SET THEORY DEFINITIONS AND NOTATION

PRELIMINARIES:
The basic kinds of statements about sets are: "Element x is in set A" , "Element y is not in set B". We denote these by $x \in A$ and $y \notin B$, respectively.

Convention: Use capital letters A, B, C, etc. for set names; use small letters a b, c, x, y, etc. for element names.

DEFINITIONS:
There are two kinds of definitions here: definitions of relationships between sets and definitions which construct new sets from given ones.
Throughout, let U be the universal set or the universe of discourse, let A, and B be sets and let C_α be an indexed collection of sets.

DEFINITION 1: "A is a subset of B" is abbreviated as $A \subseteq B$.
$A \subseteq B$ iff $(\forall x)(x \in A \Rightarrow x \in B)$ iff every element of A is also in B.

DEFINITION 2: "A is equal to B" is abbreviated as $A = B$.
$A = B$ iff $(A \subseteq B)$ and $(B \subseteq A)$;
iff $(\forall x)(x \in A \Leftrightarrow x \in B)$.
iff Every element of A is in B and vice versa.

DEFINITION 3: Some Set Constructions:

$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$; Union.

$A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$; Intersection.

$A - B = \{x \in U : x \in A \text{ and } x \notin B\}$; Set difference.

$A \times B = \{(x, y) \in U : x \in A \text{ and } y \in B\}$; Cartesian product.

$P(A) = \{B : B \subseteq A\}$; Power set.

$\overline{A} = \{x \in U : x \notin A\}$; Complement (sometimes also A^C).

$\emptyset = \{ \quad \}$, the set with no elements; called the empty set.

$\bigcup_{\alpha \in A} C_\alpha = \{x \in U \mid x \in C_\alpha \text{ for some } \alpha \in A\}$; Union of a collection of sets.

$\bigcap_{\alpha \in A} C_\alpha = \{x \in U \mid x \in C_\alpha \text{ for all } \alpha \in A\}$; Intersection of a collection of sets.

DEFINITION 4:

a) Two sets A and B are said to be **disjoint** if $A \cap B = \emptyset$.
b) Let S be a collection of sets.
S is said to be **pairwise disjoint** if $\quad \forall A, B \in S,\ A \neq B; A \cap B = \emptyset$
that is, every distinct pair of elements in S are disjoint
c) Let A be any set and S be a collection of subsets of A.
S is said to be a **partition** of A if it satisfies:

1. $\forall X \in S, \quad X \neq \emptyset$
2. $\forall X, Y \in S$, either $X = Y$ or $X \cap Y = \emptyset$
3. $\bigcup_{X \in S} X = A$

STANDARD PROPERTIES FOR SET OPERATIONS:

These include De Morgan's laws, associativity, commutativity, distributive laws, etc:

$$\begin{aligned} \overline{A \cup B} &= \overline{A} \cap \overline{B}. \\ \overline{A \cap B} &= \overline{A} \cup \overline{B}. \end{aligned}$$

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C. \\ A \cap (B \cap C) &= (A \cap B) \cap C. \end{aligned}$$

$$\begin{aligned} A \cup B &= B \cup A. \\ A \cap B &= B \cap A. \end{aligned}$$

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C). \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned}$$

STRATEGIES FOR SET THEORY PROOFS:

- the set-ups can be more complicated, with more layers to deal with; but there are 4 basic types, and a framework to use for each type.

- once the type is identified and the framework set up, the actual proof can often be done symbolically.

- very often the assumptions and/or goals with be For-all statements. General rule to follow:
- - - SAVE a for-all assumption to use at the appropriate point in the proof; don't start your proof here.
- - - when the goal is a For-all-if-then or a For-all-iff, start with "Let $x \in U$" and then do a direct, symbolic proof of the if-then or iff statement.

SET THEORY PROOF TECHNIQUES AND FRAMEWORKS

TYPE I: Subset Type: $Set1 \subseteq Set2$.

Goal: To Prove $(\forall x)(x \in Set1 \Rightarrow x \in Set2)$.
This is a For-all-if-then statement!

Framework: Let $x \in U$. Then direct proof of if-then:

$$\begin{aligned} x \in Set1 &\Rightarrow \quad \text{[USE DEFINITIONS]} \\ &\Rightarrow \\ &\Rightarrow \\ &\vdots \\ &\Rightarrow x \in Set2. \end{aligned}$$

TYPE II: Set Equality Type: $Set1 = Set2$.

Goal: To prove $(\forall x)(x \in Set1 \Leftrightarrow x \in Set2)$.
This is a For-all-iff statement!

Method 1: Split the $\Leftrightarrow$ into two $\Rightarrow$s, and do two Type I proofs.

Method 2: Do the $\Leftrightarrow$ directly.
Framework: Let $x \in U$. Then direct $\Leftrightarrow$ proof:

$$\begin{aligned} x \in Set1 &\Leftrightarrow \quad \text{[USE DEFINITIONS]} \\ &\Leftrightarrow \\ &\vdots \\ &\Leftrightarrow x \in Set2. \end{aligned}$$

Warning: Make sure your implications work in both directions at each step. Sometimes you need a different explanation for each direction!

Method 3. Use previously proven set result and the = operator :

$$\begin{aligned} Set1 &= \quad \text{[USE PREVIOUSLY PROVEN RESULTS]} \\ &= \\ &\vdots \\ &= Set2. \end{aligned}$$

TYPE III: If-Then Type: If [statement1] then [statement2].

Proceed as for standard direct proof of an if-then statement.
Assume: Statement1.
Goal: Statement2.

The assumption will be a statement of Type I or II; that is, a For-all-if-then or a For-all-iff. In either case, you can't start directly from this assumption. Keep the assumption to use at the appropriate point in the proof!
The goal will be a statement of Type I or II. Set up the proof to prove this goal, with the framework for Type I or II as appropriate. At some point in the sequence of steps you will need to use the assumption.

TYPE IV: Iff Type: [Statement1] iff [Statement2].

This should NOT be done directly (in both directions at once) unless you are very good at manipulating quantifiers. Instead, break this kind of proof into two Type III if-then sentences:
If [Statement1] then [Statement2] and If [Statement2] then [Statement1].

Now do two Type III proofs.