1. Given sets A and B, define the following concepts:

a) R is a **relation** from A to B means $R \subseteq A \times B$.

b) a relation R on A is **reflexive** means that every element of A is related to itself. Symbolic form: $(\forall x \in A)(xRx)$ OR $(\forall x \in A)(x,x) \in R$.

c) a relation R on A is **symmetric** means that for all elements a, b in A, if a is related to b, then b is related to a. Symbolic form: $(\forall a, b \in A)(aRb \Rightarrow bRa)$.

d) a relation R on A is **transitive** means that for all elements a, b, c in A, if a is related to b and b is related to c, then a is related to c.
Symbolic form: $(\forall a, b, c \in A)\ (aRb\ \wedge\ bRc \Rightarrow aRc)$.

e) Let R be an equivalence relation on A. For any $a \in A$, the **equivalence class** of a denoted $[a]$ is the set of elements of A which are related to a.
Symbolic form: $[a] = \{x \in A : xRa\}$.

f) a relation f is a **function** from A to B means that f satisfies :

1. Dom (f) $= A$
2. $\forall x \in A$, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$;

2. Let A be the set $\{1, 2, 3\}$. Find an example of a relation on A which is:
a) reflexive but not symmetric and not transitive. eg $R = \{(1,1), (2,2), (3,3), (2,3), (3,1)\}$.
b) symmetric but not reflexive and not transitive. eg $R = \{(1,1), (2,3), (3,2)\}$.
c) symmetric and transitive but not reflexive. eg $R = \{(2,3), (3,2), (2,2), (3,3)\}$.

3. Let R be a relation on the set $A = \{1, 2, 3, 4, 5\}$. Describe the equivalence relation R induced by each of these partitions:
a) The partition $\Big\{\{1,2\}, \{3,4,5\}\Big\}$ gives the equivalence relation

$$\Big\{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (3,4), (4,3), (3,5), (5,3), (4,5), (5,4)\Big\}.$$

b) The partition $\Big\{\{1\}, \{2\}, \{3,4\}, \{5\}\Big\}$ gives the equivalence relation

$$\Big\{(1,1), (2,2), (3,3), (4,4), (5,5), (3,4), (4,3)\Big\}.$$

c) The partition $\Big\{\{1\}, \{2,3,4,5\}\Big\}$ gives the equivalence relation

$$\Big\{(1,1), (2,2), (3,3), (4,4), (5,5), (2,3), (3,2), (2,4), (4,2), (2,5), (5,2), (3,4), (4,3),$$
$$(3,5), (5,3), (4,5), (5,4)\Big\}.$$

d) The partition $\Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\Big\}$ gives the equivalence relation

$$\Big\{(1,1), (2,2), (3,3), (4,4), (5,5)\Big\}.$$

4. Let $A = \{1, 2, 3, 4, 5, 6\}$. Define an equivalence relation on A by:
$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$.
Describe the partition of A generated by R.
The partition is $\Big\{\{1, 5\}, \{2, 3, 6\}, \{4\}\Big\}$

5. Let P be the set of all people. Define a relation R on P by $x \ R \ y$ iff x has the same eye colour as y. Prove that R is an equivalence relation on P.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in P$, xRx:
Let $x \in P$. Then x has the same eye colour as x, so xRx holds.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in P$, if xRy then yRx.
Let $x, y \in P$. Then

$$\begin{aligned} xRy \quad &\Rightarrow x \text{ has the same eye colour as } y \\ &\Rightarrow \ y \text{ has the same eye colour as } x \\ &\Rightarrow \ yRx. \end{aligned}$$

iii) Show that R is transitive; i.e., show that $\forall x, y, z \in P$, if xRy and yRz then xRz.
Let $x, y, z \in P$. Then

$$\begin{aligned} xRy \text{ and } yRz \Rightarrow \quad & x \text{ has the same eye colour as } y \text{ and } y \text{ has the same eye colour as } z \\ \Rightarrow \quad & x \text{ has the same eye colour as } z \\ \Rightarrow \quad & xRz. \end{aligned}$$

This proves that R is an equivalence relation on P.

6. From the text.
a) Page 193 Question 8.6 Let $S = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on S. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

R is not reflexive since $b \in S$ but $(b, b) \notin R$.
R is not symmetric since $(a, b) \in R$ but $(b, a) \notin R$.
R is transitive since the only first component is a and $(a, a) \in R$

Question 8.8 Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

There are many examples. Here is one: $R = \{(a, b), (b, c)\}$.
R is not reflexive since $b \in A$ but $(b, b) \notin R$.
R is not symmetric since $(a, b) \in R$ but $(b, a) \notin R$.
R is not transitive since $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$.

Question 8.11 Let $R = \emptyset$ be the empty relation on a nonempty set A. Which of the properties reflexive, symmetric, and transitive does the relation R possess?

R is not reflexive since A is not empty so $\exists x \in A$ but $(x, x) \notin R$.
R is vacuously symmetric and transitive.

Question 8.12 Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric, but not transitive.
$\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2), (3, 4), (4, 3)\}$

(b) reflexive and transitive, but not symmetric. $\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 4), (2, 4)\}$

(c) symmetric and transitive, but not reflexive. $\emptyset$ OR $\{(1, 2), (2, 1), (1, 1)\}$

(d) reflexive, but neither symmetric nor transitive. $\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 4)\}$

(e) symmetric, but neither reflexive nor transitive. $\{(2, 3), (3, 2)\}$

(f) transitive, but neither reflexive nor symmetric. $\{(2, 3), (3, 4), (2, 4)\}$

Question 8.18 Let $A = \{1, 2, 3, 4, 5, 6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1, 4, 5\}$, $\{2, 6\}$, and $\{3\}$. What is R?

$R = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 6), (6, 2), (6, 6), (3, 3)\}$

b) Page 213 Question 9.1 Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$.
Then $f = \{(a, y), (b, z), (c, y), (d, z)\}$ is a function from A to B.
Determine $dom f$ and $range f$.

$dom f = \{a, b, c, d\}$ and $range f = \{y, z\}$.

Question 9.2 Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Give an example of a relation R from A to B containing exactly three elements such that R is *not* a function from A to B. Explain why R is not a function.

Let $R = \{(1, a), (1, b), (2, c)\}$. Then R is not a function since 1 is related to two different elements, a and b. Also $3 \in A$ but $3 \notin dom R$.

Question 9.4 For the given subsetA_i of $\boldsymbol{R}$ and the relation $R_i(1 \leq i \leq 3)$ fromA_i to $\boldsymbol{R}$, determine whether R_i is a function from A_i to $\boldsymbol{R}$.

(a) $A_1 = \boldsymbol{R}$, $R_1 = \{(x, y) : x \in A_1, y = 4x - 3\}$
R_1 is a function since $dom R_1 = A_1$ and
(x, y) and $(x, z) \in R_1 \Rightarrow y = 4x - 3$ and $z = 4x - 3 \Rightarrow y = z$

(b) $A_2 = [0, \infty)$, $R_2 = \{(x, y) : x \in A_2, (y + 2)^2 = x\}$
R_2 is not a function since $(9, 1) \in R_2$ and $(9, -5) \in R_2$

(c) $A_3 = \boldsymbol{R}$, $R_3 = \{(x, y) : x \in A_3, (x + y)^2 = 4\}$
R_3 is not a function since $(0, 2) \in R_3$ and $(0, -2) \in R_3$

7. Let R and S be relations on a set A. Prove that:
a) If R is symmetric, then $R = R^{-1}$.

PROOF: Direct proof of if-then statement.
ASSUME: R is symmetric.
GOAL: show that $R = R^{-1}$.
Let $(x, y) \in U$.

$$\begin{aligned} (x, y) \in R \quad &\Leftrightarrow \quad (y, x) \in R, \quad \text{since } R \text{ is symmetric} \\ &\Leftrightarrow \quad (x, y) \in R^{-1}. \end{aligned}$$

b) If R and S are both transitive, then $R \cap S$ is transitive.

PROOF: Direct proof of if-then statement.
ASSUME: R is transitive and S is transitive. Save for later.
GOAL: show that $R \cap S$ is transitive.
Let $x, y, z \in A$.

$$\begin{aligned} x\ (R \cap S)\ y \quad \text{and} \quad y\ (R \cap S)\ z \Rightarrow \quad &(x, y) \in R \ \text{ and } \ (x, y) \in S \ \text{ and } \ (y, z) \in R \ \text{ and } \ (y, z) \in S \\ \Rightarrow \quad &(x, y) \in R \ \text{ and } \ (y, z) \in R \ \text{ and } \ (x, y) \in S \ \text{ and } \ (y, z) \in S \\ \Rightarrow \quad &(x, z) \in R \ \text{ and } \ (x, z) \in S \\ \Rightarrow \quad &(x, z) \in R \cap S, \quad \text{since } R \text{ and } S \text{ are transitive} \\ \Rightarrow \quad &x\ (R \cap S)\ z. \end{aligned}$$

8. Let R be the relation defined on $\boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$ by $(a,b,c)\ R\ (d,e,f)$
iff $b = e$ and $c = f$.

a) Prove that R is an equivalence relation.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall (a,b,c) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$, $(a,b,c)\ R\ (a,b,c)$:
Let $(a,b,c) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$. Then $b = b$ and $c = c$, so $(a,b,c)\ R\ (a,b,c)$.

ii) Show that R is symmetric; i.e., show that $\forall (a,b,c),\ (d,e,f) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$, if $(a,b,c)\ R\ (d,e,f)$ then $(d,e,f)\ R\ (a,b,c)$.
Let $(a,b,c),\ (d,e,f) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$. Then

$$\begin{aligned}(a,b,c)\ R\ (d,e,f) &\Rightarrow\ b = e \quad \text{and} \quad c = f\\ &\Rightarrow\ e = b \quad \text{and} \quad f = c\\ &\Rightarrow\ (d,e,f)\ R\ (a,b,c).\end{aligned}$$

iii) Show that R is transitive; i.e., show that $\forall (a,b,c),(d,e,f),(p,q,r) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$, if $(a,b,c)\ R\ (d,e,f)$ and $(d,e,f)\ R\ (p,q,r)$ then $(a,b,c)\ R\ (p,q,r)$.
Let $(a,b,c),\ (d,e,f),\ (p,q,r) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$. Then

$$\begin{aligned}(a,b,c)\ R\ (d,e,f) \text{ and } (d,e,f)\ R\ (p,q,r) &\Rightarrow\ b = e \quad \text{and} \quad c = f \quad \text{and} \quad e = q \quad \text{and} \quad f = r\\ &\Rightarrow\ b = q \quad \text{and} \quad c = r\\ &\Rightarrow\ (a,b,c)\ R\ (p,q,r).\end{aligned}$$

This proves that R is an equivalence relation on $\boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$.

b) Find $[(0,0,0)]$.

A triple (x,y,z) is R-related to $(0,0,0)$ iff $y = 0$ and $z = 0$.
Therefore $[(0,0,0)] \ = \ \{(x,0,0) : x \in \boldsymbol{Z}\}$.
This could also be written as $[(0,0,0)] \ = \ \{(x,y,z) \in \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z} \ \mid \ y = z = 0\}$.

9. Let n be a fixed natural number ≥ 2. Let R_n be the relation on the set $\boldsymbol{Z}$ of integers, defined by xR_ny iff $x \equiv y (\text{mod } n)$. Prove that R_n is an equivalence relation on $\boldsymbol{Z}$.
PROOF: We must show that R_n is reflexive, symmetric and transitive.

i) Show that R_n is reflexive; i.e., show that $\forall x \in \boldsymbol{Z}$, $x\ R_n\ x$ OR $x \equiv x (\text{mod } n)$
Let $x \in \boldsymbol{Z}$. Then $x - x = 0$, and $n \mid 0$ so $n \mid (x - x)$, so $x \equiv x(\text{ mod } n)$ so $x\ R_n\ x$

ii) Show that R_n is symmetric; i.e., show that $\forall x, y \in \boldsymbol{Z}$, if xR_ny then yR_nx.
Let $x, y \in \boldsymbol{Z}$. Then

$$\begin{aligned} xR_ny &\Rightarrow x \equiv y (\text{mod } n) \\ &\Rightarrow n \mid (x - y) \\ &\Rightarrow x - y = kn, \quad k \in \boldsymbol{Z} \\ &\Rightarrow y - x = -kn, \quad \text{where } -k \in \boldsymbol{Z} \\ &\Rightarrow y \equiv x (\text{mod } n) \\ &\Rightarrow yR_nx. \end{aligned}$$

iii) Show that R_n is transitive; i.e., show that $\forall x, y, z \in \boldsymbol{Z}$, if xR_ny and yR_nz then xR_nz. Let $x, y, x \in \boldsymbol{Z}$. Then

$$\begin{aligned} xR_ny \text{ and } yR_nz &\Rightarrow x \equiv y (\text{mod } n) \text{ and } y \equiv z (\text{mod } n) \\ &\Rightarrow n \mid (x - y) \text{ and } n \mid (y - z) \\ &\Rightarrow x - y = kn, \ k \in \boldsymbol{Z} \text{ and } y - z = tn, \ t \in \boldsymbol{Z} \\ &\Rightarrow x - y + y - z = kn + tn, \quad k, t \in \boldsymbol{Z} \\ &\Rightarrow x - z = (k + t)n \quad \text{where } k + t \in \boldsymbol{Z} \\ &\Rightarrow x \equiv z (\text{mod } n) \\ &\Rightarrow xR_nz. \end{aligned}$$

This proves that R_n is an equivalence relation on $\boldsymbol{Z}$.

10. Let R_4 be the relation of equivalence modulo 4 on the set $\boldsymbol{Z}$ of integers.
a) What class is the integer 423 in?
The number 423 has a remainder of 3 after division by 4, so it is in [3].
b) Make an addition table for the set of equivalence classes of this relation under the operation of addition of classes.

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

c) Let x, y, a, b be integers. Prove that if $x \in [a]$ and $y \in [b]$, then $x + y \in [a + b]$.

PROOF:
Let x, y, a, b be any integers. Assume that $x \in [a]$ and $y \in [b]$.
GOAL: show that $x + y \in [a + b]$.

$x \in [a]$ means that $x\ R_4 a$, so $4 \mid x - a$ so $x - a = 4k$, $k \in \boldsymbol{Z}$.
$y \in [b]$ means that $y\ R_4 b$, so $4 \mid y - b$ so $y - b = 4t$, $t \in \boldsymbol{Z}$.
Adding these two together gives $(x - a) + (y - b) = 4k + 4t$.
Therefore $(x + y) - (a + b) = 4(k + t)$, with $k + t \in \boldsymbol{Z}$.
Therefore $4 \mid (x + y) - (a + b)$ and $x + y\ R_4\ a + b$.
So $x + y \in [a + b]$.

d) Let x, y, a, b be integers. Prove that if $x \in [a]$ and $y \in [b]$, then $xy \in [ab]$.

PROOF:
Let x, y, a, b be any integers. Assume that $x \in [a]$ and $y \in [b]$.
GOAL: show that $xy \in [ab]$.

$x \in [a]$ means that $x\ R_4 a$, so $4 \mid x - a$ so $x - a = 4k$ and $x = 4k + a$, $k \in \boldsymbol{Z}$.
$y \in [b]$ means that $y\ R_4 b$, so $4 \mid y - b$ so $y - b = 4t$ and $y = 4t + b$, $t \in \boldsymbol{Z}$.
Therefore $xy = (4k + a)(4t + b) = (16kt + 4kb + 4ta + ab)$
Therefore $xy - ab = 4(4kt + kb + ta)$, with $4kt + kb + ta \in \boldsymbol{Z}$.
Therefore $4 \mid xy - ab$ and $xy\ R_4\ ab$.
Therefore $xy \in [ab]$.

11. From the text.
a) Pages 193-194. Questions 8.14 Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that aRc, cRd, dRg, and bRf. If there are three distinct equivalence classes resulting from R, then determine these equivalence classes and determine all elements of R.

$$[a] = \{a, c, d, g\} \quad [b] = \{b, f\} \quad [e] = \{e\}$$

$$\begin{aligned} R = \{ & (a,a), (a,c), (a,d), (a,g), \\ & (c,a), (c,c), (c,d), (c,g), \\ & (d,a), (d,c), (d,d), (d,g), \\ & (g,a), (g,c), (g,d), (g,g), \\ & (b,b), (b,f), (f,b), (f,f), \\ & (e,e) \} \end{aligned}$$

Question 8.16

(a) Let R be the relation defined on $\boldsymbol{Z}$ by aRb if $a+b$ is even. Show that R is an equivalence relation and determine the distinct equivalence classes.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in \boldsymbol{Z}$, xRx:
Let $x \in \boldsymbol{Z}$. Then $x + x = 2x$, which is even, so xRx.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in \boldsymbol{Z}$, if xRy then yRx.
Let $x, y \in \boldsymbol{Z}$
Assume xRy that is $x+y$ is even
Now $y + x \quad = \quad x + y$. So $y+x$ is even and yRx.

iii) Show that R is transitive; i.e., show that $\forall x, y, z \in \boldsymbol{Z}$, if xRy and yRz then xRz. Let $x, y, z \in \boldsymbol{Z}$.
Assume xRy and yRz, that is that $x + y = 2s$ and $y + z = 2t$, $s, t \in \boldsymbol{Z}$.
So $x = 2s - y$ and $z = 2t - y$, $s, t \in \boldsymbol{Z}$
And $x+z = 2s-y+2t-y = 2s+2t-2y = 2(s+t-y)$ where $s+t-y \in \boldsymbol{Z}$
So $x + z$ is even. Therefore xRz

This proves that R is an equivalence relation on $\boldsymbol{Z}$.

$[0] = \{x \ : \ xR0\} = \{x \ : \ x + 0 \text{ is even}\} = \{x \ : \ x \text{ is even}\}$
$[1] = \{x \ : \ xR1\} = \{x \ : \ x + 1 \text{ is even}\} = \{x \ : \ x \text{ is odd}\}$
So these are the two distinct equivalence classes.

(b) Suppose that "even" is replaced by "odd" in (a). Which of the properties reflexive, symmetric, and transitive does R possess?

R is not reflexive since $x + x$ is even not odd. So $x \not R x$
R is symmetric since if $x + y$ is odd, so is $y + x$
R is not transitive. Consider $x = 3,\ y = 2,\ z = 1$
Then $x + y = 5$ which is odd and $y + z = 3$ which is odd but $x + z = 4$ which is even.

Question 8.22 A relation A is defined on $\boldsymbol{N}$ byaRb if $a^2 + b^2$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in \boldsymbol{Z}$, xRx:
Let $x \in \boldsymbol{Z}$. Then $x^2 + x^2 = 2x^2$, which is even, so xRx.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in \boldsymbol{Z}$, if xRy then yRx.
Let $x, y \in \boldsymbol{Z}$
Assume xRy that is $x^2 + y^2$ is even
Now $y^2 + x^2 \quad = \quad x^2 + y^2$. So $y^2 + x^2$ is even and yRx.

iii) Show that R is transitive; i.e., show that $\forall x, y, z \in \boldsymbol{Z}$, if xRy and yRz then xRz.
Let $x, y, z \in \boldsymbol{Z}$.
Assume xRy and yRz, that is that $x^2 + y^2 = 2s$ and $y^2 + z^2 = 2t$, $s, t \in \boldsymbol{Z}$.
So $x^2 = 2s - y^2$ and $z^2 = 2t - y^2$, $s, t \in \boldsymbol{Z}$
And $x^2 + z^2 = 2s - y^2 + 2t - y^2 = 2s + 2t - 2y^2 = 2(s + t - y^2)$ where $s + t - y^2 \in \boldsymbol{Z}$
So $x^2 + z^2$ is even. Therefore xRz

This proves that R is an equivalence relation on $\boldsymbol{Z}$.

$[1] = \{x : xR1\} = \{x : x^2 + 1 \text{ is even}\} = \{x : x^2 \text{ is odd}\} = \{x : x \text{ is odd}\}$
$[2] = \{x : xR2\} = \{x : x^2 + 4 \text{ is even}\} = \{x : x^2 \text{ is even}\} = \{x : x \text{ is even}\}$
So these are the two distinct equivalence classes.

Question 8.24 Let S be a nonempty subset of $\boldsymbol{Z}$, and let R be a relation defined on S by xRy if $3 \mid (x + 2y)$.

(a) Prove that R is an equivalence relation.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in S$, xRx:
Let $x \in S$. Then $x + 2x = 3x$, and $3|3x$, so xRx.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in S$, if xRy then yRx.
Let $x, y \in S$
Assume xRy that is $3 \mid (x + 2y)$ so $x + 2y = 3k$, $k \in \boldsymbol{Z}$ and $x = 3k - 2y$
Now $y + 2x = y + 2(3k - 2y) = y + 6k - 4y = 6k - 3y$
$= 3(2k - y)$ where $2k - y \in \boldsymbol{Z}$.
So $3 \mid (y + 2x)$ and yRx.

iii) Show that R is transitive; i.e., show $\forall x, y, z \in S$, if xRy and yRz then xRz.
Let $x, y, z \in S$.
Assume xRy and yRz, that is that $3 \mid (x + 2y)$ and $3 \mid (y + 2z)$.
So $x + 2y = 3s$ and $y + 2z = 3t$, $s, t \in \boldsymbol{Z}$
Which gives $x = 3s - 2y$ and $2z = 3t - y$
Now $x + 2z = 3s - 2y + 3t - y = 3s + 3t - 3y = 3(s + t - y)$ where $s + t - y \in \boldsymbol{Z}$.
So $3 \mid (x + 2z)$. Therefore xRz.

This proves that R is an equivalence relation on S.

(b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

The distinct equivalence classes on S are:
$[0] = \{-6, 0\}$, $[1] = \{-2, 1, 4, 7\}$, $[5] = \{-7, 5\}$.

Question 8.25 A relation R is defined on $\boldsymbol{Z}$ by xRy if $3x - 7y$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in \boldsymbol{Z}$, xRx:
Let $x \in \boldsymbol{Z}$. Then $3x - 7x = -4x = 2(-2x)$, which is even, so xRx.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in \boldsymbol{Z}$, if xRy then yRx.
Assume xRy that is $3x - 7y$ is even so $3x - 7y = 2k, \quad k \in \boldsymbol{Z}$
Show yRx that is that $3y - 7x$ is even.
Now $3y - 7x \;=\; 3x - 7y - 10x + 10y \;=\; 2k - 10x + 10y$ by assumption
So $3y - 7x \;=\; 2(k - 5x + 5y)$, where $k - 5x + 5y \in \boldsymbol{Z}$
So $3y - 7x$ is even and yRx.

iii) Show that R is transitive; i.e., show that $\forall x, y, z \in \boldsymbol{Z}$, if xRy and yRz then xRz.
Let $x, y, z \in \boldsymbol{Z}$.

$$\begin{aligned}
xRy \text{ and } yRz &\Rightarrow 3x - 7y \text{ is even and } 3y - 7z \text{ is even} \\
&\Rightarrow 3x - 7y = 2k \text{ for some } k \in \boldsymbol{Z} \text{ and } 3y - 7z = 2t \text{ for some } t \in \boldsymbol{Z} \\
&\Rightarrow 3x - 7y + 3y - 7z = 2k + 2t \quad \text{for some } k, t \in \boldsymbol{Z} \\
&\Rightarrow 3x - 7z = 2k + 2t + 4y \quad \text{for some } k, t \in \boldsymbol{Z} \\
&\Rightarrow 3x - 7z = 2(k + t + 2y), \quad \text{where } k + t + 2y \in \boldsymbol{Z} \\
&\Rightarrow 3x - 7z \text{ is even} \\
&\Rightarrow xRz.
\end{aligned}$$

This proves that R is an equivalence relation on $\boldsymbol{Z}$.

$x \in [0] \;\Leftrightarrow\; xR0 \;\Leftrightarrow\; 3x - 7(0)$ is even $\;\Leftrightarrow\; 3x$ is even $\;\Leftrightarrow\; x$ is even.
Therefore $[0] = \{x \in \boldsymbol{Z} \mid x \text{ is even}\}$.
$x \in [1] \;\Leftrightarrow\; xR1 \;\Leftrightarrow\; 3x - 7(1)$ is even $\;\Leftrightarrow\; 3x$ is odd $\;\Leftrightarrow\; x$ is odd.
Therefore $[1] = \{x \in \boldsymbol{Z} \mid x \text{ is odd}\}$.
So this relation has two distinct equivalence classes, one containing all the even integers and one containing all the odd integers.

Question 8.26

(a) Prove that the intersection of two equivalence relations on a nonempty set is an equivalence relation.

NOTE: We are asked to prove : if R and S are equivalence relations on a set, say T, then $R \cap S$ is an equivalence relation on T.

PROOF: Assume that R and S are equivalence relations on T.
Show that $R \cap S$ is an equivalence relation on T.

We must show that $R \cap S$ is reflexive, symmetric and transitive.

i) Reflexive. Let $x \in T$.
$\forall x \in T, \ \ (x, x) \in R$ and $(x, x) \in S$ since R and S are reflexive.
So $(x, x) \in R \cap S$.

ii) Symmetric. Let $x, y \in T$

$$\begin{aligned}(x, y) \in R \cap S &\Rightarrow (x, y) \in R \text{ and } (x, y) \in S \\ &\Rightarrow (y, x) \in R \text{ and } (y, x) \in S \text{ since } R \text{ and } S \text{ are symmetric} \\ &\Rightarrow (y, x) \in R \cap S.\end{aligned}$$

iii) Transitive. Let $x, y, z \in T$.

$$\begin{aligned}&(x, y) \in R \cap S \text{ and } (y, z) \in R \cap S \\ \Rightarrow &(x, y) \in R \text{ and } (x, y) \in S \text{ and } (y, z) \in R \text{ and } (y, z) \in S \\ \Rightarrow &(x, y) \in R \text{ and } (y, z) \in R \text{ and } (x, y) \in S \text{ and } (y, z) \in S \\ \Rightarrow &(x, z) \in R \text{ and } (x, z) \in S \text{ since } R \text{ and } S \text{ are transitive.} \\ \Rightarrow &(x, z) \in R \cap S.\end{aligned}$$

This proves that R is an equivalence relation.

(b) Consider the equivalence relations R_2 and R_3 defined on $\boldsymbol{Z}$ by:
aR_2b if $a \equiv b(\text{mod } 2)$ and aR_3b if $a \equiv b(\text{mod } 3)$. By (a), $R_1 = R_2 \cap R_3$ is an equivalence relation on $\boldsymbol{Z}$. Determine the distinct equivalence classes in R_1.

$$\begin{aligned}(x, y) \in R_1 &\Leftrightarrow (x, y) \in R_2 \cap R_3 \\ &\Leftrightarrow (x, y) \in R_2 \text{ and } (x, y) \in R_3 \\ &\Leftrightarrow x \equiv y(\text{mod } 2) \text{ and } x \equiv y(\text{mod } 3) \\ &\Leftrightarrow 2|(x - y) \text{ and } 3|(x - y) \\ &\Leftrightarrow 6|(x - y) \\ &\Leftrightarrow x \equiv y(\text{mod } 6)\end{aligned}$$

So $R_1 = \{(x, y) \mid x \equiv y(\text{mod } 6)\}$. Thus there are 6 distinct equivalence classes, $[n] = \{x \mid x \equiv n(\text{mod } 6), 0 \leq n \leq 5\}$.

Question 8.29 A relation R is defined on $\boldsymbol{Z}$ by aRb if $3a + 5b \equiv 0 (\text{mod } 8)$. Prove that R is an equivalence relation.

PROOF: We must show that R is reflexive, symmetric and transitive.

i) Show that R is reflexive; i.e., show that $\forall x \in \boldsymbol{Z}$, xRx.
Let $x \in \boldsymbol{Z}$. $3x + 5x = 8x$ and $8 \mid 8x$. So xRx.

ii) Show that R is symmetric; i.e., show that $\forall x, y \in \boldsymbol{Z}$, if xRy then yRx.
Let $x, y \in \boldsymbol{Z}$
Assume xRy that is $3x + 5y \equiv 0(\text{mod } 8)$, that is $3x + 5y = 8k, \quad k \in \boldsymbol{Z}$
Now $3y + 5x = 7(3x + 5y) - 16x - 32y = 7(8k) - 16x - 32y$
$= 8(7k - 2x - 4y)$ where $7k - 2x - 2y \in \boldsymbol{Z}$
So $8 \mid (3y + 5x)$ and yRx.

iii) Show that R is transitive; i.e., show that $\forall x, y, z \in \boldsymbol{Z}$, if xRy and yRz then xRz.
Let $x, y, z \in \boldsymbol{Z}$.

$$\begin{aligned}
xRy \text{ and } yRz &\Rightarrow 3x + 5y \equiv 0(\text{mod } 8) \text{ and } \quad 3y + 5z \equiv 0(\text{mod } 8) \\
&\Rightarrow 3x + 5y = 8s \text{ for some } s \in \boldsymbol{Z} \text{ and } 3y + 5z = 8t \text{ for some } t \in \boldsymbol{Z} \\
&\Rightarrow 3x + 5y + 3y + 5z = 8s + 8t \quad \text{for some } s, t \in \boldsymbol{Z} \\
&\Rightarrow 3x + 5z = 8s + 8t - 8y \quad \text{for some } s, t \in \boldsymbol{Z} \\
&\Rightarrow 3x + 5z = 8(s + t - y), \quad \text{where } s + t - y \in \boldsymbol{Z} \\
&\Rightarrow 3x - 7z \equiv 0(\text{mod } 8) \\
&\Rightarrow xRz.
\end{aligned}$$

This proves that R is an equivalence relation on $\boldsymbol{Z}$.

Question 8.31 A relation R on $\boldsymbol{Z}$ defined by aRb if $a^2 \equiv b^2(\text{mod } 4)$ is known to be an equivalence relation. Determine the distinct equivalence classes.

$[0] = \{x \ : \ xR0\} = \{x \ : \ 4|(x^2 - 0^2)\} = \{x \ : \ x^2 = 4k, \ k \in \boldsymbol{Z}\}$
$= \{x \ : \ x^2 \text{ is even }\} = \{x \ : \ x \text{ is even}\}$
$[1] = \{x \ : \ xR1\} = \{x \ : \ 4|(x^2 - 1^2)\} = \{x \ : \ x^2 - 1 = 4k, \ k \in \boldsymbol{Z}\}$
$= \{x \ : \ x^2 = 4k + 1\} = \{x \ : \ x^2 \text{ is odd }\} = \{x \ : \ x \text{ is odd }\}$
So these are the two distinct equivalence classes.

b) Page 214. Question 9.6 In each of the following,
a function $f_1 : A_i \to \boldsymbol{R}$ $(1 \le i \le 5)$ is defined, where the domain A_i consists of all real numbers x for which $f_i(x)$ is defined. In each case, determine the domain of A_i and the range of f_i.

(a) $f_1(x) = 1 + x^2$ *Dom* $f_1 = \boldsymbol{R}$; *Range* $f_1 = [1, \infty)$

(b) $f_2(x) = 1 - \frac{1}{x}$ *Dom* $f_2 = \boldsymbol{R} - \{0\}$; *Range* $f_2 = \boldsymbol{R} - \{1\}$

(c) $f_3(x) = \sqrt{3x - 1}$ *Dom* $f_3 = [\frac{1}{3}, \infty)$; *Range* $f_3 = [0, \infty)$

(d) $f_4(x) = x^3 - 8$ *Dom* $f_4 = \boldsymbol{R}$; *Range* $f_4 = \boldsymbol{R}$

(e) $f_5(x) = \frac{x}{x-3}$ *Dom* $f_5 = \boldsymbol{R} - \{3\}$; *Range* $f_5 = \boldsymbol{R} - \{1\}$