

Remark 4.13. For the most part the choice of connection will be irrelevant, provided that the connection preserves both the metric and the almost CR structure, since this choice does not affect the equivariant index of the Dirac operator constructed using the connection. However, when the structure of our manifold is particularly nice, choosing the connection wisely may allow us to impart additional geometric significance to the index of the corresponding Dirac operator. For example, as noted above, when M is Kähler the Levi-Civita connection is a Hermitian connection, and we obtain the Dolbeault-Dirac operator, whose index can be identified with the Euler number of a holomorphic vector bundle $\mathcal{W}$.

If $(M, \phi, \eta, \xi, \mathsf{g})$ is a contact metric manifold one can always find a *contact connection*; that is, a connection ∇ such that $\nabla\phi = \nabla\mathsf{g} = 0$. When M is Sasakian, there is a unique contact connection whose torsion is *pure* (see [Ura94], and [DT06, Section 1.2]).

Definition 4.14. *Let $(M, E_{1,0})$ be a non-degenerate CR manifold equipped with contact form η and Reeb field ξ. The* Tanaka-Webster connection *of M is the unique linear connection on M such that*

1. *For all $X \in \Gamma(TM)$, $\nabla_X \Gamma(E) \subset \Gamma(E)$,*
2. *$\nabla J_b = \nabla\xi = \nabla\eta = 0$,*
3. *The torsion T_∇ of ∇ is* pure*; that is, for any $X, Y \in \Gamma(E)$,*

$$T_\nabla(X, Y) = d\eta(X, Y)\xi, \quad \textit{and} \quad T_\nabla(\xi, J_b Y) + J_b T_\nabla(\xi, Y) = 0.$$

It follows from the above that $\nabla\mathsf{g}_\eta = 0$, and that, in terms of sections $Z, W \in \Gamma(E_{1,0})$, we have $T_\nabla(Z, \overline{W}) = 2iL_\eta(Z, \overline{W})\xi$.

Proposition 4.15. *If M is a Sasakian manifold and ∇ is the Tanaka-Webster connection, then the operator $\not{D}_b$ given by* (4.15) *agrees with the operator*

$$\not{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*),$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator (3.11).

Proof. Let $Z_1, \ldots, Z_n$ be a local frame for $E_{1,0}$ and $\theta^1, \ldots, \theta^n$ the corresponding coframe for $E^{1,0}$. Since ∇ is not torsion-free, we cannot express the full exterior differential d in terms of ∇. However, since T_∇ is pure, it vanishes when restricted to $E_{0,1} \otimes E_{0,1}$, and we have (see [DT06, Section 1.7.6])

$$\overline{\partial}_b = \sum_{i=1}^n \varepsilon(\overline{\theta}^i)\nabla_{\overline{Z}_i}.$$

The operator $\overline{\partial}_b^*$ is the L^2 adjoint of $\overline{\partial}_b$ with respect to the inner product

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \; \eta \wedge d\eta^n,$$

where $\langle \, , \, \rangle$ denotes the pairing on $\bigwedge^k E^{0,1}$. Using a similar argument to the Kähler case (see [DT06, Section 1.7.6]; see also [Ura94]), $\overline{\partial}_b^*$ is given locally by

$$\overline{\partial}_b^* = -\sum_{i=1}^n \iota(\overline{Z}_i)\nabla_{Z_i}. \qquad \square$$

We are thus able to interpret the index of $\not{D}_b$ in terms of the Kohn-Rossi cohomology groups [KR65]

$$H^{(i)}_{KR}(M, E_{1,0}) = \frac{\ker(\overline{\partial}_b : \mathcal{A}^{(0,i)}(M) \to \mathcal{A}^{(0,i+1)}(M))}{\operatorname{ran}(\overline{\partial}_b : \mathcal{A}^{(0,i-1)}(M) \to \mathcal{A}^{(0,i)}(M))}. \tag{4.16}$$

Proposition 4.16. *As virtual G-representations we have an isomorphism*

$$\operatorname{index}^G(\not{D}_b) \cong \sum (-1)^i H^{(i)}_{KR}(M, E_{1,0}).$$

Proof. From [Koh65], we have that $H^{(i)}_{KR}(M, E_{1,0}) \cong \ker(\Box_b^i)$, where $\Box_b$ is the Kohn-Rossi Laplacian. Since $\ker(\Box_b) = \ker(\overline{\partial}_b) \cap \ker(\overline{\partial}_b^*)$ [Koh65] and $\Box_b = \not{D}_b^2$, we have $\ker(\Box_b) = \ker(\not{D}_b)$, and the result follows. $\square$

Now, let $\mathcal{W} \to M$ be a CR-holomorphic vector bundle equipped with a Hermitian metric h. There then exists a Hermitian connection $\nabla^{\mathcal{W}}$ on $\mathcal{W}$ such that

$$\nabla^{\mathcal{W}}|_{E_{0,1}} = \overline{\partial}_{\mathcal{W}}.$$

This connection is not unique, but there is a unique such connection satisfying a particular curvature condition (see [Ura94]). Let $\nabla^{\mathcal{S}}$ denote the connection induced on $\mathcal{S} = \bigwedge E^{0,1}$ by the Tanaka-Webster connection on M, and let ∇ be the tensor product connection on $\mathcal{S} \otimes \mathcal{W}$ given by (4.10). We then have the twisted Dirac operator

$$\not{D}_b^{\mathcal{W}} = (\mathbf{c} \otimes \mathrm{Id}_{\mathcal{W}}) \circ r \circ \nabla \tag{4.17}$$

acting on sections of $\mathcal{S} \otimes \mathcal{W}$. A question that we are currently investigating is whether or not the operator $\not{D}_b^{\mathcal{W}}$ can be expressed in terms of the CR -holomorphic operator $\overline{\partial}_{\mathcal{W}}$ (extended to $\mathcal{W}$-valued $(0, q)$-forms) and its L^2 adjoint, and the relationship between the index of $\not{D}_b^{\mathcal{W}}$ and the CR-holomorphic sections of $\mathcal{W}$.

Remark 4.17. In the Sasakian case, we are able to use the Tanaka-Webster connection on M. In general, since the endomorphism J_b is not defined on all of TM, we have only defined ∇ above as a connection on E preserving J_b. However, it is possible to do the above construction on all of TM in the case that our almost CR structure comes from an f-structure. By Proposition 3.37, this amounts to making a choice of complement of E. Given an f-structure such that $E = f(TM)$, the endomorphism J_b is given by $f|_E$, and we have a compatible metric g, by Lemma 3.38. In particular one has $\mathsf{g}(fX, fY) = -\mathsf{g}(X, f^2 Y) = \mathsf{g}(X, Y)$ for $X, Y \in \Gamma(E)$. It is always possible to find a connection ∇ on M adapted to f and g, in the sense that

$$\nabla f = \nabla \mathsf{g} = 0.$$

Such connections are known as (f, g)-linear connections and are characterized in [Soa97], where it is shown that there exists a canonical connection adapted to f and g related to the Levi-Civita connection and depending uniquely on f and g.

When we have a framed f-structure, such a metric is given by (3.22). Recall that an S-structure is a framed f-structure, such that the frame $\eta^1, \ldots, \eta^k$ for E^0 satisfies $d\eta^i = \Phi$ for each i, where the closed form $\Phi(X, Y) = \mathsf{g}(X, fY)$ is the fundamental 2-form. In

[LP04], an *almost S-manifold* is defined as above, but with Φ not necessarily closed, and it is shown that there exists a connection on such manifolds satisfying an extension of the definition of the Tanaka-Webster connection. Moreover, since f is normal, the almost CR structure determined by f is integrable, and the splitting $TM = E \oplus E^{\perp}$ of the tangent bundle defined by the f-structure allows us to define the $\overline{\partial}_b$ operator on such manifolds, using (3.14). It should be interesting to investigate the relationship between $\not{D}_b$ and the $\overline{\partial}_b$ operator in this setting.

4.4 The principal symbol of $\not{D}_b$

If we write $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ with respect to the $\mathbb{Z}_2$-grading given by exterior degree, and write $\gamma = \begin{pmatrix} \gamma^+ \\ \gamma^- \end{pmatrix}$, then the odd operator $\not{D}_b$ can be written as

$$\not{D}_b(\gamma) = \begin{pmatrix} 0 & \not{D}_b{}^- \\ \not{D}_b{}^+ & 0 \end{pmatrix} \begin{pmatrix} \gamma^+ \\ \gamma^- \end{pmatrix}$$

where $\not{D}_b{}^+ = \not{D}_b|_{\mathcal{S}^+} : \mathcal{S}^+ \to \mathcal{S}^-$, and $\not{D}_b{}^- = (\not{D}_b{}^+)^*$ since $\not{D}_b$ is self-adjoint by construction.

Let $\sigma_b : \pi_M^*\mathcal{S}^+ \to \pi_M^*\mathcal{S}^-$ denote the principal symbol of $\not{D}_b$, which is given for any $(x, \xi) \in T^*M$ and $\gamma \in \mathcal{S}^+$ by

$$\sigma_b(x, \xi)(\gamma) = i\mathbf{c}(r(\xi))\gamma. \tag{4.18}$$

Since $\sigma_b^2(x, \xi) = ||r(\xi)||^2$, we have $\mathrm{Supp}(\sigma_b) = E^0$. This implies that when the action of G on M is transverse to E, $\not{D}_b$ is a G-transversally elliptic differential operator, since $E^0 \cap T_G^*M = 0$ (see Remark 2.3). In Section 5.5 we will define the Berline-Vergne Chern character of a transversally elliptic symbol, and compute in particular the Chern character of σ_b.

Chapter 5

Equivariant characteristic forms

We note that much of the material in this chapter can be found in the text [BGV91]; another useful reference is [LM89].

5.1 Superspaces and superconnections

Throughout this thesis, we will make use of the language of superspaces, superbundles and superconnections, which we quickly review here. A *superspace* is simply a $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$. We refer to V^+ as the *even* part of V, and V^- as the *odd* part of V. If V is equipped with a multiplication that respects the $\mathbb{Z}_2$-grading, then we call V a *superalgebra*. The prototypical example of a superalgebra is the exterior algebra $\bigwedge W$ of an (ungraded) vector space W. We define a $\mathbb{Z}_2$-grading on the space of endomorphisms of a superspace V by

$$\begin{aligned}
\mathrm{End}^+(V) &= \mathrm{Hom}(E^+, E^+) \oplus \mathrm{Hom}(E^-, E^-),\\
\mathrm{End}^-(V) &= \mathrm{Hom}(E^+, E^-) \oplus \mathrm{Hom}(E^-, E^+).
\end{aligned}$$

A *superbundle* $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ over a manifold M is a vector bundle $\mathcal{V} \to M$ whose fibres are superspaces.

If A is a superalgebra, we can define the *supercommutator* $[a, b]$ by

$$[a, b] = ab - (-1)^{|a||b|}ba,$$

where $|a|$ is zero if a is even, and one if a is odd. The supercommutator makes A into a Lie superalgebra. For the general definition of a Lie superalgebra (which we will omit here), see [BGV91].

We will also make use of the notion of a *supertrace*, which is any linear form ϕ such that $\phi([a, b]) = 0$ for all elements a, b of a superalgebra A. In particular we will make use of the supertrace on $\mathrm{End}(V)$ (and its superbundle equivalent on $\mathrm{End}(\mathcal{V})$), given by

$$\mathrm{Str}(T) = \begin{cases} \mathrm{Tr}_{V^+} T - \mathrm{Tr}_{V^-} T, & \text{if } T \text{ is even,} \\ 0, & \text{if } T \text{ is odd.} \end{cases} \tag{5.1}$$

The space of differential forms on a manifold M with sections in a superbundle $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ has a $\mathbb{Z}_2$-grading defined by

$$\mathcal{A}^{\pm}(M, \mathcal{V}) = \mathcal{A}^{even}(M, \mathcal{V}^{\pm}) \oplus \mathcal{A}^{odd}(M, \mathcal{V}^{\mp}).$$

We end this section with the definition of a superconnection, a concept due to Quillen [Qui85] and employed by Mathai and Quillen [MQ86] in their construction of the Chern character, which we describe in Chapter 5.

Definition 5.1. *A* superconnection *on a superbundle $\mathcal{V} \to M$ is an odd first order differential operator*

$$\mathbb{A} : \mathcal{A}^{\pm}(M, \mathcal{V}) \to \mathcal{A}^{\mp}(M, \mathcal{V})$$

that satisfies the $\mathbb{Z}_2$-graded Leibniz rule

$$\mathbb{A}(\beta \wedge \gamma) = d\beta \wedge \gamma + (-1)^{|\beta|} \beta \wedge \mathbb{A}\gamma,$$

for all $\beta \in \mathcal{A}(M)$ and $\gamma \in \mathcal{A}(M, \mathcal{V})$.

A superconnection $\mathbb{A}$ is extended to elements $\tau \in \mathcal{A}(M, \mathrm{End}(\mathcal{V}))$ by setting $\mathbb{A}\tau = [\mathbb{A}, \tau]$. As noted in [BGV91], any operator that supercommutes with exterior multiplication is *local*, in the sense that it is given by an element of $\mathcal{A}(M, \mathrm{End}(\mathcal{V}))$. In particular, the operator $\mathbb{A}^2 = \mathbb{F}(\mathbb{A})$, known as the *curvature* of $\mathbb{A}$, is a local operator. When restricted to $\Gamma(M, \mathcal{V})$, a superconnection determines an operator $\mathbb{A} : \Gamma(M, \mathcal{V}^{\pm}) \to \mathcal{A}^{\mp}(M, \mathcal{V})$, that decomposes into a sum $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \cdots$, where the component $\mathbb{A}_{[i]}$ in degree i takes values in $\mathcal{A}^i(M, \mathcal{V})$. The component $\mathbb{A}_{[1]}$ is a connection in the usual sense (that is, a covariant derivative), while $\mathbb{A}_{[0]}$ is an odd endomorphism of $\mathcal{V}$.

5.2 The Cartan model of equivariant cohomology

Suppose that a Lie group G acts smoothly on a manifold M. In [Car51], Cartan introduced the G-equivariant de Rham complex as a method for studying the G-equivariant cohomology of M. Let us recall the definition.[1] We let $\mathcal{A}(\mathfrak{g}, M)$ denote the $\mathbb{Z}$-graded algebra of G-equivariant polynomial maps $\alpha : \mathfrak{g} = \mathrm{Lie}(G) \to \mathcal{A}(M)$ (see [BGV91, Chapter 7]). The $\mathbb{Z}$-grading is defined as follows: the degree of an element $p \otimes \alpha \in (S(\mathfrak{g}^*) \otimes \mathcal{A}(M))^G$ is given by $2\deg(p) + \deg(\alpha)$. Let $\{X^1, \ldots, X^n\}$ be a basis of $\mathfrak{g}$ with dual basis $\{X_1, \ldots X_n\}$ for $\mathfrak{g}^*$. We define the equivariant differential D on $\mathcal{A}(\mathfrak{g}, M)$ by the formula

$$D(p \otimes \alpha) = p \otimes d\alpha - \sum_{i=1}^{n} X_i p \otimes \iota(X^i_M)\alpha. \tag{5.2}$$

where X^i_M denotes the vector field on M denoted by the infinitesimal action of X. We will usually denote an element $X \mapsto \alpha(X)$ of $\mathcal{A}(\mathfrak{g}, M)$ by $\alpha(X)$, and write the equivariant differential D as

$$(D\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)). \tag{5.3}$$

We will typically omit the first pair of parentheses in (5.3) and write the equivariant differential of $\alpha(X)$ as $D\alpha(X)$.

[1] For the most part we will follow the notational conventions in [PV07a, PV08a, PV08b].

For any G-invariant vector field ξ, the Lie derivative and contraction operators $\mathcal{L}(\xi)$ and $\iota(\xi)$ commute with the action of G on N, and thus extend to $\mathcal{A}(\mathfrak{g}, M)$. Moreover, we see that Cartan's formula holds when d is replaced with the equivariant differential D: we have

$$\mathcal{L}(\xi) = D \circ \iota(\xi) + \iota(\xi) \circ D. \tag{5.4}$$

We note that $\deg(D\alpha(X)) = \deg(\alpha(X)) + 1$, and that

$$D^2\alpha(X) = -d\iota(X_M)\alpha(X) - \iota(X_M)d\alpha(X) = -\mathcal{L}(X)\alpha(X),$$

where $\mathcal{L}(X)$ denotes the Lie derivative in the direction of X_M. Thus, we have $D^2 = 0$ on the space $\mathcal{A}(\mathfrak{g}, M)$ of G-invariant elements, whence the equivariant cohomology $\mathcal{H}(\mathfrak{g}, M)$ can be defined as the cohomology of the complex $\mathcal{A}(\mathfrak{g}, M)$.

We will not make use of the above complex, known as the *Cartan model* of equivariant cohomology, but will work instead with its extension $\mathcal{A}^\infty(\mathfrak{g}, M)$, the space of G-equivariant differential forms on M whose coefficients depend smoothly on X in a neighbourhood of $0 \in \mathfrak{g}$. While this algebra is no longer $\mathbb{Z}$-graded, it does have a $\mathbb{Z}_2$-grading with respect to odd and even elements. The differential D extends to $\mathcal{A}^\infty(\mathfrak{g}, M)$, and is an odd operator with respect to the $\mathbb{Z}_2$-grading. It is still the case that $D^2 = 0$, and we denote by $\mathcal{H}^\infty(\mathfrak{g}, M)$ the corresponding cohomology space, which is studied in [DV93].

Finally, we recall that a G-equivariant vector bundle $\mathcal{V}$ is a vector bundle $\pi : \mathcal{V} \to M$ such that for each $g \in G$ the action $g^{\mathcal{V}} : \mathcal{V}_x \to \mathcal{V}_{g\cdot x}$ is linear, and G acts compatibly on M, in the sense that

$$g \circ \pi = \pi \circ g.$$

The corresponding action on the space of sections of $\mathcal{V}$ is given by

$$(g \cdot s)(x) = g^{\mathcal{V}} \cdot s(g^{-1}x).$$

We will denote by $\mathcal{A}(\mathfrak{g}, M; \mathcal{V}) = (C^\infty(\mathfrak{g} \otimes \mathcal{A}(M, \mathcal{V}))^G$ the space of equivariant differential forms on M that take values in a G-equivariant vector bundle $\mathcal{V}$.

5.3 Chern-Weil forms

Suppose now that $\mathcal{V} \to M$ is a given superbundle. The supertrace map (5.1) can be extended to a map

$$\mathrm{Str} : \mathcal{A}^{\pm}(M, \mathrm{End}(\mathcal{V})) \to \mathcal{A}^{\pm}(M) \tag{5.5}$$

by setting $\mathrm{Str}(\alpha \otimes \varphi) = \alpha\, \mathrm{Str}(\varphi)$ for all $\alpha \in \mathcal{A}(M)$ and $\varphi \in \Gamma(\mathrm{End}(\mathcal{V}))$. If $\mathcal{V}$ is equipped with a superconnection $\mathbb{A}$, then the curvature $\mathbb{A}^2$ is a section of $\mathcal{A}^+(M, \mathrm{End}(\mathcal{V}))$. If $f(z)$ is a polynomial, then we can form the element $f(\mathbb{A}) \in \mathcal{A}^+(M, \mathrm{End}(\mathcal{V}))$. Applying the map (5.5) we obtain the *characteristic form* (or Chern-Weil form)

$$\mathrm{Str}(f(\mathbb{A}^2)) \in \mathcal{A}^+(M) \tag{5.6}$$

associated to f and $\mathbb{A}$. For any f the form $\mathrm{Str}(f(\mathbb{A}^2))$ is closed and of even degree [BGV91]. Moreover, the de Rham cohomology class of $\mathrm{Str}(f(\mathbb{A}^2))$ is independent of the choice of superconnection $\mathbb{A}$.

When the curvature of $\mathbb{A}$ is nilpotent (for example, if $\mathbb{A}$ is an ordinary connection ∇) we can extend the above constructure to arbitrary smooth functions $f(z)$ by Taylor's formula. We define

$$f(\mathbb{A}^2) = \sum \frac{f^{(k)}(0)}{k!} \mathbb{A}^{2k}; \tag{5.7}$$

we allow the same definition for an arbitrary superconnection $\mathbb{A}$ provided that f is an entire analytic function. Let us now define those characteristic forms of which we will make use.

Definition 5.2. *Let $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ be a complex superbundle equipped with superconnection $\mathbb{A}$. The* Chern character *form of $\mathbb{A}$ is defined by*

$$\mathrm{Ch}(\mathbb{A}) = \mathrm{Str}(e^{\mathbb{A}^2}). \tag{5.8}$$

The Chern character has a number of useful properties. We have:

1. $\mathrm{Ch}(\mathbb{A}_1 \oplus \mathbb{A}_2) = \mathrm{Ch}(\mathbb{A}_1) + \mathrm{Ch}(\mathbb{A}_2)$ for the superconnection $\mathbb{A}_1 \oplus \mathbb{A}_2$ on $\mathcal{V}_1 \oplus \mathcal{V}_2$,

2. $\mathrm{Ch}(\mathbb{A}_1 \otimes 1 + 1 \otimes \mathbb{A}_2) = \mathrm{Ch}(\mathbb{A}_1) \wedge \mathrm{Ch}(\mathbb{A}_2)$ for the superconnection $\mathbb{A}_1 \otimes 1 + 1 \otimes \mathbb{A}_2$ on $\mathcal{V}_1 \otimes \mathcal{V}_2$,

3. The cohomology class of $\mathrm{Ch}(\mathbb{A})$ is independent of $\mathbb{A}$; in particular $[\mathrm{Ch}(\mathbb{A})] = [\mathrm{Ch}(\nabla)]$, where $\nabla = \mathbb{A}_{[1]}$.

Given the last of these properties, we refer to $[\mathrm{Ch}(\mathbb{A})]$ as the Chern character of the *bundle* $\mathcal{V}$, and write $[\mathrm{Ch}(\mathbb{A})] = \mathrm{Ch}(\mathcal{V})$. By decomposing ∇ into its components on $\mathcal{V}^+$ and $\mathcal{V}^-$, we have

$$\mathrm{Ch}(\mathcal{V}) = \mathrm{Ch}(\mathcal{V}^+) - \mathrm{Ch}(\mathcal{V}^-).$$

Definition 5.3. *Let $\mathcal{V} \to M$ be a real vector bundle equipped with a connection ∇ with curvature $F(\nabla)$. The $\widehat{\mathrm{A}}$-form on $\mathcal{V}$ is the characteristic form given by*

$$\widehat{\mathrm{A}}(\mathcal{V}) = \det{}^{1/2}\left(\frac{F(\nabla)/2}{\sinh(F(\nabla)/2)}\right). \tag{5.9}$$

The $\widehat{\mathrm{A}}$-genus of TM will be denoted by $\widehat{\mathrm{A}}(M)$.

Definition 5.4. *Let $\mathcal{V} \to M$ be a complex vector bundle equipped a connection ∇ with curvature $F(\nabla)$. The Todd form on $\mathcal{V}$ is given by*

$$\mathrm{Td}(\mathcal{V}) = \det\left(\frac{F(\nabla)}{e^{F(\nabla)} - 1}\right). \tag{5.10}$$

When M is an (almost) complex manifold, we denote by $\mathrm{Td}(M)$ the Todd genus of TM.

Remark 5.5. Both $\widehat{\mathrm{A}}(\nabla)$ and $\mathrm{Td}(\nabla)$ are multiplicative with respect to direct sums. They are (exponentials of) characteristic forms in the sense of (5.6) by the identity $\det(\alpha) = \exp(\mathrm{Tr}(\log\alpha))$ for $\alpha \in \mathcal{A}^+(M, \mathrm{End}(E))$. On a complex vector bundle $\mathcal{V}$ of complex rank k, we note that the Todd genus is computed using the $k \times k$ complex determinant, while the $\widehat{\mathrm{A}}$ genus is computed using the Pfaffian of the corresponding real $2k \times 2k$ matrix. When there is a possibility of confusion we will use the notation $\det_{\mathbb{R}}$ and $\det_{\mathbb{C}}$ to specify which determinant we are using.

Proposition 5.6. *With respect to the usual decomposition $\mathcal{V}\otimes\mathbb{C}=\mathcal{V}^{1,0}\oplus\mathcal{V}^{0,1}$ we have the following relationship between the $\widehat{\mathrm{A}}$ and Todd forms [LM89]:*

$$\widehat{\mathrm{A}}(\mathcal{V})^2=\mathrm{Td}(\mathcal{V}^{1,0})\wedge\mathrm{Td}(\mathcal{V}^{0,1}). \tag{5.11}$$

We also define a *Thom form* on an oriented vector bundle $p:\mathcal{V}\to M$ of rank n to be a differential form $\mathrm{Th}(\mathcal{V})$ such that $p_*\,\mathrm{Th}(\mathcal{V})=1$, where p_* denotes integration over the fibres of $\mathcal{V}$ (see [PV07a] for a thorough discussion of the Thom form). The pullback of $\mathrm{Th}(\mathcal{V})$ to M (embedded as the zero section) defines a characteristic form known as the *Euler form*, given, for any connection ∇, by

$$\mathrm{Eul}(\mathcal{V})=\det{}^{1/2}\left(\frac{-F(\nabla)}{2\pi}\right).$$

If we suppose that a Lie group G acts smoothly on M, we can extend the above construction of characteristic forms to the space of equivariant differential forms on M as follows:

Let $\mathcal{V}=\mathcal{V}^+\oplus\mathcal{V}^-\to M$ be a G-equivariant superbundle. We say a superconnection $\mathbb{A}$ on $\mathcal{V}$ is *G-invariant* if it commutes with the action of G on $\mathcal{A}(M,\mathcal{V})$. From $\mathbb{A}$ one can form the corresponding equivariant superconnection $\mathbb{A}_{\mathfrak{g}}$ on $\mathcal{A}^\infty(\mathfrak{g},M;\mathcal{V})$ given for $X\in\mathfrak{g}$ by

$$(\mathbb{A}_{\mathfrak{g}}\alpha)(X)=(\mathbb{A}-\iota(X_M))(\alpha(X)),$$

so that $\mathbb{A}_{\mathfrak{g}}(\beta\wedge\gamma)=D\beta(X)\wedge\gamma+(-1)^{|\beta|}\beta\wedge\mathbb{A}_{\mathfrak{g}}\gamma$, for each $\beta\in\mathcal{A}(\mathfrak{g},M)$ and $\gamma\in\mathcal{A}(\mathfrak{g},M;\mathcal{V})$.

Definition 5.7. *Given a superconnection $\mathbb{A}$ on $\mathcal{V}$, we define the* moment *of $\mathbb{A}$ to be the map*

$$\mu^{\mathbb{A}}:\mathfrak{g}\to\mathcal{A}^+(M,\mathrm{End}(\mathcal{V}))$$

given by

$$\mu^{\mathbb{A}}(X)=\mathcal{L}^{\mathcal{V}}(X)-[\iota(X_M),\mathbb{A}], \tag{5.12}$$

where $\mathcal{L}^{\mathcal{V}}(X)$ denotes the Lie derivative with respect to $X_{\mathcal{V}}$. We define the equivariant curvature *of $\mathbb{A}$ by*

$$\mathbb{F}(\mathbb{A})(X)=\mathbb{A}^2-\mu^{\mathbb{A}}(X). \tag{5.13}$$

The equivariant curvature is an element of $\mathcal{A}(\mathfrak{g}, M; \mathrm{End}(\mathcal{V}))$; in terms of the equivariant superconnection $\mathbb{A}_{\mathfrak{g}}$ it is given by $\mathbb{F}(\mathbb{A}) = \mathbb{A}_{\mathfrak{g}}^2 + \mathcal{L}^{\mathcal{V}}(X)$. In the case that $\mathbb{A}$ is an ordinary G-invariant connection ∇, we have $F(\nabla)(X) = F(\nabla) - \mu^{\nabla}(X)$, where μ^{∇} is given simply by $\mu^{\nabla}(X) = \mathcal{L}^{\mathcal{V}}(X) - \nabla_{X_M}$. By replacing the usual curvature in the above characteristic class definitions with the equivariant curvature, we obtain the corresponding *equivariant* characteristic classes. For example, we have

$$\mathrm{Ch}(\mathbb{A}, X) = \mathrm{Str}\left(e^{\mathbb{F}(\mathbb{A})(X)}\right). \tag{5.14}$$

For a G-invariant connection ∇ on a real vector bundle $\mathcal{V}$ and $X \in \mathfrak{g}$ sufficiently small, we can define the equivariant $\widehat{\mathrm{A}}$-form $\widehat{\mathrm{A}}(\mathcal{V}, X)$ by replacing the curvature $F(\nabla)$ by the equivariant curvature $F(\nabla)(X)$. Given a complex vector bundle $\mathcal{W}$ we can similarly define the equivariant Todd form $\mathrm{Td}(\mathcal{W}, X)$.

For each $g \in G$, we can consider the decomposition (4.4) of the restriction of TM to $M(g)$. If ∇ is an invariant connection on TM, its restriction to the fibres over $M(g)$ decomposes into components on $TM(g)$ and $\mathcal{N}(g)$ according to

$$\nabla|_{M(g)} = \nabla^g \oplus \nabla^{\mathcal{N}}.$$

We define the $G(g)$-equivariant characteristic form $\widehat{\mathrm{A}}(M(g), X)$ using the connection ∇^g, for $X \in \mathfrak{g}(g)$. If M is equipped with a G-invariant almost complex structure and ∇ preserves this structure, then we can similarly define $\mathrm{Td}(M(g))$. We also have the equivariant characteristic form $D_g(\mathcal{N}(g), X)$ associated to the normal bundle, defined for $X \in \mathfrak{g}(g)$ sufficiently small by

$$D_g(\mathcal{N}(g), X) = \det(1 - g^{\mathcal{N}} \exp(F(\nabla^{\mathcal{N}})(X))). \tag{5.15}$$

If $\mathcal{N}(g)$ is a complex vector bundle, we will use the notation $D_g^{\mathbb{R}}$ and $D_g^{\mathbb{C}}$ to denote the above form defined using real and complex determinants, respectively.

Remark 5.8. Let us briefly recall the Chern-Weil homomorphism, which is a key tool in the section to follow. We follow the conventions used in [BGV91] and [PV07a]. Let

$P \to M$ be a principal G-bundle, equipped with connection 1-form $\omega \in \mathcal{A}^1(P, \mathfrak{g})^G$ with curvature Ω. The connection form ω determines a splitting $TP = HP \oplus VP$, where $HP = \ker \omega$ is the horizontal subbundle and $VP \cong P \times \mathfrak{g}$ is the vertical bundle, spanned by the vector fields X_P, $X \in \mathfrak{g}$. We denote by $\mathcal{A}_{hor}(P) = (VP)^0$ the space of horizontal forms, and by $h : \mathcal{A}(P) \to \mathcal{A}_{hor}(P)$ the projection. We have the usual identification $\mathcal{A}_{bas}(P) \cong \mathcal{A}(M)$, where $\mathcal{A}_{bas}(P) = \mathcal{A}_{hor}(P)^G$ is the space of basic forms on P.

Now let V be a vector space on which a representation of G is given.[2] We then have the associated vector bundle $\mathcal{V} = P \times_G V$ and the identification $\mathcal{A}(\mathcal{V}) \cong \mathcal{A}_{bas}(P, V)$. The *Chern-Weil homomorphism* is the map $\phi_\omega^V : \mathcal{A}(\mathfrak{g}, V) \to \mathcal{A}(\mathcal{V})$ given by

$$\phi_\omega^V(\alpha(X)) = h(\alpha(\Omega)). \tag{5.16}$$

Using the identification $\mathcal{A}(\mathcal{V}) \cong \mathcal{A}_{bas}(P, V)$ the connection ω determines a covariant derivative ∇ on $\mathcal{V}$ as the operator corresponding to

$$d_\omega = d + \sum \omega^i \rho(X_i),$$

where $\omega = \sum \omega^i \otimes X_i$ with respect to a basis $\{X_1, \ldots, X_k\}$ for $\mathfrak{g}$, and $\rho(X_i)$ denotes the infinitesimal action of X_i on V corresponding to the G-representation. The curvature of ∇ corresponds to the action of Ω with respect to the representation ρ [MQ86].

In [PV07a] we have the following extension of the Chern-Weil homomorphism to equivariant cohomology: Let P be a principal G-bundle, and suppose that a compact Lie group H acts on P as well, commuting with the action of G, so that $H \times G$ acts on P by $(h, g) \cdot p = hpg^{-1}$. We then suppose that the connection ω is H-invariant, and let

$$\Omega(Y) = \Omega - \iota(Y_P)\omega,\ Y \in \mathfrak{h},$$

denote the H-equivariant curvature of ω. If V is a G-representation then we have a Chern-Weil map

$$\phi_\omega^V : \mathcal{A}(\mathfrak{g}, V) \to \mathcal{A}_{bas}(\mathfrak{h}, P \times V) \cong \mathcal{A}(\mathfrak{h}, \mathcal{V}), \tag{5.17}$$

[2] More generally, we may take V to be any G-manifold, but the case of a G-representation suffices for our purposes.

given by $\phi^V_\omega(f)(Y) = h(f(\Omega(Y)))$.

5.4 The Mathai-Quillen construction

Let $\pi : \mathcal{V} \to M$ be a Euclidean vector bundle of even rank, and suppose $\mathcal{W} \to M$ is a Hermitian superbundle[3] equipped with a Hermitian connection ∇. We now review the construction in [MQ86] of the Chern character associated to a symbol mapping $\sigma : \pi^*\mathcal{W}^+ \to \pi^*\mathcal{W}^-$. Our notations and sign conventions are chosen to agree with those in [PV07a, PV08a, PV08b]. Given such a σ, we may associate to it the odd endomorphism $v_\sigma \in \mathrm{End}^-(\mathcal{W})$ given by

$$v_\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}. \tag{5.18}$$

We may then form Quillen's [Qui85] superconnection $\mathbb{A}^\sigma$ on $\pi^*\mathcal{W}$, given by

$$\mathbb{A}^\sigma = \pi^*\nabla + iv_\sigma. \tag{5.19}$$

The curvature of $\mathbb{A}^\sigma$ is given by

$$\mathbb{F}(\mathbb{A}^\sigma) = (\mathbb{A}^\sigma)^2 = -v_\sigma^2 + i[\pi^*\nabla, v_\sigma] + \pi^*F_\nabla, \tag{5.20}$$

where $v_\sigma^2 = \begin{pmatrix} \sigma^*\sigma & 0 \\ 0 & \sigma\sigma^* \end{pmatrix}$. Thus v_σ^2 is non-negative, and positive-definite away from

$$\mathrm{Supp}(\sigma) = \{(x,\xi) \in \mathcal{V} | \sigma(x,\xi) \text{ is not invertible}\}.$$

For example, when σ is an elliptic symbol, v_σ^2 is positive-definite away from the zero section in $\mathcal{V}$.

In the case that $\mathcal{V}$ is a complex vector bundle, we can take $\mathcal{W}$ to be the complex spinor bundle $S = \bigwedge \mathcal{V}^{0,1}$ as described in Section 4.2, and we let σ be given by $\sigma(x,\xi) = i\mathbf{c}(\xi)$,

[3]In [MQ86] $\mathcal{V}$ is assumed to have spin structure, and $\mathcal{W}$ is given by the associated spinor bundle. For the class of symbols we will consider, $\mathcal{W}$ will always be a Hermitian superbundle, and so we discuss here the complex variant of their construction.

where $(x, \xi) \in \mathcal{V}$ and $\mathbf{c}(\xi)$ is the Clifford multiplication (4.9). The Clifford multiplication is defined with respect to the Hermitian metric such that $\mathbf{c}(\xi)^* = -\mathbf{c}(\xi)$, and thus $\sigma^* = \sigma$, whence

$$\sigma^*\sigma(x, \xi) = \sigma\sigma^*(x, \xi) = (i\mathbf{c}(\xi))^2 = ||\xi||^2. \tag{5.21}$$

Using the Chern-Weil homomorphism, one can reduce the problem of computing the Chern character to explicit computation on a vector space (which we do not repeat here). Doing so, one determines that Quillen's Chern character

$$\mathrm{Ch}_Q(\mathbb{A}^\sigma) = \mathrm{Str}(e^{\mathbb{F}(\mathbb{A}^\sigma)}) \tag{5.22}$$

can be expressed in terms of a "Gaussian-shaped" Thom form $\mathrm{Th}_Q(\mathcal{V})$ that is rapidly decreasing on the fibres of $\mathcal{V}$ thanks to the term $e^{-||\xi||^2}$. The calculations in [MQ86] have been repeated carefully in [PV07a] in several settings, and in particular for the corresponding equivariant characteristic forms. The result from [PV07a] of which we will make most use is the following extension of the Riemann-Roch formula in [MQ86] to the equivariant setting.

Proposition 5.9 ([PV07a])**.** *Let $p : \mathcal{V} \to M$ be a G-equivariant Hermitian vector bundle of complex rank n equipped with G-invariant Hermitian connection ∇. For the symbol $\sigma = i\mathbf{c}$ defined on $p^* \bigwedge \mathcal{V}^{0,1}$, we have*

$$\mathrm{Ch}_Q(\mathbb{A}^\sigma, X) = (2\pi i)^n\, \mathrm{Td}(\mathcal{V}, X)^{-1}\, \mathrm{Th}_{MQ}(\mathcal{V}, X), \tag{5.23}$$

where $\mathrm{Th}_Q(\mathcal{V}, X)$ is the equivariant version of the Mathai-Quillen Thom form.

In particular, suppose that $(M, E_{1,0})$ is an almost CR manifold of type (n, k), let $E \subset TM$ denote the Levi distribution, and consider the complex vector bundle $p : E^* \to M$. For the restriction to E^* of the symbol σ_b (4.18) we have

$$\mathrm{Ch}_Q(\mathbb{A}^{\sigma_b}, X) = (2\pi i)^n\, \mathrm{Td}(E^*, X)^{-1}\, \mathrm{Th}_{MQ}(E^*, X). \tag{5.24}$$

5.5 Chern character of a transversally elliptic symbol

Suppose now that a compact Lie group G acts on M, let $\mathcal{V} \to M$ be a G-equivariant superbundle equipped with G-invariant connection ∇. Let $\mathcal{W} = \pi^*\mathcal{V} \to T^*M$, and let $\sigma : \pi^*\mathcal{V}^+ \to \pi^*\mathcal{V}^-$ be a G-transversally elliptic symbol. In [BV96a, BV96b], Berline and Vergne define a Chern character $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ by modifying Quillen's Chern character as follows:

Definition 5.10. *Let $\theta \in \mathcal{A}^1(T^*M)$ denote the canonical 1-form on T^*M, and define the superconnection*

$$\mathbb{A}^{\sigma,\theta} = \pi^*\nabla + iv_\sigma + i\theta, \tag{5.25}$$

where we write θ as shorthand for the endomorphism $\varepsilon(\theta) \cdot \mathrm{Id}$ on $\mathcal{W}^+ \oplus \mathcal{W}^-$.

The equivariant curvature of $\mathbb{A}^{\sigma,\theta}$ is given by

$$\mathbb{F}(\mathbb{A}^{\sigma,\theta})(X) = \mathbb{F}(\mathbb{A}^{\sigma})(X) + iD\theta(X),$$

and the Chern characters $\mathrm{Ch}_Q(\mathbb{A}^\sigma, X)$ and $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ are related by

$$\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X) = e^{iD\theta(X)}\,\mathrm{Ch}_Q(\mathbb{A}^\sigma, X). \tag{5.26}$$

The Berline-Vergne Chern character is studied extensively in [BV96a] and appears in the Berline-Vergne formula for the cohomological index of σ. The fact that this index is well-defined depends upon certain growth conditions placed upon the symbol σ and the corresponding Chern character. We will mention an alternative approach due to Paradan and Vergne in Section 6.4 below, that replaces $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ by a compactly-supported form, at the expense of introducing equivariant differential forms with generalized coefficients.

Given a fixed $g \in G$, we can define a $G(g)$-equivariant Chern character on $T^*M(g)$, following [BV96a]. Let $j : T^*M(g) \to T^*M$ denote the inclusion of the g-fixed points.

Denoting by $g^{\mathcal{W}}$ the corresponding action of g on $\mathcal{W}$, we define for any superconnection $\mathbb{A}$ and $X \in \mathfrak{g}(g)$ the $G(g)$-equivariant form on $\mathcal{W}|_{T^*M(g)}$ given by

$$\mathrm{Ch}^g(\mathbb{A}, X) = \mathrm{Str}\left(g^{\mathcal{W}} \cdot j^* e^{\mathbb{F}(\mathbb{A})(X)}\right). \tag{5.27}$$

We note that when $g = e$ we obtain the usual definition of the Chern character. We have

$$j^*\mathbb{A}^{\sigma,\theta} = j^*\pi^*\nabla + i \begin{pmatrix} \theta_g & \sigma_g^* \\ \sigma_g & \theta_g \end{pmatrix},$$

where σ_g and θ_g denote the restrictions of σ and θ to $T^*M(g)$. By [BV96a, Lemma 19], we have that θ_g is the canonical 1-form on $T^*M(g)$, and if σ is G-transversally elliptic, then σ^g is $G(g)$-transversally elliptic. Let $\mathcal{V}|_{M(g)} = \mathcal{V}(g) \oplus \mathcal{N}_{\mathcal{V}}$ denote the decomposition of $\mathcal{V}$ where $\mathcal{V}(g)$ consists of the vectors in $\mathcal{V}$ fixed by the action of $g^{\mathcal{V}}$, and $\mathcal{N}_{\mathcal{V}}$ is the normal bundle to $\mathcal{V}(g)$ in $\mathcal{V}$. Let $\nabla = \nabla^g \oplus \nabla^{\mathcal{N}}$ the corresponding decomposition of ∇. By the commutative diagram

$$\begin{tikzcd} T^*M(g) \arrow[r, hook, "j"] \arrow[d, "\pi_g"] & T^*M \arrow[d, "\pi"] \\ M(g) \arrow[r, hook, "\imath"] & M, \end{tikzcd}$$

we see that

$$j^*\pi^*\nabla = \pi_g^*\imath^*\nabla = \pi_g^*\nabla^g \oplus \pi_g^*\nabla^{\mathcal{N}}.$$

Let us denote by $\mathbb{A}_g^{\sigma,\theta}$ the $G(g)$-invariant superconnection on $\pi_g^*\mathcal{V}(g)$ given by

$$\mathbb{A}_g^{\sigma,\theta} = \pi_g^*\nabla^g + i \begin{pmatrix} \theta_g & \sigma_g^* \\ \sigma_g & \theta_g \end{pmatrix}. \tag{5.28}$$

From the decomposition $j^*\mathbb{A}^{\sigma,\theta} = \mathbb{A}_g^{\sigma,\theta} \oplus \pi_g^*\nabla^{\mathcal{N}}$ we obtain

Lemma 5.11. *The Berline-Vergne Chern character of $\mathbb{A}^{\sigma,\theta}$ satisfies*

$$\mathrm{Ch}_{BV}^g(\mathbb{A}^{\sigma,\theta}, X) = \mathrm{Ch}(\mathbb{A}_g^{\sigma,\theta}, X)\,\mathrm{Str}(g^{\mathcal{V}} \cdot e^{\pi_g^* F(\nabla^{\mathcal{N}})(X)}). \tag{5.29}$$

We can refine (5.29) further in the settings with which we will be concerned, and write the second term on the right-hand side of (5.11) in terms of the form $D_g(\mathcal{N}(g), X)$:

Lemma 5.12. *Suppose that $\mathcal{V} = \bigwedge \mathcal{E}^*$, where $\mathcal{E} \to M$ is a Hermitian vector bundle, and that ∇ comes from a Hermitian connection on $\mathcal{E}$. Let $\mathcal{N}(g)$ denote the normal bundle to $\mathcal{E}(g)$ in $\mathcal{E}$. We then have*

$$\mathrm{Str}(g^{\mathcal{V}} \cdot e^{F(\nabla^{\mathcal{N}})(X)}) = D_g^{\mathbb{C}}(\overline{\mathcal{N}(g)}, X). \tag{5.30}$$

Proof. From [Ver96, Section 3], we deduce that

$$\mathrm{Str}(g^{\mathcal{V}} \cdot e^{F(\nabla^{\mathcal{N}})(X)}) = D_g^{1/2}(\mathcal{N}(g)^*, X).$$

The normal bundle $\mathcal{N}(g)$ inherits a complex structure from $\mathcal{E}$, and using the Hermitian metric we have the identification $\mathcal{N}(g)^* = \overline{N(g)}$. □

5.5.1 The contact case

Suppose that $(M, \phi, \alpha, \xi, \mathsf{g})$ is a co-oriented contact metric manifold, and that a compact Lie group G acts on M by co-orientation preserving contact transformations. Let $E^* \otimes \mathbb{C} = E^{1,0} \oplus E^{0,1}$ be the decomposition into the $\pm i$-eigenbundles of $\phi|_E$. Let ∇ be a Hermitian connection on E. As described in Section 4.3, the bundle $\mathcal{S} = \bigwedge E^{0,1}$ is a spinor module for the Clifford bundle $\mathbb{C}l(E)$, and we have the symbol

$$\sigma_b(x, \xi) = i\mathbf{c}(r(\xi)) : \pi^*\mathcal{S}^+ \to \pi^*\mathcal{S}^-$$

defined in Section 4.4. If the action of G is transverse to E, then σ_b is a G-transversally elliptic symbol, and by Proposition 4.9, we have that $(M(g), E(g))$ is a contact manifold for any $g \in G$. By averaging we can assume that α is G-invariant, so that

$$T^*M|_{M(g)} = T^*M(g) \oplus \mathcal{N}^*(g) = E^*(g) \oplus \mathbb{R}\alpha_g \oplus \mathcal{N}^*(g),$$

where α_g denotes the restriction of α to $M(g)$. Here $F(\nabla^{\mathcal{N}})$ is the curvature of $\mathcal{N}(g)$, and since $\mathcal{N}^*(g)$ carries a complex structure induced by that on E^*, by Proposition 5.11 we have $\mathrm{Ch}^g_{BV}(\mathbb{A}^{\sigma_b,\theta}, X) = \mathrm{Ch}(\mathbb{A}_g^{\sigma_b,\theta}, X) D_g^{\mathbb{C}}(\mathcal{N}(g), X)$, and using (5.24) we can write

$$\mathrm{Ch}(\mathbb{A}_g^{\sigma_b,\theta}, X) = e^{iD\theta(X)}\, \mathrm{Td}(E^*, X)^{-1} r^*\, \mathrm{Th}_{MQ}(E^*, X). \tag{5.31}$$

5.5.2 The almost CR case

The general calculation is essentially the same as the one given above in the contact case, but slightly more complicated due to the fact that we no longer have a trvialization of the annihilator $E^0 \subset T^*M$ of E, and the intersection $\mathcal{N}(g) \cap (E^0|_{M(g)})$ may be non-trivial.

Let $(M, E_{1,0})$ be an almost CR manifold with Levi distribution E. Equip M with a Riemannian metric compatible with the almost complex structure J_b on E, and equip E with a Hermitian connection ∇. We again have the morphism σ_b given by (4.18), defined using the Hermitian metric on $E^{0,1}$ induced by g. For any $g \in G$ we again have the decomposition $T^*M|_{M(g)} = T^*M(g) \oplus \mathcal{N}(g)$, but it need no longer be the case that $\mathcal{N}(g) \subset E|_{M(g)}$. Thus we let

$$\mathcal{N}_E(g) = \mathcal{N}(g) \cap (E|_{M(g)}),$$

and let $\nabla = \nabla^g \oplus \nabla^{\mathcal{N}}_E$ denote the corresponding decomposition of ∇ on $E^*|_{M(g)}$. The remainder of the calculation is as above, except that we obtain the term

$$D_g^{\mathbb{C}}(\mathcal{N}_E(g), X) = \det\nolimits_{\mathbb{C}}\left(1 - g^{\mathcal{N}} \cdot e^{F(\nabla^{\mathcal{N}}_E)(X)}\right) \tag{5.32}$$

corresponding to $\mathcal{N}_E$ rather than the full normal bundle.

Chapter 6

Equivariant cohomology with generalized coefficients

6.1 Basic Definitions

Let G be a compact, connected Lie group, and let $\pi : N \to B$ be an oriented G-equivariant vector bundle. Let us consider some variations on the complex $(\mathcal{A}^\infty(\mathfrak{g}, N), D)$ defined in Section 5.2. We denote by

$$\mathcal{A}_c^\infty(\mathfrak{g}, N) = (C^\infty(\mathfrak{g}) \otimes \mathcal{A}_c(N))^G$$

the space of equivariant differential forms with compact support, and by $\mathcal{H}_c^\infty(\mathfrak{g}, N)$ its cohomology with respect to D. When N is oriented we have the map

$$\mathcal{H}_c^\infty(\mathfrak{g}, N) \to C^\infty(\mathfrak{g})^G \tag{6.1}$$

induced by integration over N. We can also define the space $\mathcal{A}_{rd}^\infty(\mathfrak{g}, N)$ of differential forms that are rapidly decreasing on the fibres. The map

$$\pi_* : \mathcal{A}_{rd}^\infty(\mathfrak{g}, N) \to \mathcal{A}^\infty(\mathfrak{g}, B)$$

given by integration over the fibres is defined on $\mathcal{A}_{rd}^\infty(\mathfrak{g}, N)$; we also denote by π_* the corresponding map in cohomology.

Remark 6.1. As shown in [MQ86] (and [PV07a] for equivariant differential forms), if we are given an elliptic symbol $\sigma : \pi^*\mathcal{V}^+ \to \pi^*\mathcal{V}^-$ as in Section 5.4, then Quillen's Chern character form $\mathrm{Ch}_Q(\mathbb{A}^\sigma, X)$ belongs to $\mathcal{A}^\infty_{rd}(\mathfrak{g}, T^*M)$.

In [BV96a] Berline and Vergne define the space $\mathcal{A}^\infty_{rdm}(\mathfrak{g}, N)$ of equivariant differential forms that are *rapidly decreasing in* $\mathfrak{g}$*-mean*: we have $\alpha(X) \in \mathcal{A}^\infty_{rdm}(\mathfrak{g}, N)$ if for any compactly supported test function $\varphi \in C^\infty(\mathfrak{g})$ the differential form

$$\int_{\mathfrak{g}} \alpha(X)\varphi(X)\, dX$$

is rapidly decreasing on the fibres of N. An example of such a form is the Berline-Vergne Chern character form $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ on T^*M given by (5.26), as shown in [BV96a]. The map (6.1) can be defined on $\mathcal{A}^\infty_{rdm}(\mathfrak{g}, N)$, provided that we replace the image $C^\infty(\mathfrak{g})^G$ by the space $C^{-\infty}(\mathfrak{g})^G$ of G-invariant generalized functions[1] on G. We define $\displaystyle\int_N \alpha(X)$ by

$$\int_{\mathfrak{g}} \left(\int_N \alpha(X) \right) \varphi(X)\, dX = \int_N \left(\int_{\mathfrak{g}} \alpha(X)\varphi(X)\, dX \right), \tag{6.2}$$

for any compactly supported G-invariant test function $\varphi \in C^\infty(\mathfrak{g})^G$. A natural question to ask is whether or not one can still define integration over the fibres of N in $\mathcal{A}^\infty_{rdm}(\mathfrak{g}, N)$. The answer is yes, but the resulting differential forms on the base will no longer depend smoothly on $\mathfrak{g}$. Thus, if one wishes, for example, to compute the integral of $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ over the fibres of T^*M, one is required to make use of *equivariant differential forms with generalized coefficients.*

Definition 6.2. *Let N be any (not necessarily compact) manifold. The space of G-equivariant differential forms with generalized coefficients on N is defined by*

$$\mathcal{A}^{-\infty}(\mathfrak{g}, N) = (C^{-\infty}(\mathfrak{g}) \otimes \mathcal{A}(N))^G. \tag{6.3}$$

[1] We will use the term "generalized function" instead of the more common "distribution" to avoid confusion with subbundles of TM.

That is, $\alpha(X) \in \mathcal{A}^{-\infty}(\mathfrak{g}, N)$ if and only if for any compactly supported test function $\varphi \in C^\infty(\mathfrak{g})$ the pairing

$$\langle \alpha, \varphi \rangle = \int_{\mathfrak{g}} \alpha(X)\varphi(X)\, dX \tag{6.4}$$

defines an element of $\mathcal{A}(N)$.

The space $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$ was first introduced in [DV90], and studied at length in [KV93]. The de Rham differential is extended pointwise to $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$ by

$$\langle d\alpha, \varphi \rangle = d \langle \alpha, \varphi \rangle ,$$

where $\langle \cdot, \varphi \rangle$ denotes pairing against a smooth test function on $\mathfrak{g}$. We again define the equivariant differential D by (5.3). It follows that $D^2 = 0$, so that one may define the equivariant cohomology $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$.

Example 6.3. For any Lie group G acting on itself by left-translation, we find that $\mathcal{H}^{-\infty}(\mathfrak{g}, G) \cong \mathbb{R}$ [KV93, Lemma 45]. A generator is defined as follows: let $\nu \in \bigwedge^n \mathfrak{g}^*$ be a positive volume element with respect to a choice of orientation o on $\mathfrak{g}$, and let dg be the the left-invariant volume form on G that coincides with ν at $1 \in G$. Then by [KV93, Lemma 11], we have the generator

$$\alpha(X) = |\nu|^{-1}\delta_{\mathfrak{g}}(X) \otimes |\det_{\mathfrak{g}} g| dg,$$

where $|\nu|^{-1}\delta_{\mathfrak{g}}(X)$ is the Dirac delta function on $\mathfrak{g}$ with respect to the measure determined by ν.

In particular, let $G = S^1$ act on itself by rotation. Suppose $\alpha \in \mathcal{A}^{-\infty}(\mathfrak{g}, G)$ has odd degree. Then $\alpha(\xi) = f(\xi)d\eta$, where f is a generalized function on G. Thus $D\alpha(\xi) = \xi f(\xi)$, so that $D\alpha = 0$ if and only if $f(\xi)$ is a multiple of $\delta(\xi)$, where $\delta(\xi)$ denotes the Dirac delta on S^1. Since the only exact form is $\alpha = 0$, the odd part of $\mathcal{H}^{-\infty}(\mathfrak{g}, S^1)$ is generated by $\delta(\xi)d\eta$.

As noted in [Ver07], the generalized function $\delta_+(x)$ on $\mathbb{R}$ given by (6.8) below satisfies $-2\pi i x \delta_+(x) = 1$, and thus $-2\pi i (D\delta_+(\xi)d\eta) = -2\pi i \xi \delta_+(\xi) = 1$; that is,

$$1 = 0 \quad \text{in} \quad \mathcal{H}^{-\infty}(\mathfrak{g}, S^1).$$

We will have more to say about this peculiarity of equivariant cohomology with generalized coefficients below in the context of Paradan's formulation of non-Abelian localization.

Example 6.4. [PV08b] Suppose that N is a principal H-bundle over a compact base M, and suppose we are given the smooth action of a Lie group G on N commuting with the principal H-action. Let ω be a G-invariant connection form on N with curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. For any $Y \in \mathfrak{g}$, let $\Omega(Y) = \Omega - \omega(Y_P)$ denote the corresponding equivariant curvature. Since $\Omega \in \mathcal{A}^2(P)$ is nilpotent, for any smooth function $\phi \in C^\infty(\mathfrak{h})$, the form $\phi(\Omega(Y))$ is given in terms of a Taylor series expansion about $-\omega(Y_P)$. Moreover, if ϕ is H-invariant, then $\phi(\Omega(Y))$ is basic, and thus defines a smooth G-equivariant differential form on M.

We obtain an $H \times G$-equivariant differential form with generalized coefficients $\delta(X - \Omega(Y))$ defined on P by

$$\int_{\mathfrak{h}\times\mathfrak{g}} \delta(X - \Omega(Y))\phi(X, Y)dXdY = \mathrm{vol}(H, dX) \int_{\mathfrak{g}} \phi(\Omega(Y), Y)dY, \tag{6.5}$$

for any compactly supported $\phi \in C^\infty(\mathfrak{h} \times \mathfrak{g})$. (The form is smooth with respect to the variable $Y \in \mathfrak{g}$.) Using the fact that Ω is basic, we may define a corresponding form $\delta_0(X - \Omega(Y))$ on M by setting

$$\int_{\mathfrak{h}} \delta_0(X - \Omega(Y))\phi(X)dX = \mathrm{vol}(H, dX)\phi(\Omega(Y)), \tag{6.6}$$

for any H-invariant $\phi \in C^\infty(\mathfrak{h})$.

Many of the equivariant differential forms with generalized coefficients that we will consider are given in terms of well-known generalized functions on $\mathbb{R}$ (or $\mathbb{R}^k$) such as the Dirac delta distribution $\delta_0(x)$ in (6.6) above. As usual, we take the Dirac delta on $\mathbb{R}^k$ to be the generalized function $\delta_0(\mathbf{x})$ such that $\langle f(\mathbf{x}), \delta_0(\mathbf{x})\rangle = f(0)$ for any smooth function f on $\mathbb{R}^k$. We will often make use of the Fourier transform representation of $\delta_0(\mathbf{x})$ given by

$$\delta_0(\mathbf{x}) = \frac{1}{(2\pi)^k} \int_{(\mathbb{R}^k)^*} e^{i\langle \mathbf{x}, \xi\rangle}\, d\xi. \tag{6.7}$$

This is a representation of δ_0 in the sense that if $f \in C_c^\infty(\mathbb{R}^k)$ is any test function, then we have the pairing

$$\begin{aligned}
\langle f(x), \delta_0(x) \rangle &= \int_{\mathbb{R}^k} f(x) \left(\frac{1}{(2\pi)^k} \int_{(\mathbb{R}^k)^*} e^{-i<\xi,x>}\, d\xi \right) dx \\
&= \frac{1}{(2\pi)^k} \int_{(\mathbb{R}^k)^*} \left(\int_{\mathbb{R}^k} f(x) e^{-i<\xi,x>}\, dx \right) d\xi \\
&= \frac{1}{(2\pi)^k} \int_{(\mathbb{R}^k)^*} \hat{f}(\xi) e^{i<\xi,0>}\, d\xi \\
&= (\hat{f}\,)^\vee(0) = f(0).
\end{aligned}$$

It will be useful to introduce as well the following generalized functions on $\mathbb{R}$:

$$\delta_+(x) = \frac{i}{2\pi} \lim_{\epsilon \to 0} \frac{1}{x + i\epsilon}, \quad \delta_-(x) = \frac{-i}{2\pi} \lim_{\epsilon \to 0} \frac{1}{x - i\epsilon}. \tag{6.8}$$

Note that we have

$$\delta_+(x) + \delta_-(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2},$$

which we identify as the Dirac delta $\delta_0(x)$ on $\mathbb{R}$, giving the first of the following identities:

$$\delta_+ + \delta_- = \delta_0, \tag{6.9}$$

$$-2\pi i x \delta_+(x) = 2\pi i x \delta_-(x) = 1, \quad x\delta_0(x) = 0, \tag{6.10}$$

and for any $a \in \mathbb{R} \setminus \{0\}$, we have

$$a\delta_0(ax) = \begin{cases} \delta_0(x), & \text{if } a > 0 \\ -\delta_0(x), & \text{if } a < 0 \end{cases}, \quad a\delta_\pm(ax) = \begin{cases} \delta_\pm(x) & \text{if } a > 0 \\ -\delta_\mp(x) & \text{if } a < 0 \end{cases}. \tag{6.11}$$

The integral representations of these generalized functions are given by

$$\delta_+(x) = \frac{1}{2\pi} \int_0^\infty e^{ixt} dt, \quad \delta_-(x) = \frac{1}{2\pi} \int_{-\infty}^0 e^{ixt} dt. \tag{6.12}$$

Our next example is due to Paradan [Par99, Par00] and also appears in [PV08a].

Example 6.5. Suppose the action of G on a symplectic manifold (N,ω) is Hamiltonian, with proper moment map $\mu : N \to \mathfrak{g}^*$. For any test function $\varphi \in C_c^\infty(\mathfrak{g}^*)$ and family of equivariant differential forms $\alpha_t(X)$ the map

$$X \mapsto \varphi(X)\alpha_t(X)$$

is compactly supported, whence the Fourier transform

$$\widehat{\varphi\alpha_t}(t\mu) = \int_{\mathfrak{g}} \varphi(X)\alpha_t(X)e^{it\langle\mu,X\rangle}\,dX$$

is rapidly decreasing as $t \to \infty$, away from $\mu^{-1}(0)$. It follows that

$$\beta(X) = \int_0^\infty \alpha_t(X)e^{it\langle\mu,X\rangle}\,dt \tag{6.13}$$

is defined as an equivariant differential form with generalized coefficients on $N \setminus \mu^{-1}(0)$.

Remark 6.6. To show that a form such as the one defined by (6.13) is well-defined as an equivariant differential form with generalized coefficients, one typically has to check that it satisfies the relevant semi-norm estimates (see [Par99, Par00, PV08a]). In certain cases these estimates can be handled using the microlocal point of view. For example, suppose that we have a fixed equivariant differential form $\alpha(X)$, and consider the generalized function $\delta_+(x)$ on $\mathbb{R}$ defined by (6.8). Using (6.12), we can write $\beta(X) = 2\pi\alpha(X)\delta_+(\langle\mu,X\rangle)$ for any $x \in N$ such that the pullback of δ_+ to $\mathfrak{g}$ by $\mu(x)$ is well-defined. This will be the case whenever $x \notin \mu^{-1}(0)$; let us see why in general.

Let $u \in C^{-\infty}(\mathbb{R})$ be a generalized function on $\mathbb{R}$, and let $\operatorname{singsupp}(u) \subset \mathbb{R}$ denote the set of points where u fails to be smooth. We may consider the pullback of u by a smooth, proper map $h : \mathfrak{g} \to \mathbb{R}$, which will give a well-defined generalized function $h^*u = u \circ h$ on $\mathfrak{g}$ provided that whenever $h(X) \in \operatorname{singsupp}(u)$, we have $dh_X \neq 0$ [FJ98, Hör83].

For example, if β is an invariant 1-form on M, then fixing a point $m \in M$ gives us a linear map $f_\beta(m)$ (4.1) from $\mathfrak{g}$ to $\mathbb{R}$. If $f_\beta(m)$ is non-zero for all $m \in M$, then we may set

$$u(D\beta)(X) = u(d\beta + f_\beta(X)) = \sum_j \frac{u^{(j)}(f_\beta(X))}{j!}(d\beta)^j, \tag{6.14}$$

since by "microlocality", we have $\mathrm{singsupp}(u^{(j)}) \subset \mathrm{singsupp}(u)$ for each j [FJ98]. In the case of Example 6.5 we have $\mathrm{singsupp}(u) = \{0\}$, and thus away from $\mu^{-1}(0)$, $\beta(X)$ is well-defined. All of the above follows from more general considerations involving *wavefront sets*, which are necessary in the case of maps $h : \mathfrak{g} \to \mathbb{R}^k$. In terms of the wavefront set of $u \in C^{-\infty}(\mathbb{R}^k)$, the relevant condition is (see [Hör83, Mel03])

$$h^*(WF(u)) \cap (\mathfrak{g} \times \{0\}) = \emptyset \subset T^*\mathfrak{g}. \tag{6.15}$$

In the next chapter we will use a similar approach define an equivariant differential form with generalized coefficients $\mathcal{J}(E, X)$ in terms of $\delta_0(x)$. This form will be a key component of our index formula.

6.2 Equivariant cohomology with support

For any closed, G-invariant subset $F \subset N$, and for either smooth or generalized coefficients, we can form the complexes

$$\mathcal{A}^{\pm\infty}(\mathfrak{g}, N, N \setminus F) = \mathcal{A}^{\pm\infty}(\mathfrak{g}, N) \oplus \mathcal{A}^{\pm\infty}(\mathfrak{g}, N \setminus F),$$

equipped with the relative equivariant differential

$$D_{rel}(\alpha, \beta) = (D\alpha, \alpha|_{N\setminus F} - D\beta).$$

The corresponding cohomology spaces can be thought of as those equivariant cohomology classes that vanish on $N \setminus F$, since $D_{rel}(\alpha, \beta) = 0$ if and only if $D\alpha = 0$ on N, and $\alpha = D\beta$ on $N \setminus F$.

Let $U \subset N$ be any open, G-invariant subset containing F. We can define the spaces $\mathcal{A}^{\infty}_U(\mathfrak{g}, N)$ of equivariant differential forms with support contained in U, and their corresponding cohomology spaces $\mathcal{H}^{\infty}_U(\mathfrak{g}, N)$. Let $\chi \in C^\infty(N)^G$ be a cutoff function with support contained in U such that $\chi \equiv 1$ on a neighbourhood of U. We can then define a map

$$p_{U,\chi} : \mathcal{A}^{\pm\infty}(\mathfrak{g}, N, N \setminus F) \to \mathcal{A}^{\pm\infty}_U(\mathfrak{g}, N) \tag{6.16}$$

by $p_{U,\chi}(\alpha,\beta) = \chi\alpha + d\chi\wedge\beta$. Since $d\chi = 0$ on F, the form $d\chi\wedge\beta$ is defined on all of N, and the support of $\chi\alpha + d\chi\wedge\beta$ is contained in U. If $D_{rel}(\alpha,\beta) = 0$, then

$$Dp_{U,\chi}(\alpha,\beta) = \chi\wedge\alpha + \chi D\alpha - d\chi\wedge D\beta = 0,$$

since $D\alpha = 0$ on N, $d\chi = 0$ on F, and $\alpha = D\beta$ on $N\setminus F$. Thus the map $p_{U,\chi}$ descends to a map on the level of cohomology. The class $[p_{U,\chi}(\alpha,\beta)]$ does not depend on the choice of cutoff function χ [Par99, PV07a], and thus we obtain a map

$$p_U : \mathcal{H}^{\pm\infty}(\mathfrak{g}, N, N\setminus F) \to \mathcal{H}^{\pm\infty}_U(\mathfrak{g}, N). \tag{6.17}$$

If we have two open subsets V and U with $F\subset V\subset U$, then the inclusion $\mathcal{A}^{\pm\infty}_V(\mathfrak{g},N)\hookrightarrow \mathcal{A}^{\pm\infty}_U(\mathfrak{g},N)$ induces a map

$$f_{U,V} : \mathcal{H}^{\pm\infty}_V(\mathfrak{g},N) \to \mathcal{H}^{\pm\infty}_U(\mathfrak{g},N),$$

As explained in [PV07a, PV08a], one can take the inverse limit with respect to sequences of open, G-invariant neighbourhoods of F, ordered by inclusion, of the spaces $\mathcal{H}^{\pm\infty}_U(\mathfrak{g},N)$ to obtain the cohomology spaces $\mathcal{H}^{\pm\infty}_F(\mathfrak{g},N)$ *with support on* F. In particular, one obtains a well-defined map

$$p_F : \mathcal{H}^{\pm\infty}(\mathfrak{g}, N, N\setminus F) \to \mathcal{H}^{\pm\infty}_F(\mathfrak{g}, N). \tag{6.18}$$

If F is compact, there is a natural map

$$\mathcal{H}^{\pm\infty}_F(\mathfrak{g},N) \to \mathcal{H}^{\pm\infty}_c(\mathfrak{g},N). \tag{6.19}$$

In computations, an element of $\mathcal{H}^{\pm\infty}_F(\mathfrak{g},N)$ in the image of p_F can (and will) be represented by one of the forms $p^{\chi}_U(\eta,\xi)$; this is the approach taken in [PV08b].

6.3 Paradan's form

Example 6.7. This example of an equivariant differential form with generalized coefficients is due to Paradan [Par99, Par00]. Let N be a G-manifold, and let λ be a smooth,

G-invariant 1-form on N with corresponding λ-moment map $f_\lambda : N \to \mathfrak{g}^*$ given by (4.1). We recall the notation $C_\lambda = f_\lambda^{-1}(0)$. On $N \setminus C_\lambda$, the form

$$\beta_\lambda(X) = -2\pi i \lambda\, \delta_+(D\lambda(X)) \tag{6.20}$$

is well-defined as a G-equivariant form with generalized coefficients. Let U be a G-equivariant neighbourhood of C_λ in N, and let $\chi \in C^\infty(N)$ be a G-invariant cutoff function supported on U, such that $\chi \equiv 1$ on a smaller neighbourhood of N. Define

$$P_\lambda^\chi(X) = \chi + d\chi \wedge \beta_\lambda(X). \tag{6.21}$$

Then P_λ^χ is a closed equivariant differential form with generalized coefficients supported in U. It is the image of $(1, \beta_\lambda) \in \mathcal{A}^{-\infty}(\mathfrak{g}, N, N \setminus C_\lambda)$ under the map (6.16). Thus, its cohomology class in $\mathcal{H}_U^{-\infty}(\mathfrak{g}, N)$ is independent of χ and determines a class $P_\lambda(X) \in \mathcal{H}_F^{-\infty}(\mathfrak{g}, N)$, where $F = C_\lambda$.

Remark 6.8. Recall (see [BGV91, Section 7.2]) that to prove the localization formula when M is compact and G is a torus, one can construct a 1-form κ such that for a given $X \in \mathfrak{g}$, $D\kappa(X)$ is invertible outside the set of zeros of the vector field X_M. The form $\dfrac{\kappa}{D\kappa(X)}$ is then used to localize to the zero set of X_M.

In [Wit92], Witten conjectured that there should be a "non-Abelian" version of the localization theorem. This was realized by Paradan, who showed in [Par99, Par00] that the form $\beta_\lambda(X)$ can play the role of $\dfrac{\kappa}{D\kappa(X)}$: Using (6.10), we see that

$$D\beta_\lambda(X) = -2\pi i(D\lambda(X))\delta_+(D\lambda(X)) = 1 \tag{6.22}$$

away from C_λ. In other words, we have "$1 = 0$" in $\mathcal{H}^{-\infty}(\mathfrak{g}, N \setminus C_\lambda)$, and thus we can localize to C_λ. Indeed, we have

$$P_\lambda^\chi(X) = 1 + D((\chi - 1)\beta_\lambda(X)),$$

so that the image of $P_\lambda \in \mathcal{H}_F^{-\infty}(\mathfrak{g}, N)$ in $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$ with respect to the natural inclusion coincides with $1 \in \mathcal{H}^{-\infty}(\mathfrak{g}, N)$. Moreover, the action of $\mathcal{A}^\infty(\mathfrak{g}, N)$ on $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$ by exterior multiplication induces a well-defined product [PV08a]

$$\mathcal{H}^{\infty}_{F_1}(\mathfrak{g}, N) \times \mathcal{H}^{-\infty}_{F_2}(\mathfrak{g}, N) \to \mathcal{H}^{-\infty}_{F_1 \cap F_2}(\mathfrak{g}, N), \tag{6.23}$$

and thus, multiplication by $P_\lambda(X)$ defines a localization map

$$\mathcal{H}^{\infty}(\mathfrak{g}, N) \to \mathcal{H}^{-\infty}_{F}(\mathfrak{g}, N). \tag{6.24}$$

Remark 6.9. Suppose that N is a principal H-bundle equipped with an action of a Lie group G commuting with the H-action, and a G-invariant connection 1-form ω. Define a 1-form ν on $N \times \mathfrak{h}^*$ by $\nu = - < \xi, \omega >$, where ξ denotes the $\mathfrak{h}^*$ variable. We may then construct an $H \times G$-equivariant differential form with generalized coefficients $P^\chi_\nu(X, Y)$ using a sufficiently small neighbourhood U of $N \times \{0\}$, such that P^χ_ν represents the class $P_\nu \in \mathcal{H}^{-\infty}(\mathfrak{h} \times \mathfrak{g}, N)$ and we have the following result from [PV08b]:

Lemma 6.10. *Let $q : N \times \mathfrak{h}^* \to N$ denote projection onto the first factor. Let $\psi_1, \ldots, \psi_r$ denote the components of ω with respect to some choice of basis for $\mathfrak{h}$. If $\mathfrak{h}^*$ is oriented with respect to the corresponding dual basis, then*

$$q_* P^\chi_\nu(X, Y) = (2\pi i)^{\dim H} \delta(X - \Omega(Y)) \frac{\omega_r \cdots \omega_1}{\mathrm{vol}(H, dX)} \tag{6.25}$$

for any $(X, Y) \in \mathfrak{h} \times \mathfrak{g}$ and any choice of cutoff function χ.

6.4 Chern character with compact support

Suppose now that $\pi : N \to M$ is a vector bundle, and consider Quillen's Chern character $\mathrm{Ch}_Q(\mathbb{A}^\sigma, X)$ defined by (5.22), where we replace the superconnection $\mathbb{A}^\sigma$ by the family of superconnections

$$\mathbb{A}^\sigma(t) = \mathbb{A} + itv_\sigma,$$

where $\mathbb{A} = \pi^* \nabla$. For the transgression form

$$\eta(\mathbb{A}, \sigma, t)(X) = -i \,\mathrm{Str}\left(v_\sigma e^{\mathbb{F}(\mathbb{A}^\sigma(t))(X)}\right)$$

we have

$$\mathrm{Ch}(\mathbb{A},X) - \mathrm{Ch}(\mathbb{A}^\sigma(t),X) = D\left(\int_0^t \eta(\mathbb{A},\sigma,s)(X)\,ds\right). \tag{6.26}$$

As shown in [PV08a], as $t \to \infty$ both $\mathrm{Ch}(\mathbb{A}^\sigma(t),X)$ and $\eta(\mathbb{A},\sigma,t)$ go to zero exponentially quickly, and the form

$$\beta(\mathbb{A},\sigma)(X) = \int_0^\infty \eta(\mathbb{A},\sigma,t)(X)\,dt$$

is defined as an equivariant differential form with smooth coefficients on $N \setminus \mathrm{Supp}(\sigma)$. Let $F_1 = \mathrm{Supp}(\sigma)$. We have

$$\mathrm{Ch}(\mathbb{A},X)|_{N\setminus\mathrm{Supp}(\sigma)} = D\beta(\mathbb{A},\sigma)(X),$$

whence we obtain the class $[\mathrm{Ch}(\mathbb{A}_0,X),\beta(\mathbb{A},\sigma)(X)] \in \mathcal{H}^\infty(\mathfrak{g},N,N\setminus F_1)$ and the class

$$\mathrm{Ch}_{F_1}(\mathbb{A},\sigma,X) = p_{F_1}(\mathrm{Ch}(\mathbb{A}_0,X),\beta(\mathbb{A},\sigma)(X)) \in \mathcal{H}^\infty_{F_1}(\mathfrak{g},N). \tag{6.27}$$

If we make a few minor adjustments, we can re-write the above to involve Quillen's Chern character form directly. Since $\mathrm{Ch}_Q(\mathbb{A}^\sigma,X) = \mathrm{Ch}(\mathbb{A}^\sigma(1),X)$, we see that (6.26) becomes

$$\mathrm{Ch}_Q(\mathbb{A}^\sigma,X) - \mathrm{Ch}(\mathbb{A}^\sigma(t)) = D\int_1^t \eta(\mathbb{A},\sigma,s)\,ds,$$

since $D\int_0^1 \eta(\mathbb{A},\sigma,s)\,ds = \mathrm{Ch}(\mathbb{A},X) - \mathrm{Ch}_Q(\mathbb{A}^\sigma,X)$. Now when σ is elliptic, the equivariant differential form $\mathrm{Ch}_Q(\mathbb{A}^\sigma,X)$ is rapidly decreasing, and thus defines a class in $\mathcal{H}^\infty_{rd}(\mathfrak{g},N)$. Since $F_1 = \mathrm{Supp}(\sigma)$ is compact, there is a natural map

$$\mathcal{H}^\infty_{F_1}(\mathfrak{g},N) \to \mathcal{H}^\infty_{rd}(\mathfrak{g},N).$$

By [PV07a, Theorem 5.19], the image of $\mathrm{Ch}_{F_1}(\mathbb{A},\sigma,X)$ under this map coincides with the cohomology class of $\mathrm{Ch}_Q(\mathbb{A}^\sigma,X)$, and in particular, if we are interested in integrating over the fibres of $N \to M$, then the use of either form results in the same element of $\mathcal{H}^\infty(\mathfrak{g},N)$.

Let us now consider the case where σ is not necessarily elliptic. Here $\mathrm{Ch}(\mathbb{A},\sigma,X)$ may no longer be compactly supported, so we make use of non-Abelian localization as follows:

If λ is a G-invariant 1-form we can construct Paradan's form $P_\lambda(X) \in \mathcal{H}^{-\infty}_{F_2}(\mathfrak{g}, N)$, where $F_2 = C_\lambda$. Using the map (6.23) given by multiplication by $P_\lambda(X)$, we obtain the class

$$\mathrm{Ch}_{F_1}(\mathbb{A}, \sigma, X) \wedge P_\lambda(X) \in \mathcal{H}^{-\infty}_{F_1 \cap F_2}(\mathfrak{g}, N). \tag{6.28}$$

Thus, even if σ is not elliptic, it may be the case that $F_1 \cap F_2$ is compact; for example, in the case of a G-transversally elliptic symbol, where $F_1 = \mathrm{Supp}(\sigma)$ and $F_2 = C_\theta = T^*_G M$.

Alternatively, we could start from the beginning with the Berline-Vergne Chern character form (5.26) on $N = T^*M$, and proceed as above, replacing the superconnection $\mathbb{A}^{\sigma,\theta}$ by the family of superconnections

$$\mathbb{A}^{\sigma,\theta}(t) = \mathbb{A} + it(v_\sigma + \theta).$$

We again have a transgression form $\eta(\mathbb{A}, \sigma, \theta, t) = -i\,\mathrm{Str}\left((v_\sigma + \theta)e^{\mathbb{F}(\mathbb{A}^{\sigma,\theta}(t))(X)}\right)$ and the identity

$$\mathrm{Ch}(\mathbb{A}, X) - \mathrm{Ch}(\mathbb{A}^{\sigma,\theta}(t), X) = D\int_0^t \eta(\mathbb{A}, \sigma, \theta, s)\, ds.$$

The form $\beta(\mathbb{A}, \sigma, \theta) = \int_0^\infty \eta(\mathbb{A}, \sigma, \theta, s)\, ds$ is now defined on the larger open set $T^*M \setminus F$, where $F = \mathrm{Supp}(\sigma) \cap T^*_G M$, but only as an equivariant differential form with generalized coefficients. We also have $\lim\limits_{t\to\infty} \mathrm{Ch}(A^{\sigma,\theta}(t), X) = 0$ on $N \setminus F$, provided we take the limit in the space $\mathcal{A}^{-\infty}(\mathfrak{g}, N)$. Thus, we obtain a class

$$\mathrm{Ch}_F(\mathbb{A}, \sigma, \theta, X) = p_F(\mathrm{Ch}(\mathbb{A}, X), \beta(\mathbb{A}, \sigma, \theta)) \in \mathcal{H}^{-\infty}_F(\mathfrak{g}, N).$$

By [PV08a, Theorem 3.22], we have the following useful equality in $\mathcal{H}^{-\infty}(\mathfrak{g}, N)$:

$$\mathrm{Ch}_F(\mathbb{A}, \sigma, \theta, X) = P_\theta(X) \wedge \mathrm{Ch}_{\mathrm{Supp}(\sigma)}(\mathbb{A}, \sigma, X). \tag{6.29}$$

In particular, using an argument similar to the one given above, we note that when $N = T^*M$ we have $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X) = \mathrm{Ch}(\mathbb{A}^{\sigma,\theta}(1), X)$, whence

$$\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X) - \mathrm{Ch}(\mathbb{A}^{\sigma,\theta}(t), X) = D\int_1^t \eta(\mathbb{A}, \sigma, \theta, s)\, ds.$$

We note that the form $\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)$ defines a class in $\mathcal{H}^{\infty}_{rdm}(\mathfrak{g}, T^*M)$, but *only* if we impose suitable growth conditions[2] on the symbol σ. Assuming these conditions, the form $\beta_1(\mathbb{A}, \sigma, \theta) = \int_1^{\infty} \eta(\mathbb{A}, \sigma, \theta, t)\, dt$ is rapidly decreasing in $\mathfrak{g}$-mean, but has generalized coefficients. We thus define the space $\mathcal{A}^{-\infty}_{rdm}(\mathfrak{g}, T^*M)$, which is preserved by the equivariant differential D, letting us define the cohomology space $\mathcal{H}^{-\infty}_{rdm}(\mathfrak{g}, T^*M)$. There are natural maps

$$\begin{tikzcd}
\mathcal{H}^{-\infty}_{F}(\mathfrak{g}, T^*M) \arrow[dr] & & \mathcal{H}^{\infty}_{rdm}(\mathfrak{g}, T^*M) \arrow[dl] \\
& \mathcal{H}^{-\infty}_{rdm}(\mathfrak{g}, T^*M) &
\end{tikzcd} .$$

By [PV08a, Proposition 3.45], the image of $\mathrm{Ch}_F(\mathbb{A}, \sigma, \theta, X) \in \mathcal{H}^{-\infty}_F(\mathfrak{g}, T^*M)$ under the first map coincides with the image of $[\mathrm{Ch}_{BV}(\mathbb{A}^{\sigma,\theta}, X)] \in \mathcal{H}^{\infty}_{rdm}(\mathfrak{g}, T^*M)$ under the second, and by [PV08a, Proposition 3.46], integrating either form over the fibres of T^*M defines the same element of $\mathcal{H}^{-\infty}(\mathfrak{g}, M)$. This is the desired fibre integral alluded to at the beginning of this chapter.

Remark 6.11. In the case of a G-transversally elliptic symbol on T^*M, with M compact, we have that $F = \mathrm{Supp}(\sigma) \cap T^*_G M = 0$ is compact. Thus, the class $\mathrm{Ch}_F(\mathbb{A}, \sigma, \theta, X) \in \mathcal{H}^{-\infty}_F(\mathfrak{g}, T^*M)$ defines a class in $\mathcal{H}^{-\infty}_c(\mathfrak{g}, T^*M)$ under the natural map induced by inclusion. The advantage of working with this class is that it is compactly supported, regardless of the growth conditions assumed for the symbol σ, and so integration over the fibres is always defined. In order to do calculations involving the Chern character $\mathrm{Ch}_F(\mathbb{A}, \sigma, \theta, X)$, we choose a representative

$$\mathrm{Ch}_{\chi}(\mathbb{A}, \sigma, \theta, X) = \chi\, \mathrm{Ch}(\mathbb{A}, X) + d\chi \wedge \beta(\mathbb{A}, \sigma, \theta)(X).$$

(See the last paragraph of Section 6.2.) Our approach will be to use the product formula (6.29), and represent $P_\theta(X)$ by one of the forms $P^{\chi}_{\theta}(X)$. Since we are interested in the

[2]See [PV08a] or [BV96a]. These growth conditions define what Berline-Vergne call "G-transversally good" symbols.

integral over the fibres of T^*M, we can then replace $\mathrm{Ch}_{\mathrm{Supp}(\sigma)}(\mathbb{A}, \sigma, X)$ by Quillen's Chern character form. While we have to make a choice of a cutoff function χ, this choice will not affect the result when integrating over the fibres, and thus we can compute everything on the level of differential forms.

Chapter 7

The differential form $\mathcal{J}(E, X)$

7.1 The contact case

On a contact manifold (M, E) on which a compact Lie group G acts transverse to E, consider the form

$$\mathcal{J}(E, X) = \alpha \wedge \delta_0(D\alpha(X)), \tag{7.1}$$

where α is any contact form. By averaging over G, we may assume that α is G-invariant. We may view the form $\delta_0(D\alpha(X))$ as the oscillatory integral $\displaystyle\int_{-\infty}^{\infty} e^{-itD\alpha(X)}\,dt$; this expression is well-defined as a generalized equivariant form wherever the pairing $X \mapsto \alpha(X_M)$ is non-zero.

Alternatively, we may consider Remark 6.6 in the case of the contact form α. The hypothesis $E^0 \cap T^*_G M = 0$ ensures that the pairing $X \mapsto \alpha(X_M)$ is non-zero, and thus $\delta_0(D\alpha(X))$ is well-defined. Using the properties of the Dirac delta given above, we obtain the following:

Proposition 7.1. *Let (M, E) be a co-oriented contact manifold on which a Lie group G acts by co-orientation-preserving contact transformations, such that that the action is transverse to E. Then the form $\mathcal{J}(E, X)$ is equivariantly closed, and independent of the choice of contact form.*

Proof. We have:

$$D(\alpha \wedge \delta_0(D\alpha)) = D\alpha \wedge \delta_0(D\alpha) = 0 \quad \text{by (6.10)} ,$$

while if we change α to $e^f\alpha$ for some $f \in C^\infty(M)$ we have using (6.11) that

$$\begin{aligned} e^f\alpha \wedge \delta_0(D(e^f\alpha)) &= e^f\alpha \wedge \delta_0(e^f(df \wedge \alpha + D\alpha)) \\ &= \alpha \wedge \delta_0(df \wedge \alpha + D\alpha) = \alpha \wedge \delta_0(D\alpha), \end{aligned}$$

where in the last equality we have used (6.14), and the fact that $\alpha \wedge \alpha = 0$. □

The form $\mathcal{J}(E,X)$ can also be described in terms of Paradan's form $P_\lambda^\chi(X)$ defined by (6.21), where λ is a 1-form on T^*M such that $\lambda|_{E^*} = 0$, and $\lambda|_{E^0} = \imath^*\theta$, where $\imath : E^0 \hookrightarrow T^*M$ denotes inclusion, and θ is the canonical 1-form on T^*M:

Lemma 7.2. *If λ is defined as above, then we have*

$$\pi_* P_\lambda^\chi(X) = 2\pi i\, \mathcal{J}(E,X),$$

where $\pi : T^*M \to M$ denotes projection, for any choice of cutoff function χ with support in a neighbourhood of E^* such that $\chi \equiv 1$ on E^*.

Proof. We choose a representative P_λ^χ of P_λ, given by

$$P_\lambda^\chi = \chi + d\chi \wedge \beta(\lambda) = \chi - 2\pi i d\chi \wedge \lambda\delta_+(D\lambda).$$

Since λ is obtained from a form on E^0, and χ is constant on E^*, P_λ^χ is the pullback to T^*M of a form on E^0, and so it remains to calculate the integral over the fibre of $E^0 = M \times \mathbb{R}$. Let t be the coordinate along the fibre, and write $\chi = \chi(t)$. Then $\chi(t)$ is supported on a neighbourhood of $t = 0$, with $\chi(0) = 1$, and λ may be written as $\lambda = -t\alpha$, for α a contact form on M. Thus $D\lambda = D(-t\alpha) = \alpha \wedge dt - tD\alpha$, and P_λ^χ becomes

$$\begin{aligned} P_\lambda^\chi &= \chi(t) - 2\pi i \chi'(t)\, dt \wedge (-t\alpha)\delta_+(\alpha \wedge dt - tD\alpha) \\ &= \chi(t) - 2\pi i \alpha \wedge t\chi'(t)\, dt\, \delta_+(-tD\alpha). \end{aligned}$$

The integral over $\mathbb{R}$ becomes, with the help of the identities in Section 5.1,

$$\begin{aligned}\int_{-\infty}^{\infty} P_\lambda^\chi &= -2\pi i\alpha \wedge \int_{-\infty}^{\infty} \chi'(t) t\delta_+(-tD\alpha)\,dt \\ &= -2\pi i\alpha \wedge \left[\int_0^{\infty} \chi'(t)\delta_-(D\alpha)\,dt - \int_{-\infty}^{0} \chi'(t)\delta_+(D\alpha)\,dt\right] \\ &= -2\pi i\alpha \wedge [-\delta_-(D\alpha) - \delta_+(D\alpha)] \\ &= 2\pi i\alpha \wedge \delta_0(D\alpha).\end{aligned}$$

□

7.2 A "Duistermaat-Heckman measure" on E^0

Recall (see [GGK02], for example) that given a symplectic manifold (N, ω) of dimension 2n and a Hamiltonian action of a Lie group G on N with moment map Ψ we may define the Duistermaat-Heckman distribution u_{DH} on $C_c^\infty(\mathfrak{g}^*)$ by

$$< \phi, u_{DH} >= \int_{\mathfrak{g}^*} \phi u_{DH} = \frac{1}{(2\pi)^n} \int_N (\Psi^*\phi) e^\omega.$$

The Fourier transform of u_{DH} is given, for $h \in C_c^\infty(\mathfrak{g})$ by

$$< \widehat{u_{DH}}, h >=< u_{DH}, \hat{h} >= \int_{\mathfrak{g}} h(X) I(X) dX,$$

so that $\widehat{u_{DH}} = I(X)dX$, where

$$I(X) = \int_{\mathfrak{g}*} e^{-i<X,\xi>} u_{DH}(\xi) = \frac{1}{(2\pi)^n} \int_N e^{-i<X,\Psi>} e^\omega = \frac{1}{2\pi i^n} \int_N e^{i\omega(X)}. \tag{7.2}$$

Now, given a co-oriented contact manifold (M, E) of dimension $2n+1$, consider the annihilator E^0 of E. Although not quite a symplectic manifold, since the form $\omega = d(t\alpha)$ is degenerate for $t = 0$, we have the moment map $\Psi = tf_\alpha$, and if we compute $I(X)$ in this case, we find

$$I(X) = \frac{1}{(2\pi i)^{n+1}} \int_{E^0} e^{i\omega(X)} = \frac{1}{(2\pi i)^n} \int_M \mathcal{J}(E, X). \tag{7.3}$$

Similarly, on the symplectic manifold E_+^0 we obtain an expression for the Fourier transform of the Duistermaat-Heckman distribution by replacing δ_0 by δ_+.

Remark 7.3. Since $\omega(X) = \imath^* D\theta(X)$, the above calculation provides us with an alternative definition of the form $\mathcal{J}(E,X)$: if we let $q : E^0 \to M$ denote projection, then (7.3) implies the relation

$$q_*\imath^* e^{iD\theta(X)} = 2\pi i\, \mathcal{J}(E,X). \tag{7.4}$$

We will see below that this relation provides one way to define $\mathcal{J}(E,X)$ for distributions of higher corank.

7.3 The general case

Now, suppose that we are given the following data: a manifold M on which a compact Lie group G acts smoothly, and a G-invariant subbundle $E \subset TM$ such that the action of G is transverse to E: we have $E^0 \cap T^*_G M = 0$. For the general case, we still assume that E^0 is oriented, but when the rank of E^0 is greater than 1, this need no longer imply that it is trivial. Since E^0 need not be trivial, we do not have an explicit global expression of the form (7.1). Thus, to extend the definition of $\mathcal{J}(E,X)$ to this case, we look to Lemma 7.2. Let $\imath : E^0 \hookrightarrow T^*M$ denote inclusion, and let $\lambda = \imath^*\theta$, where θ is the canonical 1-form on T^*M. We consider the form $P^\chi_\lambda(X)$ given by (6.21), where χ is a cutoff function supported on a neighbourhood of E^0, with $\chi \equiv 1$ on E^0.

Definition 7.4. *We denote by $\mathcal{J}(E,X) \in \mathcal{A}^{-\infty}(\mathfrak{h}, M)$ the equivariant differential form with generalized coefficients given by*

$$\mathcal{J}(E,X) = (2\pi i)^{-k} q_* P^\chi_\lambda(X), \tag{7.5}$$

where $k = \operatorname{rank} E^0$, and $q : E^0 \to M$ denotes projection.

This integral over the fibres of E^0 is well-defined, since $P^\chi_\lambda(X)$ has compact support on the fibres of E^0.

Note: We will show in Section 7.5 below that $\mathcal{J}(E,X)$ does not depend on the choice of cutoff function. Thus, we will generally abuse notation and let $P_\lambda(X)$ denote the differential form $P^{\chi}_{\lambda}(X)$ representing the corresponding class in generalized equivariant cohomology.

Remark 7.5. An alternative definition of $\mathcal{J}(E,X)$ uses the equivariant differential form $e^{iD\lambda(X)}$, as in (7.4). This form does not have compact support on the fibres of E^0, but it is rapidly decreasing in $\mathfrak{g}$-mean (see Section 6.2 above). As noted by Paradan [Par00, Section 2.4], the pushforward morphism q_* from $\mathcal{A}_c^{-\infty}(\mathfrak{g},E^0)$ to $\mathcal{A}^{-\infty}(\mathfrak{g},M)$ defined by integration over the fibres of E^0 extends to the space of differential forms that are rapidly decreasing in $\mathfrak{g}$-mean on E^0. This morphism commutes with the equivariant differential D, and thus induces a morphism (that we will also denote by q_*) in cohomology. Since we have that $f_\lambda^{-1}(0) = E^0 \cap T^*_G M = 0$, the inclusion of M as the zero section in E^0 allows us to specialize Proposition 2.9 of [Par00] to the bundle $E^0 \to M$, giving

$$q_*[P_\lambda(X)] = q_*[\imath^* e^{iD\theta(X)}] \quad \text{in} \quad \mathcal{H}^{-\infty}(\mathfrak{g},M). \tag{7.6}$$

This suggests that we can replace $P_\lambda(X)$ in (7.5) by $\imath^* e^{iD\theta(X)} = e^{iD\lambda(X)}$, giving

$$\mathcal{J}(E,X) = (2\pi i)^{-k} q_* \imath^* e^{iD\theta(X)}. \tag{7.7}$$

There are certain advantages to using (7.7) as our definition of the form $\mathcal{J}(E,X)$: it is clear that it is equivariantly closed; that is, $D\mathcal{J}(E,X) = 0$, and that it is independent of any choices, being determined entirely by the distribution E and the action of G on M. However, unlike (7.5), it is not immediate that (7.7) is well-defined. The result [Par00, Proposition 2.9] establishes the equality of our two definitions of $\mathcal{J}(E,X)$ on the level of cohomology. We will see below that this equality indeed holds on the level of differential forms.

Remark 7.6. In the next chapter we will use the form $\mathcal{J}(E,X)$ to give our formula for the equivariant index. When computing the value of this index near a general element $g \in G$, it will be necessary to restrict to elements of the Lie subalgebra $\mathfrak{g}(g)$. Now, restriction of a generalized function to a subspace is in general not well-defined, so some care must be taken with the form $\mathcal{J}(E, X)$. Fortunately, in this case we do not consider the full subbundle E, but only the subset $E(g)$ of g-fixed points, and by Proposition 4.7, the Lie subgroup $G(g)$ acts transverse to $E(g)$, so that we can construct a well-defined form $\mathcal{J}(E(g), X)$, for $X \in \mathfrak{g}(g)$. It is then not too hard to see that this form is indeed the result of restricting $\mathcal{J}(E, X)$ to $M(g)$, and then to the subspace $\mathfrak{g}(g)$.

7.4 A local form for $\mathcal{J}(E, X)$

We now wish to proceed with a local construction of the form defined in (7.1). Although the above definition suffices to obtain our results, the structure and properties of $\mathcal{J}(E, X)$ are revealed more clearly by this local description. Let $U \subset M$ be a trivializing neighbourhood for E^0, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}^1(U) \otimes \mathbb{R}^k$ be a local oriented frame for $E^0|_U$. Given such a choice of frame, we define a map

$$f_{\boldsymbol{\alpha}} : U \to \mathrm{Hom}(\mathfrak{h}, \mathbb{R}^k)$$

by $f_{\boldsymbol{\alpha}}(X) = -\boldsymbol{\alpha}(X_M)$, for any $X \in \mathfrak{h}$, where X_M is the fundamental vector field on M generated by X. The equivariant differential of $\boldsymbol{\alpha}$ is thus $D\boldsymbol{\alpha}(X) = d\boldsymbol{\alpha} + f_{\boldsymbol{\alpha}}(X) \in \mathcal{A}^2(U) \otimes \mathbb{R}^k$.

Let $\delta_0 \in C^{-\infty}(\mathbb{R}^k)$ denote the Dirac delta function on $\mathbb{R}^k$. Since $||\boldsymbol{\alpha}|| \neq 0$ on U, the transversality assumption ensures that $f_{\boldsymbol{\alpha}}(m)$ is non-zero for all $m \in U$. Thus, for any derivative $\delta_0^{(I)}$, the composition $\delta_0^{(I)} \circ f_{\boldsymbol{\alpha}}(m)$ is well-defined as a generalized function on $\mathfrak{h}$ (see [Hör83, Mel03]). The expression $\delta_0(D\boldsymbol{\alpha}(X))$ can be described in terms of its Taylor expansion as

$$\begin{aligned}
\delta_0(D\boldsymbol{\alpha}(X)) &= \delta_0(d\boldsymbol{\alpha} + f_{\boldsymbol{\alpha}}(X)) \\
&= \sum_{|I|=0}^{\infty} \frac{\delta_0^{(I)}(f_{\boldsymbol{\alpha}}(X))}{I!} d\alpha^I,
\end{aligned}$$

where $I = (i_1, \ldots, i_k)$, $I! = i_1! \cdots i_k!$, $|I| = i_1 + \cdots + i_k$, $\delta_0^{(I)} = \left(\frac{\partial}{\partial x_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial x_k}\right)^{i_k} \delta_0$, and $d\alpha^I = d\alpha_1^{i_1} \wedge \cdots \wedge d\alpha_k^{i_k}$.

Since $\delta_0(D\boldsymbol{\alpha}(X))$ is given in terms of the pullback of the Dirac delta function on $\mathbb{R}^k$, its pairing against a test function on $\mathfrak{g}$ depends on the map $f_{\boldsymbol{\alpha}}$ and hence does not admit a simple description such as (6.6) above. However, we can give an integral representation of $\delta_0(D\boldsymbol{\alpha}(X))$ using (6.7):

$$\delta_0(D\boldsymbol{\alpha}(X)) = \frac{1}{(2\pi)^k}\int_{(\mathbb{R}^k)^*} e^{-i<\xi, D\boldsymbol{\alpha}(X)>} d\xi, \tag{7.8}$$

where $\langle \xi, D\boldsymbol{\alpha}(X)\rangle = \sum_{i=1}^k \xi^j(d\alpha_j + \alpha_j(X_M))$ and $d\xi = d\xi^1\cdots d\xi^k$ with respect to the basis for $(\mathbb{R}^k)^*$ dual to the one defined by the frame $\boldsymbol{\alpha}$. We now define an equivariant differential form with generalized coefficients on U by

$$\mathcal{J}_{\boldsymbol{\alpha}}(E,X) = \alpha_k \wedge \cdots \wedge \alpha_1 \wedge \delta_0(D\boldsymbol{\alpha}(X)).$$

Lemma 7.7. *The form $\mathcal{J}_{\boldsymbol{\alpha}}(E,X)$ does not depend on the choice of oriented frame $\boldsymbol{\alpha}$.*

Proof. Suppose that $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_k)$ is another frame for E^0 on U defining the same orientation as $\boldsymbol{\alpha}$. Then we have $\boldsymbol{\beta} = A\boldsymbol{\alpha}$ for some matrix A with positive determinant, and so

$$\begin{aligned}\beta_k \wedge\cdots\wedge\beta_1\wedge\delta_0(D\boldsymbol{\beta}(X)) &= \det(A)\alpha_k\wedge\cdots\wedge\alpha_1\wedge\delta_0(A(D\boldsymbol{\alpha}(X)) + dA\wedge\boldsymbol{\alpha})\\ &= \alpha_k\wedge\cdots\wedge\alpha_1\wedge\delta_0(D\boldsymbol{\alpha}(X) + A^{-1}(dA)\wedge\boldsymbol{\alpha})\\ &= \alpha_k\wedge\cdots\wedge\alpha_1\wedge\delta_0(D\boldsymbol{\alpha}(X)),\end{aligned}$$

since $\det(A)\delta_0(A\mathbf{x}) = \delta_0(\mathbf{x})$ if $\det(A) > 0$, and $\alpha_j \wedge \alpha_j = 0$ for all j. □

Corollary 7.8. *There exists a well-defined equivariant differential form with generalized coefficients $\tilde{\mathcal{J}}(E,X)$ whose restriction to any trivializing neighbourhood $U \subset M$ is $\mathcal{J}_{\boldsymbol{\alpha}}(E,X)$.*

Proposition 7.9. *The form $\tilde{\mathcal{J}}(E,X)$ is equivariantly closed.*

Proof. In any local frame $\boldsymbol{\alpha}$ we have

$$\begin{aligned} D\mathcal{J}_{\boldsymbol{\alpha}}(E,X) &= (D(\alpha_k \cdots \alpha_1))\delta_0(D\boldsymbol{\alpha}(X)) \\ &= \left(\sum_{j=1}^{k}(-1)^{k-i}\alpha_k \cdots D\alpha_j(X)\cdots\alpha_1\right)\delta_0(D\boldsymbol{\alpha}(X)) \\ &= 0, \end{aligned}$$

thanks to the identity $u_j\delta_0(\mathbf{u}) = 0$ for $j = 1 \dots k$. □

7.5 Local = global

We have seen in the previous section that the locally defined forms $\mathcal{J}_{\boldsymbol{\alpha}}(E,X)$ can be patched together to define a differential form with generalized coefficients $\tilde{\mathcal{J}}(E,X)$. We now show that this differential form agrees with the global definition (7.5). We include the computation of both fibre integrals (7.5) and (7.7), which shows that [Par00, Proposition 2.9] holds on the level of differential forms, and gives a good demonstration of the relationship between $e^{iD\theta(X)}$ and $P_\theta(X)$.

Proposition 7.10. *We have the following equality of equivariant differential forms with generalized coefficients on M:*

$$\tilde{\mathcal{J}}(E,X) = \mathcal{J}(E,X). \tag{7.9}$$

In particular, the form $\mathcal{J}(E,X)$ is independent of the choice of representative $P_\lambda^\chi(X)$ of $P_\lambda(X)$.

Proof. We prove (7.9) by showing that it holds on any choice of trivializing neighbourhood U. Let $N = U \times (\mathbb{R}^k)^*$ denote the trivialization of the open subset $q^{-1}(U) \subset E^0$, and let ξ denote the coordinate on $(\mathbb{R}^k)^*$. Define a 1-form λ on N by $\lambda = - < \xi, \boldsymbol{\alpha} >$. It follows from the definition of the canonical 1-form on T^*M that λ coincides with $\imath^*\theta|_N$ under the identification $N \cong q^{-1}(U)$. Let $t^1, \dots, t^k$ be the basis for $\mathbb{R}^k$ with respect to which we have $\boldsymbol{\alpha} = \sum \alpha_j t^j$. If $\xi = \sum \xi^j t_j$ with respect to the corresponding dual basis for $(\mathbb{R}^k)^*$, then we have $\lambda = -\sum \xi^j \alpha_j$, and thus

$$D\lambda(X) = \alpha_1 \wedge d\xi^1 + \cdots + \alpha_k \wedge d\xi^k - < \xi, d\boldsymbol{\alpha} + f_{\boldsymbol{\alpha}}(X) > .$$

Define $P_\lambda^\chi(X)$ as in (6.21), with $\chi(\xi)$ any arbitrary cutoff function supported on an open neighbourhood of $U \times \{0\}$ in N. The contribution to the integral of P_λ^χ over $(\mathbb{R}^k)^*$ comes from the term of maximum degree in the $d\xi^j$. We have

$$P_\lambda^\chi(X) = \chi(\xi) - id\chi(\xi) \wedge \lambda \int_0^\infty e^{itD\lambda(X)} dt,$$

where $d\chi(\xi) = \sum \frac{\partial \chi}{\partial \xi^i}(\xi) d\xi^i$, and

$$\begin{aligned} e^{it<\boldsymbol{\alpha}, d\xi>} &= \prod_{j=1}^k (1 + it\alpha_j \wedge d\xi^j) \\ &= (it)^k \alpha_1 \wedge d\xi^1 \wedge \cdots \wedge \alpha_k d\xi^k \\ &\qquad + (it)^{k-1} \sum_{i=1}^k \alpha_1 \wedge d\xi^1 \wedge \cdots \wedge \widehat{\alpha_j \wedge d\xi^j} \wedge \cdots \wedge \alpha_k \wedge d\xi^k \\ &\qquad\qquad + \text{terms of lower degree.} \end{aligned}$$

We are thus interested in the top-degree part of $d\chi(\xi) \wedge \lambda\, e^{it<\boldsymbol{\alpha}, d\xi>}$, which is given by

$$\begin{aligned} &\left(\sum_{i=1}^k \frac{\partial \chi}{\partial \xi^i} d\xi^i\right)\left(-\sum_{j=1}^k \xi^j \alpha_j\right)\left((it)^{k-1} \sum_{l=1}^k \alpha_1 \wedge d\xi^1 \wedge \cdots \wedge \widehat{\alpha_l \wedge d\xi^l} \wedge \cdots \wedge \alpha_k \wedge d\xi^k\right) \\ &= -(it)^{k-1} \left(\sum_{j=1}^k \frac{\partial \chi}{\partial \xi^j} \xi^j\right) \alpha_1 \wedge d\xi^1 \wedge \cdots \wedge \alpha^k \wedge d\xi^k. \end{aligned}$$

Making the change of variables $\zeta^i = t\xi^i$, the top-degree part of $-id\chi \wedge \lambda \int_0^\infty e^{itD\lambda(X)}$ becomes

$$\begin{aligned} &i^k \int_0^\infty \left(\frac{1}{t} \sum_{j=1}^k \frac{\partial \chi}{\partial \zeta^j}\left(\frac{\zeta}{t}\right) \zeta^j\right) \alpha_1 \wedge d\zeta_1 \cdots \alpha_k \wedge d\zeta^k e^{-i<\zeta, D\boldsymbol{\alpha}(X)>} \\ =&i^k \int_0^\infty \frac{d}{dt}\left(-\chi\left(\frac{\zeta}{t}\right)\right) \alpha_k \cdots \alpha_1 e^{-i<\zeta, D\alpha(X)>} d\zeta^1 \cdots d\zeta^k \\ =&i^k \alpha_k \cdots \alpha_1 e^{-<\zeta, D\boldsymbol{\alpha}(X)>} d\zeta^1 \cdots d\zeta^k. \end{aligned}$$

Integrating over $(\mathbb{R}^k)^*$ and using (7.8), we obtain our result. $\square$

Theorem 7.11. *We have the following equality of differential forms with generalized coefficients on M:*

$$q_*i^*P_\theta(X) = (2\pi i)^k \mathcal{J}(E,X) = q_*i^*e^{iD\theta(X)}. \tag{7.10}$$

Proof. Let λ denote the 1-form defined on $N = U \times (\mathbb{R}^k)^*$ as above. As in the proof of Proposition 7.10, we have

$$D\lambda(X) = \alpha_1 \wedge d\xi^1 + \cdots + \alpha_k \wedge d\xi^k - < \xi, d\boldsymbol{\alpha} + f_{\boldsymbol{\alpha}}(X) >,$$

whence

$$e^{iD\lambda(X)} = i^k \alpha_k \cdots \alpha_1 e^{-i\langle \xi, D\boldsymbol{\alpha}(X)\rangle} d\xi_1 \cdots d\xi_k.$$

Thus, using (7.8), we have

$$\mathcal{J}(E,X)|_U = (2\pi i)^{-k} q_* e^{iD\theta_0(X)}|_U = (2\pi i)^{-k} \int_{(\mathbb{R}^k)^*} e^{iD\lambda(X)} = \mathcal{J}_{\boldsymbol{\alpha}}(E,X) = \tilde{\mathcal{J}}(E,X)|_U.$$

□

Remark 7.12. Let us consider the $H \times G$ equivariant form $\mathcal{J}(E,(X,Y))$ in the setting of Remark 6.9, where $E \subset TN$ is the space of horizontal vectors with respect to the connection 1-form θ. By (6.25), the form $\mathcal{J}(E,(X,Y))$ is given by:

$$\mathcal{J}(E,(X,Y)) = \delta(X - \Psi(Y)) \frac{\psi_r \cdots \psi_1}{\mathrm{vol}(H,dX)}.$$

Now, the coefficients of the above equivariant differential form are generalized functions on $\mathfrak{h}$ supported at the origin, whence the pairing of this form against a smooth function of arbitrary support is well-defined. Thus, given any invariant $f \in C^\infty(\mathfrak{h})^H$, we have the pairing

$$\int_{\mathfrak{h}} \mathcal{J}(E,(X,Y)) f(X)\, dX = f(\Psi(Y))\psi_r \cdots \psi_1. \tag{7.11}$$

Integrating over the fibres of $\pi : N \to M$, gives

$$\pi_* \langle \mathcal{J}(E,(X,Y)), f(X) \rangle = \mathrm{vol}(H,dX) f(\Psi(Y)).$$

As a result, the equivariant Chern-Weil characteristic forms can be obtained as the fibre integral of the pairing of invariant polynomials on $\mathfrak{h}$ against $\mathcal{J}(E,(X,Y))$.

Note: The reader is warned that in this case, the form $\delta_0(D\boldsymbol{\psi}(X))$ defined by (7.8) differs from the form $\delta(X-\Psi)$ defined in [PV08b] by a factor of $\mathrm{vol}(H,dX)$: we have

$$\delta(X-\Psi) = \mathrm{vol}(H,dX)\delta_0(D\boldsymbol{\psi}(X)).$$

7.6 A product formula

Suppose that a compact Lie group G acts on (M,E) transverse to E, so that the form $\mathcal{J}(E,X)$ is defined. Now suppose that $\pi : P \to M$ is a principal H-bundle, with connection 1-form $\omega \in \mathcal{A}^1(P)\otimes\mathfrak{h}$. Using ω, we can define horizontal lifts of TM, $E \subset TM$, and the G-action. (We will denote horizontal lifts by π^*.) It follows that $G\times H$ acts on P transverse to π^*E, so that there is a well-defined equivariant form with generalized coefficients $\mathcal{J}(\pi^*E,(X,Y))$, where $(X,Y)\in\mathfrak{g}\times\mathfrak{h}$. Since H acts on P transverse to the horizontal distribution $HP\cong\pi^*TM$, we can define

$$\mathcal{J}(\pi^*TM,Y) = \frac{\omega_l\wedge\cdots\wedge\omega_1}{\mathrm{vol}(H,dY)}\delta(\Omega - Y)$$

as in the remark above, where $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ is the curvature form of ω. (Since the lift of the G-action is horizontal, it leaves the connection form ω invariant and thus does not appear in this expression.) We then have the following:

Proposition 7.13. *The forms $\mathcal{J}(E,X)$, $\mathcal{J}(\pi^*TM,Y)$ and $\mathcal{J}(\pi^*E,(X,Y))$ given as above satisfy the relationship*

$$\mathcal{J}(\pi^*E,(X,Y)) = \pi^*\mathcal{J}(E,X)\mathcal{J}(\pi^*TM,Y).$$

Proof. For simplicity, we assume that E^0 can be trivialized, and let $(\alpha_1,\ldots,\alpha_k)$ denote a global frame. The annihilator in T^*P of the horizontal lift π^*E is given by

$$(\pi^*E)^0 = \pi^*(E^0)\oplus(P\times\mathfrak{h}),$$

and a frame for $(\pi^* E)^0$ is given by $(\pi^*\alpha_1, \dots, \pi^*\alpha_k, \omega_1, \dots, \omega_l)$. In terms of this frame we have

$$\begin{aligned}
\mathcal{J}(\pi^* E, (X, Y)) &= \pi^*\alpha_k \cdots \pi^*\alpha_1 \omega_l \cdots \omega_1 \delta_0(\pi^* D\boldsymbol{\alpha}(X) \oplus D\boldsymbol{\omega}(Y)) \\
&= \pi^*(\alpha_k \cdots \alpha_1 \delta_0(D\boldsymbol{\alpha}(X))) \frac{\omega_l \cdots \omega_1}{\mathrm{vol}(H)} \delta(Y - \Omega) \\
&= \pi^* \mathcal{J}(E, X) \mathcal{J}(\pi^* TM, Y). \qquad \square
\end{aligned}$$

Chapter 8

The index formula

We are now ready to compute the equivariant index formulas that form the main results of this thesis. Before proceeding with the proof of the index formula in the most general of settings, we will give a separate treatment of the case of a contact manifold. Following the statement of the index formula in the contact case, we will consider the basic examples of S^1 and S^3.

8.1 The contact case

When (M, E) is a contact manifold, we can construct a differential operator $\not{D}_b$, following the procedure in Section 4.3. Suppose then that $E \subset TM$ is a co-oriented contact distribution, and let E^0_+ denote the choice of positive connected component of E^0. Let $\alpha \in \mathcal{A}^1(M)$ be a contact 1-form compatible with the co-orientation (that is, $\alpha(M) \subset E^0_+$). Since the fibres of E can be equipped with a complex structure compatible with the symplectic structure $d\alpha$, the contact case is a special case of the almost CR case described below. However, since the annihilator of E in T^*M is the trivial line bundle $M \times \mathbb{R}$, the index formula is simpler, and so we begin with the contact case. We give only a sketch of the proof, since the methods are similar to the general case.

Theorem 8.1. *Let (M, E) be a compact, co-oriented contact manifold of dimension $2n+1$, and let a compact Lie group G act on M by co-orientation-preserving contact transformations, such that the G-action is transverse to E. Let $g \in G$, and let $k(g)$ be the locally constant function defined by* $\dim M(g) = 2k(g) + 1$.

The G-equivariant index of $\not{D}_b$ is the G-invariant generalized function on G whose germ at $g \in G$, is given, for $X \in \mathfrak{g}(g)$ sufficiently small, by

$$\operatorname{index}^G(\not{D}_b)(ge^X) = \int_{M(g)} (2\pi i)^{-k(g)} \frac{\operatorname{Td}(E(g), X)\mathcal{J}(E(g), X)}{D_g^{\mathbb{C}}(\mathcal{N}(g), X)}. \tag{8.1}$$

Proof. Substitute (5.31) into the Berline-Vergne formula (2.7), and integrate over the fibres of T^*M, using (7.4). □

In particular, we have the following formula near the identity element in G:

For $X \in \mathfrak{g}$ sufficiently small,

$$\operatorname{index}^G(\not{D}_b)(e^X) = \frac{1}{(2\pi i)^n} \int_M \operatorname{Td}(E, X)\mathcal{J}(M, X). \tag{8.2}$$

Remark 8.2. It is often useful in practice to twist the complex $\mathcal{S} = \bigwedge E^{0,1}$ by some G-equivariant Hermitian line bundle $\mathcal{W}$. If we use the operator $\not{D}_b^{\mathcal{W}}$ (4.17) on sections of $\mathcal{S} \otimes \mathcal{W}$, then we immediately obtain the following extension to (8.1) above: for $X \in \mathfrak{g}(g)$ sufficiently small,

$$\operatorname{index}^G(\not{D}_b^{\mathcal{W}})(ge^X) = \int_{M(g)} (2\pi i)^{-k(g)} \frac{\operatorname{Td}(E(g), X)\operatorname{Ch}_g(\mathcal{W}, X)\mathcal{J}(E(g), X)}{D_g^{\mathbb{C}}(\mathcal{N}(g), X)}. \tag{8.3}$$

In the case that α is a *regular* contact form, then by the Boothby-Wang theorem (see Theorem 3.9), the flow of the Reeb field ξ generates a free, effective $U(1)$-action on M. Since ξ can be assumed to be G-invariant, this action commutes with the G-action on M, and thus the quotient $M/U(1)$ is a Hamiltonian G-manifold (B, ω, Φ), that is *prequantizable* in the usual sense (see [GGK02], for example). That is, the symplectic form ω is integral, and if $\omega(X) = \omega - \Phi(X)$ denotes the equivariant symplectic form, then

$$\pi^*\omega(X) = -D\alpha(X), \quad \text{for all} \quad X \in \mathfrak{g}. \tag{8.4}$$

The associated line bundle $\mathbb{L} \to B$ is then a G-equivariant prequantum line bundle, equipped with a G invariant Hermitian metric h and connection ∇, such that $F_\nabla(X) = \frac{1}{2\pi}\omega(X)$, and $\pi : M \to B$ is the unit circle bundle inside of $\mathbb{L}$ with respect to h.

Let $\not{D}$ denote the Dolbeault-Dirac operator on sections of $\bigwedge T^{0,1}B$, and let $\not{D}_m$ denote the operator defined by (4.11), for $\mathcal{W} = \mathbb{L}^{\otimes m}$, on sections of $\bigwedge T^{0,1}B \otimes \mathbb{L}^{\otimes m}$. In this case the operator $\not{D}_b$ on sections of $\mathcal{S} \to M$ is the pullback to M of $\not{D}$, and from (8.3) we obtain:

Corollary 8.3. *We have the following equality of generalized functions on $G \times U(1)$:*

$$\operatorname{index}^{G\times U(1)}(\not{D}_b)(g,u) = \sum_{m\in\mathbb{Z}} u^{-m} \operatorname{index}^G(\not{D}_m)(g). \tag{8.5}$$

We note that the terms $\operatorname{index}^G(\not{D}_m)(g)$ appearing on the right-hand side of (8.5) are given by the equivariant Riemann-Roch number of the line bundle $\mathbb{L}^{\otimes m} \to B$: for $X \in \mathfrak{g}(g)$ sufficiently small,

$$\operatorname{index}^G(\not{D}_m)(ge^X) = \int_B (2\pi i)^{-\dim B/2} \frac{\operatorname{Td}(B,X)\operatorname{Ch}_g(\mathbb{L}^{\otimes m}, X)}{D_g^{\mathbb{C}}(\mathcal{N}_B(g), X)}$$

Proof. With the right identifications, this result can be viewed as a special case of Théorème 25 in [BV96b] for $H = U(1)$, and the details of the proof are similar.

We need to check that, for any fixed $(g,u) \in G \times U(1)$, the formula holds in a sufficiently small neighbourhood of (g,u) in $G(g) \times U(1)$. That is, for $X \in \mathfrak{g}(g)$ and $\phi \in \mathbb{R}$ sufficiently small, we need to show that

$$\operatorname{index}^{G\times U(1)}(\not{D}_b)(ge^X, ue^{i\phi}) = \sum_{m\in\mathbb{Z}} u^{-m}e^{-im\phi} \operatorname{index}^G(\not{D}_m)(ge^X). \tag{8.6}$$

For any $v \in U(1)$, we have $M(g,v) = \{y \in M | g\cdot y = y \cdot v\}$. When $M(g,v)$ is non-empty, $U(1)$ acts freely on $M(g,v)$, and we denote $B(g)^v = M(g,v)/U(1)$. The fixed-point set $B(g)$ is a (finite) disjoint union of the spaces $B(g)^v$.

Since $\mathbb{L} \cong M \times_{U(1)} \mathbb{C}$, the action of $g \in G$ on the fibres of $\mathbb{L}|_{B(g)^v}$ is scalar multiplication by $v \in U(1)$. Thus, $\mathrm{Ch}_g(\mathbb{L}^{\otimes m}, X)|_{B(g)^v} = v^m e^{im\omega(X)}$, and we have

$$\mathrm{index}^G(\not{D}_m)(ge^X) = \sum_{\substack{v\in U(1)\\ M(g,v)\neq\emptyset}} \int_{B(g)^v} (2\pi i)^{-k(g)} \frac{\mathrm{Td}(B(g)^v, X)}{D_g^{\mathbb{C}}(\mathcal{N}_B(g), X)} v^m e^{im\omega(X)}.$$

Thus, the only contribution to the right-hand side of (8.6) comes from $B(g)^u$ (provided $M(g,u)$ is non-empty), in which case we can apply the Poisson summation formula to obtain

$$\sum_{m\in\mathbb{Z}} u^{-m} e^{-im\phi}\, \mathrm{index}^G(\sigma_m)(ge^X) = \int_{B(g)^u} (2\pi i)^{-k(g)} \frac{\mathrm{Td}(B(g)^u, X)\delta_0(\omega^g(X) - \phi)}{D_g^{\mathbb{C}}(\mathcal{N}_B(g), X)}.$$

Using the index formula (8.1), the left-hand side of (8.6) is given by

$$\int_{M(g,u)} (2\pi i)^{-k(g)} \frac{\mathrm{Td}(E(g,u),(X,\phi))\, \alpha^{g,u}\delta_0(D\alpha^{g,u}(X) - \phi)}{D_g^{\mathbb{C}}(\mathcal{N}_M(g,u),(X,\phi))}.$$

The prequantization condition implies that $D\alpha^{g,u}(X, i\phi) = \pi^*\omega^g(X) - \phi$, and since the forms $\mathrm{Td}(E(g,u))$ and $D_g^{\mathbb{C}}(\mathcal{N}_M(g,u))$ are the pullback to $M(g,u)$ of the corresponding forms on $B(g)^u$, the result follows. $\square$

8.1.1 Examples

The two simplest examples of contact manifolds are the circle S^1 (which is already present in [Ati74]) and 3-sphere S^3, and thus we have found it instructive to explicitly compute the invariants defined by (8.1).

Example 8.4. Consider the circle $S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. The form $d\theta$ is a contact form on S^1, with the zero section as the contact distribution. The group $U(1) = \{e^{i\phi}\}$ acts freely on S^1 by multiplication. The action is elliptic, since $T^*_G S^1 = 0$ (while $E^0 = T^*S^1$).

Here, our operator is $\not{D}_b = 0$, and since $T^*_G S^1 = 0$, even the zero operator on S^1 is $U(1)$-transversally elliptic. The $U(1)$-equivariant index is given simply by

$$\mathrm{index}^G(0)(e^{i\phi}) = \int_{S^1} \mathcal{J}(\phi) = 2\pi\delta_0(\phi) = \sum_{m\in\mathbb{Z}} e^{im\phi},$$

where the last equality is valid for ϕ sufficiently small, using the Poisson summation formula for δ_0.

More generally, if $U(1)$ acts on S^1 by $e^{i\phi}\cdot z = e^{ik\phi}z$, for $k\in\mathbb{N}$, then S^1 is fixed by the elements of $U(1)$ of the form $g_l = e^{2\pi i l/k}$, for $0\leq l\leq k$, and we obtain a similar formula to the above near each of the points g_l.

Example 8.5. Let $M = S^3$ be given as the unit sphere in $\mathbb{R}^4$ in terms of coordinates (x_1, y_1, x_2, y_2), and consider the frame $\{X, Y, T\}$ for TS^3 given by

$$\begin{aligned} X &= x_2\frac{\partial}{\partial x_1} - y_2\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_2}\\ Y &= -y_2\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial y_2}\\ T &= y_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial y_2}. \end{aligned}$$

A contact structure is given by $E = TS^3/\mathbb{R}T$. If we let $\{\xi, \zeta, \alpha\}$ denote the corresponding coframe, then α is a contact form on S^3. In coordinates we have

$$\alpha = y_1dx_1 - x_1dy_1 + y_2dx_2 - x_2dy_2,$$

and one readily sees that $\alpha(T) = x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$, so that T is the Reeb field for α.

We let $U(1)$ act on S^3, with action given in complex coordinates as follows: identify $\mathbb{R}^4\cong\mathbb{C}^2$ via $z_j = x_j + iy_j$, $j = 1, 2$. The action of $e^{i\phi}\in U(1)$ on $\mathbb{C}^2$ by $e^{i\phi}\cdot(z_1, z_2) = (e^{i\phi}z_1, e^{i\phi}z_2)$ restricts to an action of $U(1)$ on S^3. Let $\mathfrak{g} = i\mathbb{R}$ denote the Lie algebra of G, and note that the infinitesimal action of $\mathfrak{g}$ on M is given by $i\phi\mapsto\phi T$. The orbits of the action are thus transverse to the contact distribution E, whence the action of $U(1)$ on S^3 is elliptic.

The almost CR structure on M is given by taking $E_{1,0} = \mathbb{C}Z$, where $Z = \frac{1}{\sqrt{2}}(X + iY)$. The corresponding covector in $E^*\otimes\mathbb{C}$ is $\theta = \frac{1}{\sqrt{2}}(\xi - i\zeta)$. The associated complex structure on E comes from the complex structure on $\mathbb{C}^2$, and is given by $J(X) = -Y$

and $J(Y) = X$, so that $J(\xi) = \zeta$, and $J(\zeta) = -\xi$ on E^*. Since this structure is integrable, M is a CR manifold, and if we choose the Tanaka-Webster connection, then $\not{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*)$.

Writing $\eta \in T_x^* S^3$ as $\eta = a\xi + b\zeta + c\alpha$, the symbol of $\not{D}_b$ is given by

$$\sigma_b(x, \eta) = -i\sqrt{2}((a + ib)\iota(\overline{Z}) - (a - ib)\epsilon(\overline{\theta})),$$

from which we see that $\sigma_b^2(x, \eta) = a^2 + b^2$.

Finally, for ϕ sufficiently small, the $U(1)$-equivariant index of $\not{D}_b$ is given by

$$\begin{aligned} \operatorname{index}^{U(1)}(\not{D}_b)(e^{i\phi}) &= \frac{1}{2\pi i} \int_{S^3} \mathrm{Td}(E, \phi)\mathcal{J}(E, \phi) = 2\pi(\delta_0(\phi) - i\delta_0'(\phi)) \\ &= \sum_{m \in \mathbb{Z}} (1 - m)e^{im\phi} = \sum_{m \in \mathbb{Z}} e^{im\phi} \frac{1}{2\pi i} \int_{S^2} \mathrm{Td}(S^2)e^{-im\omega}, \end{aligned} \tag{8.7}$$

since

$$\mathcal{J}(E, \phi) = \alpha \wedge \delta_0(d\alpha - \phi) = \alpha(\delta_0(-\phi) + \delta_0'(-\phi)d\alpha),$$

while, if $\pi : S^3 \to S^2$ denotes the projection onto the orbit space, then

$$\begin{aligned} \mathrm{Td}(E, \phi) &= \mathrm{Td}(\pi^* TS^2, \phi) = \pi^* \mathrm{Td}(S^2) \\ &= \pi^*(1 + i\omega) = 1 + id\alpha. \end{aligned}$$

8.2 General formula

We wish to consider the following situation: suppose $E \subset TM$ is a sub-bundle of even rank, and that a compact Lie group G acts on M transverse to E. We suppose given G-equivariant Hermitian vector bundles $\mathcal{W}^\pm \to M$, and an G-equivariant morphism $\sigma : \pi^*\mathcal{W}^+ \to \pi^*\mathcal{W}^-$, where $\pi : T^*M \to M$ is the projection mapping. We suppose that our symbol σ "depends only on E^*" in the following sense: we have the exact sequence of vector bundles

$$0 \longrightarrow E^0 \overset{\iota}{\hookrightarrow} T^*M \overset{r}{\twoheadrightarrow} E^* \longrightarrow 0,$$

and σ is such that $\sigma = r^*\sigma_E$, for some symbol $\sigma_E : s^*\mathcal{W}^+ \to s^*\mathcal{W}^-$, where $s : E^* \to M$ denotes projection onto M.

We note that if σ_E is an elliptic symbol on E^*, then the support of σ is E^0, and the transversality condition on the action of G gives $E^0 \cap T^*_G M = 0$, which means that σ is G-transversally elliptic, in the sense of Atiyah [Ati74]. We may then compute its equivariant index using the formula of Paradan-Vergne [PV08b]:

$$\operatorname{index}^G(\sigma)(ge^X) = \int_{T^*M(g)} (2\pi i)^{-\dim M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)}{D_g(\mathcal{N}, X)} \operatorname{Ch}^g_F(\mathbb{A}, \sigma, X) P_{\theta^g}(X). \tag{8.8}$$

Here, the 1-form θ^g is the restriction of the canonical 1-form on T^*M to $T^*M(g)$, and $P_{\theta^g}(X)$ Paradan's form (6.21). The form $\operatorname{Ch}^g_F(\mathbb{A}, \sigma, X)$ is the Chern character with support $F = \operatorname{Supp}(\sigma)$ (6.27), constructed using Quillen's Chern character $\operatorname{Ch}^g_Q(\mathbb{A}^\sigma, X)$ (5.22), where $\mathbb{A}$ is a superconnection on the $\mathbb{Z}_2$-graded bundle $\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^-$ with no term in exterior degree zero. The cohomology class of $\operatorname{Ch}^g_Q(\mathbb{A}^\sigma, X)$ depends only on the symbol σ (see [BV96a], Section 4.5), and as noted in Section 6.4, this class coincides with that of $\operatorname{Ch}^g_F(\mathbb{A}, \sigma, X)$ in an appropriate cohomology space [PV07a, PV08a].

Remark 8.6. We are allowing an abuse of notation in our statement of the formula (8.8) above. The G-equivariant index of σ is an G-invariant generalized function on G. The right hand side of the above formula in fact defines an G-invariant generalized function $\operatorname{index}^G(\sigma)_g(X)$ on the tubular neighbourhood $G \times_{G(g)} \mathcal{U}_g \hookrightarrow G$, where $\mathcal{U}_g$ is an open $G(g)$-invariant neighbourhood of 0 in $\mathfrak{g}(g)$. For smooth functions the equality $f(ge^X) = f_g(X)$ follows from the localization formula in equivariant cohomology, but in the case of generalized functions one must carefully check compatibility conditions using the descent method of Duflo-Vergne [DV93]. When the index formula of Berline and Vergne is used, this checking was done in [BV96a, BV96b]. Paradan and Vergne solve this problem in [PV08b] by proving a localization formula in equivariant cohomology with generalized coefficients that allows the right-hand sides of (8.8) to be patched together to give a generalized function on G.

Our goal is to compute the pushforward of the formula (8.8) on T^*M to obtain a formula as an integral over M. Using a transgression argument similar to that in [PV08a, Proposition 3.38],[1] we have:

Proposition 8.7. *If the 1-forms $\alpha, \beta \in \mathcal{A}(T^*M)$ agree on* $\mathrm{Supp}(\sigma)$*, then the following equality holds in $\mathcal{H}^{-\infty}(\mathfrak{g}, T^*M)$:*

$$\mathrm{Ch}_F(\mathbb{A}, \sigma, X)P_\alpha(X) = \mathrm{Ch}_F(\mathbb{A}, \sigma, X)P_\beta(X). \tag{8.9}$$

Now, let $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ be a $\mathbb{Z}_2$-graded G-equivariant Hermitian vector bundle, and suppose $\sigma_E : s^*\mathcal{V}^+ \to s^*\mathcal{V}^-$ is an G-invariant elliptic symbol on $E^* \subset T^*M$. We let s continue to denote the restriction of s to $T^*M(g)$. If we let $\sigma = r^*\sigma_E$, then we have:

Theorem 8.8. *If the Lie group G acts on M transverse to the distribution $E \subset TM$, then the symbol $\sigma = r^*\sigma_E$ is transversally elliptic, and the G-equivariant index of σ is the generalized function on G whose germ at $g \in G$ is given, for $X \in \mathfrak{g}(g)$ sufficiently small, by*

$$\mathrm{index}^G(\sigma)(ge^X) = \int_{M(g)} (2\pi i)^{-\operatorname{rank} E(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)}{D_g(\mathcal{N}(g), X)} \mathcal{J}(E(g), X) s_* \mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X). \tag{8.10}$$

Proof. Since the symbol σ is the pullback to T^*M of the elliptic symbol σ_E on E^*, we have $\mathrm{Ch}_F^g(\mathbb{A}, \sigma, X) = r^*(\mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X))$. Denote by θ^g the restriction of the canonical 1-form θ to $T^*M(g)$. The restriction of σ_E to $E^*(g)$ is again elliptic, and $\sigma|_{T^*M(g)}$ has support $E^0(g)$. By Proposition 4.7, we know that the action of $G(g)$ on $M(g)$ is transverse to $E(g) \subset TM(g)$, whence the restriction of σ to $T^*M(g)$ is transversally elliptic. Moreover, since the action of $G(g)$ is transverse to $E(g)$, the form $\mathcal{J}(E(g), X)$ is well-defined as a $G(g)$-equivariant differential form with generalized coefficients on $M(g)$. We choose a

[1] This proposition first appears in [Par99, Proposition 3.11] and is generalized in [Par00, Section 2.3].

G-invariant splitting $T^*M = E^* \oplus E^0$, giving us the commutative diagram

$$\begin{tikzcd} & T^*M(g) \arrow[dl, "p"'] \arrow[dr, "r"] \arrow[dd, "\pi"] & \\ E^0(g) \arrow[dr, "q"'] & & E^*(g) \arrow[dl, "s"] \\ & M(g) & \end{tikzcd}$$

Since θ^g and $\imath^*\theta^g$ agree on $E^0(g)$, we may use Proposition 8.7 to obtain

$$\mathrm{Ch}_F^g(\mathbb{A}, \sigma, X) P_{\theta^g}(X) = r^* \,\mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X) p^* \imath^* P_{\theta^g}(X)$$

Using Theorem 7.11 and the commutative diagram, we see that

$$r_* p^* \imath^* P_{\theta^g}(X) = s^* q_* \imath^* P_{\theta^g}(X)) = (2\pi i)^{\mathrm{rank}\, E^0(g)} s^* \mathcal{J}(E(g), X),$$

and thus,

$$\begin{aligned} \pi_* \,\mathrm{Ch}_F^g(\mathbb{A}, \sigma, X) P_{\theta^g}(X) &= s_* r_* \left(r^* \,\mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X) p^* \imath^* P_{\theta^g}(X)\right) \\ &= s_* \left(\mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X) r_* p^* \imath^* P_{\theta^g}(X)\right) \\ &= (2\pi i)^{\mathrm{rank}\, E^0} s_* \left(\mathrm{Ch}_F^g(\mathbb{A}, \sigma_E, X) s^* \mathcal{J}(E(g), X)\right). \end{aligned}$$

By integrating over the fibres in (8.8) and substituting the above, the result follows. □

Remark 8.9. Suppose $\mathcal{W}$ is some G-equivariant vector bundle on M, and consider the symbol $\sigma_{\mathcal{W}} = \sigma \otimes \mathrm{Id}_{\mathcal{W}} : \pi^*\mathcal{V}^+ \otimes \mathcal{W} \to \pi^*\mathcal{V}^- \otimes \mathcal{W}$. Using the multiplicativity of the Chern character [PV07b, PV07a], we have $\mathrm{Ch}_F(\mathbb{A}, \sigma_{\mathcal{W}}, X) = \mathrm{Ch}_F(\mathbb{A}, \sigma, X)\,\mathrm{Ch}(\mathcal{W})$. Since $\mathrm{Ch}(\mathcal{W})$ is a form on M, we obtain the following extension to 8.10:

Proposition 8.10. *The G-equivariant index of $\sigma_{\mathcal{W}}$ is given, for $X \in \mathfrak{g}$ sufficiently small, by*

$$\mathrm{index}^G(\sigma_{\mathcal{W}})(e^X) = \frac{1}{(2\pi i)^{\mathrm{rank}\, E}} \int_M \widehat{\mathrm{A}}^2(M, X) \mathcal{J}(E, X)\,\mathrm{Ch}(\mathcal{W}, X) s_* \,\mathrm{Ch}_F(\mathbb{A}, \sigma_E, X), \tag{8.11}$$

with similar formulas near other elements $g \in G$.

8.3 The almost CR case

We now consider the case where the subbundle E is the Levi distribution of an almost CR structure $E_{1,0} \subset T^*_{\mathbb{C}}M$ of type (n,k). As in Section 4.3, we equip $E \otimes \mathbb{C}$ with a Hermitian connection h and Hermitian connection ∇. The connection ∇ induces a connection $\nabla^{\mathcal{S}}$ on the bundle $\mathcal{S} = \bigwedge E^{0,1}$ (4.13). We recall that the Clifford multiplication $\mathbf{c} : \mathbb{C}l(E) \to \mathrm{End}(\mathcal{S})$ given by (4.14) makes $\mathcal{S}$ into a spinor module for the Clifford bundle $\mathbb{C}l(E) \to M$. We are then able to construct the Dirac operator

$$\not{D}_b : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$$

given by (4.15). As shown in Section 4.4, the principal symbol $\sigma_b = \sigma(\not{D}_b) : \pi^*\mathcal{S}^+ \to \pi^*\mathcal{S}^-$ is given by

$$\sigma_b(x, \xi) = i\mathbf{c}(r(\xi)),$$

where $r(\xi) \in E^*_x$ is the projection of $\xi \in T^*_x M$. Since σ_b depends only on the projection onto E^*, we may let $\sigma_E = \sigma_b$ in (8.10), and choose the Quillen superconnection

$$\mathbb{A}^{\sigma_b} = \pi^*\nabla^{\mathcal{S}} + iv_{\sigma_b} \tag{8.12}$$

on $\pi^*\mathcal{S}$. As we saw in Section 5.4, σ_b^* is defined with respect to the Hermitian metric h such that $\sigma_b^* = \sigma_b$, so that $v^2_{\sigma_b}(x,\xi) = ||r(\xi)||^2_x \,\mathrm{Id}_{\mathcal{S}}$ (5.21).

From Section 6.4, we have that

$$s_* \,\mathrm{Ch}^g_F(\mathbb{A}, \sigma_b, X) = s_* \,\mathrm{Ch}^g_Q(\mathbb{A}^{\sigma_b}, X), \tag{8.13}$$

where $\mathrm{Ch}^g_Q(\mathbb{A}^{\sigma_b}, X) = \mathrm{Str}(g^{\mathcal{S}} \cdot j^* e^{\mathbb{F}(\mathbb{A}^{\sigma_b})(X)})$ is Quillen's Chern character on the fixed point set $\mathcal{S}(g)$. Let $\mathcal{N}_E(g) = \mathcal{N}(g) \cap E(g)$, and note that $\mathcal{N}_E(g)$ inherits a complex structure from E. Following Section 5.5, we have

$$Ch^g_Q(\mathbb{A}^{\sigma_b}, X) = \mathrm{Ch}_Q(\mathbb{A}^{\sigma_b}_g, X) D^{\mathbb{C}}_g(\overline{\mathcal{N}_E(g)}, X), \tag{8.14}$$

where $\mathbb{A}^{\sigma_b}_g$ is given by (5.28). We recall the identity (5.24), which gives

$$\mathrm{Ch}_Q(\mathbb{A}^{\sigma_b}_g, X) = (2\pi i)^n \,\mathrm{Td}(E_{0,1}, X)^{-1} \,\mathrm{Th}_{MQ}(E^*, X), \tag{8.15}$$

where we have used the Hermitian metric to identify E^* with $E_{0,1}$.

By the splitting principle, we have $D_g(\mathcal{N}(g),X) = D_g(\mathcal{N}_E(g),X)D_g(\mathcal{N}_0(g),X)$, where $\mathcal{N}_0(g) = \mathcal{N}(g)\cap E^0(g)$. Using the complex structure on $\mathcal{N}_E(g)$, we may write (see [AS68c], or [LM89, Chapter 14])

$$\begin{aligned} D_g(\mathcal{N}(g),X) &= D_g(\mathcal{N}_0(g),X)D_g^{\mathbb{R}}(\mathcal{N}_E(g),X) \\ &= D_g(N_0(g),X)D_g^{\mathbb{C}}(N_E(g),X)D_g^{\mathbb{C}}(\overline{\mathcal{N}_E(g)},X). \end{aligned} \tag{8.16}$$

Finally, we similarly have, using the isomorphism $TM/E \cong E^0$ and (5.11), the identity

$$\widehat{\mathrm{A}}^2(M,X) = \widehat{\mathrm{A}}^2(E,X)\widehat{\mathrm{A}}^2(E^0,X) = \mathrm{Td}(E_{1,0},X)\,\mathrm{Td}(E_{0,1},X)\widehat{\mathrm{A}}^2(E^0,X). \tag{8.17}$$

Substituting (8.12) - (8.17) into (8.10) and simplifying, we obtain:

Theorem 8.11. *Suppose that a compact Lie group G acts on an almost CR manifold $(M,E_{1,0})$ preserving the CR structure, and such that the G-orbits are transverse to the Levi distribution $E \subset TM$. The Dirac operator $\not{D}_b$ is then a G-transversally elliptic operator, and its equivariant index is the generalized function on G given near $g \in G$, for $X \in \mathfrak{g}(g)$ sufficiently small, by*

$$\mathrm{index}^G(\not{D}_b)(ge^X) = \int_{M(g)} (2\pi i)^{-\operatorname{rank} E(g)/2}\frac{\mathrm{Td}(E(g),X)}{D_g^{\mathbb{C}}(\mathcal{N}_E,X)}\frac{\widehat{\mathrm{A}}^2(E^0(g),X)}{D_g(\mathcal{N}_0,X)}\mathcal{J}(E(g),X). \tag{8.18}$$

Remark 8.12. Given a G-equivariant Hermitian vector bundle $\mathcal{W}$, we may form the operator $\not{D}_b^{\mathcal{W}}$ of the form (4.17). The symbol of this operator is $\sigma_b \otimes \mathrm{Id}_W$, and thus we can given an extension to (8.18) of the form (8.11) including the additional term $\mathrm{Ch}(\mathcal{W},X)$.

Chapter 9

Examples and Applications

Since all of the geometries we considered in Chapter 3 can be viewed as special cases of an almost CR structure, we can in each case construct a differential operator of the form $\not{D}_b$ (4.15). Provided we are given the action of a compact Lie group preserving the geometric structure, such that the orbits are transverse to the distribution E defined by that structure, the operator $\not{D}_b$ provides us with an example of a G-transversally elliptic operator, and its equivariant index is given by (8.18).

In certain cases, such as an (almost) contact structure or framed f-structure, the annihilator bundle E^0 is trivial, and we have the simpler formula given by (8.1). From the discussion at the end of Section 4.3, we are motivated to consider the situation where one assumes additional structure, such as a Sasakian structure (or S-structure, for higher corank).

9.1 Sasakian Manifolds

Let M be a Sasakian manifold, equipped with contact form α, Reeb field ξ, endomorphism field ϕ and metric g. Let $E_{1,0}$ denote the $+i$-eigenbundle of $\phi|_E$. We recall that $(M, E_{1,0})$ is then a strongly pseudoconvex CR manifold, and ξ is a Killing vector field with respect to the metric g. As noted in Section 4.3, we can construct $\not{D}_b$ using the Tanaka-Webster connection associated to the CR structure, giving

$$\not{D}_b = \sqrt{2}(\overline{\partial}_b + \overline{\partial}_b^*),$$

where $\overline{\partial}_b$ is the tangential CR operator (see Section 3.4.1). Because the Reeb field is Killing, it generates a one-parameter group G of isometries, whence $\not{D}_b$ is G-transversally elliptic. By Proposition 4.16, we obtain the virtual representation of G

$$\sum(-1)^i H^{(i)}_{KR}(M, E_{1,0})$$

given by the Kohn-Rossi cohomology of the $\overline{\partial}_b$ operator. The character of this representation is given by the index formula (8.1), and we note that this depends only on the Sasakian structure. That is, (8.1) defines an invariant of the Sasakian structure.

The similarities between this result and the index of the operator $\not{D}$ (4.11) prompt us to consider the following application of our index formula.

9.2 Almost CR quantization

Let us explain briefly why our index formula (8.1) deserves to be described as a *quantization* of a contact manifold (M, E). The traditional quantization problem was the attempt to improve the mathematical understanding of the relationship between classical mechanical phase space T^*X and the corresponding quantum system $L^2(X)$, in particular in the case $X = \mathbb{R}^n$. For reasons not necessarily physical, the phase space T^*X is often replaced by a compact symplectic manifold (M, ω); the expected corresponding quantum object is then a finite-dimensional Hilbert space $\mathcal{H}$. As noted by Kirillov, [Kir04], there is no canonical classical-quantum correspondence; however, there have been many attempts to define a quantization procedure, such as path integral quantization, deformation quantization, and geometric quantization. A good overview of geometric quantization (the method we follow) is the expository article [Sja96]; a more detailed treatment can be found in the text [GGK02].

One way to describe the quantization problem is as an attempt to define a functor Q that associates to each symplectic manifold (M, ω) a Hilbert space $\mathcal{H} = Q(M)$ such that to each "classical observable" $f \in C^\infty(M)$ there corresponds a "quantum observable", a skew-adjoint operator $\hat{f} : \mathcal{H} \to \mathcal{H}$. The map $f \mapsto \hat{f}$ is required to be a Lie algebra homomorphism from the Poisson algebra[1] $(C^\infty(M), \{\,,\,\})$ to the Lie algebra of skew-adjoint operators on $\mathcal{H}$ (with respect to the commutator Lie bracket), normalized such that $\hat{1} = i\,\mathrm{Id}$. There is an additional axiom, the *minimality* axiom, which requires that every complete family of functions corresponds to a complete family of operators.

What these conditions tell us is that to the Lie algebra action of the Poisson algebra there should correspond an action of the Lie algebra of skew-adjoint operators. The trouble with all of the above is that it is impossible in general to define a correspondence that satisfies all of the above properties. If we suppose instead that we are given instead a Hamiltonian action of a Lie group G on M with moment map Φ, we may try to quantize with respect to the Lie subalgebra generated by the moment map components Φ^X, $X \in \mathfrak{g}$, and ask that the quantization procedure associate to this action a unitary representation of G on $Q(M)$, such that the infinitesimal action of $\mathfrak{g}$ on $Q(M)$ is given by the operators $\widehat{\Phi^X}$. In this context, the minimality condition requires that a transitive G-action on M correspond to an irreducible unitary representation on $Q(M)$[2]. It then becomes possible to satisfy the minimality condition in certain cases by imposing a polarization.

Let us describe the procedure of geometric quantization. Suppose we are given a Hamiltonian G-space (M, ω, Φ), such that the equivariant cohomology class of $\omega(X) = \omega - \Phi(X)$ is integral. (That is, $[\omega(X)]$ lies in the image of the coefficient homomorphism $\mathcal{H}(\mathfrak{g}, M; \mathbb{Z}) \to \mathcal{H}(\mathfrak{g}, M; \mathbb{R})$.) Then there exists a G-equivariant complex line bundle $\mathbb{L} \to M$, equipped with G-invariant Hermitian metric h and connection ∇ with equivariant curvature $F_\nabla(X) = \omega(X)$.

[1]The Poisson bracket is given by $\{f, g\} = \omega(df, dg)$.

[2]In the case that M is a coadjoint orbit of G in $\mathfrak{g}^*$, the relevant construction is the orbit method of Kirillov [Kir04]

Definition 9.1. *A G-invariant complex line bundle $\mathbb{L} \to M$ is a G*-equivariant prequantum line bundle *for (M, ω, Φ) if the equivariant curvature of $\mathbb{L}$ is equal to $[\omega(X)]$.*

Remark 9.2. The unit circle bundle P (with respect to the metric h) inside of $\mathbb{L}$ is a principal $U(1)$-bundle, and is known as the prequantum circle bundle. A choice of connection α on P such that $D\alpha(X) = -\omega(X)$ determines a *G-equivariant prequantization* of M [GGK02]. Since the line bundle $\mathbb{L}$ can be recovered as the associated line bundle $P \times_{U(1)} \mathbb{C}$, a prequantization is equivalently specified by either $(\mathbb{L}, \mathsf{h}, \nabla)$ or (P, α). We recall from Example 3.8 that (P, α) is a contact manifold, with contact form α.

The action of G on M induces an action of G on the space of sections of $\mathbb{L}$ by bundle automorphisms. This action, and the metric h, determine a unitary representation of G on the space of L^2-sections of $\mathbb{L}$. From the infinitesimal action of $\mathfrak{g}$ on the space of sections, we obtain the desired correspondence $\Phi^X \mapsto \widehat{\Phi^X}$ via

$$\widehat{\Phi^X} = -\nabla_{X_M} + i\pi^*\Phi^X.$$

This correspondence determines a Lie algebra homomorphism, normalized such that $1 \mapsto i\,\mathrm{Id}$, but it does not satisfy the minimality axiom, because the space of L^2-sections is too big. To cut down the space of sections, we apply a (complex) polarization (see Section 4.2). A polarization determines a subspace of the space of L^2 sections of $\mathbb{L}$ by requiring $\nabla_{\overline{X}} s = 0$ for all $X \in \mathcal{P}$; these are the so-called *polarized* sections; the space of polarized sections is then a candidate for the space $Q(M)$. For the sake of simplicity, let's assume that M is Kähler, and $\mathbb{L}$ is holomorphic. In this case, we can take the polarized sections of $\mathbb{L}$ to be its holomorphic sections.

The only problem with this definition of $Q(M)$ is that it is often the case that $Q(M) = 0$. By the Kodaira vanishing theorem (see [BGV91, Proposition 3.72]), if M is compact and the curvature of $\mathbb{L}$ is sufficiently negative, then the space of holomorphic sections is trivial. There are two ways to save ourselves from having all of our work be for nothing: we can replace $\mathbb{L}$ by some higher tensor power of $\mathbb{L}$, or instead of considering the space of holomorphic sections of $\mathbb{L}$, we can consider the sheaf of holomorphic sections $\mathcal{O}_{\mathbb{L}}$, and recover a vector space via the cohomology groups $H^k(M;\mathcal{O}_{\mathbb{L}})$. The definition of $Q(M)$ is then taken to be

$$Q(M) = \sum(-1)^k H^k(M;\mathcal{O}_{\mathbb{L}}). \tag{9.1}$$

This latter approach might seem to be the less satisfactory of the two, since the resulting object is a "virtual vector space". There are certain advantages to this approach, however. For one, when the curvature of $\mathbb{L}$ is sufficiently positive, Kodaira's theorem tells us that only the space $H^0(M;\mathcal{O}_{\mathbb{L}})$ is non-zero, and we recover the space of holomorphic sections. The other advantage to this approach is that the cohomology $H^k(M;\mathcal{O}_{\mathbb{L}})$ is isomorphic to the cohomology $H^{0,k}(M;\mathbb{L})$ of the complex of differential forms on M with values in $\mathbb{L}$, with respect to holomorphic $\overline{\partial}_{\mathbb{L}}$ operator of $\mathbb{L}$. If we consider the corresponding rolled-up complex

$$\mathcal{A}^{0,even}(M,\mathbb{L}) \xrightarrow{\overline{\partial}_{\mathbb{L}}+\overline{\partial}_{\mathbb{L}}^*} \mathcal{A}^{0,odd}(M,\mathbb{L}),$$

then the dimension of $Q(M)$ is given by the index of $\overline{\partial}_{\mathbb{L}}+\overline{\partial}_{\mathbb{L}}^*$, which is in turn given by the Riemann-Roch formula. We also note that this operator is equal to the Dolbeault-Dirac operator $\not{D}$ given by (4.11) (see [BGV91, Section 3.6]).

In the presence of an action of G on M preserving the polarization on M, we obtain a G-representation on the space of holomorphic sections of $\mathbb{L}$, and hence $Q(M) = \sum(-1)^k H^{0,k}(M;\mathbb{L})$ is a virtual G-representation. The character of this representation is then given by the G-equivariant index of $\not{D}$, and hence, by the equivariant Riemann-Roch formula.

Finally, the discussion in the above few paragraphs remains valid if we choose an *almost* complex structure J compatible with the symplectic form ω. The operator $\not{D}$ can still be defined, and although it no longer squares to zero, it remains an elliptic first-order differential operator with a well-defined index. The (equivariant) quantization $Q(M)$ then can still be defined via the (equivariant) index of $\not{D}$.

The notion of contact quantization comes from the analogy between the construction of the operator $\not{D}_b$ using the almost CR structure on a contact manifold, and the operator $\not{D}$. Instead of an almost complex structure on M, we have an almost CR structure. In either case we have a bundle of spinors $\mathcal{S}$ defined with the aid of a polarization; in the almost complex case we take $\mathcal{P} = T^{1,0}M$, and in the almost CR case we take $\mathcal{P} = E_{1,0}(M)$. Thus, for any almost CR manifold, we are motivated to make the following:

Definition 9.3. *Let $(M, E_{1,0})$ be an almost CR manifold, and suppose that a compact Lie group G acts on M preserving $E_{1,0}$. We define the* almost CR quantization *of $(M, E_{1,0})$ to be the virtual G-representation*

$$Q(M) = \ker \not{D}_b - \ker \not{D}_b^*.$$

When the action of G is transverse to the Levi distribution E, then $\not{D}_b$ is transversally elliptic, and the character of the G-representation $Q(M)$ is given by the generalized function $\mathrm{index}^G(\not{D}_b)$ on G. In the case of a Sasakian manifold (see the previous section) we can take our analogy further, by writing $\not{D}_b$ in terms of the $\overline{\partial}_b$ operator. Using the Hodge theory for the Kohn-Rossi Laplacian $\Box_b = \not{D}_b^2$ [Koh65], the space $Q(M)$ can be identified with the alternating sum of the Kohn-Rossi cohomology groups, by Proposition 4.16.

Given an Hermitian vector bundle $\mathcal{W} \to M$, we can consider the operator $\not{D}_b^{\mathcal{W}}$ on $\mathcal{S} \otimes \mathcal{W}$. We then expect that the index of $\not{D}_b^{\mathcal{W}}$ has an interpretation in terms of the polarized sections of $\mathcal{W}$. For example, if $\mathcal{W}$ is a CR-holomorphic vector bundle over a strongly pseudoconvex CR manifold, the polarized sections of $\mathcal{W}$ should be the CR-holomorphic sections of $\mathcal{W}$, for a suitable choice of connection.

One of our goals, upon the completion of this thesis, is to investigate this relationship precisely. We also hope to be able to prove analogous results in the case of almost S-structures considered in [LP04] using the notion of a generalized Tanaka-Webster connection. We note that without a symplectic form, there does not seem to be any obvious analogue of the prequantum condition for contact or CR manifolds. One could consider the fundamental 2-form Φ defined for any f-structure with compatible metric g by $\Phi(X,Y) = \mathsf{g}(X, fY)$. However in most interesting cases this form is exact, so perhaps the CR analogue of a prequantum line bundle is simply a trivial bundle. (In the contact case this would be consistent with the notion that the quantization of a contact manifold should agree with the geometric quantization of its symplectization.)

9.3 Principal bundles

Suppose that $\pi : P \to M$ is a principal H-bundle, where H is compact. Suppose also that a compact Lie group G acts from the left on P, commuting with the right H-action. The G-action thus descends to an action on the quotient M. We have the $H \times G$-equivariant differential form with generalized coefficients $\delta(X - \Omega(Y))$ on P defined by (6.5), where $\Omega(Y)$ is the G-equivariant curvature of a G-invariant connection form $\omega \in \mathcal{A}(P, \mathfrak{h})$. (See Example 6.4.) We choose a basis $X_1, \ldots X_l$ for $\mathfrak{h}$, and write $\omega = \sum \omega^j \otimes X_j$ with respect to this basis. As in Remark 6.9, we define the form $\nu = - < \omega, \xi >$ on $P \times \mathfrak{h}^*$, where $\xi^1, \ldots, \xi^l$ denotes the corresponding dual basis, and construct the $H \times G$-equivariant differential form with generalized coefficients $P_\nu(X,Y)$. By (6.25), we have

$$q_* P_\nu(X,Y) = \frac{(2\pi i)^l}{\mathrm{vol}(H, dX)} \omega^k \cdots \omega^1 \delta(X - \Omega(Y)),$$

where $q : P \times \mathfrak{h}^* \to P$ denotes projection. Now, let us suppose that the G-action on P is the horizontal lift of a G-action on M with respect to the connection ω, so that $\Omega(Y) = \Omega$. Let $T_{hor}P \subset TP$ denote the horizontal bundle with respect to the connection ω. Since H acts on P transverse to $T_{hor}P$, we see that

$$\delta(X - \Omega) = \mathrm{vol}(H, dX)\delta_0(D\boldsymbol{\omega}(X)), \tag{9.2}$$

where $\boldsymbol{\omega} = (\omega^1, \ldots, \omega^l)$, since in this case (see Remark 7.12) we have

$$\mathcal{J}(T_{hor}P, X) = (2\pi i)^{-l} q_* P_\nu(X) = \omega^k \cdots \omega^1 \delta_0(D\boldsymbol{\omega}(X)).$$

Suppose that σ is a G-transversally elliptic symbol on T^*M, such that σ is given by an elliptic symbol on some subbundle $E^* \subset T^*M$, and that G acts on M transverse to E. We can then compute the G-equivariant index of σ by our methods, in terms of the form $\mathcal{J}(E, Y)$.

The horizontal lift of σ to T^*P determines an $H \times G$-transversally elliptic symbol $\sigma_\flat$ (see [BV96b, Ver96]). The support of $\sigma_\flat$ is $(\widetilde{E})^0 = \widetilde{E^0} \oplus (P \times \mathfrak{h}^*)$, where $\widetilde{E^0} \subset T^*_H P$ is the horizontal lift of $E^0 \subset T^*M$, since $\sigma_\flat$ is elliptic along the subbundle $\widetilde{E} \subset TP$ given by the horizontal lift of E. Since $H \times G$ acts on P transverse to $\widetilde{E}$, we can construct the $H \times G$-equivariant differential form with generalized coefficients $\mathcal{J}(\widetilde{E}, (X, Y))$. By Proposition 7.13, we have

$$\mathcal{J}(\widetilde{E}, (X, Y)) = \pi^*(\mathcal{J}(E, Y))\mathcal{J}(T_{hor}P, Y). \tag{9.3}$$

Let us see how this result fits in with our index formula. For simplicity we compute the case $(h, g) = (1, 1)$; the general case follows the same procedures as in [BV96b]. By Theorem 8.8, we have

$$\operatorname{index}_P^{H\times G}(\sigma_\flat)(e^X, e^Y) = \int_P (2\pi i)^{-k} \widehat{\mathrm{A}}^2(P, (X, Y)) s_* \operatorname{Ch}(\sigma_\flat), (X, Y))\mathcal{J}(\widetilde{E}, (X, Y)). \tag{9.4}$$

As in [BV96b] we may assume that $\operatorname{Ch}(\sigma_\flat, (X, Y)) = \pi^* \operatorname{Ch}(\sigma, Y)$, and we have

$$\widehat{\mathrm{A}}^2(P, (X, Y)) = \pi^* \widehat{\mathrm{A}}^2(M, Y) j_{\mathfrak{h}}^{-1}(X)$$

and

$$\delta_0(X - \Omega) = j_{\mathfrak{h}}(X) \sum_{\tau \in \hat{H}} \operatorname{Tr} \tau(e^X) \operatorname{Tr} \tau^*(e^\Omega)$$

where

$$j_{\mathfrak{h}}(X) = \det_{\mathfrak{h}} \frac{e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2}}{\operatorname{ad} X}. \tag{9.5}$$

Thus,

$$\widehat{\mathrm{A}}^2(P, (X, Y))\mathcal{J}(\widetilde{E}, (X, Y)) = \widehat{\mathrm{A}}^2(P, (X, Y))\pi^*\mathcal{J}(E, X)\frac{\omega^l \cdots \omega^1}{\operatorname{vol}(H, dX)}\delta(X - \Omega),$$

which descends to the form

$$\sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(e^X)\widehat{\mathrm{A}}^2(M,Y)\mathcal{J}(E,X)\operatorname{Tr}\tau^*(e^\Omega)$$

on the quotient $M = P/H$. Substituting this into (9.4), we obtain

$$\operatorname{index}_P^{H\times G}(\sigma_b)(e^X,e^Y) = \sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(e^X)\operatorname{index}_M^G(\sigma)(e^Y)\operatorname{Tr}\tau^*(e^\Omega).$$

9.4 Orbifolds

We can generalize the results of the previous section slightly, to include the case of a orbifold $M = P/H$. As mentioned in Remark 4.4 above, we have a locally free action whenever $\operatorname{rank} E^0 = \dim H$. This case is already present in [Ati74], and is further expanded upon in [Ver96]. As in the case of a free action, we can take $E \subset TP$ to be the space of horizontal vectors with respect to some choice of connection form ω on M, and the annihilator E^0 becomes the trivial bundle $E^0 = M \times \mathfrak{h}^*$. We then have the global expression

$$\mathcal{J}(E,X) = \omega_r\cdots\omega_1\delta_0(D\boldsymbol{\omega}(X)) = \frac{\omega_r\cdots\omega_1}{\operatorname{vol}(H)}\delta(X-\Omega).$$

Let $\pi : P \to M = P/H$ denote the quotient mapping. We have the Schur orthogonality formula [PV08b]

$$\delta_0(X-\Omega) = j_{\mathfrak{h}}(X)\sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(e^X)\operatorname{Tr}\tau^*(e^\Omega),$$

where $\delta_0(X-\Omega)$ denotes the form (6.6) on M corresponding to $\delta(X-\Omega)$, and the identity $\widehat{\mathrm{A}}^2(P,X) = j_{\mathfrak{h}}^{-1}(X)\pi^*\widehat{\mathrm{A}}^2(M)$, where $j_{\mathfrak{h}}(X)$ is given by (9.5). Let σ_E be defined on $E^* = \pi^*T^*M$, whence $s_*\operatorname{Ch}(\sigma_E)$ is the pull-back of a form on M.

Combining the above, when the H-action is locally free, we obtain the expansion

$$\begin{aligned}\operatorname{index}^H(\sigma)(e^X) &= \frac{1}{(2\pi i)^{\operatorname{rank}E}}\int_M \widehat{\mathrm{A}}^2(M,X)s_*\operatorname{Ch}(\sigma_E)(X)\mathcal{J}(E,X)\\ &= \sum_{\tau\in\hat{H}}\operatorname{Tr}\tau(e^X)\frac{1}{(2\pi i)^{\dim M/H}}\int_{M/H}\widehat{\mathrm{A}}^2(M/H)|S|^{-1}s_*\operatorname{Ch}(\sigma_E)\operatorname{Tr}\tau^*(e^\Psi),\end{aligned}$$

with similar formulas near other elements of H. From [Ver96], we have the formula

$$\operatorname{index}_B(\sigma_E) = \sum_{\gamma\in(C)} \int_{B(\gamma)} (2\pi i)^{-\dim B(\gamma)} |S|^{-1} \frac{\widehat{\mathrm{A}}^2(B(\gamma))}{D_\gamma(T_{B(\gamma)}B)} s_* \operatorname{Ch}_\gamma(\sigma_E)$$

for the index of σ_E on the orbifold M/H, where $B(\gamma) = M(\gamma)/H(\gamma)$, $T_{B(\gamma)}B$ is the (orbifold) normal bundle, and $C = \{h \in H | M(h) \neq \emptyset\}$, and (C) is the (finite) set of conjugacy classes in C. Now, the sum

$$\sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(e^X) \int_{B(\gamma)} (2\pi i)^{-\dim B(\gamma)} |S|^{-1} \frac{\widehat{\mathrm{A}}^2(B(\gamma))}{D_\gamma(T_{B(\gamma)}B)} s_* \operatorname{Ch}_\gamma(\sigma_E) \operatorname{Tr}\tau^*(\gamma e^\Psi)$$

vanishes unless $\gamma = 1$ (see [BV96b, Equations 44 and 45]), and thus we have

$$\operatorname{index}_M^H(\sigma)(e^X) = \sum_{\tau\in\hat{H}} \operatorname{Tr}\tau(e^X) \operatorname{index}_B(\sigma_E) \operatorname{Tr}\tau^*(e^\Psi). \tag{9.6}$$

Remark 9.4. For simplicity of notation we have considered the case $G = \{1\}$, so the formula we give here is a special case of Vergne's formula originally due to Kawasaki [Kaw79]. The case of a compact Lie group G acting on M/H can be handled similarly using [Ver96, Theorem 1] and the form $\mathcal{J}(E,(X,Y))$ of Remark 7.12.

9.5 Induced representations

We now consider the following setting: Suppose G is a compact semi-simple Lie group, and H is a closed subgroup of G. We let $M = G/H$, on which G acts transitively. Suppose $\tau : H \to \operatorname{End}(V)$ is a finite-dimensional, irreducible unitary representation of H. Denote by $\mathcal{V}_\tau = G \times_\tau V$ the corresponding vector bundle over M. One may then define the induced representation $\operatorname{ind}_H^G(\tau)$ of G on the L^2-sections of $\mathcal{V}_\tau \to M$ [Kna02]. The character of this representation is a generalized function on G. Berline and Vergne [BV92] gave a formula for this character as an equivariant index, as follows:

Since G acts transitively on M, every differential operator is transversally elliptic, including the zero operator

$$0_\tau : \Gamma_{L^2}(\mathcal{V}_\tau) \to 0.$$

Its index, in terms of the Berline-Vergne index formula [BV96a, BV96b], is given near $g \in G$ by

$$\operatorname{index}^G(0_\tau)(ge^X) = \int_{T^*M(g)} (2\pi i)^{-\dim M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)}{D_g(\mathcal{N}, X)} \operatorname{Ch}_g(\mathcal{V}_\tau, X) e^{iD\theta^g(X)}, \tag{9.7}$$

where θ is the canonical 1-form on T^*M, and θ^g denotes its restriction to $T^*M(g)$.

The main result of [BV92] is the identity

$$\chi(\operatorname{ind}_H^G(\tau))(g) = \operatorname{index}^G(0_\tau)(g). \tag{9.8}$$

Now suppose that $M = G/H$ is Hermitian; that is, we suppose that M is a complex manifold. We may equivalently write $M = G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ denotes the complexification of G, and P is a parabolic subgroup [Kna02]. (We may, for example, take H to be a maximal torus T.)

As shown by Bott [Bot57], if $\tau : H \to \operatorname{End}(V)$ is a unitary representation of H on a finite-dimensional complex vector space V, then $\mathcal{V}_\tau = G \times_\tau V$ is a holomorphic vector bundle over $M = G/H$. In this setting we may define the "holomorphic induced representation" of G on the space of holomorphic sections of $\mathcal{V}_\tau$, which we denote by hol-$\operatorname{ind}_H^G(\tau)$, following [Kna02].

The G-action on $\mathcal{V}_\tau$ induces a G-module structure on the spaces $H^q(M, \mathcal{O}(\mathcal{V}_\tau))$ of cohomology with values in the sheaf of holomorphic sections of $\mathcal{V}_\tau$. Bott showed that if τ is irreducible, then the above cohomology spaces vanish in all but one degree, and that the non-vanishing space $H^p(M, \mathcal{O}(\mathcal{V}_\tau))$ is an irreducible G-representation. The character of this representation can be computed using the equivariant Riemann-Roch theorem on the complex manifold M. If σ denotes the symbol of the Dolbeault-Dirac operator on sections of $\mathcal{S} = \bigwedge T^{0,1}M$ then the character of hol-$\operatorname{ind}_H^G(\tau)$ is the index of the operator $\not{D}_\tau$ on $\Gamma(\mathcal{S} \otimes \mathcal{V}_\tau)$ defined by (4.11), for $\mathcal{W} = \mathcal{V}_\tau$:

$$\begin{aligned}\chi(\text{hol-}\operatorname{ind}_H^G(\tau))(ge^X) &= \operatorname{index}^G(\not{D}_\tau)(ge^X)\\ &= \int_{M(g)} (2\pi i)^{-\dim M(g)/2} \frac{\operatorname{Td}(TM(g), X)}{D_g^{\mathbb{C}}(\mathcal{N}, X)} \operatorname{Ch}_g(\mathcal{V}_\tau, X).\end{aligned}$$

The two cases given above represent two extremes of transversally elliptic operators: the zero operator in the first case, and an elliptic operator in the second. We may also consider the following intermediate possibility: we suppose there exists a G-invariant complex subbundle $E \subset TM$, and a Dirac operator $\not{D}_b$ whose symbol has support $E^0 \subset T^*M$.

For example, if G/H is Hermitian, then $TM = G \times_H \mathfrak{h}^\perp$, and $\mathfrak{h}^\perp$ is a complex vector space. We choose some complex, H-invariant subspace W of $\mathfrak{h}^\perp$, and let $E = G \times_H W \subset TM$. (If H is a maximal torus, then we may take W to be a sum of root spaces.)

We are now in the setting of Section 8.3 above: $E \subset TM$ is G-invariant and equipped with a complex structure. Since G acts transitively on M, the action is automatically transverse to E. We let $\mathcal{S} = \bigwedge E^{0,1}$. Suppose $\tau : H \to \mathrm{End}(V)$ is a finite-dimensional unitary irreducible H-representation, and let $\mathcal{V}_\tau = G \times_\tau V$. If we consider the operator $\not{D}_b^{\mathcal{V}_\tau}$ on $\Gamma(\mathcal{W} \otimes \mathcal{V}_\tau)$, then Proposition 8.11 and Theorem 8.11 give

$$\mathrm{index}^G(\not{D}_b^{\mathcal{V}_\tau})(ge^X) = \int_{M(g)} (2\pi i)^{-k} \frac{\mathrm{Td}(E(g), X)}{D_g^{\mathbb{C}}(\mathcal{N}_E, X)} \frac{\widehat{\mathrm{A}}^2(E^0(g), X)}{D_g(\mathcal{N}_0, X)} \mathcal{J}(E(g), X)\,\mathrm{Ch}_g(\mathcal{V}_\tau, X), \tag{9.9}$$

where $k = \mathrm{rank}\, E/2$. By varying the rank of E (and the corresponding projection $r : T^*M \to E^*$ in the definition of $\not{D}_b^{\mathcal{V}_\tau}$) we interpolate between the following two special cases:

1. $E = 0$: This is the case of the zero operator on sections of $\mathcal{V}_\tau$. We have $E^0 = T^*M$, $\mathcal{N}_0 = \mathcal{N}$, and $\mathcal{N}_E = \{0\}$. If we let $\mathcal{J}(M, X) = (2\pi i)^{-\dim M} \pi_* e^{iD\theta(X)}$ denote the form corresponding to the zero section, then (9.9) becomes

$$\begin{aligned}
\mathrm{index}^G(0)(ge^X) &= \int_{M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)}{D_g(\mathcal{N}, X)}\,\mathrm{Ch}_g(\mathcal{V}_\tau, X)\mathcal{J}(M(g), X)\\
&= \int_{T^*M(g)} (2\pi i)^{-\dim M(g)} \frac{\widehat{\mathrm{A}}^2(M(g), X)}{D_g(\mathcal{N}, X)}\,\mathrm{Ch}_g(\mathcal{V}_\tau, X) e^{iD\theta^g(X)}\\
&= \chi(\mathrm{ind}_H^G(\tau))(ge^X),
\end{aligned}$$

by the Berline-Vergne character formula (9.8).

2. $E = TM$: This is the case of the Dolbeault-Dirac operator on sections of $\bigwedge T^{0,1}M$, twisted by the bundle $\mathcal{V}_\tau$. In this case we have $E^0 = 0$, and so $\mathcal{N}_0 = \{0\}$, $\mathcal{N}_E = \mathcal{N}$, and $\mathcal{J}(E, X) = 1$, and thus (9.9) becomes

$$\operatorname{index}^G(\not{D}_\tau)(ge^X) = \int_{M(g)} (2\pi i)^{-\dim M(g)/2} \frac{\operatorname{Td}(TM(g), X)}{D_h^{\mathbb{C}}(\mathcal{N}, X)} \operatorname{Ch}_g(\mathcal{V}_\tau, X),$$

and we recover the Riemann-Roch formula for the character of the holomorphic induced representation.

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